MCSS manuscript No.
(will be inserted by the editor)

Inner-Outer Factorization for Differential-Algebraic Systems

Timo Reis · Matthias Voigt

Received: date / Accepted: date

Abstract We consider transfer functions of linear time-invariant differential-algebraic systems. Based on the stabilizing solutions of certain differential-algebraic Lur'e equations, we will derive simple formulas for realizations of inner-outer factorizations. We show that the existence of a stabilizing solution only requires behavioral stabilizability of the system. We neither assume properness nor (proper) invertibility of the transfer function. We briefly discuss numerical aspects for the determination of such factorizations.

Keywords differential-algebraic systems, inner-outer factorizations, Lur'e equations, Riccati equations

1 Introduction

We consider differential-algebraic systems

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1b}$$

T. Reis
Universität Hamburg, Fachbereich Mathematik, Bundesstraße 55 (Geomatikum), 20146 Hamburg, Germany
Tel.: +49-40-42838-5111
Fax: +49-40-42838-5117
E-mail: timo.reis@math.uni-hamburg.de

M. Voigt
Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany
Tel.: +49-30-314-24767
Fax: +49-30-314-79706
E-mail: mvoigt@math.tu-berlin.de

This work was supported by the *International Max Planck Research School for Advanced Methods in Process and Systems Engineering* and in the framework of the Einstein Center for Mathematics (ECMath) supported by the *Einstein Foundation Berlin*.

where $E,\ A \in \mathbb{K}^{n\times n}$ (for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we refer to p. 3 for the notation) are such that the pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is *regular*, i. e., $\det(sE - A)$ is not the zero polynomial, and $B \in \mathbb{K}^{n\times m}$, $C \in \mathbb{K}^{p\times n}$, $D \in \mathbb{K}^{p\times m}$. The functions $x : \mathbb{R} \to \mathbb{K}^n$, $u : \mathbb{R} \to \mathbb{K}^m$, and $y : \mathbb{R} \to \mathbb{K}^p$ are called *(generalized) state*, *input*, and *output* of the system, respectively. The *transfer function* of (1) is

$$G(s) = C(sE - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}. \tag{2}$$

Conversely, we call (1) a *realization of* $G(s) \in \mathbb{K}(s)^{p\times m}$, if (2) holds true.

In this paper we discuss the construction of *inner-outer factorizations* of $G(s)$, that is

$$\boxed{G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s),}$$

where the rational matrix $G_{\mathrm{i}}(s) \in \mathbb{K}(s)^{p\times q}$ is inner and $G_{\mathrm{o}}(s) \in \mathbb{K}(s)^{q\times m}$ is outer (see Def. 5).

The crucial role of inner-outer factorizations was recognized in the early days of $\mathcal{H}_\infty$-controller design; see the pioneering textbook by Bruce A. Francis [5]. This has led to various publications on inner-outer factorization for systems governed by ordinary differential equations (i. e., for $E = I_n$) [6,3,28,8,23,26,27,7].

The simplest situation is when the transfer function $G(s)$ of a system with $E = I_n$ has no zeros (see Def. 4) on the imaginary axis and D has full column rank; in this case, rather simple realizations of $G_{\mathrm{i}}(s)$ and $G_{\mathrm{o}}(s)$ can be constructed by using the stabilizing solution of a certain algebraic Riccati equation (see also Remark 2). If the latter conditions are not fulfilled, the situation becomes more involved. Inner-outer factorization for right-invertible transfer functions is considered in [3,28]; the articles [8,7] treat the case of strictly proper transfer functions, where the first one determines a descriptor system realization of the inner factor using a generalized algebraic Riccati equation. Another class of methods uses zero dislocation techniques [12,4,23]. There a product of inner factors is determined such that all zeros of $G(s)$ in $\mathbb{C}_+$ are successively incorporated into $G_{\mathrm{i}}(s)$ and reflected in $G_{\mathrm{o}}(s)$.

The general factorization problem for transfer functions governed by differential-algebraic systems has been examined in the works by Oară and Varga [24,14], where the factorization problem is considered from the numerical point of view. Several successive orthogonal transformations and reductions are carried out to reduce the problem to a factorization problem for the transfer function of a system of the form (1) where E is nonsingular. This allows to apply the above mentioned methods for the standard case. However, no closed formulas for the realizations of the factors in terms of the given descriptor system realization are given.

The novelty of our approach is that we exploit recent results on Lur'e equations for differential-algebraic systems [20] to derive such simple formulas for realizations of the inner and outer factors. This yields also new results for systems described by ordinary differential equations; this is possible with the larger framework of DAEs. The only assumption on the realization of the transfer function to be factored will be behavioral stabilizability. Stability, properness, or proper invertibility are not required. Note that parts of this work have already been discussed in the thesis of the second author [25] under the stronger assumption of strong stabilizability.

Notation

We use the notation $\mathbb{N}_0$ for the set of natural numbers including zero. The symbol $\mathbb{K}$ stands for either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers. The closure of $S \subset \mathbb{C}$ is denoted by $\overline{S}$. The set of complex numbers with positive real part is denoted by $\mathbb{C}_+$. The symbols $\imath$, $\overline{\lambda}$, A^*,,I_n, $0_{m\times n}$ respectively stand for the imaginary unit, the complex conjugate of $\lambda \in \mathbb{C}$, the conjugate transpose of a complex matrix, the identity matrix of size $n \times n$ and the zero matrix of size $m \times n$ (subscripts are omitted, if clear from context). Further, $\|x\|$ is the Euclidean norm of $x \in \mathbb{K}^n$ and $\|A\|$ denotes the maximum singular value of $A \in \mathbb{K}^{m\times n}$.

By writing $A \geq B$ we mean that for two Hermitian matrices $A, B \in \mathbb{K}^{n\times n}$, the matrix $A-B$ is positive semidefinite. The following concept, namely equality and semidefiniteness on some subspace will be frequently used.

Definition 1 (Equality and semidefiniteness on a subspace) Let $\mathcal{V} \subseteq \mathbb{K}^n$ be a subspace and $A, B \in \mathbb{K}^{n\times n}$ be Hermitian. Then we write

$$A =_{\mathcal{V}} B, \quad \text{if } v^*(A-B)v = 0 \text{ for all } v \in \mathcal{V}, \text{ and}$$
$$A \geq_{\mathcal{V}} B, \quad \text{if } v^*(A-B)v \geq 0 \text{ for all } v \in \mathcal{V}.$$

The following sets are further used in this article:

$\mathbb{K}[s]$, $\mathbb{K}(s)$	the ring of polynomials and, resp., the field of rational functions over the field $\mathbb{K}$,
$\Lambda(E, A)$	set of zeros of $\det(sE - A)$ for a pencil $sE - A \in \mathbb{K}[s]^{n\times n}$,
$\mathrm{Gl}_n(\mathbb{K}(s))$	the group of invertible $n\times n$ matrices with entries in $\mathbb{K}(s)$,
$\mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n)$	the set of locally square Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{K}^n$.

Further, for $G(s) \in \mathbb{K}(s)^{p\times m}$, we respectively use $\operatorname{rank}_{\mathbb{K}(s)} G(s)$, $\operatorname{im}_{\mathbb{K}(s)} G(s)$ and $\ker_{\mathbb{K}(s)} G(s)$ for the rank, image and kernel of the linear mapping $\mathbb{K}(s)^m \to \mathbb{K}(s)^p$, $v(s) \mapsto G(s)v(s)$. Note that $\operatorname{rank}_{\mathbb{K}(s)} G(s)$ coincides with the generic rank of $G(s)$.

2 Preliminaries

2.1 Basic systems theoretic concepts

We denote by $\Sigma_{n,m,p}(\mathbb{K})$ the set of systems (1) with $E, A \in \mathbb{K}^{n\times n}$ such that the pencil $sE-A \in \mathbb{K}[s]^{n\times n}$ is regular and $B \in \mathbb{K}^{n\times m}$, $C \in \mathbb{K}^{p\times n}$, $D \in \mathbb{K}^{p\times m}$, and we write $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$. The set of control systems (1a) with E, A and B as above is denoted by $\Sigma_{n,m}(\mathbb{K})$, and we write $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$.

The *behavior* of $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$ is the set of all solutions of (1a), i.e.

$$\mathfrak{B}_{[E,A,B]} := \left\{(x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^m) : \tfrac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu\right\},$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the distributional derivative. Note that $(x,u) \in \mathfrak{B}_{[E,A,B]}$ implies that Ex is absolutely continuous, hence the evaluation $Ex(0) := (Ex)(0)$ is well-defined. The behavior of $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ is defined by

$$\mathfrak{B}_{[E,A,B,C,D]} := \left\{(x,u,y) \in \mathfrak{B}_{[E,A,B]} \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^p) : y = Cx + Du\right\}.$$

Next we consider the notions of behavioral stabilizability and controllability which have been introduced for a larger class of systems in [15].

Definition 2 (Behavioral stabilizability/controllability) The system $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$ is called

(i) *behaviorally stabilizable*, if for all $(x_1,u_1) \in \mathfrak{B}_{[E,A,B]}$, there exists some $(x,u) \in \mathfrak{B}_{[E,A,B]}$ with

$$(x(t),u(t)) = (x_1(t),u_1(t)) \ \text{ for } t < 0 \text{ and } \lim_{t\to\infty}(x(t),u(t)) = 0;$$

(ii) *behaviorally controllable*, if for all $(x_1,u_1),(x_2,u_2) \in \mathfrak{B}_{[E,A,B]}$, there exists some $(x,u) \in \mathfrak{B}_{[E,A,B]}$ and some $T > 0$ with

$$(x(t),u(t)) = \begin{cases} (x_1(t),u_1(t)), & \text{for } t < 0,\\ (x_2(t),u_2(t)), & \text{for } t > T.\end{cases}$$

Behavioral stabilizability and controllability have simple algebraic characterizations [1, Cor. 4.3], see also [15, Thm. 5.2.30].

Proposition 1 (Algebraic conditions) *The system* $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$ *is*

(i) behaviorally stabilizable $\iff$ $\forall\,\lambda \in \overline{\mathbb{C}_+}: \ \operatorname{rank}\begin{bmatrix}\lambda E - A & B\end{bmatrix} = n;$

(ii) behaviorally controllable $\iff$ $\forall\,\lambda \in \mathbb{C}: \ \operatorname{rank}\begin{bmatrix}\lambda E - A & B\end{bmatrix} = n.$

Next we introduce two fundamental spaces of a system $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$, namely the system space and the space of consistent initial differential variables.

Definition 3 (System space, space of consistent initial differential variables) Let $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$ be given.

(i) The *system space* of $[E,A,B]$ is the smallest subspace $\mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \subseteq \mathbb{K}^{n+m}$ such that for all $(x,u) \in \mathfrak{B}_{[E,A,B]}$ we have

$$\begin{pmatrix} x(t)\\ u(t)\end{pmatrix} \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \quad \text{for almost all } t \in \mathbb{R}.$$

(ii) The *space of consistent initial differential variables* of $[E,A,B]$ is

$$\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} := \left\{x_0 \in \mathbb{K}^n : \exists (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0\right\}.$$

For a geometric characterization of $\mathcal{V}^{\text{sys}}_{[E,A,B]}$ and $\mathcal{V}^{\text{diff}}_{[E,A,B]}$, we refer to [20, Prop. 2.9 & Prop. 3.3].

Next we introduce some facts and properties of rational matrices. Many properties will be analyzed by means of the Smith-McMillan form; it is a canonical form on $\mathbb{K}(s)^{p\times m}$ under the group action of multiplication from the left and right with unimodular matrices (i.e., units of the ring of square polynomial matrices).

Theorem 1 (Smith-McMillan form [11, Sec. 6.5.2]) *Let a rational matrix $G(s) \in \mathbb{K}(s)^{p\times m}$ be given and let $q = \operatorname{rank}_{\mathbb{K}(s)} G(s)$. Then there exist unimodular $U(s) \in \mathbb{K}[s]^{p\times p}$ and $V(s) \in \mathbb{K}[s]^{m\times m}$, such that*

$$U^{-1}(s)G(s)V^{-1}(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$D(s) = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \dots, \frac{\varepsilon_q(s)}{\psi_q(s)}\right)$$

with unique monic and coprime polynomials $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]\backslash\{0\}$ such that $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$ and $\psi_{i+1}(s) \mid \psi_i(s)$ for all $i \in \{1, \dots, q-1\}$.

Theorem 1 gives rise to the following (standard) definitions.

Definition 4 (Poles and zeros [29]) Let $G(s) \in \mathbb{K}(s)^{p\times m}$ be given with $\operatorname{rank}_{\mathbb{K}(s)} G(s) = q$. Using the notation of Theorem 1, $\lambda \in \mathbb{C}$ is called

(i) a *zero* of $G(s)$, if $\varepsilon_r(\lambda) = 0$;
(ii) a *pole* of $G(s)$, if $\psi_1(\lambda) = 0$.

Next we introduce the main concepts for this work.

Definition 5 (Outer and inner rational functions) A rational function $G(s) \in \mathbb{K}(s)^{p\times m}$ is called

(i) *outer*, if $p = \operatorname{rank}_{\mathbb{K}(s)} G(s)$ and $G(s)$ has no zeros in $\mathbb{C}_+$;
(ii) *inner*, if $G(s)$ has no poles in $\mathbb{C}_+$ and $G^*(-\overline{s})G(s) = I_m$.

Remark 1 (Inner and outer functions)

(i) If $G(s) \in \mathbb{K}(s)^{p\times m}$ is inner, then $G(s)$ is bounded in $\mathbb{C}_+$. Inner functions fulfill $G^*(\imath\omega)G(\imath\omega) = I_m$ for all $\omega \in \mathbb{R}$ with $\imath\omega \notin \Lambda(E, A)$. This means that for a realization $[E, A, B, C, D]$ of $G(s)$, all frequencies pass equally in gain. For this reason, realizations of inner functions are also called *all-pass filters* [29].
(ii) The transfer function $G(s) \in \mathbb{K}(s)^{p\times m}$ of $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ is outer, if

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ C & D \end{bmatrix} = n + p \quad \forall\, \lambda \in \mathbb{C}_+.$$

Further properties of realizations of outer transfer functions have been considered in [9].

The following lemma about realizations of certain fractions of transfer functions will be essential for the construction of the factorizations considered in this article.

Lemma 1 *[21, Lem. 3.5] Consider the systems* $[E,A,B,C_1,D_1] \in \Sigma_{n,m,m}(\mathbb{K})$ *and* $[E,A,B,C_2,D_2] \in \Sigma_{n,m,p}(\mathbb{K})$ *with transfer functions*

$$\begin{aligned} G_1(s) &= C_1(sE-A)^{-1}B + D_1 \in \mathrm{Gl}_m(\mathbb{K}(s)), \\ G_2(s) &= C_2(sE-A)^{-1}B + D_2 \in \mathbb{K}(s)^{p\times m}. \end{aligned}$$

Then the pencil $\left[\begin{smallmatrix} sE-A & -B \\ -C_1 & -D_1 \end{smallmatrix}\right]$ *is regular. Moreover, the transfer function of*

$$\begin{aligned} &[E_\mathrm{e}, A_\mathrm{e}, B_\mathrm{e}, C_\mathrm{e}, D_\mathrm{e}] \\ &\quad := \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} 0 \\ -I_m \end{bmatrix}, \begin{bmatrix} C_2 & D_2 \end{bmatrix}, 0_{p\times m}\right] \in \Sigma_{n+m,m,p}(\mathbb{K}) \end{aligned}$$

is $G_\mathrm{e}(s) = G_2(s)G_1^{-1}(s)$.

2.2 Lur'e equations

The key ingredient for our inner-outer factorizations are solutions of Lur'e equations for differential-algebraic systems which have been developed in [20].

Definition 6 (Lur'e equation) For $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$ and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$, the *Lur'e equation* is given by

$$\begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*. \tag{3}$$

A triple $(X,K,L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ for some $q \in \mathbb{N}_0$ is called *solution* of (3), if (3) holds with

$$\operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} -sE+A & B \\ K & L \end{bmatrix} = n+q.$$

Further, a solution (X,K,L) of the Lur'e equation (3) is called

(i) *stabilizing*, if

$$\operatorname{rank} \begin{bmatrix} -\lambda E+A & B \\ K & L \end{bmatrix} = n+q \quad \forall\, \lambda \in \mathbb{C}_+;$$

(ii) *nonnegative*, if $E^*XE \geq_{\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} 0$.

Note that (3) is a generalization of the standard Lur'e equation for ordinary differential equations [16] which is, on the other hand, a generalization of the famous algebraic Riccati equation. The same is true for the concept of a stabilizing solution.

The following theorem gives a sufficient condition for the existence of a stabilizing solution and summarizes some further implications.

Theorem 2 *[20, Thm. 5.3(b), Thm. 5.5(a), Rem. 5.7] Consider a behaviorally stabilizable control system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. Further, let the matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$ be given. Assume that there exists some $P \in \mathbb{K}^{n\times n}$ that satisfies the* Kalman-Yakubovich-Popov (KYP) inequality

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \geq_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} 0, \quad P = P^*. \tag{4}$$

Then the Lur'e equation (3) has a stabilizing solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$. This solution has the following properties:

(i) It is maximal in the sense that it holds that

$$E^*XE \geq_{\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} E^*PE$$

for all $P \in \mathbb{K}^{n\times n}$ fulfilling the KYP inequality (4).

(ii) It realizes a spectral factorization *of the* Popov function

$$\Phi(s) := \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{K}(s)^{m\times m}, \tag{5}$$

in the sense that $\Phi(s) = W^*(-\overline{s})W(s)$ for the outer function

$$W(s) = K(sE - A)^{-1}B + L \in \mathbb{K}(s)^{q\times m}.$$

(iii) The number q with $K \in \mathbb{K}^{q\times n}$, $L \in \mathbb{K}^{q\times m}$ and the Popov function are related by $q = \operatorname{rank}_{\mathbb{K}(s)} \Phi(s)$.

In the following we show that Lur'e equations can be further used to characterize when a rational function is inner.

Theorem 3 *Let $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ with transfer function $G(s) \in \mathbb{K}(s)^{p\times m}$ be given. Then the following statements are satisfied:*

(i) If there exists some Hermitian matrix $P \in \mathbb{K}^{n\times n}$ with

$$\begin{bmatrix} A^*PE + E^*PA + C^*C & E^*PB + C^*D \\ B^*PE + D^*C & D^*D - I_m \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} 0,$$
$$E^*PE \geq_{\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} 0, \tag{6}$$

then $G(s)$ is inner.

(ii) Conversely, if $[E, A, B]$ is behaviorally controllable and $G(s)$ is inner, then there exists some Hermitian matrix $P \in \mathbb{K}^{n\times n}$ which fulfills (6).

Proof (i) We obtain from the differential-algebraic bounded real lemma [21, Thm. 4.4(a)] that

$$I_m - G^*(\lambda)G(\lambda) \geq 0 \quad \forall\, \lambda \in \mathbb{C}_+ \setminus \Lambda(E, A).$$

This implies that $G(s)$ has no poles in $\mathbb{C}_+$. By [20, Lem. 3.5] we have

$$\operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1} B \\ I_m \end{bmatrix} \subseteq \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \quad \forall \lambda \in \mathbb{C} \setminus \Lambda(E,A),$$

and [20, Eq. (4.12)] yields

$$\begin{bmatrix} (-\overline{s}E - A)^{-1} B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*PE + E^*PA & E^*PB \\ B^*PE & 0 \end{bmatrix} \begin{bmatrix} (sE - A)^{-1} B \\ I_m \end{bmatrix} = 0.$$

This results in

$$\begin{aligned} 0 = & \begin{bmatrix} (-\overline{s}E - A)^{-1} B \\ I_m \end{bmatrix}^* \\ & \cdot \begin{bmatrix} A^*PE + E^*PA + C^*C & E^*PB + C^*D \\ B^*PE + D^*C & D^*D - I_m \end{bmatrix} \begin{bmatrix} (sE - A)^{-1} B \\ I_m \end{bmatrix} \\ = & \begin{bmatrix} (-\overline{s}E - A)^{-1} B \\ I_m \end{bmatrix}^* \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - I_m \end{bmatrix} \begin{bmatrix} (sE - A)^{-1} B \\ I_m \end{bmatrix} \\ = & G^*(-\overline{s})G(s) - I_m, \end{aligned}$$

which shows that $G(s)$ is inner.

(ii) Since $G(s)$ is inner, we have for all $\omega \in \mathbb{R}$ with $\imath\omega \notin \Lambda(E,A)$ that

$$\|G(\imath\omega)\|^2 = \|G^*(\imath\omega)G(\imath\omega)\| = \|G^*(\imath\omega)G(-\overline{\imath\omega})\| = \|I_m\| = 1$$

Since $G(s)$ has no poles in $\mathbb{C}_+$, we can infer from the maximum principle for holomorphic functions [13, Ch. III, §1, Cor. 1.4] that $\|G(\lambda)\| \leq 1$ for all $\lambda \in \mathbb{C}_+$. Thus, for all $v \in \mathbb{C}^m$, $\lambda \in \mathbb{C}_+$ we have

$$\begin{aligned} v^*\big(I_m - G^*(\lambda)G(\lambda)\big)v &= \|v\|^2 - \|G(\lambda)v\|^2 \\ &\geq \|v\|^2 - \|G(\lambda)\|^2 \cdot \|v\|^2 \geq \|v\|^2 - \|v\|^2 = 0, \end{aligned}$$

whence $I_m - G^*(\lambda)G(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$. Consequently, $G(s)$ is bounded real in the sense of [21, Def. 4.1(b)]. From behavioral controllability of $[E,A,B]$, it follows from [21, Thm. 4.5(b)] (by replacing E by $-E$ and reversing the sign of the whole inequality) that there exists some Hermitian matrix $\widetilde{P} \in \mathbb{K}^{n \times n}$ with $E^*\widetilde{P}E \geq_{\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} 0$ such that

$$\begin{bmatrix} A^*\widetilde{P}(-E) + (-E)^*\widetilde{P}A - C^*C & (-E)^*\widetilde{P}B - C^*D \\ B^*\widetilde{P}(-E) - D^*C & I_m - D^*D \end{bmatrix} \geq_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} 0.$$

Since $[E,A,B]$ is behaviorally controllable, we can conclude from Proposition 1 that $[-E,A,B]$ is behaviorally stabilizable. Now using Theorem 2 it follows that the Lur'e equation

$$\begin{aligned} &\begin{bmatrix} A^*P(-E) + (-E)^*PA - C^*C & (-E)^*PB - C^*D \\ B^*P(-E) - D^*C & I_m - D^*D \end{bmatrix} \\ &\qquad\qquad = _{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} \begin{bmatrix} M^* \\ N^* \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix} \end{aligned} \tag{7}$$

has a stabilizing solution $(P, M, N) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ with q being the rank of the associated Popov function

$$\begin{aligned}\Phi(s) := &\begin{bmatrix}(-\overline{s}E - A)^{-1}B \\ I_m\end{bmatrix}^* \begin{bmatrix}-C^*C & -C^*D \\ -D^*C & I_m - D^*D\end{bmatrix} \begin{bmatrix}(sE - A)^{-1}B \\ I_m\end{bmatrix} \\ &= I_m - \big(C(-\overline{s}E - A)^{-1}B + D\big)^* \cdot \big(C(sE - A)^{-1}B + D\big) \\ &= I_m - G^*(-\overline{s})G(s).\end{aligned}$$

Since $G(s)$ is inner, we have for all $\omega \in \mathbb{R}$ that

$$\Phi(\imath\omega) = I_m - G^*(-\overline{\imath\omega})G(\imath\omega) = I_m - G^*(\imath\omega)G(\imath\omega) = 0.$$

Hence $q = \operatorname{rank}_{\mathbb{K}(s)} \Phi(s) = 0$, which implies that the right hand side of (7) vanishes. Invoking this, a multiplication of (7) with -1 gives the first equation in (6). By further using Theorem 2 (i), we obtain

$$E^*PE \geq_{\mathcal{V}^{\text{diff}}_{[E,A,B]}} E^*\widetilde{P}E \geq_{\mathcal{V}^{\text{diff}}_{[E,A,B]}} 0,$$

which completes the proof.

3 Construction of inner-outer factorizations

We construct inner-outer factorizations of arbitrary rational matrices. The basis for such a construction will be the Lur'e equation

$$\begin{bmatrix}A^*XE + E^*XA + C^*C & E^*XB + C^*D \\ B^*XE + D^*C & D^*D\end{bmatrix} =_{\mathcal{V}^{\text{sys}}_{[E,A,B]}} \begin{bmatrix}K^* \\ L^*\end{bmatrix} \begin{bmatrix}K & L\end{bmatrix}, \qquad X = X^*. \tag{8}$$

First we present the general idea for our approach: The Popov function corresponding to the Lur'e equation (8) is

$$\Phi(s) = \begin{bmatrix}(-\overline{s}E - A)^{-1}B \\ I_m\end{bmatrix}^* \begin{bmatrix}C^*C & C^*D \\ D^*C & D^*D\end{bmatrix} \begin{bmatrix}(sE - A)^{-1}B \\ I_m\end{bmatrix} = G^*(-\overline{s})G(s). \tag{9}$$

For a stabilizing solution (X, K, L) of (8) we obtain from Theorem 2 that $\Phi(s) = W^*(-\overline{s})W(s)$ for the outer function $W(s) = K(sE - A)^{-1}B + L$. Assume for convenience that an inner-outer factorization $G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s)$ with $G_{\mathrm{i}}(s) \in \mathbb{K}(s)^{p\times q}$ and $G_{\mathrm{o}}(s) \in \mathbb{K}(s)^{q\times m}$ exists. Then (9) and the property $G_{\mathrm{i}}^*(-\overline{s})G_{\mathrm{i}}(s) = I_q$ imply that

$$G^*(-\overline{s})G(s) = G_{\mathrm{o}}^*(-\overline{s}) \underbrace{G_{\mathrm{i}}^*(-\overline{s})G_{\mathrm{i}}(s)}_{=I_q} G_{\mathrm{o}}(s) = G_{\mathrm{o}}^*(-\overline{s})G_{\mathrm{o}}(s).$$

This justifies the ansatz $G_{\mathrm{o}}(s) = W(s) = K(sE - A)^{-1}B + L$. The inner factor will be constructed by $G_{\mathrm{i}}(s) = G(s)G_{\mathrm{o}}(s)^-$, where $G_{\mathrm{o}}(s)^-$ denotes a right inverse of $G_\mathrm{o}(s)$. Thereby, we will construct a right inverse of $G_\mathrm{o}(s)$ by $Z(G_\mathrm{o}(s)Z)^{-1}$, where $Z \in \mathbb{R}^{m\times q}$ is a matrix such that $G_\mathrm{o}(s)Z$ is invertible. The realization of $G_\mathrm{i}(s) = G(s)G_\mathrm{o}(s)^{-} = G(s)Z(G_\mathrm{o}(s)Z)^{-1}$ will be constructed by using Lemma 1.

We will show in Theorem 4 that the above outlined idea can indeed be used to construct inner-outer factorizations. Note that our construction will be purely based on a realization $[E, A, B, C, D]$ of $G(s)$; no inversions of transfer functions will be involved. As the above idea illustrates, the key ingredients will be the stabilizing solution (X, K, L) of the Lur'e equation (8) and a matrix $Z \in \mathbb{R}^{m\times q}$ such that $G_\mathrm{o}(s)Z$ is invertible. Before we present our main result on the construction of inner-outer factorizations, we first show that these key ingredients exist.

Proposition 2 *Let* $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ *be behaviorally stabilizable. Then the Lur'e equation* (8) *has a stabilizing solution* (X, K, L)*; this solution is nonnegative (in the sense of Definition 6(ii)).*

Proof Since $P = 0$ solves the KYP inequality associated to (8), the result follows immediately from Theorem 2.

Proposition 3 *Let a system* $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ *with transfer function* $G(s) \in \mathbb{K}(s)^{p\times m}$ *be given. Denote* $q = \operatorname{rank}_{\mathbb{K}(s)} G(s)$. *Then there exists some matrix* $Z \in \mathbb{K}^{m\times q}$ *such that* $\operatorname{rank}_{\mathbb{K}(s)} G(s)Z = q$.
If $q = p$ *and* Z *has the above property, then the following statements are satisfied:*

(i) The pencil $\left[\begin{smallmatrix} -sE+A & BZ \\ C & DZ \end{smallmatrix}\right] \in \mathbb{K}[s]^{(n+p)\times(n+p)}$ *is regular.*
(ii) The rational function $\mathcal{P}(s) = Z(G(s)Z)^{-1}G(s) \in \mathbb{K}(s)^{p\times p}$ *is a projector with* $\ker_{\mathbb{K}(s)} G(s) = \ker_{\mathbb{K}(s)} \mathcal{P}(s)$.

Proof Denote the k-th canonical unit vector by $e_k \in \mathbb{R}^m$. Since the column vectors of $G(s)$ can be reduced to a basis of $\operatorname{im}_{\mathbb{K}(s)} G(s)$ and $\dim \operatorname{im}_{\mathbb{K}(s)} G(s) = q$, there exist $i_1, \ldots, i_q \in \{1, \ldots, m\}$ such that $\left\{G(s)e_{i_1}, \ldots, G(s)e_{i_q}\right\}$ is a basis of $\operatorname{im}_{\mathbb{K}(s)} G(s)$. Then the matrix $Z = [e_{i_1}, \ldots, e_{i_q}]$ has the desired property.

Consequently, if $q = p$ then $G(s)Z \in \mathrm{Gl}_p(\mathbb{K}(s))$. Statement (i) can now be concluded from

$$\begin{aligned}\operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} -sE + A & BZ \\ C & DZ \end{bmatrix} &= \operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} I_n & 0 \\ C(sE - A)^{-1} & I_q \end{bmatrix} \begin{bmatrix} -sE + A & BZ \\ C & DZ \end{bmatrix} \\ &= \operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} -sE + A & BZ \\ 0 & G(s)Z \end{bmatrix} = n + q.\end{aligned}$$

Statement (ii) follows by simple calculations.

Now we formulate our main result on the construction of inner-outer factorizations.

Theorem 4 *Let* $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ *be behaviorally stabilizable with transfer function* $G(s) \in \mathbb{K}(s)^{p\times m}$. *Let* $q = \operatorname{rank}_{\mathbb{K}(s)} G(s)$ *and* $Z \in \mathbb{K}^{m\times q}$ *be a matrix with* $\operatorname{rank}_{\mathbb{K}(s)} G(s)Z = q$ *(which exists by Proposition 3). Let* (X, K, L) *be a stabilizing solution of the Lur'e equation* (8) *(which exists by Proposition 2). Then an inner-outer factorization is* $G(s) = G_{\rm i}(s)G_{\rm o}(s)$, *where* $G_{\rm i}(s) \in \mathbb{K}(s)^{p\times q}$ *is the transfer function of*

$$[E_{\rm i}, A_{\rm i}, B_{\rm i}, C_{\rm i}, D_{\rm i}] := \left[\begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix} A & BZ\\ K & LZ\end{bmatrix}, \begin{bmatrix} 0\\ -I_q\end{bmatrix}, \begin{bmatrix} C & DZ\end{bmatrix}, 0_{p\times q}\right] \in \Sigma_{n+q,q,p}(\mathbb{K}), \quad (10)$$

and $G_{\rm o}(s) \in \mathbb{K}(s)^{q\times m}$ *is the transfer function of*

$$[E_{\rm o}, A_{\rm o}, B_{\rm o}, C_{\rm o}, D_{\rm o}] := [E, A, B, K, L] \in \Sigma_{n,m,q}(\mathbb{K}). \quad (11)$$

Proof We proceed in several steps.
Step 1: $G_{\rm o}(s)$ *is outer:*
This follows from Theorem 2 (ii).
Step 2: $\ker_{\mathbb{K}(s)} G(s) = \ker_{\mathbb{K}(s)} G_{\rm o}(s)$*:*
Theorem 2 (ii) together with (9) yields

$$G_{\rm o}^*(\imath\omega)G_{\rm o}(\imath\omega) = G^*(\imath\omega)G(\imath\omega) \quad \forall \omega \in \mathbb{R} \text{ with } \imath\omega \notin \Lambda(E, A). \quad (12)$$

First we show that $\ker_{\mathbb{K}(s)} G(s) \subseteq \ker_{\mathbb{K}(s)} G_{\rm o}(s)$: Assume that $v(s) \in \ker_{\mathbb{K}(s)} G(s)$. Let $\Gamma \subset \mathbb{C}$ be the (finite) set of poles of $v(s) \in \mathbb{K}(s)^m$. Then, we obtain from (12) that for all $\omega \in \mathbb{R}$ with $\imath\omega \notin \Gamma \cup \Lambda(E, A)$ we have

$$\|G_{\rm o}(\imath\omega)v(\imath\omega)\|^2 = \|G(\imath\omega)v(\imath\omega)\|^2 = 0.$$

Hence, $\lambda \mapsto G_{\rm o}(\lambda)v(\lambda)$ is a vector-valued rational function which vanishes on the infinite set $\imath\mathbb{R} \setminus (\Gamma \cup \Lambda(E, A))$. This gives $G_{\rm o}(s)v(s) = 0$, i.e., $v(s) \in \ker_{\mathbb{K}(s)} G_{\rm o}(s)$. The proof of the reverse inclusion $\ker_{\mathbb{K}(s)} G_{\rm o}(s) \subseteq \ker_{\mathbb{K}(s)} G(s)$ can be done by simply interchanging $G_{\rm o}(s)$ and $G(s)$ in the above argumentation; it is therefore omitted.
Step 3: $G_{\rm o}(s)Z$ *is invertible:*
We obtain from Step 2 and the outerness of $G_{\rm o}(s)$ that

$$\operatorname{rank}_{\mathbb{K}(s)} G(s) = \operatorname{rank}_{\mathbb{K}(s)} G_{\rm o}(s) = q.$$

The outerness of $G_{\rm o}(s)$ further implies $G_{\rm o}(s) \in \mathbb{K}(s)^{q\times m}$. By (12), we obtain

$$(G_{\rm o}(\imath\omega)Z)^*(G_{\rm o}(\imath\omega)Z) = (G(\imath\omega)Z)^*(G(\imath\omega)Z) \quad \forall \omega \in \mathbb{R} \text{ with } \imath\omega \notin \Lambda(E, A).$$

The assumption $\operatorname{rank}_{\mathbb{K}(s)} G(s)Z = q$ then leads to $G_{\rm o}(s)Z \in \mathrm{Gl}_q(\mathbb{K}(s))$.
Step 4: $G_{\rm i}(s)G_{\rm o}(s) = G(s)$*:*
Using the statement in Step 1 and the fact that $G_{\rm o}(s)Z$ is realized by the system $[E, A, BZ, K, LZ]$, Proposition 3 (i) leads to regularity of the pencil

$$\begin{bmatrix} -sE + A & BZ\\ K & LZ\end{bmatrix} \in \mathbb{K}[s]^{(n+q)\times(n+q)}.$$

Lemma 1 then gives rise to

$$G_{\mathrm{i}}(s) = \begin{bmatrix} C & DZ \end{bmatrix} \begin{bmatrix} sE-A & -BZ \\ -K & -LZ \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I_q \end{bmatrix} = G(s)Z(G_{\mathrm{o}}(s)Z)^{-1}.$$

Proposition 3 (ii) yields that $Z(G_{\mathrm{o}}(s)Z)^{-1}G_{\mathrm{o}}(s) \in \mathbb{K}(s)^{q\times q}$ is a projector along $\ker_{\mathbb{K}(s)} G_{\mathrm{o}}(s)$. Since further, by Step 2, $\ker_{\mathbb{K}(s)} G_{\mathrm{o}}(s) = \ker_{\mathbb{K}(s)} G(s)$, we obtain

$$\begin{aligned} G_{\mathrm{i}}(s)\cdot G_{\mathrm{o}}(s) &= G(s)Z(G_{\mathrm{o}}(s)Z)^{-1}\cdot G_{\mathrm{o}}(s) \\ &= G(s) - \underbrace{G(s)(I_m - Z(G_{\mathrm{o}}(s)Z)^{-1}G_{\mathrm{o}}(s))}_{=0} = G(s). \end{aligned}$$

Step 5: $G_{\mathrm{i}}(s)$ is inner:

Using Theorem 3 and invoking $D_{\mathrm{i}} = 0_{p\times q}$, it suffices to show that there exists some Hermitian matrix $P_{\mathrm{i}} \in \mathbb{K}^{(n+q)\times(n+q)}$ such that

$$\begin{bmatrix} A_{\mathrm{i}}^*P_{\mathrm{i}}E_{\mathrm{i}} + E_{\mathrm{i}}^*P_{\mathrm{i}}A_{\mathrm{i}} + C_{\mathrm{i}}^*C_{\mathrm{i}} & E_{\mathrm{i}}^*P_{\mathrm{i}}B_{\mathrm{i}} \\ B_{\mathrm{i}}^*P_{\mathrm{i}}E_{\mathrm{i}} & -I_q \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E_{\mathrm{i}},A_{\mathrm{i}},B_{\mathrm{i}}]}} 0, \quad E_{\mathrm{i}}^*P_{\mathrm{i}}E_{\mathrm{i}} \geq_{\mathcal{V}^{\mathrm{diff}}_{[E_{\mathrm{i}},A_{\mathrm{i}},B_{\mathrm{i}}]}} 0,$$

with $[E_{\mathrm{i}}, A_{\mathrm{i}}, B_{\mathrm{i}}, C_{\mathrm{i}}, D_{\mathrm{i}}] \in \Sigma_{n+q,q,p}(\mathbb{K})$ as in (10). By using the block matrix structure in (10), the system space and space of consistent initial differential variable of this system may be represented as

$$\mathcal{V}^{\mathrm{sys}}_{[E_{\mathrm{i}},A_{\mathrm{i}},B_{\mathrm{i}}]} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} \in \mathbb{K}^{n+2q} : \begin{pmatrix} x_1 \\ Zx_2 \end{pmatrix} \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \text{ and } Kx_1 + LZx_2 = u \right\}, \tag{13}$$

$$\mathcal{V}^{\mathrm{diff}}_{[E_{\mathrm{i}},A_{\mathrm{i}},B_{\mathrm{i}}]} = \mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \times \mathbb{K}^m. \tag{14}$$

Now consider $P_{\mathrm{i}} = \left[\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Proposition 2 gives $E^*XE \geq_{\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} 0$. The representation (14) of $\mathcal{V}^{\mathrm{diff}}_{[E_{\mathrm{i}},A_{\mathrm{i}},B_{\mathrm{i}}]}$ leads to

$$E_{\mathrm{i}}^*P_{\mathrm{i}}E_{\mathrm{i}} = \begin{bmatrix} E^*XE & 0 \\ 0 & 0 \end{bmatrix} \geq_{\mathcal{V}^{\mathrm{diff}}_{[E_{\mathrm{i}},A_{\mathrm{i}},B_{\mathrm{i}}]}} 0. \tag{15}$$

Let $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E_{\mathrm{i}},A_{\mathrm{i}},B_{\mathrm{i}}]}$. Partitioning

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{16}$$

according to the block structure (10) of the system $[E_{\mathrm{i}}, A_{\mathrm{i}}, B_{\mathrm{i}}, C_{\mathrm{i}}, D_{\mathrm{i}}]$, we obtain from the representation (13) of $\mathcal{V}^{\mathrm{sys}}_{[E_{\mathrm{i}},A_{\mathrm{i}},B_{\mathrm{i}}]}$ that

$$\begin{pmatrix} x_1 \\ Zx_2 \end{pmatrix} \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \tag{17}$$

and

$$u = Kx_1 + LZx_2. \tag{18}$$

Further, (15) and the construction (10) of the system $[E_\mathrm{i}, A_\mathrm{i}, B_\mathrm{i}, C_\mathrm{i}, D_\mathrm{i}]$ imply

$$\begin{aligned} A_\mathrm{i}^* P_\mathrm{i} E_\mathrm{i} + E_\mathrm{i}^* P_\mathrm{i} A_\mathrm{i} + C_\mathrm{i}^* C_\mathrm{i} &= \begin{bmatrix} A & BZ \\ K & LZ \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & BZ \\ K & LZ \end{bmatrix} + \begin{bmatrix} C^* \\ Z^* D^* \end{bmatrix} \begin{bmatrix} C & DZ \end{bmatrix} \\ &= \begin{bmatrix} A^* XE + E^* XA + C^* C & E^* XBZ + C^* DZ \\ Z^* B^* XE + Z^* D^* C & Z^* D^* DZ \end{bmatrix}, \\ E_\mathrm{i}^* P_\mathrm{i} B_\mathrm{i} &= \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -I_q \end{bmatrix} = 0_{(n+q)\times q}, \\ C_\mathrm{i}^* D_\mathrm{i} &= 0_{(n+q)\times q}. \end{aligned}$$

This gives rise to

$$\begin{aligned} &\begin{bmatrix} A_\mathrm{i}^* P_\mathrm{i} E_\mathrm{i} + E_\mathrm{i}^* P_\mathrm{i} A_\mathrm{i} + C_\mathrm{i}^* C_\mathrm{i} & E_\mathrm{i}^* P_\mathrm{i} B_\mathrm{i} + C_\mathrm{i}^* D_\mathrm{i} \\ D_\mathrm{i}^* C_\mathrm{i} + B_\mathrm{i}^* P_\mathrm{i} E_\mathrm{i} & D_\mathrm{i}^* D_\mathrm{i} - I_q \end{bmatrix} \\ &\qquad = \begin{bmatrix} A^* XE + E^* XA + C^* C & E^* XBZ + C^* DZ & 0 \\ Z^* B^* XE + Z^* D^* C & Z^* D^* DZ & 0 \\ 0 & 0 & -I_q \end{bmatrix}. \end{aligned} \tag{19}$$

Then we compute

$$\begin{aligned} &\begin{pmatrix} x \\ u \end{pmatrix}^* \begin{bmatrix} A_\mathrm{i}^* P_\mathrm{i} E_\mathrm{i} + E_\mathrm{i}^* P_\mathrm{i} A_\mathrm{i} + C_\mathrm{i}^* C_\mathrm{i} & E_\mathrm{i}^* P_\mathrm{i} B_\mathrm{i} + C_\mathrm{i}^* D_\mathrm{i} \\ D_\mathrm{i}^* C_\mathrm{i} + B_\mathrm{i}^* P_\mathrm{i} E_\mathrm{i} & D_\mathrm{i}^* D_\mathrm{i} - I_q \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ \overset{(16)\&(18)\&(19)}{=} &\begin{pmatrix} x_1 \\ x_2 \\ Kx_1 + LZx_2 \end{pmatrix}^* \\ &\cdot \begin{bmatrix} A^* XE + E^* XA + C^* C & E^* XBZ + C^* DZ & 0 \\ Z^* B^* XE + Z^* D^* C & Z^* D^* DZ & 0 \\ 0 & 0 & -I_q \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ Kx_1 + LZx_2 \end{pmatrix} \\ = &\begin{pmatrix} x_1 \\ Zx_2 \end{pmatrix}^* \left(\begin{bmatrix} A^* XE + E^* XA + C^* C & E^* XB + C^* D \\ B^* XE + D^* C & D^* D \end{bmatrix} - \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \right) \begin{pmatrix} x_1 \\ Zx_2 \end{pmatrix} \\ \overset{(8)\&(17)}{=} &\ 0. \end{aligned}$$

The latter together with (15) means that $[E_\mathrm{i}, A_\mathrm{i}, B_\mathrm{i}, C_\mathrm{i}, D_\mathrm{i}]$ meets the preliminaries of Theorem 3 and is therefore inner.

Now we present two toy examples to illustrate the presented theory.

Example 1 Consider the system $[E, A, B, C, D] \in \Sigma_{2,1,2}(\mathbb{R})$ with

$$sE - A = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \quad D = 0_{2\times 1}.$$

The system $[E, A, B, C, D]$ is behaviorally stabilizable with transfer function

$$G(s) = \begin{bmatrix} s+1 \\ s \end{bmatrix} \in \mathbb{R}(s)^{2\times 1}.$$

A stabilizing solution of the Lur'e equation (8) is

$$(X, K, L) = \left(\begin{bmatrix} \sqrt{2}-1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} & -1 \end{bmatrix}, 0 \right).$$

Since $G(s)$ has full column rank over $\mathbb{R}(s)$, we can choose $Z = 1$. By using (10) and (11) we obtain

$$G_{\mathrm{o}}(s) = \begin{bmatrix} -\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sqrt{2}s + 1 \in \mathbb{R}(s),$$

$$G_{\mathrm{i}}(s) = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & s & 0 \\ 0 & -1 & -1 \\ \sqrt{2} & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{1+\sqrt{2}s} \\ \frac{s}{1+\sqrt{2}s} \end{bmatrix} \in \mathbb{R}(s)^{2\times 1}.$$

It can be verified that $G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s)$ and, moreover, that $G_{\mathrm{i}}(s)$ is inner and $G_{\mathrm{o}}(s)$ is outer.

Example 2 Consider the system $[E, A, B, C, D] \in \Sigma_{2,2,1}(\mathbb{R})$ with

$$sE - A = \begin{bmatrix} -1 & s & 0 \\ 0 & -1 & 0 \\ 0 & 0 & s \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}, \quad D = 0_{1\times 2}.$$

The system $[E, A, B, C, D]$ is behaviorally stabilizable with transfer function

$$G(s) = \begin{bmatrix} s-1, & 1 - \frac{1}{s} \end{bmatrix} \in \mathbb{R}(s)^{1\times 2}.$$

A stabilizing solution of the Lur'e equation (8) is

$$(X, K, L) = \left(\begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}, 0_{1\times 2} \right),$$

and we obtain from (11) that the outer factor is given by

$$G_{\mathrm{o}}(s) = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & s & 0 \\ 0 & -1 & 0 \\ 0 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s+1 & 1 + \frac{1}{s} \end{bmatrix} \in \mathbb{R}(s)^{1\times 2}.$$

The matrix Z in Theorem 4 can be chosen to be $Z = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Then, in view of (10), we obtain that the inner factor reads

$$G_{\mathrm{i}}(s) = \begin{bmatrix} -1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & s & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & s & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \frac{s-1}{s+1} \in \mathbb{R}(s).$$

Remark 2 (Inner-outer factorization)

(i) It clearly holds that the transfer function of $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ fulfills $G(s) \in \mathbb{K}(s)^{p\times m}$. On the other hand, any $G(s) \in \mathbb{K}(s)^{p\times m}$ has a realization $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ by the results of realization theory for differential-algebraic systems [22]. Therefore, as a consequence of Theorem 4, any $G(s) \in \mathbb{K}(s)^{p\times m}$ has an inner-outer factorization.

(ii) If $\det(sE - A)$ has no zeros in $\overline{\mathbb{C}_+}$, then the outer factor $G_{\mathrm{o}}(s)$ in (10) has no poles in in $\overline{\mathbb{C}_+}$. If $G(s)$ is proper (that is, $\lim_{\lambda\to\infty} \|G(\lambda)\| < \infty$), then it follows from

$$G_{\mathrm{o}}(s) = G_{\mathrm{i}}^*(-\overline{s})G(s)$$

that $G_{\mathrm{o}}(s)$ is proper as well.

(iii) As we can see from Example 1 and Example 2, the realizations (10) and (11) of the inner and outer factors are in general not minimal. Of course, minimal realizations can be obtained by a transformation into Kalman decomposition [2, Thm. 8.1] and a subsequent elimination of the uncontrollable and unobservable parts.

(iv) In [10], transfer functions of single-input single-output systems (that is, $m = p = 1$) are considered. It is shown that inner-outer factorizations can be obtained in a rather simple way: The transfer function $g(s) \in \mathbb{K}(s)$ is first factorized as

$$g(s) = \frac{d^+(s) \cdot d^-(s)}{n(s)}$$

for polynomials $d^+(s)$, $d^-(s)$, $n(s) \in \mathbb{K}[s]$ with the property that all roots of $d^+(s)$ are in $\mathbb{C}^+$ and all roots of $d^-(s)$ are in $\mathbb{C} \setminus \mathbb{C}^+$. An inner-outer factorization is then given by $g(s) = g_{\mathrm{i}}(s)g_{\mathrm{o}}(s)$, where

$$g_{\mathrm{i}}(s) = \frac{d^+(s)}{\overline{d^+(-\overline{s})}}, \quad g_{\mathrm{o}}(s) = \frac{\overline{d^+(-\overline{s})} \cdot d^-(s)}{n(s)}.$$

This approach is called *Hurwitz reflection.*

(v) If $E = I_n$ and $\operatorname{rank} D = m$, then $\mathcal{V}^{\mathrm{sys}}_{[E,A,B]} = \mathbb{K}^{n+m}$ and we may choose $Z = I_m$. The Lur'e equation (8) can be reformulated as an algebraic Riccati equation

$$A^*X + XA + C^*C - \big(XB + C^*D\big)\big(D^*D\big)^{-1}\big(XB + C^*D\big)^* = 0$$

which has to be solved for the Hermitian matrix $X \in \mathbb{K}^{n\times n}$, and we may choose $L = \big(D^*D\big)^{1/2}$ and $K = L^{-*}\big(B^*X + D^*C\big)$. In this case we see that the realization of $G_{\mathrm{i}}(s)$ reduces to

$$[I_n, A - BL^{-1}K, BL^{-1}, C - DL^{-1}K, DL^{-1}] \in \Sigma_{n,m,p}(\mathbb{K}).$$

This coincides with the realization obtained in [29, Sect. 13.7].

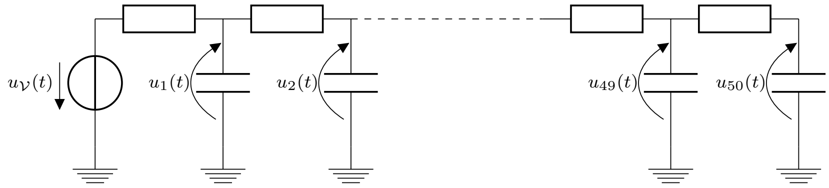


Fig. 1: An RC ladder network

The bottleneck in the determination of inner-outer factorization is the computation of the stabilizing solution of the Lur'e equation (3). The numerical solution of Lur'e equations has been treated in [18,19], especially for the large-scale and sparse case. The numerical method in [18,19] for the numerical solution of Lur'e equations basically consists of three steps: In the first step, the system space is determined by so-called "Wong sequences". In the second step, some further singularities of the Lur'e equation are eliminated which leads to an algebraic Riccati equation on a subspace, whose stabilizing solution is desired. The latter is solved via the Newton-ADI algorithm, which yields a low rank factorization $X \approx SS^*$ for some $S \in \mathbb{K}^{n\times r}$, where typically $r \ll n$. We will use this numerical technique for the subsequently presented numerical example.

Example 3 Consider the RC ladder network with as in Figure 1. All capacitors have a capacity of 0.02 F, the resistances are 0.02 Ω. The input is generated by the voltage source at the left. The output consists of

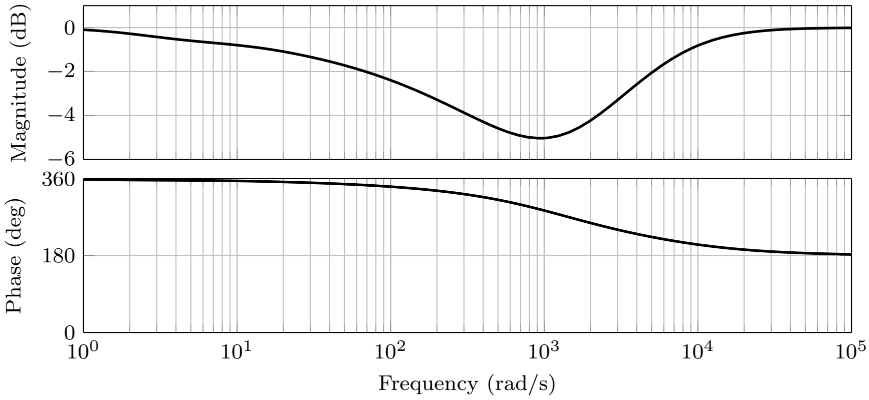
$$y(t) = u_1 - 2u_2.$$

The system is modeled by a modified nodal analysis [17], which results in a state space dimension of $n = 52$. The transfer function $G(s)$ has a zero at $\lambda = 1250\,\mathrm{s}^{-1}$, whence the system is not outer. The Lur'e equation is solved via the method presented in [19]. All calculations are done in Matlab 17a on a laptop computer with Intel® Core™i5-7300U CPU with 2.70 GHz and 8 GB RAM. The overall computation time is 0.61 seconds.

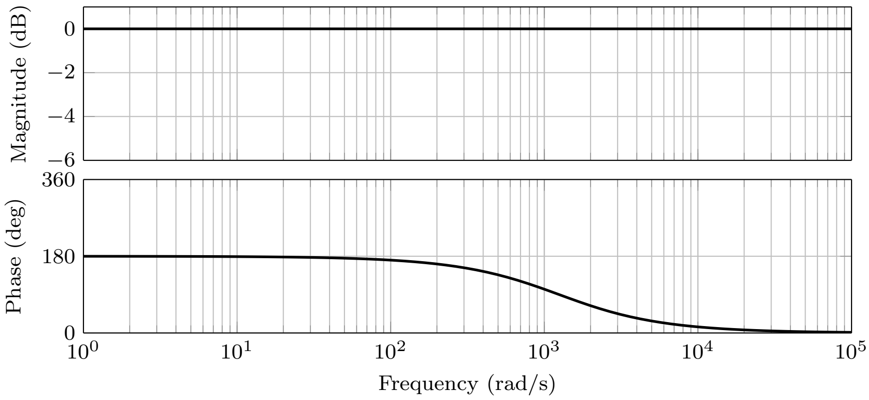
The Bode plots of the transfer function $G(s)$ of the system, the inner factor $G_{\mathrm{i}}(s)$, and the outer factor $G_{\mathrm{o}}(s)$ are depicted in Figure 2. In particular, the figure shows that the magnitudes of the system and its outer factor coincide; the magnitude of the inner factor is constantly one.

Acknowledgment

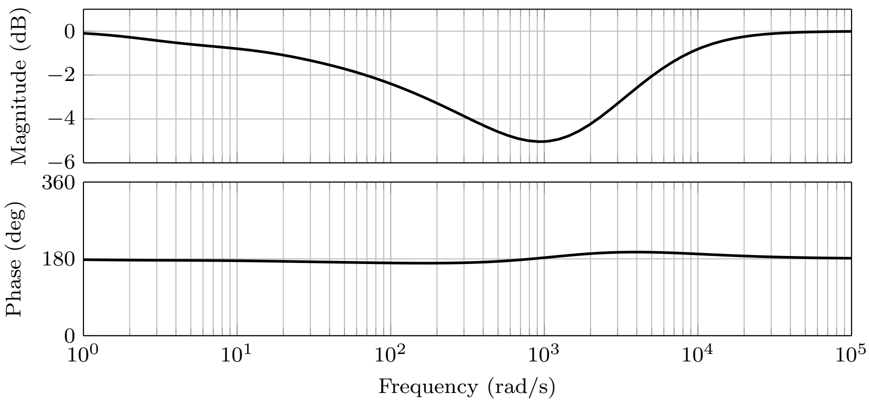
The authors thank Olaf Rendel for the support in dealing with the numerical examples.



Bode plot of $G(s)$.



Bode plot of $G_{\mathrm{i}}(s)$.



Bode plot of $G_{\mathrm{o}}(s)$.

Fig. 2: Bode plots of $G(s)$, $G_{\mathrm{i}}(s)$, and $G_{\mathrm{o}}(s)$.

References

1. T. Berger and T. Reis. Controllability of linear differential-algebraic systems – a survey. In A. Ilchmann and T. Reis, editors, *Surveys in Differential-Algebraic Equations I*, Differ.-Algebr. Equ. Forum, pages 1–61. Springer-Verlag, Berlin, Heidelberg, 2013.
2. T. Berger, T. Reis, and S. Trenn. Observability of linear differential-algebraic systems: A survey. In A. Ilchmann and T. Reis, editors, *Surveys in Differential-Algebraic Equations IV*, Differ.-Algebr. Equ. Forum, pages 161–219. Springer International Publishing AG, Cham, Switzerland, 2017.
3. T. Chen and B.A. Francis. Spectral and inner-outer factorizations of rational matrices. *SIAM J. Matrix Anal. Appl.*, 10(1):1–17, 1989.
4. P. Van Dooren. Rational and polynomial matrix factorizations via recursive pole-zero cancellation. *Linear Algebra Appl.*, 137–138:663–697, 1990.
5. B. A. Francis. *A Course in H_∞ Control Theory*, volume 88 of *Lecture Notes in Control and Inform. Sci.* Springer-Verlag, Heidelberg, 1987.
6. M. Green. On inner-outer factorization. *Systems Control Lett.*, 11(2):93–97, 1988.
7. G. Gu. Inner-outer factorization for strictly proper transfer matrices. *IEEE Trans. Automat. Control*, 47(11):1915–1919, 2002.
8. S. Hara and T. Sugie. Inner-outer factorization for strictly proper functions with $j\omega$-axis zeros. *Systems Control Lett.*, 16(3):179–185, 1991.
9. A. Ilchmann and T. Reis. Outer transfer functions of differential-algebraic systems. *ESAIM Control Optim. Calc. Var.*, 23(2):391–425, 2017.
10. A. Ilchmann and F. Wirth. On minimum phase. *at-Automatisierungstechnik*, 61(12):805–817, 2013.
11. T. Kailath. *Linear Systems.* Prentice Hall, Englewood Cliffs, NJ, 1980.
12. H. Kimura. Conjugation, interpolation, and model matching in H^∞. *Internat. J. Control*, 49(1):269–307, 1989.
13. S. Lang. *Complex Analysis*, volume 103 of *Grad. Texts in Math.* Springer-Verlag, New York, 2th edition, 1985.
14. C. Oară and A. Varga. Computation of general inner-outer and spectral factorizations. *IEEE Trans. Automat. Control*, 45(12):2307–2325, 2000.
15. J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory. A Behavioral Approach.* Springer-Verlag, New York, 1998.
16. T. Reis. Lur'e equations and even matrix pencils. *Linear Algebra Appl.*, 434:152–173, 2011.
17. T. Reis. Mathematical modeling and analysis of nonlinear time-invariant RLC circuits. In P. Benner, R. Findeisen, D. Flockerzi, U. Reichl, and K. Sundmacher, editors, *Large-Scale Networks in Engineering and Life Sciences*, Model. Simul. Sci. Eng. Technol., chapter 2, pages 125–198. Birkäuser, Basel, 2014.
18. T. Reis and O. Rendel. Numerical solution of singular Lur'e equations. *Proc. Appl. Math. Mech.*, 14(1):829–830, 2014.
19. T. Reis and O. Rendel. Numerical solution of matrix equations arising in linear-quadratic optimal control of differential-algebraic equations. Hamburger Beiträge zur Angewandten Mathematik 2016-36, Universität Hamburg, Fachbereich Mathematik, 2016. Available from `https://preprint.math.uni-hamburg.de/public/papers/hbam/hbam2016-36.pdf`.
20. T. Reis, O. Rendel, and M. Voigt. The Kalman-Yakubovich-Popov inequality for differential-algebraic systems. *Linear Algebra Appl.*, 485:153–193, 2015.
21. T. Reis and M. Voigt. The Kalman-Yakubovich-Popov inequality for differential-algebraic systems: existence of nonpositive solutions. *Systems Control Lett.*, 86:1–8, 2015.
22. S. Tan and J. Vandewalle. Realization algorithm for determining generalized state-space representations. *Internat. J. Control*, 45(4):1137–1146, 1987.
23. M. C. Tsai and L. W. Chen. A numerical algorithm for innter-outer factorizations for real-rational matrices. *Systems Control Lett.*, 20(3):209–217, 1993.
24. A. Varga. Computation of inner-outer factorizations of rational matrices. *IEEE Trans. Automat. Control*, 43(5):684–688, 1998.

25. M. Voigt. *On Linear-Quadratic Optimal Control and Robustness of Differential-Algebraic Systems*. Logos-Verlag, Berlin, 2015. also as Dissertation, Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik, 2015.
26. M. Weiss. Spectral and inner-outer factorizations through the constrained Riccati equations. *IEEE Trans. Automat. Control*, 39(3):677–681, 1994.
27. X. Xin and T. Mita. Inner-outer factorization for non-square proper functions with infinite and finite $j\omega$-axis zeros. *Internat. J. Control*, 71(1):145–161, 1998.
28. F.-B. Yeh and L.-F. Wei. Inner-outer factorizations of right-invertible real-rational matrices. *Systems Control Lett.*, 14(1):31–36, 1990.
29. K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Englewood Cliffs, NJ, 1996.