LINEAR-QUADRATIC OPTIMAL CONTROL OF DIFFERENTIAL-ALGEBRAIC SYSTEMS: THE INFINITE TIME HORIZON PROBLEM WITH ZERO TERMINAL STATE[‡]

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Abstract. In this work we revisit the linear-quadratic optimal control problem for differential-algebraic systems on the infinite time horizon with zero terminal state. Based on the recently developed Lur'e equation for differential-algebraic equations we obtain new equivalent conditions for feasibility. These are related to the existence of a stabilizing solution of the Lur'e equation. This approach also allows to determine optimal controls if they exist. In particular, we can characterize regularity of the optimal control problem. The latter refers to existence and uniqueness of optimal controls for any consistent initial condition.

Key words. descriptor systems, differential-algebraic equations, linear-quadratic optimal control, Lur'e equation, Riccati equation, Kalman-Yakubovich-Popov inequality

AMS subject classifications. Primary, 49N10; Secondary, 15A24, 49J21, 93B52

1. Introduction.

We consider differential-algebraic systems

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{1.1}$$

where $E,\, A \in \mathbb{R}^{n\times n}$ are such that the pencil $sE - A \in \mathbb{R}[s]^{n\times n}$ is *regular*, i. e., $\det(sE - A)$ is not the zero polynomial, and $B \in \mathbb{R}^{n\times m}$. For an interval $\mathcal{I} \subseteq \mathbb{R}$, the functions $x : \mathcal{I} \to \mathbb{R}^n$ and $u : \mathcal{I} \to \mathbb{R}^m$ are called *generalized state* and *input* of the system, respectively. We denote the set of systems (1.1) by $\Sigma_{n,m}$, and we write $[E, A, B] \in \Sigma_{n,m}$.

Here $\mathcal{L}^2(\mathcal{I}, \mathcal{V})$ and $\mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}, \mathcal{V})$ denote the spaces of measurable and (locally) square integrable functions $f : \mathcal{I} \to \mathcal{V}$ on the interval $\mathcal{I} \subseteq \mathbb{R}$ with values in a finite-dimensional normed space $\mathcal{V}$, where we identify functions that are identical almost everywhere. We call $(x, u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}, \mathbb{R}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}, \mathbb{R}^m) \simeq \mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}, \mathbb{R}^{n+m})$ a *solution of* $[E, A, B]$ on $\mathcal{I}$, if (1.1) holds in the weak sense. That is, for all smooth $\varphi : \mathbb{R} \to \mathbb{R}$ that are compactly supported in the interior of $\mathcal{I}$ it holds that

$$-\int_{\mathcal{I}} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\right) Ex(t)\mathrm{d}t = \int_{\mathcal{I}} \varphi(t)(Ax(t) + Bu(t))\mathrm{d}t.$$

We further call (x, u) a *solution of* $[E, A, B]$, if it is a solution of $[E, A, B]$ on $\mathbb{R}$.

Note that (x, u) being a solution of $[E, A, B]$ on $\mathcal{I}$ implies that $Ex,\, \tfrac{\mathrm{d}}{\mathrm{d}t}Ex \in \mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}, \mathbb{R}^n)$ and thus, Ex has an absolutely continuous representative [1, Thm. 10.23]. The evaluation $Ex(t) := (Ex)(t)$ is therefore well-defined for all $t \in \mathcal{I}$. We further

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consider the vector space of *consistent initial differential variables* of $[E, A, B]$, which is given by

$$\mathcal{V}^{\text{diff}}_{[E,A,B]} := \{x_0 \in \mathbb{R}^n \,:\, \exists\, \text{a solution } (x,u) \text{ of } [E,A,B] \text{ with } Ex(0) = Ex_0\}\,. \tag{1.2}$$

Note that any solution (x, u) of $[E, A, B]$ satisfies $Ex(0) \in E\mathcal{V}^{\text{diff}}_{[E,A,B]}$.

For an interval $\mathcal{I} \subseteq \mathbb{R}$ and matrices $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, and $R = R^\top \in \mathbb{R}^{m\times m}$ we introduce the *cost functional*

$$\mathcal{J}(x,u,\mathcal{I}) := \int_{\mathcal{I}} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t. \tag{1.3}$$

For a given $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$ we consider the following optimal control problems:

(OC+)

Minimize $\mathcal{J}(x,u,\mathbb{R}_{\geq 0})$
subject to $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$.

(OC–)

Minimize $\mathcal{J}(x,u,\mathbb{R}_{\leq 0})$
subject to $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu$ with $Ex(0) = Ex_0$ and $Ex(-\infty) = 0$.

Note that the above optimal control problems are also subject to the terminal conditions $Ex(\pm\infty) = 0$ which is a short-hand notation for $\lim_{t\to\pm\infty} Ex(t) = 0$.

This article is devoted to an analysis of the above optimal control problems. We will first study *feasibility*. Loosely speaking, this refers to the property that for each $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$ there exists a solution (x, u) of $[E, A, B]$ on $\mathbb{R}_{\geq 0}$ (resp. on $\mathbb{R}_{\leq 0}$) with $Ex(\pm\infty) = 0$ and additionally, the cost functionals cannot be made arbitrarily negative. We analyze existence and construction of *optimal controls*, i. e., minimizers of the above optimization problems. Further, we characterize *regularity*, which is a property referring to the existence and uniqueness of optimal controls for any consistent initial condition. In our analysis of the optimal control problems **(OC+)** and **(OC–)**, we present an approach similar to the one of Willems in his seminal article [47]. Namely, our findings are based on quadratic storage functions and certain matrix equations which can be solved for a Hermitian matrix expressing the optimal cost.

In this article we use the standard notations $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$, $\imath$, $A^\top$, I_n, $0_{m\times n}$ for the natural numbers including zero, the real numbers, the complex numbers, the imaginary unit, the transpose of a matrix, the identity matrix of size $n \times n$, and the zero matrix of size $m \times n$ (subscripts are omitted, if clear from context). Further, $\mathbb{C}_+$ and $\mathbb{C}_-$ are the open sets of complex numbers with positive and negative real part, whereas $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$ denote the sets of nonnegative and nonpositive real numbers, respectively. The symbol $\mathbb{R}[s]$ stands for the ring of real polynomials, whereas $\operatorname{im}_{\mathcal{R}} A$, $\ker_{\mathcal{R}} A$, $\operatorname{rank}_{\mathcal{R}} A$ denote the image, kernel, and rank of a matrix $A \in \mathcal{R}^{m\times n}$ over the ring $\mathcal{R}$. Further, the restriction of $f : \mathcal{I} \to M$ to $\widetilde{\mathcal{I}} \subseteq \mathcal{I}$ is denoted by $f|_{\widetilde{\mathcal{I}}}$.

The optimal control problems **(OC+)** and **(OC–)** motivate the introduction of the *value functions*

$$V_+,\, V_- : E\mathcal{V}^{\text{diff}}_{[E,A,B]} \to \mathbb{R} \cup \{-\infty, \infty\},$$

expressing the *optimal costs*, which are defined by

$$V_+(Ex_0) = \inf\{\mathcal{J}(x,u,\mathbb{R}_{\geq 0}) \,:\, (x,u) \text{ is a solution of } [E,A,B] \text{ on } \mathbb{R}_{\geq 0} \text{ with } Ex(0)=Ex_0 \text{ and } Ex(\infty)=0\}, \tag{1.4}$$

$$V_-(Ex_0) = -\inf\{\mathcal{J}(x,u,\mathbb{R}_{\leq 0}) \,:\, (x,u) \text{ is a solution of } [E,A,B] \text{ on } \mathbb{R}_{\leq 0} \text{ with } Ex(0)=Ex_0 \text{ and } Ex(-\infty)=0\}. \tag{1.5}$$

Next we define some notions which are, loosely speaking, related to the solvability of the optimal control problems. These concepts are crucial for all considerations in this article.

Definition 1.1 (Feasibility, regularity, optimal control). *Let $[E,A,B]\in\Sigma_{n,m}$ and $Q=Q^\top\in\mathbb{R}^{n\times n}$, $S\in\mathbb{R}^{n\times m}$, and $R=R^\top\in\mathbb{R}^{m\times m}$ be given.*

*a) The optimal control problem **(OC+)** (resp. **(OC−)**) is called* feasible*, if for all $x_0\in\mathcal{V}^{\text{diff}}_{[E,A,B]}$ it holds that*

$$-\infty < V_+(Ex_0) < \infty, \quad (\textit{resp. } -\infty < V_-(Ex_0) < \infty).$$

b) A solution (x_*,u_*) of $[E,A,B]$ on $\mathbb{R}_{\geq 0}$ (resp. on $\mathbb{R}_{\leq 0}$) with $Ex(0)=Ex_0$ and $Ex_*(\infty)=0$ (resp. $Ex_*(-\infty)=0$) is called an optimal control *for **(OC+)** (resp. **(OC−)**), if*

$$V_+(Ex_0)=\mathcal{J}(x_*,u_*,\mathbb{R}_{\geq 0}) \quad (\textit{resp. } V_-(Ex_0)=\mathcal{J}(x_*,u_*,\mathbb{R}_{\leq 0})).$$

*c) The optimal control problem **(OC+)** (resp. **(OC−)**) is called* regular*, if for all $x_0\in\mathcal{V}^{\text{diff}}_{[E,A,B]}$, there exists a unique optimal control for **(OC+)** (resp. **(OC−)**).*

To formulate our main results, we present some notions related to controllability and stabilizability of differential-algebraic systems. Algebraic characterizations can be found in [8].

Definition 1.2 (Controllability and stabilizability). *The system $[E,A,B]\in\Sigma_{n,m}$ is called*

a) behaviorally stabilizable*, if for all solutions (x,u) of $[E,A,B]$ on $\mathbb{R}_{\leq 0}$ there exists a solution $(\widetilde{x},\widetilde{u})$ of $[E,A,B]$ with $(\widetilde{x},\widetilde{u})|_{\mathbb{R}_{\leq 0}}=(x,u)$ and*

$$\lim_{t\to\infty} \operatorname*{ess\,sup}_{\tau>t} \|(\widetilde{x}(\tau),\widetilde{u}(\tau))\| = 0;$$

b) behaviorally controllable*, if for all solutions (x_1,u_1), (x_2,u_2) of $[E,A,B]$ there exists some $T>0$ and a solution (x,u) of $[E,A,B]$ with*

$$(x(t),u(t)) = \begin{cases} (x_1(t),u_1(t)) & \textit{for } t<0,\\ (x_2(t),u_2(t)) & \textit{for } t>T. \end{cases}$$

2. Curiosities in Optimal Control of Differential-Algebraic Equations. We present some examples which emphasize the main differences between optimal control of ordinary differential equations and differential-algebraic equations. Whereas in optimal control of ordinary differential equations, positive semi-definiteness of the input weight R is necessary for feasibility of the optimal control problem [47], this is not necessarily true in the case of differential-algebraic equations.

EXAMPLE 2.1 (Feasibility of the optimal control problem **(OC+)** does not imply $R \geq 0$.). *Consider the optimal control problem*

$$\begin{array}{c} \text{Minimize } \mathcal{J}(x,u,\mathbb{R}_{\geq 0}) = \int_0^\infty x_2^2(t) - \tfrac{1}{2}u^2(t)\mathrm{d}t \\ \text{subject to } \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix}u \\ \text{with } x_1(0) = x_{01} \text{ and } x_1(\infty) = 0. \end{array} \tag{2.1}$$

For this optimal control problem we have – in the notation of (1.3) and ***(OC+)*** *– $R = -\frac{1}{2} < 0$. However, resolving the algebraic constraint $x_2 = u$ yields that this optimal control problem is equivalent to*

$$\begin{array}{c} \text{Minimize } \mathcal{J}(x_1,u,\mathbb{R}_{\geq 0}) = \int_0^\infty \tfrac{1}{2}u^2(t)\mathrm{d}t \\ \text{subject to } \tfrac{\mathrm{d}}{\mathrm{d}t}x_1 = -x_1 + u \text{ with } x_1(0) = x_{01} \text{ and } x_1(\infty) = 0. \end{array} \tag{2.2}$$

together with $x_2 = u$. The optimal control problem (2.2) is even regular. Namely, the non-negative cost functional can be made zero by setting $u_* = 0$. Then we have indeed $x_{*,1}(t) = \mathrm{e}^{-t} \cdot x_{01}$ with $x_{*,1}(\infty) = 0$. Therefore, the differential-algebraic optimal control problem (2.1) is regular with optimal control $\left(\left(\begin{smallmatrix}x_{*,1}\\ x_{*,2}\end{smallmatrix}\right), u_*\right)$ where $x_{*,1}(t) = \mathrm{e}^{-t} \cdot x_{01}$ and $x_{*,2} = u_* = 0$. In particular, (2.1) is a feasible optimal control problem.

In optimal control of ordinary differential equations, regularity of the input weight R is necessary for regularity of the optimal control problem [13]. This is not necessarily true for differential-algebraic equations, as the following example shows.

EXAMPLE 2.2 (Regularity of the optimal control problem **(OC+)** does not imply that R is invertible.). *Consider the optimal control problem*

$$\begin{array}{c} \text{Minimize } \mathcal{J}(x,u,\mathbb{R}_{\geq 0}) = \int_0^\infty \tfrac{1}{2}x_2^2(t)\mathrm{d}t \\ \text{subject to } \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix}u \\ \text{with } x_1(0) = x_{01} \text{ and } x_1(\infty) = 0. \end{array} \tag{2.3}$$

For this optimal control problem we have – in the notation of (1.3) and ***(OC+)*** *– $R = 0$. However, resolving the algebraic constraint $x_2 = u$ yields that this optimal control problem is again equivalent to (2.2), now together with $x_2 = u$. The latter is even a regular optimal control problem with – as previously shown – optimal control $(x_{*,1}, u_*)$ with $x_{*,1}(t) = \mathrm{e}^{-t} \cdot x_{01}$ and $u_* = 0$. Therefore, the differential-algebraic optimal control problem is again regular with optimal control $\left(\left(\begin{smallmatrix}x_{*,1}\\ x_{*,2}\end{smallmatrix}\right), u_*\right)$ with $x_{*,1}(t) = \mathrm{e}^{-t} \cdot x_{01}$ and $x_{*,2} = u_* = 0$.*

Next we show that the *index* of the differential-algebraic equation $[E, A, B]$ does not necessarily cause singularity of the optimal control problem **(OC+)**. In our context, the index is defined by the nilpotency index of the nilpotent matrix N in a *quasi-Weierstraß form*

$$W(sE-A)T = \begin{bmatrix} sI - A_1 & 0 \\ 0 & sN - I \end{bmatrix} \tag{2.4}$$

for some invertible matrices $W, T \in \mathbb{R}^{n\times n}$, see [9].

Example 2.3 (Differential-algebraic equations with arbitrary index can lead to a regular optimal control problem.). *Consider the system* $[E, A, B] \in \Sigma_{n,1}$ *with*

$$E = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then, since $A^{-1}E$ *is nilpotent with index* n, $A^{-1}(sE-A)$ *is in quasi-Weierstraß form and the index of this differential-algebraic equation is* n.

We further define a cost functional (1.3) *with the matrices* $R = 0$, $S = 0$, *and* $Q \in \mathbb{R}^{n\times n}$ *which has the entry 1 at the lower right position and zeros elsewhere. This yields the optimal control problem*

$$\begin{gathered} \text{Minimize } \mathcal{J}(x,u,\mathbb{R}_{\geq 0}) = \int_0^\infty x_n^2(t)\mathrm{d}t \\ \text{subject to } \begin{cases} 0 = x_1 + u, \\ \frac{\mathrm{d}}{\mathrm{d}t}x_1 = -x_1 + x_2, \\ \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t}x_{n-1} = -x_{n-1} + x_n, \end{cases} \\ \text{with } x_1(0) = x_{01}, \ldots, x_{n-1}(0) = x_{0,n-1} \text{ and} \\ x_1(\infty) = \ldots = x_{n-1}(\infty) = 0. \end{gathered}$$

Since u *only enters in the algebraic equation* $u + x_1 = 0$, *we see that the optimal control problem is equivalent to* $u = -x_1$ *together with*

$$\begin{gathered} \text{Minimize } \mathcal{J}(x,u,\mathbb{R}_{\geq 0}) = \int_0^\infty x_n^2(t)\mathrm{d}t \\ \text{subject to } \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_1 = -x_1 + x_2, \\ \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t}x_{n-1} = -x_{n-1} + x_n, \end{cases} \\ \text{with } x_1(0) = x_{01}, \ldots, x_{n-1}(0) = x_{0,n-1} \text{ and} \\ x_1(\infty) = \ldots = x_{n-1}(\infty) = 0. \end{gathered} \tag{2.5}$$

Since the ordinary differential equation in (2.5) *with the additional constraint* $x_n = 0$ *is asymptotically stable, we obtain that the nonnegative cost functional* $\mathcal{J}(x,u,\mathbb{R}_{\geq 0})$ *can indeed be made zero. Thus, an optimal control* (x_*, u_*) *has to fulfill* $x_{*,n} = 0$ *and* *the optimal control* (x_*, u_*) *is uniquely determined by* $x_{*,n} = 0$*, and*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} x_{*,1} &= \quad - x_{*,1} + x_{*,2}, \\ &\vdots \\ \tfrac{\mathrm{d}}{\mathrm{d}t} x_{*,n-2} &= - x_{*,n-2} + x_{*,n-1}, \\ \tfrac{\mathrm{d}}{\mathrm{d}t} x_{*,n-1} &= - x_{*,n-1}\,, \end{aligned}$$

with the initial conditions $x_{*,1}(0) = x_{01}, \ldots, x_{*,n-1}(0) = x_{0,n-1}$*, and* $u_* = -x_{*,1}$*. In particular, the optimal control problem is regular.*

3. Feasibility, Storage Functions, and the Kalman-Yakubovich-Popov Inequality. The key ingredient for our considerations are so-called *storage functions*. This concept has been introduced for ordinary differential equations in [47, 18].

DEFINITION 3.1 (Storage function, dissipation inequality). *Let* $[E, A, B] \in \Sigma_{n,m}$ *and* $Q = Q^\top \in \mathbb{R}^{n\times n}$*,* $S \in \mathbb{R}^{n\times m}$*,* $R = R^\top \in \mathbb{R}^{m\times m}$ *be given. A function* $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$ *is called a* storage function*, if it is continuous,* $V(0) = 0$*, and the cost functional* $\mathcal{J}(\cdot,\cdot,\cdot)$ *as in* (1.3) *fulfills the* dissipation inequality

$$\mathcal{J}(x, u, [t_0, t_1]) + V(Ex(t_1)) \geq V(Ex(t_0)) \tag{3.1}$$

for all solutions (x, u) *of* $[E, A, B]$ *and for all* $t_0, t_1 \in \mathbb{R}$ *with* $t_0 \leq t_1$*.*

Our special emphasis will be put on *quadratic storage functions*. This means that there exists some Hermitian matrix $P \in \mathbb{R}^{n\times n}$ such that V attains the form

$$V(Ex_0) = x_0^\top E^\top P E x_0 \quad \forall\, x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}. \tag{3.2}$$

To further characterize quadratic storage functions, we introduce and characterize of the *system space* of $[E, A, B]$. Therefore, denote by $\mathrm{VS}(\mathbb{R}^{n+m})$ the set of linear subspaces of $\mathbb{R}^{n+m}$ and define the set

$$\mathcal{F}_{[E,A,B]} := \big\{ \mathcal{V} \in \mathrm{VS}(\mathbb{R}^{n+m}) \ : \ \text{for all solutions } (x,u) \text{ of } [E,A,B] \\ \text{it holds that } (x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{V}) \big\}\,.$$

PROPOSITION 3.2. *The set* $\mathcal{F}_{[E,A,B]}$ *is non-empty lower semi-lattice relative to intersection and subspace inclusion as partial order. It contains a unique smallest element* $\mathcal{V}_*$ *which satisfies*

$$\mathcal{V}_* = \bigcap \big\{ \mathcal{V} \ : \ \mathcal{V} \in \mathcal{F}_{[E,A,B]} \big\}\,. \tag{3.3}$$

Proof. Obviously, since $\mathbb{R}^{n+m} \in \mathcal{F}_{[E,A,B]}$, the set $\mathcal{F}_{[E,A,B]}$ is non-empty. By the definition of $\mathcal{F}_{[E,A,B]}$ and since the intersection of two subspaces of $\mathbb{R}^{n+m}$ is again a subspace in $\mathbb{R}^{n+m}$, we see that $\mathcal{F}_{[E,A,B]}$ is also closed under intersection, i. e., $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{F}_{[E,A,B]}$ implies $\mathcal{V}_1 \cap \mathcal{V}_2 \in \mathcal{F}_{[E,A,B]}$. Now there exists an element $\widetilde{\mathcal{V}} \in \mathcal{F}_{[E,A,B]}$ such that

$$\dim \widetilde{\mathcal{V}} \leq \dim \mathcal{V} \quad \text{for all } \mathcal{V} \in \mathcal{F}_{[E,A,B]}.$$

Then it holds that

$$\dim \widetilde{\mathcal{V}} \cap \mathcal{V} \leq \dim \widetilde{\mathcal{V}} \quad \text{for all } \mathcal{V} \in \mathcal{F}_{[E,A,B]}. \tag{3.4}$$

This yields $\widetilde{\mathcal{V}} \cap \mathcal{V} = \widetilde{\mathcal{V}}$ for all $\mathcal{V} \in \mathcal{F}_{[E,A,B]}$. But then $\widetilde{\mathcal{V}} \subseteq \mathcal{V}$ holds for all $\mathcal{V} \in \mathcal{F}_{[E,A,B]}$. Thus, $\mathcal{V}_* := \widetilde{\mathcal{V}}$ is the unique smallest element of $\mathcal{F}_{[E,A,B]}$. Moreover, by (3.4) we get

$$\begin{aligned} \bigcap \{\mathcal{V} \,:\, \mathcal{V} \in \mathcal{F}_{[E,A,B]}\} &= \mathcal{V}_* \cap \bigcap \{\mathcal{V} \,:\, \mathcal{V} \in \mathcal{F}_{[E,A,B]}\} \\ &= \bigcap \{\mathcal{V}_* \cap \mathcal{V} \,:\, \mathcal{V} \in \mathcal{F}_{[E,A,B]}\} = \mathcal{V}_*, \end{aligned}$$

and therefore, (3.3) is satisfied. ◻

Definition 3.3. *Let* $[E, A, B] \in \Sigma_{n,m}$ *be given. Then the space* $\mathcal{V}^{\text{sys}}_{[E,A,B]} := \mathcal{V}_*$ *with* $\mathcal{V}_*$ *as in Proposition 3.2 is called the* system space *of* $[E, A, B]$.

Remark 3.4. *The* system space *of* $[E, A, B] \in \Sigma_{n,m}$ *is the smallest subspace* $\mathcal{V}^{\text{sys}}_{[E,A,B]} \subseteq \mathbb{R}^{n+m}$ *such that*

$$\forall \textit{ solutions } (x, u) \textit{ of } [E, A, B]: \quad (x, u) \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathcal{V}^{\text{sys}}_{[E,A,B]}).$$

A geometric characterization of the system space can be found in [42]. For the theory presented in this article, it is crucial to introduce what we mean by equality and positive semi-definiteness on some subspace.

Definition 3.5. *Let* $\mathcal{V} \subseteq \mathbb{R}^n$ *be a subspace and* M, $N \in \mathbb{R}^{n\times n}$ *be Hermitian matrices. Then we write*

$$\begin{aligned} M =_{\mathcal{V}} N \quad &:\Longleftrightarrow \quad x^\top M x = x^\top N x \quad \forall\, x \in \mathcal{V}, \\ M \geq_{\mathcal{V}} N \quad &:\Longleftrightarrow \quad x^\top M x \geq x^\top N x \quad \forall\, x \in \mathcal{V}. \end{aligned}$$

The previous definitions enable us to introduce the *Kalman-Yakubovich-Popov (KYP) inequality.*

Definition 3.6 (Kalman-Yakubovich-Popov inequality). *Let* $[E, A, B] \in \Sigma_{n,m}$ *and* $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ *be given. We call* $P \in \mathbb{R}^{n\times n}$ *a* solution of the Kalman-Yakubovich-Popov (KYP) inequality, *if*

$$\begin{bmatrix} A^\top P E + E^\top P A + Q & E^\top P B + S \\ B^\top P E + S^\top & R \end{bmatrix} \geq_{\mathcal{V}^{\text{sys}}_{[E,A,B]}} 0, \quad P = P^\top. \tag{3.5}$$

In this section we analyze the relationship of storage functions with solutions of the KYP inequality and their connection to feasibility of **(OC+)**. An overview of the results that we will show in this section is given in Figure 3.1.

First we consider the special case where the storage function is differentiable. For this we need an auxiliary result which basically states that we can often restrict to smooth solutions.

Lemma 3.7. *Let* $[E, A, B] \in \Sigma_{n,m}$ *be given. Then the following holds:*

i) For all $\binom{x_0}{u_0} \in \mathcal{V}^{\text{sys}}_{[E,A,B]}$, *there exists some infinitely differentiable solution* (x, u) *of* $[E, A, B]$ *with* $x(0) = x_0$ *and* $u(0) = u_0$. *In particular,* $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$.

ii) For all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$, *there exists some infinitely differentiable solution* (x, u) *of* $[E, A, B]$ *with* $Ex(0) = Ex_0$. *In particular, there exists some* $\binom{x_{01}}{u_{01}} \in$

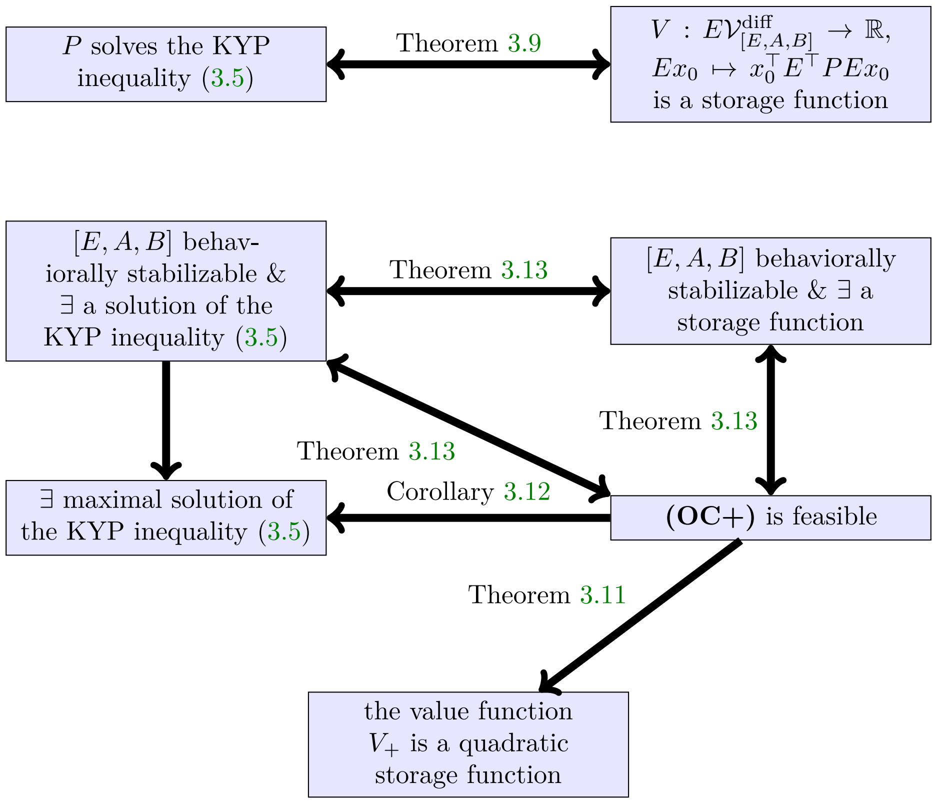

FIG. 3.1. *Results of Section 3 and their relation*

$\mathcal{V}^{\text{sys}}_{[E,A,B]}$ *with* $Ex_0 = Ex_{01}$.

Proof. This follows by an application of [9, Thm. 3.2] to the differential-algebraic equation $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w(t) = \mathcal{A}w(t)$ with $\mathcal{E} = \begin{bmatrix} E & 0 \end{bmatrix}$, $\mathcal{A} = \begin{bmatrix} A & B \end{bmatrix}$, and $w(t) = \left(\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right)$. ◻

PROPOSITION 3.8. *Let* $[E, A, B] \in \Sigma_{n,m}$ *and* $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ *be given. Then a differentiable function* $V : E\mathcal{V}^{\text{diff}}_{[E,A,B]} \to \mathbb{R}$ *with* $V(0) = 0$ *is a storage function, if and only if*

$$\begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \geq -(\nabla V(Ex_0))^\top (Ax_0 + Bu_0) \quad \forall \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \in \mathcal{V}^{\text{sys}}_{[E,A,B]}, \tag{3.6}$$

where $\nabla V(Ex_0) \in \mathbb{R}^n$ *denotes the gradient of* V *in* Ex_0.

Proof. First assume that $V : E\mathcal{V}^{\text{diff}}_{[E,A,B]} \to \mathbb{R}$ is a differentiable storage function. Suppose that $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in \mathcal{V}^{\text{sys}}_{[E,A,B]}$. Then, by Lemma 3.7, there exists some infinitely differentiable solution (x, u) of $[E, A, B]$ with $x(0) = x_0$ and $u(0) = u_0$. Consequently, the real-valued function $t \mapsto V(Ex(t))$ is continuously differentiable. The dissipation inequality yields that for all $h > 0$ we have

$$\frac{1}{h}(V(Ex(h)) - V(Ex(0))) \geq -\frac{1}{h}\mathcal{J}(x, u, [0, h])$$

$$= -\frac{1}{h}\int_0^h \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t.$$

Now taking the limit $h \to 0$, we see that the right hand side converges to

$$-\begin{pmatrix} x(0) \\ u(0) \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x(0) \\ u(0) \end{pmatrix} = -\begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}.$$

Then (3.6) is a consequence of the fact that the left hand side tends to

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} V(Ex(t))\big|_{t=0} &= (\nabla V(Ex(0)))^\top \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t)\big|_{t=0} \\ &= (\nabla V(Ex(0)))^\top (Ax(0) + Bu(0)) = (\nabla V(Ex_0))^\top (Ax_0 + Bu_0). \end{aligned}$$

To prove the reverse implication, assume that (3.6) is satisfied. Let (x, u) be a solution of $[E, A, B]$ and let $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$. Then, by using the fundamental theorem of calculus for weakly differentiable functions [1, Sec. E3.6] and the chain rule for weak derivatives [33], together with $(x, u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{V}^{\mathrm{sys}}_{[E,A,B]})$, we obtain

$$\begin{aligned} V(Ex(t_1)) - V(Ex(t_0)) &= \int_{t_0}^{t_1} \tfrac{\mathrm{d}}{\mathrm{d}t} V(Ex(t))\mathrm{d}t \\ &= \int_{t_0}^{t_1} (\nabla V(Ex(t)))^\top \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t)\mathrm{d}t \\ &= \int_{t_0}^{t_1} (\nabla V(Ex(t)))^\top (Ax(t) + Bu(t))\mathrm{d}t \\ &\geq -\int_{t_0}^{t_1} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t = -\mathcal{J}(x, u, [t_0, t_1]), \end{aligned}$$

i. e., the dissipation inequality (3.1) is fulfilled. ◻

Next we show that the set of quadratic storage functions corresponds to the set of solutions of the KYP inequality.

THEOREM 3.9. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Then the following two statements are equivalent for $P = P^\top \in \mathbb{R}^{n\times n}$:*

i) It holds that $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$, $Ex_0 \mapsto x_0^\top E^\top PEx_0$ is a storage function.

ii) The matrix P solves the KYP inequality (3.5).

Proof. First note that for $P = P^\top \in \mathbb{R}^{n\times n}$, the function $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$, $Ex_0 \mapsto x_0^\top E^\top PEx_0$ fulfills $\nabla V(Ex_0) = 2PEx_0$. Thus for all $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ we have

$$\begin{aligned} (\nabla V(Ex_0))^\top (Ax_0 + Bu_0) &= 2x_0^\top E^\top P(Ax_0 + Bu_0) \\ &= \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^\top \begin{bmatrix} A^\top PE + E^\top PA & E^\top PB \\ B^\top PE & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}. \end{aligned} \tag{3.7}$$

Now we show that "i)⇒ii)": Assume that $P = P^\top \in \mathbb{R}^{n\times n}$ and that $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$, $Ex_0 \mapsto x_0^\top E^\top PEx_0$ is a storage function. Let $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ be given. By combining (3.7) with Proposition 3.8, we obtain that the KYP inequality (3.5) is satisfied.

Next we show "ii)⇒i)": If P fulfills (3.5), then for all $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ we have

$$\begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \geq - \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^\top \begin{bmatrix} A^\top PE + E^\top PA & E^\top PB \\ B^\top PE & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}.$$

By further using that $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$, $Ex_0 \mapsto x_0^\top E^\top PEx_0$ fulfills (3.7), we obtain from Proposition 3.8 that V is a storage function. □

Before we prove that the value functions are quadratic storage functions, we show the connection between stabilizability and feasibility of the optimal control problem **(OC+)**.

Lemma 3.10. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. If **(OC+)** is feasible, then $[E, A, B]$ is behaviorally stabilizable.*

Proof. Assume that **(OC+)** is feasible. Then the value function V_+ defined in (1.4) satisfies $V_+(Ex_0) < \infty$ for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. In particular, the set of all solutions (x, u) of $[E, A, B]$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ is non-empty for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. This gives rise to stabilizability of $[E, A, B]$. □

Next we show that V_+ is a quadratic storage function.

Theorem 3.11. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. If **(OC+)** is feasible, then the value function V_+ defined in (1.4) is a quadratic storage function.*

Proof. Assume that **(OC+)** is feasible. Then $[E, A, B]$ is behaviorally stabilizable by Lemma 3.10. For $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, consider the set of trajectories of $[E, A, B]$ which are square integrable on $\mathbb{R}_{\geq 0}$ and with initial differential value x_0, i. e.,

$$\mathfrak{B}_{\mathcal{L}^2}(x_0) := \big\{(x, u) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \times \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \ : \ (x, u) \text{ is a solution of } [E, A, B] \text{ on } \mathbb{R}_{\geq 0} \text{ with } Ex(0) = Ex_0\big\}.$$

Note that $\mathfrak{B}_{\mathcal{L}^2}(x_0)$ is non-empty for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ since $[E, A, B]$ is behaviorally stabilizable. An element in $\mathfrak{B}_{\mathcal{L}^2}(x_0)$ can be constructed by a transformation of $[E, A, B]$ to feedback equivalence form [42, Prop. 2.9] and a subsequent construction of a stabilizing feedback on the ordinary differential equation part of the transformed system. The stabilizing input and resulting state are then in $\mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and $\mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, respectively.

Now consider the functional $\widetilde{V}_+ : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R} \cup \{-\infty, \infty\}$ with

$$\widetilde{V}_+(Ex_0) = \inf \left\{\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) \ : \ (x, u) \in \mathfrak{B}_{\mathcal{L}^2}(x_0)\right\}.$$

<u>Step 1:</u> We show that

$$V_+(Ex_0) \leq \widetilde{V}_+(Ex_0) < \infty \quad \forall\, x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}. \tag{3.8}$$

Assume that $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. Then, by stabilizability of $[E, A, B]$, there exists some $(x, u) \in \mathfrak{B}_{\mathcal{L}^2}(x_0)$. This gives rise to $\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) \in \mathbb{R}$, whence $\widetilde{V}_+(Ex_0) < \infty$. On the other hand, $(x, u) \in \mathfrak{B}_{\mathcal{L}^2}(x_0)$ implies that x, $\frac{\mathrm{d}}{\mathrm{d}t}Ex \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and therefore, we obtain from [14, Thm. 3], a variant of Barbălat's lemma, that $Ex(\infty) = 0$. Hence, $\widetilde{V}_+(Ex_0)$ is the infimum over a set which is contained in the set whose infimum is

$V_+(Ex_0)$. This implies (3.8).

Step 2: We show that $\widetilde{V}_+$ is quadratic (similarly to [2, p. 21]): To this end, we need to show that for all $\lambda \in \mathbb{R}$ and $x_0,\ x_{01},\ x_{02} \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ it holds that

$$\widetilde{V}_+(\lambda \cdot Ex_0) = |\lambda|^2 \cdot \widetilde{V}_+(Ex_0), \tag{3.9a}$$

$$\widetilde{V}_+(E(x_{01} - x_{02})) + \widetilde{V}_+(E(x_{01} + x_{02})) = 2 \cdot \widetilde{V}_+(Ex_{01}) + 2 \cdot \widetilde{V}_+(Ex_{02}). \tag{3.9b}$$

An expansion of the products in the integral yields that for all $\lambda \in \mathbb{R}$ and solutions (x_1, u_1), (x_2, u_2) of $[E, A, B]$ with (x_1, u_1), $(x_2, u_2) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \times \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, the cost function fulfills

$$\mathcal{J}(\lambda x_1, \lambda u_1, \mathbb{R}_{\geq 0}) = |\lambda|^2 \cdot \mathcal{J}(x_1, u_1, \mathbb{R}_{\geq 0}), \tag{3.10a}$$

$$\begin{aligned} 2 \cdot \mathcal{J}(x_1, u_1, \mathbb{R}_{\geq 0}) + 2 \cdot \mathcal{J}(x_2, u_2, \mathbb{R}_{\geq 0}) = \mathcal{J}&(x_1 + x_2, u_1 + u_2, \mathbb{R}_{\geq 0}) \\ &+ \mathcal{J}(x_1 - x_2, u_1 - u_2, \mathbb{R}_{\geq 0}). \end{aligned} \tag{3.10b}$$

We first prove (3.9a): We have $\widetilde{V}_+(0) \leq 0$, since $\mathcal{J}(0, 0, \mathbb{R}_{\geq 0}) = 0$. On the other hand, the existence of a solution (x, u) of $[E, A, B]$ with $Ex(0) = Ex(\infty) = 0$ with with $\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) < 0$ would imply, by taking scalar multiples of (x, u), that $V_+(0) \leq \widetilde{V}_+(0) = -\infty$. Hence, feasibility of **(OC+)** gives rise to $\widetilde{V}_+(0) = 0$. Thus, we have

$$\widetilde{V}_+(0 \cdot Ex_0) = 0 = |0|^2 \cdot \widetilde{V}_+(Ex_0).$$

Further, for all $\lambda \in \mathbb{R} \setminus \{0\}$, $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, and $\varepsilon > 0$, the definition of $\widetilde{V}_+$ leads to the existence of some $(x, u) \in \mathfrak{B}_{\mathcal{L}^2}(x_0)$ with

$$\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) \leq \widetilde{V}_+(Ex_0) + \tfrac{\varepsilon}{\lambda^2},$$

and therefore we have

$$\begin{aligned} \widetilde{V}_+(E(\lambda x_0)) &\leq \mathcal{J}(\lambda x, \lambda u, \mathbb{R}_{\geq 0}) = |\lambda|^2 \cdot \mathcal{J}(x, u, \mathbb{R}_{\geq 0}) \\ &\leq |\lambda|^2 \cdot \left(\widetilde{V}_+(Ex_0) + \tfrac{\varepsilon}{|\lambda|^2}\right) = |\lambda|^2 \cdot \widetilde{V}_+(Ex_0) + \varepsilon. \end{aligned}$$

Since the above inequality holds for all $\varepsilon > 0$ it follows that

$$\widetilde{V}_+(E(\lambda x_0)) \leq |\lambda|^2 \cdot \widetilde{V}_+(Ex_0). \tag{3.11}$$

The reverse inequality follows from

$$\widetilde{V}_+(Ex_0) = \widetilde{V}_+\left(E\left(\tfrac{1}{\lambda} \cdot \lambda x_0\right)\right) \overset{(3.11)}{\leq} \tfrac{1}{|\lambda|^2} \cdot \widetilde{V}_+(E(\lambda x_0)).$$

Altogether we obtain that (3.9a) is satisfied.

Next we show (3.9b): Assume that $x_{01},\ x_{02} \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ and $\varepsilon > 0$. The definition of $\widetilde{V}_+$ implies that there exist $(x_1, u_1) \in \mathfrak{B}_{\mathcal{L}^2}(x_{01})$, $(x_2, u_2) \in \mathfrak{B}_{\mathcal{L}^2}(x_{02})$ and

$$\mathcal{J}(x_1, u_1, \mathbb{R}_{\geq 0}) \leq \widetilde{V}_+(Ex_{01}) + \tfrac{\varepsilon}{4}, \quad \mathcal{J}(x_2, u_2, \mathbb{R}_{\geq 0}) \leq \widetilde{V}_+(Ex_{02}) + \tfrac{\varepsilon}{4}. \tag{3.12}$$

Then we obtain

$$\begin{aligned}
&\widetilde{V}_+\left(E(x_{01}+x_{02})\right)+\widetilde{V}_+\left(E(x_{01}-x_{02})\right) \\
&\qquad\qquad \leq \ \mathcal{J}(x_1+x_2,u_1+u_2,\mathbb{R}_{\geq 0})+\mathcal{J}(x_1-x_2,u_1-u_2,\mathbb{R}_{\geq 0}) \\
&\qquad\qquad \overset{(3.10b)}{=} 2\cdot\mathcal{J}(x_1,u_1,\mathbb{R}_{\geq 0})+2\cdot\mathcal{J}(x_2,u_2,\mathbb{R}_{\geq 0}) \\
&\qquad\qquad \overset{(3.12)}{\leq} \ 2\cdot\widetilde{V}_+(Ex_{01})+2\cdot\widetilde{V}_+(Ex_{02})+\varepsilon.
\end{aligned}$$

Since the above inequality holds for all $\varepsilon>0$ we have

$$\widetilde{V}_+\left(E(x_{01}+x_{02})\right)+\widetilde{V}_+\left(E(x_{01}-x_{02})\right)\leq 2\cdot\widetilde{V}_+(Ex_{01})+2\cdot\widetilde{V}_+(Ex_{02}). \tag{3.13}$$

Now we prove the reverse inequality: For $\widetilde{x}_{01}=\frac{1}{2}(x_{01}+x_{02})$ and $\widetilde{x}_{02}=\frac{1}{2}(x_{01}-x_{02})$ we have $\widetilde{x}_{01}+\widetilde{x}_{02}=x_{01}$ and $\widetilde{x}_{01}-\widetilde{x}_{02}=x_{02}$. Then (3.10a) is satisfied due to

$$\begin{aligned}
&2\cdot\widetilde{V}_+(Ex_{01})+2\cdot\widetilde{V}_+(Ex_{02}) \\
&\qquad\qquad = \ 2\cdot\widetilde{V}_+(E(\widetilde{x}_{01}+\widetilde{x}_{02}))+2\cdot\widetilde{V}_+(E(\widetilde{x}_{01}-\widetilde{x}_{02})) \\
&\qquad\qquad \overset{(3.13)}{\leq} \ 4\cdot\widetilde{V}_+(E\widetilde{x}_{01})+4\cdot\widetilde{V}_+(E\widetilde{x}_{02}) \\
&\qquad\qquad = \ 4\cdot\widetilde{V}_+\left(E\left(\tfrac{1}{2}(x_{01}+x_{02})\right)\right)+4\cdot\widetilde{V}_+\left(E\left(\tfrac{1}{2}(x_{01}-x_{02})\right)\right) \\
&\qquad\qquad \overset{(3.9a)}{=} \ \widetilde{V}_+(E(x_{01}+x_{02}))+\widetilde{V}_+(E(x_{01}-x_{02})).
\end{aligned}$$

Step 3: We prove that $\widetilde{V}_+$ is a storage function. Since $\widetilde{V}_+$ is quadratic by Step 2, it is continuous with $\widetilde{V}(0)=0$. Now assume that $t\geq 0$ and (x,u) is a solution of $[E,A,B]$ with $Ex(0)=Ex_0$. By definition of $\widetilde{V}_+$, there exists some $(\widetilde{x},\widetilde{u})\in\mathfrak{B}_{\mathcal{L}^2}(x(t))$ with

$$\mathcal{J}(\widetilde{x},\widetilde{u},\mathbb{R}_{\geq 0})\leq\widetilde{V}_+(Ex(t))+\varepsilon. \tag{3.14}$$

Consider the concatenation $(\overline{x},\overline{u})$ with $(\overline{x}(\tau),\overline{u}(\tau))=(x(\tau),u(\tau))$ for all $\tau\in[0,t]$, and $(\overline{x}(\tau),\overline{u}(\tau))=(\widetilde{x}(\tau-t),\widetilde{u}(\tau-t))$ for all $\tau\in[t,\infty)$. Then $(\overline{x},\overline{u})$ is a solution of $[E,A,B]$ with $Ex(0)=Ex_0$. In particular, we have $(\overline{x},\overline{u})\in\mathfrak{B}_{\mathcal{L}^2}(x_0)$. Then, by using time-invariance, we obtain

$$\begin{aligned}
\widetilde{V}_+(Ex_0)\leq\mathcal{J}(\overline{x},\overline{u},\mathbb{R}_{\geq 0}) \ &= \ \mathcal{J}(\overline{x},\overline{u},[0,t])+\mathcal{J}(\overline{x},\overline{u},[t,\infty)) \\
&= \ \mathcal{J}(\overline{x},\overline{u},[0,t])+\mathcal{J}(\overline{x}(\cdot+t),\overline{u}(\cdot+t),\mathbb{R}_{\geq 0}) \\
&= \ \mathcal{J}(x,u,[0,t])+\mathcal{J}(\widetilde{x},\widetilde{u},\mathbb{R}_{\geq 0}) \\
&\overset{(3.14)}{\leq} \ \mathcal{J}(x,u,[0,t])+\widetilde{V}_+(Ex(t))+\varepsilon.
\end{aligned}$$

The result follows now by time-invariance of $[E,A,B]$ and by the fact that $\varepsilon>0$ can be made arbitrarily small.

Step 4: We show that $\widetilde{V}_+=V_+$. Assume that $x_0\in\mathcal{V}^{\text{diff}}_{[E,A,B]}$. The inequality $V_+(Ex_0)\leq\widetilde{V}_+(Ex_0)$ has already been proven in Step 1. To show the reverse inequality, consider a solution (x,u) of $[E,A,B]$ with $Ex(0)=Ex_0$, $Ex(\infty)=0$ and $\mathcal{J}(x,u,\mathbb{R}_{\geq 0})\in\mathbb{R}$. Since by Step 3, $\widetilde{V}_+$ is a storage function, we obtain that for all

$t \geq 0$ it holds that

$$\widetilde{V}_+(Ex_0) - \widetilde{V}_+(Ex(t)) \leq \mathcal{J}(x, u, [0, t]).$$

Taking the limit $t \to \infty$ and using $Ex(\infty) = 0$ together with the continuity of $\widetilde{V}_+$ and $\widetilde{V}_+(0) = 0$, we obtain

$$\widetilde{V}_+(Ex_0) \leq \mathcal{J}(x, u, \mathbb{R}_{\geq 0}).$$

This implies $\widetilde{V}_+(Ex_0) \leq V_+(Ex_0)$. ◻

As an immediate consequence, we have that the value functions define special solutions of the KYP inequality.

COROLLARY 3.12. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$, $R = R^\top \in \mathbb{R}^{m \times m}$ be given. If* ***(OC+)*** *is feasible, then there exists a solution $P_+ \in \mathbb{R}^{n \times n}$ of the KYP inequality (3.5) with $V_+(Ex_0) = x_0^\top E^\top P_+ Ex_0$ for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$.*

Proof. Assume that **(OC+)** is feasible. Theorem 3.11 implies that there exists some Hermitian $P \in \mathbb{R}^{n \times n}$ with $V_+(Ex_0) = x_0^\top E^\top P_+ Ex_0$ for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. Since further by Theorem 3.11, V_+ is a storage function, Theorem 3.9 then gives rise to the fact that P_+ solves the KYP inequality (3.5). ◻

Now we present some characterizations for feasibility, and we show that V_+ has a certain extremality condition among all storage functions.

THEOREM 3.13. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$, $R = R^\top \in \mathbb{R}^{m \times m}$ be given. Then the following statements are equivalent:*

i) The problem ***(OC+)*** *is feasible, i. e., $V_+(Ex_0) \in \mathbb{R}$ for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$.*

ii) The system $[E, A, B]$ is behaviorally stabilizable and there exists a storage function V.

iii) The system $[E, A, B]$ is behaviorally stabilizable and the KYP inequality (3.5) has a solution $P \in \mathbb{R}^{n \times n}$.

Further, in case of feasibility of ***(OC+)****, all storage functions V fulfill*

$$V(Ex_0) \leq V_+(Ex_0) \quad \forall\, x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}. \tag{3.15}$$

Proof. "i)⇒iii)": Assume that **(OC+)** is feasible: Then $[E, A, B]$ is behaviorally stabilizable by Lemma 3.10. Further, the existence of a solution of the KYP inequality follows from Corollary 3.12.

"iii)⇒ii)": This follows from Theorem 3.9.

"ii)⇒i)": Assume that $[E, A, B]$ is behaviorally stabilizable and there exists a continuous $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$ with $V(0) = 0$ such that the dissipation inequality (3.1) is satisfied. Assume that $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. Behavioral stabilizability of $[E, A, B]$ yields the existence of some of a solution (x, u) on $\mathbb{R}_{\geq 0}$ with $Ex(0) = x_0$, $Ex(\infty) = 0$ and $\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) < \infty$, and thus $V(Ex_0) < \infty$. Further, by taking the limit $t \to \infty$, we obtain

$$\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) = \lim_{t \to \infty} \mathcal{J}(x, u, [0, t]) \geq \lim_{t \to \infty} V(Ex(0)) - V(Ex(t)) = V(Ex_0).$$

By taking the infimum over all solutions (x, u) with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$, we obtain that $V_+(Ex_0) \geq V(Ex_0) > -\infty$. This proves feasibility of **(OC+)** as well as the inequality (3.15). ◻

The next result shows that if **(OC+)** is feasible, then the solution set of the KYP inequality (3.5) has a maximal element with respect to definiteness on the subspace $E\mathcal{V}^{\text{diff}}_{[E,A,B]}$.

Corollary 3.14. *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. If **(OC+)** is feasible, then there exists a maximal solution $P_+ \in \mathbb{R}^{n\times n}$ of the KYP inequality (3.5) in the sense that all solutions $P \in \mathbb{R}^{n\times n}$ of the KYP inequality (3.5) satisfy*

$$P \leq_{E\mathcal{V}^{\text{diff}}_{[E,A,B]}} P_+. \tag{3.16}$$

Further, the maximal solution fulfills $V_+(Ex_0) = x_0^\top E^\top P_+ E x_0$ for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$.

Proof. If **(OC+)** is feasible, then Corollary 3.12 implies that there exists a solution $P_+ \in \mathbb{R}^{n\times n}$ of the KYP inequality (3.5) with $V_+(Ex_0) = x_0^\top E^\top P_+ E x_0$ for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$. Assume that $P \in \mathbb{R}^{n\times n}$ is a further solution of the KYP inequality (3.5). Then, by Theorem 3.9, $V : E\mathcal{V}^{\text{diff}}_{[E,A,B]} \to \mathbb{R}$, $x_0 \mapsto V(Ex_0) = x_0^\top E^\top P E x_0$ is a storage function. Then (3.16) can be concluded from (3.15). □

4. Lur'e Equations, Regularity, and Optimal Controls. In this section we take a closer look at the optimal control problem **(OC+)**. We will consider so-called *Lur'e equations*, which are - loosely speaking - derived from the KYP inequality by factorizing its left hand side.

Definition 4.1 (Lur'e equation). *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Then for some $q \in \mathbb{N}_0$, we call a triple $(P,K,L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$* solution of the Lur'e equation*, if*

$$\begin{bmatrix} A^\top PE + E^\top PA + Q & E^\top PB + S \\ B^\top PE + S^\top & R \end{bmatrix} =_{\mathcal{V}^{\text{sys}}_{[E,A,B]}} \begin{bmatrix} K^\top \\ L^\top \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^\top \tag{4.1}$$

is satisfied with

$$\operatorname{rank}_{\mathbb{R}[s]} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q. \tag{4.2}$$

A solution $(P,K,L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ of the Lur'e equation is called stabilizing*, if additionally*

$$\operatorname{rank}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall\, \lambda \in \mathbb{C}_+. \tag{4.3}$$

Note that it follows immediately that, if (P,K,L) solves the Lur'e equation, then P is a solution of the KYP inequality. Lur'e equations have been analyzed in detail in [42, 39] from a linear algebraic point of view. The numerical determination of stabilizing solutions has been considered in [40, 41].

In the previous section we have seen that the value function V_+ can be expressed by maximal solutions of the KYP inequality. In this section we will see that the stabilizing solutions of the Lur'e equation are related to the maximal solutions of the KYP inequality. We will further show that feasibility of **(OC+)** is equivalent to the existence of stabilizing solutions of the Lur'e equation. In other words, we have necessary and sufficient conditions on feasibility of the optimal control problem **(OC+)**. The stabilizing solutions of the Lur'e equations will further be used to characterize regularity and to design optimal controls. An overview of the results presented here is given in Figure 4.1.

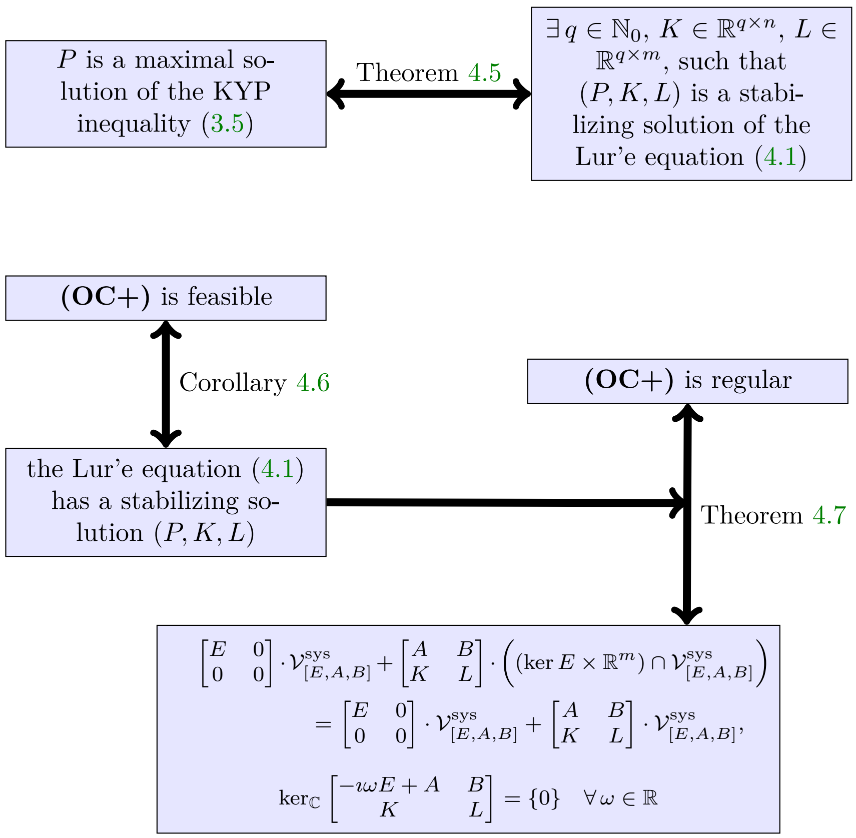

FIG. 4.1. *Results of Section 4 and their relation*

Next we show that the right hand side of the KYP inequality can be factored in a special way for which we need the following auxiliary result. This factorization will be useful to give an interpretation of the matrices K and L in (4.1).

LEMMA 4.2. *Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace and $M \in \mathbb{R}^{n\times n}$ with $M \geq_{\mathcal{V}} 0$. Then there exists some $\ell \in \mathbb{N}_0$ and $N \in \mathbb{R}^{\ell\times n}$ such that $N\mathcal{V} = \mathbb{R}^\ell$ and $M =_{\mathcal{V}} N^\top N$.*

Proof. Let $r := \dim \mathcal{V}$ and an invertible $T \in \mathbb{R}^{n\times n}$ be such that $T(\mathbb{R}^r \times \{0\}) = \mathcal{V}$. Partition

$$T^\top M T = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix}$$

with $M_{11} \in \mathbb{R}^{r\times r}$, $M_{12} \in \mathbb{R}^{r\times(n-r)}$, and $M_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$. Then for all $x \in \mathbb{R}^r$ it holds that $T\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \in \mathcal{V}$, and thus

$$x^\top M_{11} x = \begin{pmatrix} x \\ 0 \end{pmatrix}^\top \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}^\top T^\top M T \begin{pmatrix} x \\ 0 \end{pmatrix} \geq 0.$$

As a consequence, $M_{11} \geq 0$. Define $\ell := \operatorname{rank}_{\mathbb{R}} M_{11}$. From the positive semidefiniteness of M_{11}, we obtain that there exists some $N_1 \in \mathbb{R}^{\ell\times r}$ with $M_{11} = N_1^\top N_1$.

In particular, N_1 has full row rank. Now define $N := \begin{bmatrix} N_1 & 0 \end{bmatrix} T^{-1}$. Then

$$N\mathcal{V} = \begin{bmatrix} N_1 & 0 \end{bmatrix} T^{-1}\mathcal{V} = \begin{bmatrix} N_1 & 0 \end{bmatrix} (\mathbb{R}^r \times \{0\}) = \operatorname{im}_{\mathbb{R}} N_1 = \mathbb{R}^\ell.$$

Next we show that $M =_{\mathcal{V}} N^\top N$: Assume that $v \in \mathcal{V}$. Then there exists some $x \in \mathbb{R}^r$ with $v = T\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$. Further we have

$$\begin{aligned}\begin{pmatrix} x \\ 0 \end{pmatrix}^\top T^\top M T \begin{pmatrix} x \\ 0 \end{pmatrix} = x^\top M_{11} x = x^\top N_1^\top N_1 x = \begin{pmatrix} x \\ 0 \end{pmatrix}^\top \begin{bmatrix} N_1 & 0 \end{bmatrix}^\top \begin{bmatrix} N_1 & 0 \end{bmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \\ = v^\top \left(T^{-\top}\right) \begin{bmatrix} N_1 & 0 \end{bmatrix}^\top \begin{bmatrix} N_1 & 0 \end{bmatrix} T^{-1} v = v^\top N^\top N v.\end{aligned}$$ □

Lemma 4.3. *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Assume that $P \in \mathbb{R}^{n\times n}$ solves the KYP inequality (3.5). Then there exist $q \in \mathbb{N}_0$, $K \in \mathbb{R}^{q\times n}$, and $L \in \mathbb{R}^{q\times m}$ such that (4.1) and*

$$\begin{bmatrix} K & L \end{bmatrix} \mathcal{V}^{\text{sys}}_{[E,A,B]} = \mathbb{R}^q. \tag{4.4}$$

Further, the dissipation inequality can be reformulated to

$$\begin{aligned}\mathcal{J}(x,u,[t_0,t_1]) + x(t_1)^\top E^\top P E x(t_1) = x(t_0)^\top E^\top P E x(t_0) + \|Kx + Lu\|^2_{\mathcal{L}^2([t_0,t_1],\mathbb{R}^q)} \\ \text{for all solutions } (x,u) \text{ of } [E,A,B] \text{ and all } t_0,\, t_1 \in \mathbb{R} \text{ with } t_0 \le t_1.\end{aligned} \tag{4.5}$$

Proof. The existence of $q \in \mathbb{N}_0$, $K \in \mathbb{R}^{q\times n}$, and $L \in \mathbb{R}^{q\times m}$ such that (4.1) and (4.4) are satisfied follows from an application of Lemma 4.2 to the matrix on the left hand side of the KYP inequality (3.5) and the subspace $\mathcal{V}^{\text{sys}}_{[E,A,B]}$. The assertion in (4.5) follows by an argument as in the proof of statement "ii)⇒i)" from Theorem 3.9. □

Willems called $Kx+Lu$ the *dissipation rate* in his pivotal article [47] on optimal control of ordinary differential equations. We can conclude the following for the dissipation inequality on the whole positive time axis.

Corollary 4.4. *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Assume that $P \in \mathbb{R}^{n\times n}$ solves the KYP inequality (3.5) and. Suppose further that there exist $q \in \mathbb{N}_0$, $K \in \mathbb{R}^{q\times n}$, and $L \in \mathbb{R}^{q\times m}$, such that (4.1) holds. Assume that $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$ and let a solution (x,u) of $[E,A,B]$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ on $\mathbb{R}_{\ge 0}$ be given. Then $\mathcal{J}(x,u,\mathbb{R}_{\ge 0})$ is finite, if and only if $Kx + Lu \in \mathcal{L}^2(\mathbb{R}_{\ge 0},\mathbb{R}^q)$. In this case we have*

$$\mathcal{J}(x,u,\mathbb{R}_{\ge 0}) = x_0^\top E^\top P E x_0 + \|Kx + Lu\|^2_{\mathcal{L}^2(\mathbb{R}_{\ge 0},\mathbb{R}^q)}. \tag{4.6}$$

Proof. Let $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$ and (x,u) be a solution of $[E,A,B]$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$. Then $x(\infty)^\top E^\top P E x(\infty) = 0$ and we see that, by taking the limit $t \to \infty$, that the left hand side in (4.5) converges, if and only if the right hand side in (4.5) converges. This limiting process further gives rise to equation (4.6). □

Now we show that the stabilizing solutions of the Lur'e equation represent the value function for **(OC+)** and moreover, that they determine the optimal controls. This is done in the following theorem. The condition $Kx_* + Lu_* = 0$ obtained below results in a closed-loop system that is "outer", see [20].

THEOREM 4.5. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Further, assume that the problem **(OC+)** is feasible. Then the following two statements are equivalent for $P_+ \in \mathbb{R}^{n\times n}$:*

i) The matrix P_+ is a maximal solution of the KYP inequality (3.5).

ii) There exist $q \in \mathbb{N}_0$, $K \in \mathbb{R}^{q\times n}$, and $L \in \mathbb{R}^{q\times m}$ such that (P_+, K, L) is a stabilizing solution of the Lur'e equation (4.1).

In this case, the value function fulfills $V_+(Ex_0) = x_0^\top E^\top P_+ E x_0$ for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. Further, a solution (x_*, u_*) of $[E, A, B]$ on $\mathbb{R}_{\geq 0}$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ is an optimal control, if and only if $Kx_* + Lu_* = 0$.

Proof. Assume that **(OC+)** is feasible.

First we show "i)⇒ii)": Let $P_+ \in \mathbb{R}^{n\times n}$ be a maximal solution of the KYP inequality (3.5). By Corollary 3.14, the value function fulfills $V_+(Ex_0) = x_0^\top E^\top P_+ E x_0$ for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. Moreover, by Lemma 4.3, we obtain that there exist $q \in \mathbb{N}_0$, $K \in \mathbb{R}^{q\times n}$, $L \in \mathbb{R}^{q\times m}$ such that $\begin{bmatrix} K & L \end{bmatrix} \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} = \mathbb{R}^q$ and (4.1) is satisfied for $P = P_+$. Now we show that (P, K, L) is a stabilizing solution of the Lur'e equation. To this end, we have to prove that (4.3) is fulfilled. According to [20, Thm. 6.6 a)], this is the case if the following two statements are valid:

1) If for $y_0 \in \mathbb{R}^q$ we have $y_0^\top (Kx + Lu) \equiv 0$ for all solutions (x, u) of $[E, A, B]$, then $y_0 = 0$.
2) For all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ and $\varepsilon > 0$, there exists a solution (x, u) of $[E, A, B]$ on $\mathbb{R}_{\geq 0}$ with $Ex(0) = Ex_0$, $Ex(\infty) = 0$ and $\|Kx + Lu\|_{\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^q)} < \varepsilon$.

First we show 1): Assume that for $y_0 \in \mathbb{R}^q$ we have $y_0^\top (Kx + Lu) \equiv 0$ for all solutions (x, u) of $[E, A, B]$. Since $\begin{bmatrix} K & L \end{bmatrix} \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} = \mathbb{R}^q$, there exists some $(x_0, u_0) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ with $Kx_0 + Lu_0 = y_0$. By Lemma 3.7, there exists some infinitely differentiable solution (x, u) of $[E, A, B]$ with $x(0) = x_0$ and $u(0) = u_0$. Then $0 = y_0^\top (Kx(0) + Lu(0)) = y_0^\top (Kx_0 + Lu_0) = y_0^\top y_0 = \|y_0\|^2$, which implies $y_0 = 0$.

Next we show 2): Suppose that $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ and $\varepsilon > 0$. By definition of the value function, there exists some solution (x, u) of $[E, A, B]$ on $\mathbb{R}_{\geq 0}$ with $Ex(0) = Ex_0$, $Ex(\infty) = 0$ and $\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) - V_+(Ex_0) < \varepsilon^2$. On the other hand, by using $V_+(Ex_0) = x_0^\top E^\top P E x_0$ and Corollary 4.4, we obtain

$$\|Kx + Lu\|^2_{\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^q)} = \mathcal{J}(x, u, \mathbb{R}_{\geq 0}) - x_0^\top E^\top P E x_0 = \mathcal{J}(x, u, \mathbb{R}_{\geq 0}) - V_+(Ex_0) < \varepsilon^2,$$

and the proof of "i)⇒ii)" is complete.

Next we show "ii)⇒i)": Assume that $(P_+, K, L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is a stabilizing solution of the Lur'e equation (4.1). By Theorem 3.13, feasibility of **(OC+)** implies that $[E, A, B]$ is behaviorally stabilizable. Further, P_+ is a solution of the KYP inequality (3.5). Then the value function fulfills $x_0^\top E^\top P_+ E x_0 \leq V_+(Ex_0)$ by Theorem 3.13. To prove that also $x_0^\top E^\top P_+ E x_0 \geq V_+(Ex_0)$, let $\varepsilon > 0$. Since (P_+, K, L) is a stabilizing solution, we know that (4.3) is satisfied. Then, by [20, Thm. 6.6], there exists a solution (x, u) of $[E, A, B]$ on $\mathbb{R}_{\geq 0}$ with $Ex(0) = Ex_0$, $Ex(\infty) = 0$ and $\|Kx + Lu\|^2_{\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^q)} < \varepsilon$. By using Corollary 4.4, we obtain that this trajectory fulfills

$$\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) = x_0^\top E^\top P_+ E x_0 + \|Kx + Lu\|^2_{\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^q)} < x_0^\top E^\top P_+ E x_0 + \varepsilon,$$

and thus $x_0^\top E^\top P_+ E x_0 \geq V_+(Ex_0)$. Altogether, this implies that $x_0^\top E^\top P_+ E x_0 = V_+(Ex_0)$, whence, by Corollary 3.14, $P \leq_{E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} P_+$ for all solutions $P \in \mathbb{R}^{n\times n}$ of the KYP inequality (3.5).

Now we prove the remaining statements: The fact that $V_+(Ex_0) = x_0^\top E^\top P_+ E x_0$ for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ has already been shown in Corollary 3.14. Assume that $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. To complete the proof, we need to show that a solution (x_*, u_*) of $[E,A,B]$ on $\mathbb{R}_{\geq 0}$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ is an optimal control, if and only if $Kx_* + Lu_* = 0$: If a solution (x_*, u_*) of $[E,A,B]$ on $\mathbb{R}_{\geq 0}$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ is an optimal control, then $V_+(Ex_0) = \mathcal{J}(x_*, u_*, \mathbb{R}_{\geq 0})$. Then by using $V_+(Ex_0) = x_0^\top E^\top P_+ E x_0$ and Corollary 4.4, we obtain

$$\begin{aligned} x_0^\top E^\top P_+ E x_0 = V_+(Ex_0) &= \mathcal{J}(x_*, u_*, \mathbb{R}_{\geq 0}) \\ &= x_0^\top E^\top P_+ E x_0 + \|Kx_* + Lu_*\|^2_{\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^q)}, \end{aligned}$$

and thus $Kx_* + Lu_* = 0$. On the other hand, by the same argument, we see that $x_0^\top E^\top P_+ E x_0 = V_+(Ex_0) = \mathcal{J}(x_*, u_*, \mathbb{R}_{\geq 0})$, if $Kx_* + Lu_* = 0$ is satisfied. ∎

As a consequence, we can show that the existence of a stabilizing solution is equivalent to the feasibility of the underlying optimal control problem.

Corollary 4.6. *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Then the following two statements are equivalent:*

i) The system $[E,A,B]$ is behaviorally stabilizable and the Lur'e equation (4.1) has a stabilizing solution $(P,K,L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$.

*ii) The optimal control problem **(OC+)** is feasible.*

Proof. We show "i)⇒ii)": Assume that $[E,A,B]$ is behaviorally stabilizable and assume that $(P,K,L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is a stabilizing solution of the Lur'e equation (4.1). Then, P solves the KYP inequality (3.5) and Theorem 3.13 gives rise to the feasibility of the optimal control problem **(OC+)**.

We conclude "ii)⇒i)": If the optimal control problem **(OC+)** is feasible, then Corollary 3.14 yields that the KYP inequality (3.5) has a maximal solution P. Then, by Theorem 4.5, there exist $q \in \mathbb{N}_0$, $K \in \mathbb{R}^{q\times n}$, and $L \in \mathbb{R}^{q\times m}$ such that (P,K,L) is a stabilizing solution of the Lur'e equation (4.1). ∎

Theorem 4.5 shows that a solution (x_*, u_*) of $[E,A,B]$ on $\mathbb{R}_{\geq 0}$ with $Ex_*(0) = Ex_0$ and $Ex_*(\infty) = 0$ is an optimal control, if and only if it fulfills the differential-algebraic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_* \\ u_* \end{pmatrix} = \begin{bmatrix} A & B \\ K & L \end{bmatrix}\begin{pmatrix} x_* \\ u_* \end{pmatrix}. \tag{4.7}$$

As a consequence, regularity corresponds to the unique solvability of the differential-algebraic equation (4.7) for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. This is characterized in the following theorem.

Theorem 4.7. *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. If **(OC+)** is feasible and $(P,K,L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is a stabilizing solution of the Lur'e equation (4.1), then the following two statements are equivalent:*

*i) The problem **(OC+)** is regular.*

ii) The conditions

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \cdot \mathcal{V}^{\text{sys}}_{[E,A,B]} + \begin{bmatrix} A & B \\ K & L \end{bmatrix} \cdot \left((\ker E \times \mathbb{R}^m) \cap \mathcal{V}^{\text{sys}}_{[E,A,B]} \right) = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \cdot \mathcal{V}^{\text{sys}}_{[E,A,B]} + \begin{bmatrix} A & B \\ K & L \end{bmatrix} \cdot \mathcal{V}^{\text{sys}}_{[E,A,B]} \tag{4.8}$$

and

$$\ker_{\mathbb{C}} \begin{bmatrix} -\imath\omega E + A & B \\ K & L \end{bmatrix} = \{0\} \quad \forall\, \omega \in \mathbb{R} \tag{4.9}$$

are satisfied.

Proof. Since Theorem 4.5 implies that optimal controls are exactly those solutions of $[E, A, B]$ on $\mathbb{R}_{\geq 0}$ which fulfill (4.7), we know that i) is equivalent to

i') For all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$, the differential-algebraic equation (4.7) with initial condition $Ex_*(0) = Ex_0$ and final condition $Ex_*(\infty) = 0$ has a unique solution.

The rest of the proof proceeds in several steps.

Step 1: We show that i') implies

$$\ker_{\mathbb{R}[s]} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = \{0\}. \tag{4.10}$$

Assuming the opposite, then we obtain from [8, Cor. 5.2] that the solution set of the differential-algebraic equation (4.7) defines a non-autonomous system in the sense of [38, Def. 3.2.1]. Since, by [38, Thm. 5.2.14], the solution set of (4.7) can be represented as a direct sum of an autonomous and a behaviorally controllable part, the latter is nontrivial. Then we can conclude from [38, Thm. 5.2.14] that there exist some nontrivial solution (x_*, u_*) of (4.7) with $(x_*, u_*)|_{\mathbb{R}_{\leq 0}} = 0$ and some $T > 0$ such that $(x_*, u_*)|_{[T,\infty)} = 0$. This is a contradiction to i').

Step 2: We show that i') implies (4.9). Aiming at a contradiction, assume that i') is satisfied and that $\ker_{\mathbb{C}} \left[\begin{smallmatrix} -\imath\omega E+A & B \\ K & L \end{smallmatrix}\right] \neq \{0\}$ for some $\omega \in \mathbb{R}$. Let $\left(\begin{smallmatrix} x_{c0} \\ u_{c0} \end{smallmatrix}\right) \in \mathbb{C}^n \times \mathbb{C}^m \setminus \{(0,0)\}$ be an element of this nullspace. It is no loss of generality to assume that the component-wise real part $\begin{pmatrix} \operatorname{Re}(x_{c0}) \\ \operatorname{Re}(u_{c0}) \end{pmatrix}$ does not vanish (otherwise, replace $\left(\begin{smallmatrix} x_{c0} \\ u_{c0} \end{smallmatrix}\right)$ by $\imath \left(\begin{smallmatrix} x_{c0} \\ u_{c0} \end{smallmatrix}\right)$). The function $t \mapsto \begin{pmatrix} x_c(t) \\ u_c(t) \end{pmatrix} := \left(\begin{smallmatrix} x_{c0} \\ u_{c0} \end{smallmatrix}\right) \cdot \mathrm{e}^{\imath\omega t}$ is a solution of the complex differential-algebraic equation (4.7). Since E, A, B, K, and L are real, we have that the component- and pointwise real part

$$t \mapsto \begin{pmatrix} x_*(t) \\ u_*(t) \end{pmatrix} := \operatorname{Re}\left(\begin{pmatrix} x_c(t) \\ u_c(t) \end{pmatrix} \right) = \begin{pmatrix} \operatorname{Re}(x_{c0})\cos(\omega t) - \operatorname{Im}(x_{c0})\sin(\omega t) \\ \operatorname{Re}(u_{c0})\cos(\omega t) - \operatorname{Im}(u_{c0})\sin(\omega t) \end{pmatrix}$$

solves the real differential-algebraic equation (4.7). By (4.10) and [8, Cor. 5.2], this is moreover the unique solution of (4.7) with $Ex_*(0) = Ex_0$ for $x_0 := \operatorname{Re}(x_{c0})$. The limit $Ex_*(\infty)$ does not exist. However, the optimal control with $Ex_*(0) = Ex_0$ and $Ex_*(\infty) = 0$ should satisfy (4.7). This is again a contradiction.

Step 3: Let $V \in \mathbb{R}^{(n+m)\times k}$ be a matrix with full column rank and $\operatorname{im}_{\mathbb{R}} V =$

$\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$. We show that (4.8) is equivalent to

(4.11)
$$\operatorname{im}_{\mathbb{R}} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V + \begin{bmatrix} A & B \\ K & L \end{bmatrix} V \cdot \ker_{\mathbb{R}} \left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V \right) = \operatorname{im}_{\mathbb{R}} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V + \operatorname{im}_{\mathbb{R}} \begin{bmatrix} A & B \\ K & L \end{bmatrix} V.$$

By using $\operatorname{im}_{\mathbb{R}} V = \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$, we see that (4.8) is satisfied, if we can show that

$$V \cdot \ker_{\mathbb{R}} \left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V \right) = (\ker_{\mathbb{R}} E \times \mathbb{R}^m) \cap \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}.$$

To show "$\subseteq$", assume that $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in V \cdot \ker_{\mathbb{R}} (\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right] V)$. Then there exists some $w_0 \in \mathbb{R}^k$ with $Vw_0 = \left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right)$ and $\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right] Vw_0 = 0$. This gives rise to

$$\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = Vw_0 \in (\ker_{\mathbb{R}} E \times \mathbb{R}^m) \cap \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}.$$

For the inclusion "$\supseteq$", assume that $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in (\ker_{\mathbb{R}} E \times \mathbb{R}^m) \cap \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$. Then, by $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$, there exists some $w_0 \in \mathbb{R}^k$ with $Vw_0 = \left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right)$. By $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in (\ker_{\mathbb{R}} E \times \mathbb{R}^m)$, we obtain

$$0 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw_0 = 0,$$

and thus

$$\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = Vw_0 \in V \cdot \ker_{\mathbb{R}} \left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V \right).$$

Step 4: Let $V \in \mathbb{R}^{(n+m)\times k}$ be a matrix with full column rank and $\operatorname{im}_{\mathbb{R}} V = \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$. We show that $w \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$ fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw = \begin{bmatrix} A & B \\ K & L \end{bmatrix} Vw, \tag{4.12}$$

if and only if $\left(\begin{smallmatrix} x_* \\ u_* \end{smallmatrix}\right) := Vw$ fulfills (4.7): If w fulfills (4.12), then $\left(\begin{smallmatrix} x_* \\ u_* \end{smallmatrix}\right) = Vw$ clearly satisfies (4.7). On the other hand, if (x_*, u_*) fulfills (4.7), then (x_*, u_*) is a solution of $[E, A, B]$, and thus $(x_*, u_*) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}) = \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \operatorname{im}_{\mathbb{R}} V)$. Then there exists some $w \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$ such that $\left(\begin{smallmatrix} x_* \\ u_* \end{smallmatrix}\right) = Vw$. Plugging this into (4.7), we obtain (4.12).

Step 5: We show that i') implies (4.8): Assume that i') holds and let $V \in \mathbb{R}^{(n+m)\times k}$ be a matrix with full column rank and $\operatorname{im}_{\mathbb{R}} V = \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$. We first show that for all $w_0 \in \mathbb{R}^k$ there exists a $w \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$ such that (4.12) is fulfilled with

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw(0) = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw_0. \tag{4.13}$$

Assume that $w_0 \in \mathbb{R}^k$ is given. Then, by Lemma 3.7, for $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) := Vw_0 \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ there exists some infinitely differentiable solution (x_*, u_*) of $[E, A, B]$ with $x_*(0) = x_0$ and $u_*(0) = u_0$. As a consequence, $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ and i') implies that the differential-algebraic equation (4.7) has a unique solution with $Ex_*(0) = Ex_0$. By the result from Step 4, we obtain that there exists some $w \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$ such that (4.12) is fulfilled with $Vw = \left(\begin{smallmatrix} x_* \\ u_* \end{smallmatrix}\right)$. We further have

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V w_0 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} Ex_0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} Ex(0) \\ 0 \end{pmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_*(0) \\ u_*(0) \end{pmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V w(0).$$

We have shown that the differential-algebraic equation (4.12) with initial condition (4.13) has a solution for all $w_0 \in \mathbb{R}^n$. By using [8, Cor. 4.3], we obtain that (4.11) is satisfied. Then, by the result from Step 3, (4.8) is fulfilled.

Step 6: We show that (4.9) implies that all solutions (x_*, u_*) of (4.7) satisfy $Ex_*(\infty) = 0$: Equation (4.9) implies that $\operatorname{rank}_{\mathbb{R}[s]} \left[\begin{smallmatrix} -sE+A & B \\ K & L \end{smallmatrix}\right] = n + m$. On the other hand, since (P, K, L) is a solution of the Lur'e equation, we have (4.2) which implies $q = m$. This fact together with (4.9) and since (P, K, L) is a stabilizing solution of the Lur'e equation, implies that

$$\ker_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = \{0\} \quad \forall\, \lambda \in \mathbb{C} \setminus \mathbb{C}_-.$$

Then we see from [38, Thm. 7.2.2] that all solutions (x_*, u_*) of (4.7) fulfill $Ex_*(\infty) = 0$.

Step 7: We show that (4.8) implies that for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$ there exists a solution (x_*, u_*) of (4.7) with $Ex_*(0) = Ex_0$: Assume that (4.8) is satisfied and let $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$. Then, by Lemma 3.7, there exists some $\left(\begin{smallmatrix} x_{01} \\ u_{01} \end{smallmatrix}\right) \in \mathcal{V}^{\text{sys}}_{[E,A,B]}$ with $Ex_0 = Ex_{01}$. Let $w_0 \in \mathbb{R}^k$ with $\left(\begin{smallmatrix} x_{01} \\ u_{01} \end{smallmatrix}\right) = V w_0$. Since, by Step 3, (4.8) implies (4.11), [8, Cor. 4.3] implies that there exists a solution $w \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^k)$ of the differential-algebraic equation (4.12) with initial condition (4.13). Then, by Step 4, we obtain that $\left(\begin{smallmatrix} x_* \\ u_* \end{smallmatrix}\right) := Vw$ fulfills (4.7). Further, we have

$$Ex_0 = Ex_{01} = \begin{bmatrix} E & 0 \end{bmatrix} \begin{pmatrix} x_{01} \\ u_{01} \end{pmatrix} = \begin{bmatrix} E & 0 \end{bmatrix} V w_0$$
$$= \begin{bmatrix} E & 0 \end{bmatrix} V w(0) = \begin{bmatrix} E & 0 \end{bmatrix} \begin{pmatrix} x_*(0) \\ u_*(0) \end{pmatrix} = Ex_*(0).$$

Step 8: We deduce the overall statement: By the initial statement in this proof, it suffices two prove the equivalence between i') and (4.8), (4.9). We obtain from Step 2 that i') implies (4.9) and from Step 5 that i') implies (4.8). This yields the implication "i)⇒ii)".

Now we show the converse. By Step 6, we obtain that, if (4.9) is satisfied, then all solutions (x_*, u_*) of (4.7) fulfill $Ex_*(\infty) = 0$. If (4.8) is fulfilled, then by Step 7, for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$ there exists a solution (x_*, u_*) of (4.7) with $Ex_*(0) = Ex_0$. To deduce that (4.8), (4.9) imply i'), it suffices to justify that the prescription of the initial condition $Ex_*(0) = Ex_0$ yields a unique solution of (x_*, u_*) of (4.7). This is however a consequence of [8, Cor. 5.2] and (4.10) proven in Step 1. ∎

5. Linear-Quadratic Optimal Control on the Negative Time Horizon.

In this section, we briefly discuss results for the optimal control problem **(OC–)**. To this end, we need the following concepts:

Definition 5.1 (Anti-stabilizability). *The system $[E, A, B] \in \Sigma_{n,m}$ is called* behaviorally anti-stabilizable*, if for all solutions (x, u) of $[E, A, B]$ on $\mathbb{R}_{\geq 0}$ there exists* *a solution* $(\widetilde{x},\widetilde{u})$ *of* $[E,A,B]$ *with* $(\widetilde{x},\widetilde{u})|_{\mathbb{R}_{\geq 0}} = (x,u)$ *and*

$$\lim_{t\to-\infty} \operatorname*{ess\,sup}_{\tau<t} \|(\widetilde{x}(\tau),\widetilde{u}(\tau))\| = 0.$$

Definition 5.2 (Anti-stabilizing solution of the Lur'e equation). *Let* $[E,A,B] \in \Sigma_{n,m}$ *and* $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ *be given. Then for some* $q \in \mathbb{N}_0$, *we call a triple* $(P,K,L) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{q\times n}\times\mathbb{R}^{q\times m}$ *an* anti-stabilizing solution of the Lur'e equation (4.1), *if it fulfills* (4.1) *with, additionally*

$$\operatorname{rank}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall\, \lambda \in \mathbb{C}_-. \tag{5.1}$$

We will not prove the following statements for **(OC–)** in detail, since they can be shown completely analogously to the results for **(OC+)**. The main arguments that are used for the proofs are the following two facts:

- Replacing E by $-E$ reflects the solutions, i. e.,

$$(x(\cdot),u(\cdot)) \text{ is a solution of } [E,A,B] \iff (x(-\cdot),u(-\cdot)) \text{ is a solution of } [-E,A,B].$$

- The functional V_- is the value function for the optimal cost in **(OC–)**, if and only if $\widehat{V}_+ := -V_-$ is the value function corresponding to the optimal control problem for the system $[-E,A,B]$ on the positive time axis.

Now the following statements can be concluded:

a) If **(OC–)** is feasible, then the value function V_- is a quadratic storage function. In this case, for all storage functions $V : E\mathcal{V}^{\text{diff}}_{[E,A,B]} \to \mathbb{R}$ it holds that

$$V_-(Ex_0) \leq V(Ex_0) \quad \forall\, x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]},$$

see also Theorem 3.11 and Theorem 3.13. Then we further have $V_-(Ex_0) = x_0^\top E^\top P_- E x_0$, where $P_- \in \mathbb{R}^{n\times n}$ is a minimal solution of the KYP inequality (3.5). That is, for each solution $P \in \mathbb{R}^{n\times n}$ of the KYP inequality (3.5) it holds that

$$P_- \leq_{E\mathcal{V}^{\text{diff}}_{[E,A,B]}} P,$$

see also Corollary 3.14.

b) The problem **(OC–)** is feasible, if and only if $[E,A,B]$ is behaviorally anti-stabilizable and there exists a storage function, cf. Theorem 3.13.

c) The problem **(OC–)** is feasible, if and only if $[E,A,B]$ is behaviorally anti-stabilizable and there exists a solution of the KYP inequality, cf. Theorem 3.13.

d) The matrix $P_- \in \mathbb{R}^{n\times n}$ is a minimal solution of the KYP inequality, if and only if there exist $q \in \mathbb{N}_0$, $K \in \mathbb{R}^{q\times n}$, and $L \in \mathbb{R}^{q\times m}$ such that (P_-,K,L) is an anti-stabilizing solution of the Lur'e equation, cf. Theorem 4.5.

e) The problem **(OC–)** is feasible, if and only if $[E,A,B]$ is behaviorally anti-stabilizable and the Lur'e equation has an anti-stabilizing solution, cf. Corollary 4.6.

f) If (P,K,L) is an anti-stabilizing solution of the Lur'e equation, then (x_*,u_*)

is an optimal control for **(OC–)**, if and only if it satisfies the optimality differential-algebraic equation (4.7) with $Ex_*(0) = Ex_0$ and $Ex_*(-\infty) = 0$, cf. Theorem 4.5.

g) If (P, K, L) is an anti-stabilizing solution of the Lur'e equation, then the optimal control problem **(OC–)** is regular, if and only if (4.8) and (4.9) are satisfied, cf. Theorem 4.7.

Now we present a consequence of the previous results on behaviorally controllable systems.

THEOREM 5.3. *Let* $[E, A, B] \in \Sigma_{n,m}$ *be* behaviorally controllable *and let* $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ *be given. Then the following statements are equivalent:*

i) For all solutions (x, u) *of* $[E, A, B]$ *with* $Ex(\infty) = Ex(-\infty) = 0$ *and* $-\infty < \mathcal{J}(x, u, \mathbb{R}) < \infty$ *it holds that* $\mathcal{J}(x, u, \mathbb{R}) \geq 0$.

*ii) The problem **(OC+)** is feasible.*

*iii) The problem **(OC–)** is feasible.*

iv) There exists a storage function $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$.

v) The KYP inequality (3.5) *has a solution* $P \in \mathbb{R}^{n\times n}$.

In the case where the above assertions are valid, we have

$$-\infty < V_-(Ex_0) \leq V(Ex_0) \leq V_+(Ex_0) < \infty \quad \forall\, x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}. \tag{5.2}$$

Proof. The equivalences between ii)–v) and (5.2) follow immediately from Theorem 3.13 and the above statements a) and b) on **(OC–)**.

Next we show that i) implies ii). Assume that $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. Since $[E, A, B]$ is behaviorally controllable, it follows from [8, Def. 2.1 & Cor. 3.4] that there exists some $T < 0$ and a solution (x_-, u_-) of $[E, A, B]$ on $\mathbb{R}_{\leq 0}$ such that $(x_-, u_-)|_{(-\infty,-T]} = 0$ and $Ex_-(0) = Ex_0$. Now let (x, u) be a solution of $[E, A, B]$ on $\mathbb{R}_{\geq 0}$ with $\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) < \infty$ and $Ex(\infty) = 0$. Then the definition of weak solutions implies that $(\widetilde{x}, \widetilde{u}) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{n+m})$ with $(\widetilde{x}, \widetilde{u})|_{\mathbb{R}_{\leq 0}} = (x_-, u_-)$ and $(\widetilde{x}, \widetilde{u})|_{\mathbb{R}_{\geq 0}} = (x, u)$ is a solution of $[E, A, B]$. Moreover, by

$$\mathcal{J}(\widetilde{x}, \widetilde{u}, \mathbb{R}) = \mathcal{J}(x_-, u_-, [T, 0]) + \mathcal{J}(x, u, \mathbb{R}_{\geq 0}) \in \mathbb{R},$$

we obtain from i) that $\mathcal{J}(x_-, u_-, [T, 0]) + \mathcal{J}(x, u, \mathbb{R}_{\geq 0}) \geq 0$, whence

$$\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) \geq -\mathcal{J}(x_-, u_-, [T, 0]).$$

Now taking the infimum over all solutions (x, u) of $[E, A, B]$ on $\mathbb{R}_{\geq 0}$ with $Ex(\infty) = 0$ and $\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) < \infty$, we obtain

$$V_+(Ex_0) \geq -\mathcal{J}(x_-, u_-, [T, 0]) > -\infty.$$

Since behavioral controllability implies behavioral stabilizability, we further have

$$V_+(Ex_0) < \infty,$$

and we obtain feasibility of **(OC+)**.

We finally prove that iv) implies i). Let (x, u) be a solution of $[E, A, B]$ with $Ex(\infty) = Ex(-\infty) = 0$ and $-\infty < \mathcal{J}(x, u, \mathbb{R}) < \infty$. Then the continuity of V

together with $V(0) = 0$ and the dissipation inequality (3.1) gives rise to

$$\mathcal{J}(x,u,\mathbb{R}) = \lim_{T\to\infty} \mathcal{J}(x,u,[-T,T]) \geq \lim_{T\to\infty} V(Ex(-T)) - V(Ex(T)) = 0. \qquad \square$$

6. Illustrative Examples. In this section we will discuss a few examples in order to illustrate our theory.

Example 6.1 (Example 2.1 revisited). *Consider again Example 2.1. There we have*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R = -\frac{1}{2}.$$

With the state vector $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ *and the input* u *we have the algebraic constraint* $0 = x_2 - u$*, thus a basis matrix of the system space of* $[E,A,B] \in \Sigma_{2,1}$ *is*

$$M_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{6.1}$$

Then with $P = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$ *we get*

$$M^{\top}_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} \begin{bmatrix} A^\top PE + E^\top PA + Q & E^\top PB + S \\ B^\top PE + S^\top & R \end{bmatrix} M_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} = \begin{bmatrix} -2p_{11} & p_{11} \\ p_{11} & \frac{1}{2} \end{bmatrix}.$$

Hence, a maximal solution of the KYP inequality (3.5) is given by $P_+ = 0$*. Moreover, we have*

$$\begin{bmatrix} Q & S \\ S & R \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

so $(P_+, K, L) = \left(0_{2\times 2}, 0_{2\times 1}, \frac{1}{\sqrt{2}}\right)$ *is a stabilizing solution of the corresponding Lur'e equation (4.1). The optimality differential-algebraic system (4.7) is*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} x_{*,1} &= -x_{*,1} + x_{*,2}, \quad x_{*,1}(0) = x_{01}, \\ 0 &= x_{*,2} - u_*, \\ 0 &= \frac{1}{\sqrt{2}} u_*, \end{aligned}$$

which yields $u_* = 0$*, hence* $x_{*,2} = 0$ *and therefore,* $x_{*,1}(t) = \mathrm{e}^{-t} \cdot x_{01}$ *as already shown in Example 2.1. By the uniqueness of the the optimal control, the problem is obviously regular. This can also be seen by checking the conditions (4.8) and (4.9). We obtain*

$$\begin{aligned} &\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \cdot \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} + \begin{bmatrix} A & B \\ K & L \end{bmatrix} \cdot \left((\ker E \times \mathbb{R}^m) \cap \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \right) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} + \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} + \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$
$$= \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \cdot \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} + \begin{bmatrix} A & B \\ K & L \end{bmatrix} \cdot \mathcal{V}^{\mathrm{sys}}_{[E,A,B]},$$

so (4.8) *is satisfied. Moreover, the matrix pencil* $\left[\begin{smallmatrix} -sE+A & B \\ K & L \end{smallmatrix}\right] \in \mathbb{R}[s]^{3\times 3}$ *has only the eigenvalue* -1 *(and two infinite eigenvalues), i. e.,* (4.9) *holds also true.*

Example 6.2. *Consider the optimal control problem*

$$\begin{gathered} \textit{Minimize } \mathcal{J}(x,u,\mathbb{R}_{\geq 0}) = \int_0^\infty -x_2^2(t) + u^2(t)\mathrm{d}t \\ \textit{subject to } \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \textit{with } x_2(0) = x_{02} \textit{ and } x_2(\infty) = 0, \end{gathered}$$

i. e., we have

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R = 1.$$

Using a similar argument as in Example 2.1 we see that a basis matrix of the system space of $[E,A,B] \in \Sigma_{2,1}$ *is given by* (6.1)*. Then with* $P = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$ *we get*

$$M^\top_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} \begin{bmatrix} A^\top PE + E^\top PA + Q & E^\top PB + S \\ B^\top PE + S^\top & R \end{bmatrix} M_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} = \begin{bmatrix} 0 & -p_{11} \\ -p_{11} & 0 \end{bmatrix}.$$

Hence, a maximal solution of the KYP inequality (3.5) *is* $P_+ = 0$*. Moreover, we have*

$$\begin{bmatrix} Q & S \\ S & R \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} 0,$$

so $(P_+, K, L) = (0_{2\times 2}, 0_{2\times 0}, 0_{1\times 0})$ *is a stabilizing solution of the corresponding Lur'e equation* (4.1) *(note that* K *and* L *are void in this example). So in fact, every solution of the differential-algebraic system* $[E,A,B]$ *is an optimal control, hence the problem is not regular. This can also be inferred from* (4.9)*, since* $\begin{bmatrix} -\imath\omega E + A & B \end{bmatrix}$ *has a nontrivial kernel for each* $\omega \in \mathbb{R}$*.*

7. Notes and References.

In this section we discuss the relation of our new results to work that was previously done. Our approach via storage and value functions is motivated by Jan C. Willems' article [47], where the optimal control problems discussed here have been introduced for systems governed by ordinary differential equations. Feasibility conditions in terms of the solvability of the KYP inequality, an algebraic Riccati inequality, and the algebraic Riccati equation have been developed under the additional assumption of controllability. In order to solve the optimal control problem (in the case where R is invertible, which is for instance the case if it is regular) one employs the *algebraic Riccati equation (ARE)* [47, 32]

$$A^\top X + XA - \big(XB + S\big)R^{-1}\big(B^\top X + S^\top\big) + Q = 0, \quad X = X^\top. \tag{7.1}$$

In the literature there exist various attempts to generalize the KYP inequality and the ARE to differential-algebraic systems. Many works [17, 29, 30, 21] focus on the case where the weight matrix is positive semi-definite, i. e.,

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \geq 0. \tag{7.2}$$

In this case, the $P = 0$ solves KYP inequality and, as a consequence of Theorem 3.13, feasibility of the optimal control problem **(OC+)** is equivalent to behavioral stabilizability of $[E, A, B]$. In [17], the optimal control problem with the additional assumption (7.2) is approached by considering a generalized KYP inequality

$$\begin{bmatrix} A^\top PE + E^\top PA + Q & E^\top PB + S \\ B^\top PE + S^\top & R \end{bmatrix} \geq 0, \quad P = P^\top. \tag{7.3}$$

That is, the KYP inequality is considered on the whole space $\mathbb{R}^{n+m}$ instead of the system space as in our work. In [17, Thm. 4.7] it is shown that if the system $[E, A, B]$ is impulse controllable (see, e. g., [8]) and behaviorally stabilizable, then there exists a maximal solution of (7.3) such that we obtain

$$V_+(Ex_0) = x_0^\top E^\top PEx_0 \geq 0 \quad \forall\, x_0 \in \mathbb{R}^n.$$

In contrast to that, our approach can also handle systems that are not impulse controllable and which appear frequently in applications such as flow control (see, e. g., [7]). In the case where the positivity condition (7.2) on the cost functional is not satisfied, it is possible that the KYP inequality (7.3) has no solution at all even if the linear-quadratic optimal control problem is feasible. The latter has been observed in [15] in the context of passive systems. This is due to the fact that the inequality (7.3) is not restricted to the system space. On the other hand, existence and uniqueness results for optimal controls with positive semi-definite cost functional are presented in [17]. These conditions are based on rank conditions for the pencil $\left[\begin{smallmatrix} -sE+A & B \\ C & D \end{smallmatrix}\right]$.

Another approach which was originally designed for behavior systems is presented in [11, 10]. There, also specializations to differential-algebraic systems are given, however under the additional assumption that $[E, A, B]$ is completely controllable (a much stronger condition than behavioral controllability, see [8]). These considerations are based on the linear matrix inequality

$$\begin{bmatrix} A^\top H + H^\top A + Q & A^\top J + H^\top B + S \\ B^\top H + J^\top A + S^\top & B^\top J + J^\top B + R \end{bmatrix} \geq 0, \quad E^\top H = H^\top E, \quad E^\top J = 0, \tag{7.4}$$

which has to be solved for a pair $(H, J) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. It is shown that for impulse controllable systems, $P = H$ is a solution of (3.5), if and only if there exists a matrix J such that (H, J) solves (7.4), see also [42].

Other approaches are based on generalizations of the ARE to differential-algebraic equations. One possibility is presented in [29, 30], where

$$A^\top X + X^\top A - (X^\top B + S)R^{-1}(B^\top X + S^\top) + Q = 0, \quad E^\top X = X^\top E \tag{7.5}$$

is considered under the assumptions (7.2) and $R > 0$. It has been proven that

$$V_+(Ex_0) = x_0^\top E^\top XEx_0 \geq 0 \quad \forall\, x_0 \in \mathbb{R}^n,$$

where X is a stabilizing solution of (7.5), meaning that $sE-(A-BR^{-1}(B^\top X+S^\top))$ is of index at most one and all its eigenvalues are contained in $\mathbb{C}_-$. By [12], a necessary condition for the existence of such a solution X is impulse controllability. On the other hand, a sufficient condition for the existence of a solution under the above mentioned assumptions is

$$\operatorname{rank}_{\mathbb{C}} \begin{bmatrix} -\imath\omega E+A & B \\ Q & S \\ S^\top & R \end{bmatrix} = n+m \quad \forall\, \omega \in \mathbb{R},$$

which has been shown in [21]. In [42] it is shown that if X is a stabilizing solution of (7.5), then (X,K,L) with $K=R^{-1/2}(B^\top X+S^\top)$ and $L=R^{1/2}$ is a stabilizing solution of the Lur'e equation (4.1).

A further solvability analysis of this type of equation is given in [23, 22] where the *generalized ARE*

$$A^\top X+X^\top A+Q+X^\top RX=0, \quad E^\top X=X^\top E$$

is considered for E, A, Q, $R\in\mathbb{R}^{n\times n}$ with $Q=Q^\top$ and $R=R^\top$. A solution $X\in\mathbb{R}^{n\times n}$ is called *stabilizing*, if the pencil $-sE+A+RX$ has index at most one and all its eigenvalues are in $\mathbb{C}_-$, which requires impulse controllability of $[E,A,R]\in\Sigma_{n,n}$. It is proven in [23] that solvability of the generalized ARE requires the solvability of a so-called *quadratic matrix equation*. Moreover, in [22] stabilizing solutions are constructed using deflating subspaces of *Hamiltonian matrix pencils*.

In [34], the optimal control problem **(OC+)** for systems of index at most one with $R\geq 0$, $Q\geq 0$, and $S=0$ is studied. In the case $R>0$, the value function can be again expressed by the stabilizing solution X (i. e., the pencil $sE-A+BR^{-1}B^\top X$ has index at most one and all its eigenvalues are in $\mathbb{C}_-$) of the *generalized ARE*

$$A^\top XE+E^\top XA-E^\top XBR^{-1}B^\top XE+Q=0, \quad X=X^\top. \tag{7.6}$$

Again, in [42] it is discussed that if $X\in\mathbb{R}^{n\times n}$ is a stabilizing solution of (7.6), then (X,K,L) with $K=R^{-1/2}(B^\top X+S^\top)$ and $L=R^{1/2}$ is a stabilizing solution of the Lur'e equation (4.1).

One of the disadvantages of the generalization of the ARE is the need for invertibility of R, which is neither necessary for feasibility nor for regularity of the optimal control problem, see Sec. 2. If **(OC+)** is regular, then it is possible to transform the system $[E,A,B]\in\Sigma_{n,m}$ by certain feedback transformations to so-called SVD coordinates and then extract a regular optimal control problem governed by an ordinary differential equation (see [6] and [37]). Such transformations however require impulse controllability of the system.

An alternative approach to optimal control of differential-algebraic equations with scalar input has recently been published in [19]. The key ingredient of this approach is an a priori transformation to quasi-Weierstraß form (2.4) leading to an equivalent optimal control problem for ordinary differential equations.

Boundary value problems for the solution of linear-quadratic optimal control problems and the associated *even matrix pencils* have also been studied intensively in the literature. In [37], the problem of constructing solutions is mainly considered from the numerical point of view. The spectral structure of these pencils for the case $E=I_n$ and their relation to the Lur'e equation are considered in [39], whereas [45, 42] extend this analysis to the case of differential-algebraic systems. Moreover, in [45] also feasibility of the optimal control problems as well as existence and uniqueness of optimal controls have been studied for impulse controllable systems. For the latter, equivalent conditions have been given in terms of the spectrum of the matrix pencil $\left[\begin{smallmatrix} -sE+A & B \\ K & L \end{smallmatrix}\right]$.

To complete the literature review we briefly discuss some generalizations into the direction of time-varying and nonlinear differential-algebraic equations. In [31], linear-quadratic optimal control problems for time-varying differential-algebraic equations and time-varying weights Q, S, and R in the cost functional are treated. Then a time-varying boundary value problem is constructed. Necessary and sufficient conditions for feasibility of the optimal control problems are derived via an inherent Hamiltonian ordinary differential equation system which is obtained by applying certain projectors to the boundary value problem. We also refer to [3, 4] where differential-algebraic equations of index two are considered.

A different approach for time-varying optimal control problems has been developed in [27, 25, 26]. In [26] two optimality boundary value problems are considered, one constructed from the original system and another one based on a so-called *strangeness-free* formulation of the differential-algebraic system (which corresponds to impulse controllability in our context). Then the solvability conditions and solutions of both boundary value problems are studied and they are related to each other. Moreover, in the recent works [28, 36], structured global condensed forms for the optimality system are derived which allow to analyze its properties.

Control problems subject to nonlinear differential-algebraic equations are mainly treated in the works by Kunkel and Mehrmann, see for instance [24, 25]. The optimal control problem is approached in [25] by using local linearizations of the nonlinear equation which usually result in time-varying linear differential-algebraic equations and allow the application of the previously mentioned techniques.

Further possible extensions of the theory presented in this paper are manifold. An extension of our results to non-regular differential-algebraic systems is likely to be possible, some partial analysis of the KYP inequality for this case has already been given in [17], but the analysis of the Lur'e equation is still open. For the linear-quadratic optimal control of ordinary differential equations, several modified cost functionals have also been studied, e. g., the *finite time-horizon* control problem without terminal constraints [47] as well as the *infinite time-horizon* optimal control problem with free endpoint conditions [16, 44] and with terminal states converging to a linear subspace [43, 46]. It would be interesting to further study these problems for differential-algebraic equations and the singular case. Moreover, a generalization of the KYP inequality and Lur'e equation to differential-algebraic equations with time-varying coefficients seems possible. For example, we believe that the system space turns to a time-varying manifold that could be treated by the theory elaborated in [5]. This could be even further extended to nonlinear differential-algebraic equations, but in this case, also general (non-quadratic) cost functionals are of interest. In this situation, an extension of the *Hamilton-Jacobi-Bellman equation* [35] would be more beneficial.

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