Technische Universität Berlin WS 2016/17
Institut für Mathematik
LECTURER: LENA SCHOLZ

5. Exercise "Differential-Algebraic Equations"

(Due date: 30.01.2017)

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1. Exercise

Let $\mathbb{I} := [t_0, T] \subset \mathbb{R}$ be a compact time interval, $t \in \mathbb{I}$ and $f : \mathbb{I} \to \mathbb{R}^4$. Consider the DAE

$$\begin{aligned} x_1 &= f_1(t) \\ \dot{x}_1(t) + x_2(t) &= f_2(t) \\ \dot{x}_2(t) + x_3(t) &= f_3(t) \\ \dot{x}_3(t) + x_4(t) &= f_4(t) \end{aligned} \tag{1}$$

with $f(t) = [\sin(t), 0, 0, 0]^T$.

a) Determine the exact solution for $x(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]^T$.

b) Complete the DAE (1) to an initial value problem by specifying a consistent initial value x_0 such that $x_0 = x(t_0)$ at $t_0 = 0$.

c) Implement (preferably in Python or Matlab) the implicit Euler method.

d) Compute approximations x_i of $x(t_i)$ at $t_i = i \cdot h$ for $i = 0, \ldots, N$ using your implementation with $t_0 = 0$, $T = 100$ and $h = 10^{-5}$ (i.e. $N = 10^7$).
(This could take some time. Test your implementation with larger step size (e.g. 10^{-2}) beforehand!)

e) Compute the global error e_i for $i = 0, \ldots, N$ and visualize e_i if possible.
(Plot the errors component-wise, i.e. $(t_i, e_{k,i})_{i=0,\ldots,N}$ for the components $k = 1, \ldots, 4$).

f) What can you say about the magnitude of the error for the different components? How can you explain your observations?

2. Exercise

Let $\mathbb{I} := [0, 3] \subset \mathbb{R}$, $t \in \mathbb{I}$ and $-\frac{1}{3} < \eta < 0$. Consider the DAE

$$\begin{aligned} \dot{x}_1 + \eta t \dot{x}_2 + (1 + \eta)x_2 &= 0 \\ x_1 + \eta t x_2 &= exp(-t) \end{aligned} \tag{2}$$

a) For the solution $x(t)$ it holds that $x(t) = \begin{bmatrix} e^{-t} + u(t) \\ e^{-t} \end{bmatrix}$ for a function $u : \mathbb{I} \to \mathbb{R}$.
Determine the exact solution of the DAE (2) by determining $u(t)$.

b) Determine a consistent initial value for (2) at $t_0 = 0$.

c) Implement (preferably in Python or Matlab) a Runge-Kutta method for DAEs of the form

$$E(t)\dot{x}(t) = A(t)x(t) + f(t)$$

using the following Butcher tableau (Radau IIA):

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array}$$

d) Compute approximations x_i of $x(t_i)$ for $\eta = -0.28$, where $t_i = i \cdot h$ for $i = 0, \ldots, N(= 60)$ using your implementation of the Runge-Kutta method with step size $h = 0.05$.

e) Compute the global error e_i and visualize e_i if possible.

vi) Compute approximations and errors for the DAE (2) using your implementation of the implicit Euler method. Choose the same setting as in d) and comment on your observations.

Please hand in your results, printed plots and comments and send your implementations by email (lscholz@math.tu-berlin.de).