NUMERICS OF PARTIAL DIFFERENTIAL EQUATIONS

Series 1

1. A fundamental tool we will need is, that we can extend linear operators to the closure of a function space under some norm. For example given an L_2-bounded operator on the set of smooth functions C^∞, we can extend it to L_2, more precisely:

Let B be a Banach space with norm $|\cdot|$ and A be a vector space that is dense in B. Let $\mathrm{F} : A \to A$ be linear and bounded/continuous (i.e. $\exists C > 0 : \forall v \in A : |\mathrm{F}(v)| \leq C \cdot |v|$).

a) Show that there is a continuous and linear $\hat{\mathrm{F}} : B \to B$ that agrees with F on A. (Hint: Cauchy sequences and limits)

b) Show that $\hat{\mathrm{F}}$ is unique (Hint: Triangle inequality)

2. Consider the following boundary value problem (BVP) in strong formulation:

Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Find $u \in C^2([-1, 1], \mathbb{R})$ such that

$$-u'' = f$$

$$u(\pm 1) = 0$$

Find the analytical solution for $f \equiv 2$.

However usually we consider the weak version of the BVP:

Find $u \in C^2([-1, 1], \mathbb{R})$ such that

$$-\int_{-1}^{1} u'' \cdot v = \int_{-1}^{1} f \cdot v \quad \forall v \in C_0^\infty$$

$$u(\pm 1) = 0$$

The v are called *trial functions* and the space C_0^p is the space of p-times continuously differentiable functions that vanish at the boundary of our domain.

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Show that the integrals are well-defined.

Show that if u is a solution to the BVP in strong formulation above it also solves the BVP in weak formulation. Is the converse true? Prove it.

Show that we can reformulate the weak formulation into

$$\int_{-1}^{1} u' \cdot v' = \int_{-1}^{1} f \cdot v \quad \forall v \in C_0^\infty$$

Note that now we can also admit functions $u \in C^1$ and even weakly differentiable functions (will be covered in the lectures soon).

Show that C_0^∞ lies densely in C_0^1 with respect to the maximum norm.

Can we relax the requirement on v to be $v \in C_0^1$?

(Hint: You may extend the result from exercise 1 and define a suitable operator, that is linear and bounded with respect to certain norms)

3. Next we want to discretise the weak BVP from exercise 2 above. We want to consider only *trial functions* which are affine, except at the nodes $\frac{1}{2^n} \cdot i$, i.e.

$$\hat{u} \in \hat{H}_n := \left\{ f \in C : f(\pm 1) = 0 \, and \, f|_{[\frac{1}{2^n} i, \frac{1}{2^n}(i+1)]} \, is \, affine \right\}$$

As we cannot hope that the exact solution u is piecewise affine, we need to relax the problem.

Show that, if we admit as solution only a linear combination of *test functions* $v \in \hat{H}_n$, then the BVP becomes a linear system of equations with a quadratic coefficient matrix. As a basis for $\hat{H}_n$ use the hat functions H_i such that $H_i(jh) = \delta_{i,j+2^n}$, $i = 1, ..., 2^{n+1} - 1$, $j \in \mathbb{Z}$ where $h := \frac{1}{2^n}$:

What happens to the demand $u(\pm 1) = 0$?

Solve this discretised problem for $f \equiv 2$ and $n = 0, 1$ by hand.

Implement an algorithm in matlab that finds the solution of the discretized BVP for $f \equiv 2$ and $n = 1, ..., 8$. You may use the backslash operator to solve the linear system of equations. How does your program perform for $n = 20$?

How can it be improved?

What can be said about the discrete solutions at the nodes $\frac{1}{2^n} \cdot i$ if f is a polynomial of degree k? (Extra points for neat solution)

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To be handed in by: October 30th, 2014 (2.00 pm)
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