Technische Universität Berlin – Institut für Mathematik WiSe 2014/15
Prof. Dr. Reinhold Schneider

Numerics of Partial Differential Equations

Series 2

1. Is $H^1_0([a,b]) := \{f \in H^1 : f(a) = f(b) = 0\}$ dense in $H^1([a,b])$ under the norm

$$||f||^2_{H^1} := \int_a^b |f|^2 + |f'|^2 \,?$$

2. (Exercise 1 of Dirk Klindworth's Exercise sheet 1 of Numerics of partial differential equations from Summer 2011)

a) By using the definitions introduced in the Skript show that the heat transfer equation is parabolic and that the wave equation is hyperbolic.

b) Consider the PDE

$$-u_{xx} - yu_{yy} = f$$

in $\Omega = (-1,1)^2$. Discuss the type of the equation for different $(x,y) \in \Omega$.

3. The Fourier transformation $F : \mathcal{S} \to \mathcal{S} : u \mapsto \frac{1}{\sqrt{2\pi}} \int u(x) \cdot e^{-ix\omega} d\omega$ is well-defined for rapidly decreasing functions from $\mathcal{S} := \{f : C^\infty(\mathbb{R}) : \forall \alpha, \beta \in \mathbb{N} : ||x^\alpha f^{(\beta)}||_\infty < \infty\}$ and has the inverse $F^{-1} : \mathcal{S} \to \mathcal{S} : u \mapsto \frac{1}{\sqrt{2\pi}} \int u(x) \cdot e^{ix\omega} d\omega$. By using exercise 1 from the first homework sheet F can be extended to L_2 with the fact that F is isometric (i.e. bounded) with respect to the L_2-norm. To make sense of the definition of the *symbol* $\sigma(x,\xi) := -\sum \xi_i a_{ij} \xi_j$ of a PDE prove the following relation for $u \in L_2$ with $\omega \mapsto u(\omega)$:

$$\frac{\partial}{\partial x} Fu = F(-i\omega u)$$

Do we need any further assumptions for moving the derivative under the integral?

How can the n-th derivative also be written with the help of the just developed formula? Use the operator S defined by

$$Su(x) := u(-x).$$

See next page!

4. Check the product rule

$$\operatorname{div}(uf) = u \operatorname{div} f + < \operatorname{grad} u, f >$$

5. Try to show that in one dimension, functions from H^1 are continuous in a certain sense. This can actually be shown for dimension n, H^s and C^m respectively with $s < m + \frac{n}{2}$ and is called Sobolev lemma.

6. (Exercise 2 of Dirk Klindworth's Exercise sheet 3 of Numerics of partial differential equations from Summer 2011)

For $\Omega = (0, 1)$ give an example of a function $u \in C^1(\Omega)$ that does not possess a weak gradient bounded in $L^2(\Omega)$.

7. (Exercise 3 of Dirk Klindworth's Exercise sheet 3 of Numerics of partial differential equations from Summer 2011)

Show that for $\mathbf{f} \in (C^1(\Omega))^d \cap (C^0(\overline{\Omega}))^d$

$$\int_\Omega \mathbf{curl}\,\mathbf{f}\,\mathrm{d}\boldsymbol{\xi} = \int_\Gamma \mathbf{n} \times \mathbf{f}\,\mathrm{d}S$$

Hint: Use

$$\int_\Omega \langle \mathbf{curl}\,\mathbf{u}, \mathbf{f}\rangle - \langle \mathbf{u}, \mathbf{curl}\,\mathbf{f}\rangle\,\mathrm{d}\boldsymbol{\xi} = \int_\Gamma \langle \mathbf{u} \times \mathbf{f}, \mathbf{n}\rangle\,\mathrm{d}S$$

for $\mathbf{u} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the canonical basis of $\mathbb{R}^3$.

To be handed in by: November 6th, 2014 (2.00 pm)
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