Vandermonde Varieties, Mirrored Spaces and the Homology of symmetric semialgebraic sets

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For $1 \leq I \leq n$ denote by $p_i := \sum_{j=1}^{k} X_j^I$ the power-sums polynomials. For $d \in \mathbb{N}$ and $y \in \mathbb{R}^d$ the algebraic set $V_d(y) = \{x \in \mathbb{R}^n : p_1(x) = y_1, \ldots, p_d(x) = y_d\}$ is called a Vandermonde Variety. These algebraic sets have been studied by Arnold and Giventhal in connection to hyperbolic polynomials. In this talk we will generalise some results of Arnold and Giventhal on the Homology of Vandermonde Varieties, more concretely: let $H^*(V_d, \mathbb{Q})$ denote the cohomology group of a Vandermonde variety. Since V_d is invariant by the natural action of the symmetric group $\mathfrak{S}_n$ the cohomology group $H^*(V_d, \mathbb{Q})$ has the structure of an $\mathfrak{S}_n$ module. We prove that for all $\lambda \vdash k$, the multiplicity of the Specht-module $\mathbb{S}^\lambda$ in $H^i(V_{d,y}^{(k)}, \mathbb{Q})$, is zero if $\mathrm{length}(\lambda) \geq i+2d-3$. This vanishing result allows us to prove similar vanishing result for arbitrary symmetric semi-algebraic sets defined by symmetric polynomials of degrees bounded by d. As a result, we obtain for each fixed $\ell \geq 0$, an algorithm for computing the first $\ell + 1$ Betti numbers of such sets, whose complexity is polynomially bounded (for fixed d and ℓ).