About the “Condition”-lecture on 14 May 2019
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This note shall clarify the notion of simple eigenvector, used in [1, section 14.3.1]. Let us
directly start with the upshot of this note.

Fact: We need a simple eigenvalue in the sense of algebraic multiplicity one.

The proofs of Proposition 14.15 and Lemma 14.17 in [1] go through, except with respect to
the (crucial) assumption that the solution map $G$ exists. We assumed the latter to prove e.g.
that $\langle u, v \rangle \neq 0$. I guess all confusion that appeared in the lecture, was due to the fact, that we
wanted to understand where the proofs fail, but:
All arguments are correct, although we need to ensure that $G$ exists. The latter fails, if $\lambda$ has
not algebraic multiplicity one (even if the geometric multiplicity is one). First, let us illustrate
this fact (see Proposition 1 below) with the example from the lecture. Set

$$B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let $\pi_1: V \to \mathbb{C}^{n \times n}, (A, [v], \lambda) \mapsto A$

be the projection, where

$$V = \{(A, [v], \lambda) \in \mathbb{C}^{n \times n} \times \mathbb{P}^{n-1} \times \mathbb{C} \mid Av = \lambda v\}$$

is the “solution” manifold.

Claim: For $n = 2$, $\pi_1$ cannot be locally inverted around $(B, [e_1], 0) =: x \in V$, i.e. $G$ does not exist.

Proof of Claim. Let $\varepsilon > 0$ and consider

$$x_{1,\varepsilon} := \left( \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}, [e_1], \varepsilon \right) \in V \quad \text{and} \quad x_{2,\varepsilon} := \left( \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}, [e_1 - \varepsilon e_2], 0 \right) \in V.$$

For any open neighborhood $U$ of $x$ in $V$ we find some $\varepsilon > 0$ (dependent on $U$) such that $x_{1,\varepsilon} \in U$
and $x_{2,\varepsilon} \in U$. Clearly, $\pi_1(x_{1,\varepsilon}) = \pi_1(x_{2,\varepsilon})$. This proves the claim. \qed

To tackle the general situation, recall the construction of $V$ (see also the proof of Lemma
14.17 during the lecture). Let $\tilde{V}$ be the zero set of

$$F: \mathbb{C}^{n \times n} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^{n \times n}, (A, v, \lambda) \mapsto Av - \lambda v.$$

$\tilde{V}$ is a manifold, because $DF$ has full rank at any point (i.e. $F$ is a so-called submersion). Set
$V := \text{pr}(\tilde{V})$, where

$$\text{pr}: \mathbb{C}^{n \times n} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^{n \times n}, (A, v, \lambda) \mapsto (A, [v], \lambda)$$

is the canonical projection. Note that $V$ is a manifold, since $\text{pr}$ is a smooth surjective submersion.
Remember that $\pi_1$ is locally invertible around $(A, [v], \lambda) \in V$ if and only if $D\pi_1(A, [v], \lambda)$
is surjective.

Let us phrase the latter in terms of $\tilde{V}$ rather than $V$. Let $\tilde{\pi}_1: \tilde{V} \to \mathbb{C}^{n \times n}, (A, v, \lambda) \mapsto A$
be the projection. Then we have $\tilde{\pi}_1 = \pi_1 \circ \tilde{\text{pr}}$ with $\tilde{\text{pr}}: \tilde{V} \to V$ the restriction of $\text{pr}$. Hence, for all
$(A, v, \lambda) \in \tilde{V}$,

$$D\tilde{\pi}_1(A, v, \lambda) = D\pi_1(A, [v], \lambda) \circ D\tilde{\text{pr}}(A, v, \lambda).$$

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Since $D\tilde{\phi}(A,v,\lambda)$ is always surjective, we see that $D\tilde{\pi}_1(A,v,\lambda)$ is surjective if and only if $D\pi_1(A,[v],\lambda)$ is surjective. On the other hand, since $\tilde{V}$ is the zero set of $F$, the lecture tells us that $D\tilde{\pi}_1(A,v,\lambda)$ is surjective if and only if

$$
\frac{\partial F}{\partial (v,\lambda)}(A,v,\lambda) : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n, \ (\dot{v},\dot{\lambda}) \mapsto (A-\lambda I)\dot{v} - \dot{\lambda}v
$$

is surjective. All together, we showed that $\pi_1$ is locally invertible around $(A,[v],\lambda) \in V$ if and only if the linear map in (1) is surjective.

**Proposition 1.** Let $(A,[v],\lambda) \in V$.

(a) If $\lambda$ has at least algebraic multiplicity 2, then $\pi_1$ is not locally invertible around $(A,[v],\lambda)$.

(b) If $\lambda$ has algebraic multiplicity 1, then $\pi_1$ is locally invertible around $(A,[v],\lambda)$.

In other words, $\pi_1$ is locally invertible around $(A,[v],\lambda)$ if and only if $\lambda$ is an eigenvalue of algebraic multiplicity 1.

**Proof.** First, recall that $\mathbb{C}^n = T_v \oplus \mathbb{C}v$ is an orthogonal decomposition by definition of $T_v$. Denote the $\mathbb{C}$-linear map from equation (1) by $f$. Note that $\mathbb{C}v \subseteq \ker(A-\lambda I)$ implies

$$
\text{im}(f) = (A-\lambda I)(T_v) + (A-\lambda I)(\mathbb{C}v) = (A-\lambda I)(T_v) + \mathbb{C}v.
$$

Hence, it suffices to consider $\dot{v} \in T_v$ in equation (1).

(a) The vector space $T_v$ has complex dimension $n - 1$. Therefore, if $\ker(A-\lambda I)$ has complex dimension $\geq 2$, then $T_v \cap \ker(A-\lambda I)$ has dimension $\geq 1$. Hence $f$ cannot be surjective for dimensional reasons.

Thus, we may assume that $\ker(A-\lambda I) = \mathbb{C}v$. By assumption we necessarily have

$$
\dim_{\mathbb{C}} \ker((A-\lambda I)^2) \geq 2 \text{, so } \ker((A-\lambda I)^2) \cap T_v \neq \{0\}.
$$

Let $w \in \ker((A-\lambda I)^2) \cap T_v$ with $w \neq 0$. As $\mathbb{C}v \cap T_v = \{0\}$ and $\ker(A-\lambda I) = \mathbb{C}v$, we see that $w' := (A-\lambda I)w \neq 0$. Moreover, we have $w' \in \ker(A-\lambda I) = \mathbb{C}v$, where we used $w \in \ker((A-\lambda I)^2)$. All together we showed

$$
\mathbb{C}v \subseteq (A-\lambda I)(T_v), \text{ hence } \text{im}(f) = (A-\lambda I)(T_v).
$$

Thus, $f$ cannot be surjective, since $T_v$ is $n - 1$ dimensional.

(b) By assumption, we have $\mathbb{C}v = \ker(A-\lambda I)$. Thus $(A-\lambda I)(T_v) \subseteq \mathbb{C}^n$ is $n - 1$ dimensional, where we used $\mathbb{C}v \cap T_v = \{0\}$. Therefore, to show surjectivity of $f$ it suffices to prove

$$
(A-\lambda I)(T_v) \cap \mathbb{C}v = \{0\}.
$$

Assume the contrary, i.e. there exists $w' \in (A-\lambda I)(T_v) \cap \mathbb{C}v$ such that $w' \neq 0$. This gives $w \in T_v \setminus \{0\}$ with $w \in \ker((A-\lambda I)^2)$. But $w \in T_v \setminus \{0\}$ implies $w \notin \mathbb{C}v = \ker(A-\lambda I)$. Thus, we obtain

$$
\mathbb{C}w \oplus \mathbb{C}v \subseteq \ker((A-\lambda I)^2),
$$

which contradicts the fact that $\lambda$ has algebraic multiplicity 1. \qed

**References**