About the “Condition”-lecture on 14 May 2019

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This note shall clarify the notion of *simple* eigenvector, used in [1, section 14.3.1]. Let us directly start with the upshot of this note.

Fact: We need a simple eigenvalue in the sense of **algebraic** multiplicity one.

The proofs of Proposition 14.15 and Lemma 14.17 in [1] go through, except with respect to the (crucial) assumption that the solution map G *exists*. We assumed the latter to prove e.g. that $\langle u, v\rangle \neq 0$. I guess all confusion that appeared in the lecture, was due to the fact, that we wanted to understand where the proofs fail, but:
All arguments are correct, although we need to ensure that G exists. The latter fails, if λ has not algebraic multiplicity one (even if the geometric multiplicity is one). First, let us illustrate this fact (see Proposition 1 below) with the example from the lecture. Set

$$B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and let} \qquad \pi_1 \colon V \to \mathbb{C}^{n\times n},\ (A,[v],\lambda) \mapsto A$$

be the projection, where

$$V = \{(A,[v],\lambda) \in \mathbb{C}^{n\times n} \times \mathbb{P}^{n-1} \times \mathbb{C} \mid Av = \lambda v\}$$

is the “solution” manifold.

Claim: For $n = 2$, π_1 cannot be locally inverted around $(B,[e_1],0) =: x \in V$, i.e. G does not exist.

Proof of Claim. Let $\varepsilon > 0$ and consider

$$x_{1,\varepsilon} := \left(\begin{pmatrix} \varepsilon & 1 \\ 0 & 0 \end{pmatrix}, [e_1], \varepsilon \right) \in V \quad \text{and} \quad x_{2,\varepsilon} := \left(\begin{pmatrix} \varepsilon & 1 \\ 0 & 0 \end{pmatrix}, [e_1 - \varepsilon e_2], 0 \right) \in V\,.$$

For any open neighborhood U of x in V we find some $\varepsilon > 0$ (dependent on U) such that $x_{1,\varepsilon} \in U$ and $x_{2,\varepsilon} \in U$. Clearly, $\pi_1(x_{1,\varepsilon}) = \pi_1(x_{2,\varepsilon})$. This proves the claim. □

To tackle the general situation, recall the construction of V (see also the proof of Lemma 14.17 during the lecture). Let $\tilde{V}$ be the zero set of

$$F \colon \mathbb{C}^{n\times n} \times \mathbb{C}^n \backslash \{0\} \times \mathbb{C} \to \mathbb{C}^n,\ (A,v,\lambda) \mapsto Av - \lambda v\,.$$

$\tilde{V}$ is a manifold, because DF has full rank at any point (i.e. F is a so-called submersion). Set $V := pr(\tilde{V})$, where

$$pr \colon \mathbb{C}^{n\times n} \times \mathbb{C}^n \backslash \{0\} \times \mathbb{C} \to \mathbb{C}^{n\times n} \times \mathbb{P}^{n-1} \times \mathbb{C},\ \ (A,v,\lambda) \mapsto (A,[v],\lambda)$$

is the canonical projection. Note that V is a manifold, since pr is a smooth surjective submersion. Remember that π_1 is locally invertible around $(A,[v],\lambda) \in V$ if and only if $D\pi_1(A,[v],\lambda)$ is surjective.
Let us phrase the latter in terms of $\tilde{V}$ rather than V. Let $\tilde{\pi}_1 \colon \tilde{V} \to \mathbb{C}^{n\times n},\ (A,v,\lambda) \mapsto A$ be the projection. Then we have $\tilde{\pi}_1 = \pi_1 \circ \bar{pr}$ with $\bar{pr} \colon \tilde{V} \to V$ the restriction of pr. Hence, for all $(A,v,\lambda) \in \tilde{V}$,

$$D\tilde{\pi}_1(A,v,\lambda) \ = \ D\pi_1(A,[v],\lambda) \ \circ \ D\,\bar{pr}(A,v,\lambda)\,.$$

Since $D\,\bar{pr}(A,v,\lambda)$ is always surjective, we see that $D\tilde{\pi}_1(A,v,\lambda)$ is surjective if and only if $D\pi_1(A,[v],\lambda)$ is surjective. On the other hand, since $\tilde{V}$ is the zero set of F, the lecture tells us that $D\tilde{\pi}_1(A,v,\lambda)$ is surjective if and only if

$$\frac{\partial F}{\partial(v,\lambda)}(A,v,\lambda):\ \mathbb{C}^n\times\mathbb{C}\to\mathbb{C}^n,\ (\dot{v},\dot{\lambda})\mapsto(A-\lambda I)\dot{v}-\dot{\lambda}v \tag{1}$$

is surjective. All together, we showed that π_1 is locally invertible around $(A,[v],\lambda)\in V$ if and only if the linear map in (1) is surjective.

Proposition 1. *Let $(A,[v],\lambda)\in V$.*

(a) If λ has at least algebraic multiplicity 2, then π_1 is not *locally invertible around $(A,[v],\lambda)$.*

(b) If λ has algebraic multiplicity 1, then π_1 is *locally invertible around $(A,[v],\lambda)$.*

In other words, π_1 is locally invertible around $(A,[v],\lambda)$ if and only if λ is an eigenvalue of algebraic multiplicity 1.

Proof. First, recall that $\mathbb{C}^n=T_v\oplus\mathbb{C}v$ is an orthogonal decomposition by definition of T_v. Denote the $\mathbb{C}$-linear map from equation (1) by f. Note that $\mathbb{C}v\subseteq\ker(A-\lambda I)$ implies

$$\mathrm{im}(f)=(A-\lambda I)(T_v)+(A-\lambda I)(\mathbb{C}v)+\mathbb{C}v=(A-\lambda I)(T_v)+\mathbb{C}v\,.$$

Hence, it suffices to consider $\dot{v}\in T_v$ in equation (1).

(a) The vector space T_v has complex dimension $n-1$. Therefore, if $\ker(A-\lambda I)$ has complex dimension ≥ 2, then $T_v\cap\ker(A-\lambda I)$ has dimension ≥ 1. Hence f cannot be surjective for dimensional reasons.

Thus, we may assume that $\ker(A-\lambda I)=\mathbb{C}v$. By assumption we necessarily have

$$\dim_{\mathbb{C}}\ker\big((A-\lambda I)^2\big)\geq 2\,,\ \text{so}\ \ \ker\big((A-\lambda I)^2\big)\cap T_v\neq\{0\}\,.$$

Let $w\in\ker\big((A-\lambda I)^2\big)\cap T_v$ with $w\neq 0$. As $\mathbb{C}v\cap T_v=\{0\}$ and $\ker(A-\lambda I)=\mathbb{C}v$, we see that $w':=(A-\lambda I)w\neq 0$. Moreover, we have $w'\in\ker(A-\lambda I)=\mathbb{C}v$, where we used $w\in\ker\big((A-\lambda I)^2\big)$. All together we showed

$$\mathbb{C}v\subseteq(A-\lambda I)(T_v),\ \text{hence}\ \ \mathrm{im}(f)=(A-\lambda I)(T_v)\,.$$

Thus, f cannot be surjective, since T_v is $n-1$ dimensional.

(b) By assumption, we have $\mathbb{C}v=\ker(A-\lambda I)$. Thus $(A-\lambda I)(T_v)\subseteq\mathbb{C}^n$ is $n-1$ dimenional, where we used $\mathbb{C}v\cap T_v=\{0\}$. Therefore, to show surjectivity of f it suffices to prove

$$(A-\lambda I)(T_v)\ \cap\ \mathbb{C}v=\{0\}\,.$$

Assume the contrary, i.e. there exists $w'\in(A-\lambda I)(T_v)\cap\mathbb{C}v$ such that $w'\neq 0$. This gives $w\in T_v\backslash\{0\}$ with $w\in\ker\big((A-\lambda I)^2\big)$. But $w\in T_v\backslash\{0\}$ implies $w\notin\mathbb{C}v=\ker(A-\lambda I)$. Thus, we obtain

$$\mathbb{C}w\oplus\mathbb{C}v\ \subseteq\ \ker\big((A-\lambda I)^2\big)\,,$$

which contradicts the fact that λ has algebraic multiplicity 1. □

References

[1] Peter Bürgisser and Felipe Cucker, *Condition: The Geometry of Numerical Algorithms*, Springer, 2013.