Varieties of Sums of Powers, Stiefel Manifolds and their degrees

Fulvio Gesmundo

December 16, 2020

Max-Planck-Institut für Mathematik in den Naturwissenschaften

We study:

- Homogeneous polynomials

$$f \in \mathbb{C}[x_1, \ldots, x_k]_d = S^d\mathbb{C}^k.$$

We study:

- Homogeneous polynomials

$$f \in \mathbb{C}[x_1, \ldots, x_k]_d = S^d\mathbb{C}^k.$$

- Waring rank:

$$R(f) = \min\{r : f = \ell_1^d + \cdots + \ell_r^d \text{ for some } \ell_i \in S^1\mathbb{C}^k\};$$

equivalently

$$R(f) = \min\{r : [f] \in \langle [\ell_1^d], \ldots, [\ell_r^d] \rangle \text{ for some } [\ell_1], \ldots, [\ell_r] \in \mathbb{P}\mathbb{C}^{k-1}\}$$

We study:

- Homogeneous polynomials

$$f \in \mathbb{C}[x_1, \dots, x_k]_d = S^d\mathbb{C}^k.$$

- Waring rank:

$$R(f) = \min\{r : f = \ell_1^d + \cdots + \ell_r^d \text{ for some } \ell_i \in S^1\mathbb{C}^k\};$$

  equivalently

$$R(f) = \min\{r : [f] \in \langle [\ell_1^d], \dots, [\ell_r^d] \rangle \text{ for some } [\ell_1], \dots, [\ell_r] \in \mathbb{PC}^{k-1}\}$$

- Hilbert scheme of points

$$\mathrm{Hilb}_r^{sm}(\mathbb{PC}^{k-1}) = \overline{\{S \subseteq \mathbb{PC}^{k-1} : S \text{ is a set of } r \text{ points}\}}$$

- Variety of sums of powers

$$\mathrm{VSP}_r(f) = \overline{\left\{ \{[\ell_1], \dots, [\ell_r]\} : [f] \in \langle [\ell_1^d], \dots, [\ell_r^d] \rangle \right\}} \subseteq \mathrm{Hilb}_r^{sm}(\mathbb{PC}^{k-1})$$

Alternative description of variety of decompositions

We introduce the following space

$$\mathcal{VSP}_r(f) = \left\{(\ell_1, \ldots, \ell_r) : f = \sum \ell_i^d\right\} \subseteq (\mathbb{C}^k)^{\times r}$$

Alternative description of variety of decompositions

We introduce the following space

$$\mathcal{VSP}_r(f) = \left\{(\ell_1, \dots, \ell_r) : f = \sum \ell_i^d \right\} \subseteq (\mathbb{C}^k)^{\times r}$$

The condition $f = \sum \ell_i^d$ determines a set of $\dim S^d \mathbb{C}^k$ polynomial equations of degree d in the $k \cdot r$ coordinates of $(\mathbb{C}^k)^{\times r}$.

So $\mathcal{VSP}_r(f)$ is an affine subvariety of $(\mathbb{C}^k)^{\times r}$.

Alternative description of variety of decompositions

We introduce the following space

$$\mathcal{VSP}_r(f) = \left\{(\ell_1, \ldots, \ell_r) : f = \sum \ell_i^d\right\} \subseteq (\mathbb{C}^k)^{\times r}$$

The condition $f = \sum \ell_i^d$ determines a set of $\dim S^d\mathbb{C}^k$ polynomial equations of degree d in the $k \cdot r$ coordinates of $(\mathbb{C}^k)^{\times r}$.

So $\mathcal{VSP}_r(f)$ is an affine subvariety of $(\mathbb{C}^k)^{\times r}$.

The map

$$\begin{aligned} \mathcal{VSP}_r(f) &\dashrightarrow \mathrm{VSP}_r(f) \\ (\ell_1, \ldots, \ell_r) &\mapsto \{[\ell_1], \ldots, [\ell_r]\} \end{aligned}$$

is rational dominant (in the cases we care about).

In some sense, $\mathcal{VSP}_r(f)$ records the same information as $\mathrm{VSP}_r(f)$.

Alternative description of variety of decompositions

We introduce the following space

$$\mathcal{VSP}_r(f) = \left\{(\ell_1, \dots, \ell_r) : f = \sum \ell_i^d\right\} \subseteq (\mathbb{C}^k)^{\times r}$$

The condition $f = \sum \ell_i^d$ determines a set of dim $S^d\mathbb{C}^k$ polynomial equations of degree d in the $k \cdot r$ coordinates of $(\mathbb{C}^k)^{\times r}$.

So $\mathcal{VSP}_r(f)$ is an affine subvariety of $(\mathbb{C}^k)^{\times r}$.

The map

$$\begin{aligned} \mathcal{VSP}_r(f) &\dashrightarrow \mathrm{VSP}_r(f) \\ (\ell_1, \dots, \ell_r) &\mapsto \{[\ell_1], \dots, [\ell_r]\} \end{aligned}$$

is rational dominant (in the cases we care about).

In some sense, $\mathcal{VSP}_r(f)$ records the same information as $\mathrm{VSP}_r(f)$.

What can we say about $\mathcal{VSP}_r(f)$? Dimension? Degree?

Quadratic forms

What if $\deg(f) = 2$?

There are normal forms: $f = q_k = x_1^2 + \cdots + x_k^2 = \mathbf{x}^T \cdot \mathbb{I}_k \cdot \mathbf{x}$.

Quadratic forms

What if $\deg(f) = 2$?

There are normal forms: $f = q_k = x_1^2 + \cdots + x_k^2 = \mathbf{x}^T \cdot \mathbb{I}_k \cdot \mathbf{x}$.

Rewrite the decomposition: if $\ell_i = \mathbf{c}_i^T \mathbf{x}$ then

$$q_k = \sum \ell_i^2 = \sum (\mathbf{c}_i^T \mathbf{x})^2 = \mathbf{x}^T \cdot [\sum (\mathbf{c}_i \cdot \mathbf{c}_i^T)] \cdot \mathbf{x} = \mathbf{x}^T \cdot CC^T \cdot \mathbf{x}$$

where $C = [\mathbf{c}_1 | \cdots | \mathbf{c}_r] \in Mat_{k \times r} = (\mathbb{C}^k)^{\times r}$.

So $(\ell_1, , \ldots, \ell_k) \in \mathcal{VSP}_r(q_k)$ if and only if $CC^T = \mathbb{I}_k$.

Quadratic forms

What if $\deg(f) = 2$?

There are normal forms: $f = q_k = x_1^2 + \cdots + x_k^2 = \mathbf{x}^T \cdot \mathbb{I}_k \cdot \mathbf{x}$.

Rewrite the decomposition: if $\ell_i = \mathbf{c}_i^T \mathbf{x}$ then

$$q_k = \sum \ell_i^2 = \sum (\mathbf{c}_i^T \mathbf{x})^2 = \mathbf{x}^T \cdot [\sum (\mathbf{c}_i \cdot \mathbf{c}_i^T)] \cdot \mathbf{x} = \mathbf{x}^T \cdot CC^T \cdot \mathbf{x}$$

where $C = [\mathbf{c}_1 | \cdots | \mathbf{c}_r] \in Mat_{k \times r} = (\mathbb{C}^k)^{\times r}$.

So $(\ell_1, , \ldots, \ell_k) \in \mathcal{VSP}_r(q_k)$ if and only if $CC^T = \mathbb{I}_k$.

Therefore:

$$\mathcal{VSP}_r(q_k) = \{C \in Mat_{k \times r} : CC^T = \mathbb{I}_k\} = St(k, r)$$

the Stiefel manifold of k frames in $\mathbb{C}^r$.

Quadratic forms

What if $\deg(f) = 2$?

There are normal forms: $f = q_k = x_1^2 + \cdots + x_k^2 = \mathbf{x}^T \cdot \mathbb{I}_k \cdot \mathbf{x}$.

Rewrite the decomposition: if $\ell_i = \mathbf{c}_i^T \mathbf{x}$ then

$$q_k = \sum \ell_i^2 = \sum (\mathbf{c}_i^T \mathbf{x})^2 = \mathbf{x}^T \cdot [\sum (\mathbf{c}_i \cdot \mathbf{c}_i^T)] \cdot \mathbf{x} = \mathbf{x}^T \cdot CC^T \cdot \mathbf{x}$$

where $C = [\mathbf{c}_1 | \cdots | \mathbf{c}_r] \in Mat_{k \times r} = (\mathbb{C}^k)^{\times r}$.

So $(\ell_1, , \ldots, \ell_k) \in \mathcal{VSP}_r(q_k)$ if and only if $CC^T = \mathbb{I}_k$.

Therefore:

$$\mathcal{VSP}_r(q_k) = \{C \in Mat_{k \times r} : CC^T = \mathbb{I}_k\} = St(k, r)$$

the Stiefel manifold of k frames in $\mathbb{C}^r$.

We compute dimension and degree of $St(k, n)$.

Dimension is easy!

When $n = k$, then

$$St(k,k) = \{C \in Mat_{k\times k} : CC^T = \mathbb{I}_k\} = O(k)$$

is the orthogonal group, so $\dim St(k,k) = \binom{k}{2}$.

Dimension is easy!

When $n = k$, then

$$St(k,k) = \{C \in Mat_{k\times k} : CC^T = \mathbb{I}_k\} = O(k)$$

is the orthogonal group, so $\dim St(k,k) = \binom{k}{2}$.

If $n > k$ consider the action of $SO(n)$ given by

$$\begin{aligned} Mat_{k\times n} \times SO(n) &\to Mat_{k\times n} \\ (C,g) &\mapsto Cg. \end{aligned}$$

Dimension is easy!

When $n = k$, then

$$St(k,k) = \{C \in Mat_{k\times k} : CC^T = \mathbb{I}_k\} = O(k)$$

is the orthogonal group, so $\dim St(k,k) = \binom{k}{2}$.

If $n > k$ consider the action of $SO(n)$ given by

$$\begin{aligned} Mat_{k\times n} \times SO(n) &\to Mat_{k\times n} \\ (C, g) &\mapsto Cg. \end{aligned}$$

- this action maps $St(k,n)$ to itself: $(Cg)(Cg)^T = Cgg^TC^T = CC^T = \mathbb{I}_k$.
- this action is transitive on $St(k,n)$;
- the stabilizer of an element is a copy of $SO(n-k)$.

Dimension is easy!

When $n = k$, then

$$St(k,k) = \{C \in Mat_{k\times k} : CC^T = \mathbb{I}_k\} = O(k)$$

is the orthogonal group, so $\dim St(k,k) = \binom{k}{2}$.

If $n > k$ consider the action of $SO(n)$ given by

$$\begin{aligned} Mat_{k\times n} \times SO(n) &\to Mat_{k\times n} \\ (C,g) &\mapsto Cg. \end{aligned}$$

- this action maps $St(k,n)$ to itself: $(Cg)(Cg)^T = Cgg^TC^T = CC^T = \mathbb{I}_k$.
- this action is transitive on $St(k,n)$;
- the stabilizer of an element is a copy of $SO(n-k)$.

So $St(k,n) = SO(n)/SO(n-k)$ is a homogeneous space and

$$\dim St(k,n) = \binom{n}{2} - \binom{n-k}{2}$$

Dimension is easy!

When $n = k$, then

$$St(k,k) = \{C \in Mat_{k\times k} : CC^T = \mathbb{I}_k\} = O(k)$$

is the orthogonal group, so $\dim St(k,k) = \binom{k}{2}$.

If $n > k$ consider the action of $SO(n)$ given by

$$\begin{aligned} Mat_{k\times n} \times SO(n) &\to Mat_{k\times n} \\ (C, g) &\mapsto Cg. \end{aligned}$$

- this action maps $St(k,n)$ to itself: $(Cg)(Cg)^T = Cgg^TC^T = CC^T = \mathbb{I}_k$.
- this action is transitive on $St(k,n)$;
- the stabilizer of an element is a copy of $SO(n-k)$.

So $St(k,n) = SO(n)/SO(n-k)$ is a homogeneous space and

$$\dim St(k,n) = \binom{n}{2} - \binom{n-k}{2} = nk - \binom{k+1}{2}.$$

Degree is harder

k \ n	1	2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2
2	*	4	8	8	8	8	8	8
3	*	*	16	40	64	64	64	64
4	*	*	*	80	384	704	1024	1024
5	*	*	*	*	768	4768	14848	23808
6	*	*	*	*	*	9536	111616	420736
7	*	*	*	*	*	*	223232	3433600
8	*	*	*	*	*	*	*	6867200

Degree is harder

k \ n	1	2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2
2	*	4	8	8	8	8	8	8
3	*	*	16	40	64	64	64	64
4	*	*	*	80	384	704	1024	1024
5	*	*	*	*	768	4768	14848	23808
6	*	*	*	*	*	9536	111616	420736
7	*	*	*	*	*	*	223232	3433600
8	*	*	*	*	*	*	*	6867200

- Diagonal: $St(k,n) = O(n)$. Degree computed by [Brandt-Bruce-Brysiewicz-Krone-Robeva - 2017].

Degree is harder

k \ n	1	2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2
2	*	4	8	8	8	8	8	8
3	*	*	16	40	64	64	64	64
4	*	*	*	80	384	704	1024	1024
5	*	*	*	*	768	4768	14848	23808
6	*	*	*	*	*	9536	111616	420736
7	*	*	*	*	*	*	223232	3433600
8	*	*	*	*	*	*	*	6867200

- Diagonal: $St(k, n) = O(n)$. Degree computed by [Brandt-Bruce-Brysiewicz-Krone-Robeva - 2017].
- Green: $n \geq 2k - 1$: proof is geometric.

Degree is harder

k \ n	1	2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2
2	*	4	8	8	8	8	8	8
3	*	*	16	40	64	64	64	64
4	*	*	*	80	384	704	1024	1024
5	*	*	*	*	768	4768	14848	23808
6	*	*	*	*	*	9536	111616	420736
7	*	*	*	*	*	*	223232	3433600
8	*	*	*	*	*	*	*	6867200

- Diagonal: $St(k,n) = O(n)$. Degree computed by [Brandt-Bruce-Brysiewicz-Krone-Robeva - 2017].
- Green: $n \geq 2k-1$: proof is geometric.
- Blue: $n \leq 2k-1$: proof is representation theoretic.

Degree is harder

k \ n	1	2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2
2	*	4	8	8	8	8	8	8
3	*	*	16	40	64	64	64	64
4	*	*	*	80	384	704	1024	1024
5	*	*	*	*	768	4768	14848	23808
6	*	*	*	*	*	9536	111616	420736
7	*	*	*	*	*	*	223232	3433600
8	*	*	*	*	*	*	*	6867200

- Diagonal: $St(k,n) = O(n)$. Degree computed by [Brandt-Bruce-Brysiewicz-Krone-Robeva - 2017].
- Green: $n \geq 2k-1$: proof is geometric.
- Blue: $n \leq 2k-1$: proof is representation theoretic.
- Easy: first row $St(1,n) = \{\mathbf{c} \in \mathbb{C}^n : \mathbf{c}^T\mathbf{c} = 1\}$ hypersurface of degree 2.

Main Theorem [Brysiewicz-G. 2019]

Let $n > k$ and let $s = \left\lfloor \frac{n}{2} \right\rfloor$.

If $n \geq 2k - 1$ then

$$\deg St(k, n) = 2^{\binom{k+1}{2}}.$$

Main Theorem [Brysiewicz-G. 2019]

Let $n > k$ and let $s = \left\lfloor \frac{n}{2} \right\rfloor$.

If $n \geq 2k - 1$ then

$$\deg St(k, n) = 2^{\binom{k+1}{2}}.$$

If $n \leq 2k - 1$ then

$$\deg St(k, n) = 2^k \cdot L_{n,k}$$

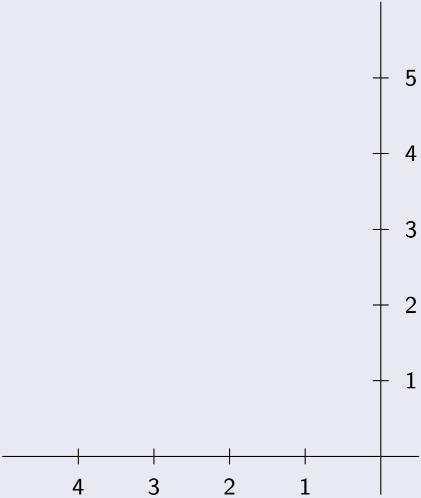
where $L_{n,k}$ is the number of non-intersecting lattice paths configuration from $A = \{(-a_i, 0) : i = 1, \ldots, s\}$ to $B = \{(0, b_i) : i = 1, \ldots, s\}$, defined by

$$(a_1, \ldots, a_s) = (\underbrace{k-1, k-2, \ldots, k-(n-k)}_{n-k}, \underbrace{2k-n-2, 2k-n-4, n-2s}_{s-(n-k)})$$

$$(b_1, \ldots, b_s) = (n-2, n-4, \ldots, n-2s).$$

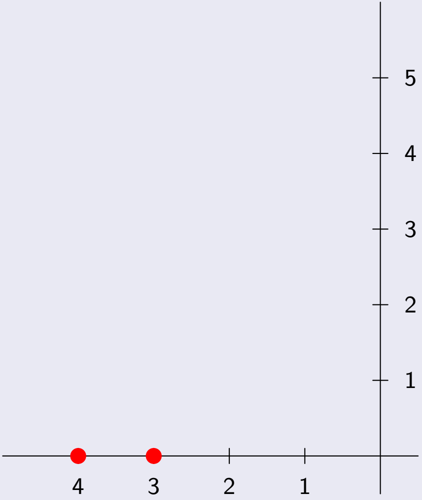
Non-intersecting lattice paths

- Fix $n = 7$ so $s = \lfloor 7/2 \rfloor = 3$;
- $k = 5$, so $n - k = 2$.



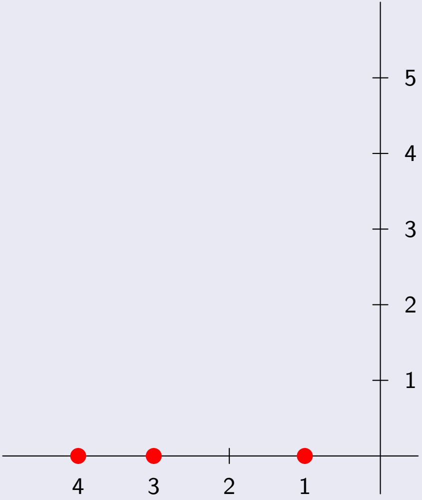
Non-intersecting lattice paths

- Fix $n = 7$ so $s = \lfloor 7/2 \rfloor = 3$;
- $k = 5$, so $n - k = 2$.



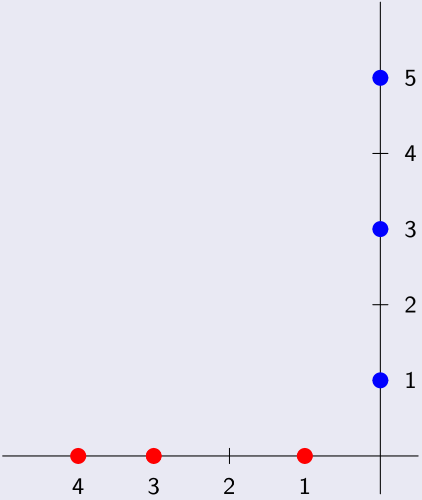
Non-intersecting lattice paths

- Fix $n = 7$ so $s = \lfloor 7/2 \rfloor = 3$;
- $k = 5$, so $n - k = 2$.



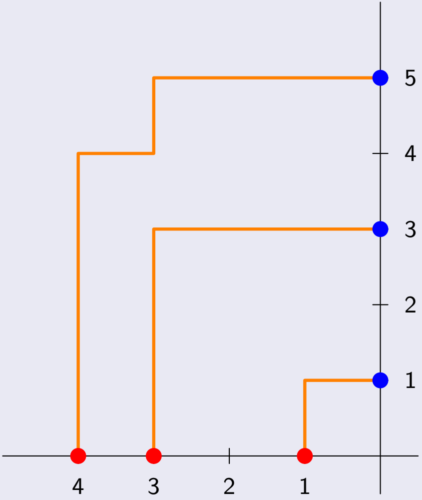
Non-intersecting lattice paths

- Fix $n = 7$ so $s = \lfloor 7/2 \rfloor = 3$;
- $k = 5$, so $n - k = 2$.



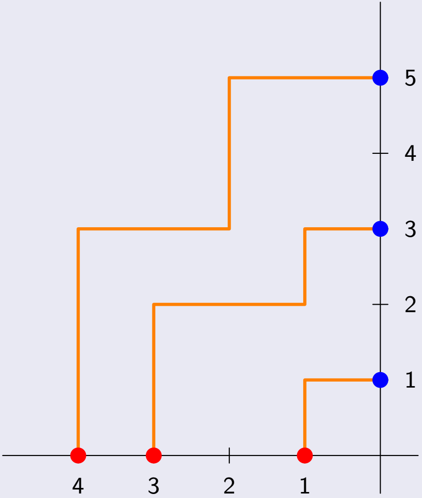
Non-intersecting lattice paths

- Fix $n = 7$ so $s = \lfloor 7/2 \rfloor = 3$;
- $k = 5$, so $n - k = 2$.



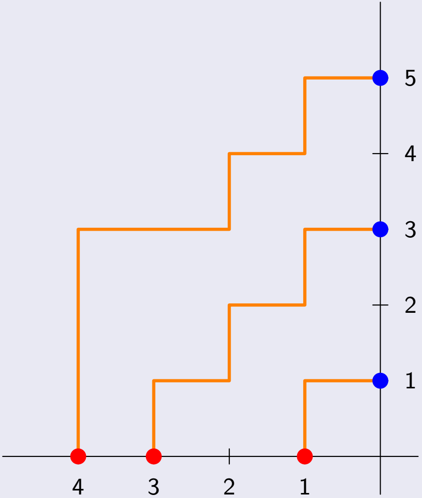
Non-intersecting lattice paths

- Fix $n = 7$ so $s = \lfloor 7/2 \rfloor = 3$;
- $k = 5$, so $n - k = 2$.



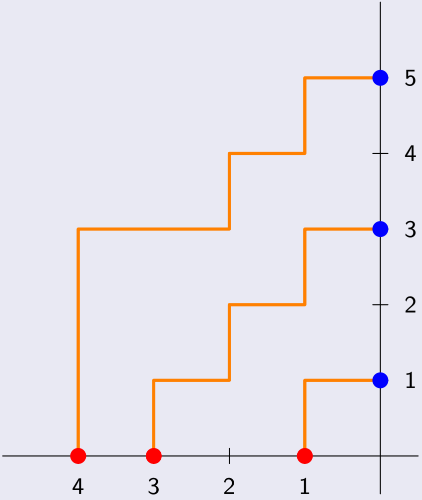
Non-intersecting lattice paths

- Fix $n = 7$ so $s = \lfloor 7/2 \rfloor = 3$;
- $k = 5$, so $n - k = 2$.



Non-intersecting lattice paths

- Fix $n = 7$ so $s = \lfloor 7/2 \rfloor = 3$;
- $k = 5$, so $n - k = 2$.



$L_{n,k} = 464$

$$\deg(St(5,7)) = 2^5 \cdot 464 = 14848$$

Degree of affine and projective varieties

Consider $X \subseteq \mathbb{A}^N$ or $X \subseteq \mathbb{P}^N$ of dimension m; let $c = N - m$ be the codimension.

Let L be a generic c-dimensional linear space.

$$\deg(X) = \#\{X \cap L\}.$$

Degree of affine and projective varieties

Consider $X \subseteq \mathbb{A}^N$ or $X \subseteq \mathbb{P}^N$ of dimension m; let $c = N - m$ be the codimension.

Let L be a generic c-dimensional linear space.

$$\deg(X) = \#\{X \cap L\}.$$

Note: $\deg(X)$ does not see components of dimension smaller than m.

Degree of affine and projective varieties

Consider $X \subseteq \mathbb{A}^N$ or $X \subseteq \mathbb{P}^N$ of dimension m; let $c = N - m$ be the codimension.

Let L be a generic c-dimensional linear space.

$$\deg(X) = \#\{X \cap L\}.$$

Note: $\deg(X)$ does not see components of dimension smaller than m.

Fix coordinates $z_0, \ldots, z_N$ on $\mathbb{P}^N$.
Identify $\mathbb{A}^N \subseteq \mathbb{P}^N$ with the affine patch $\{z_0 \neq 0\}$.

If $X \subseteq \mathbb{A}^N$ is an affine variety, let $\overline{X} \subseteq \mathbb{P}^N$ be the closure.
Then $\deg(X) = \deg(\overline{X})$.

Complete Intersections

Suppose $X \subseteq \mathbb{P}^N$ of codimension c.

Special case: the ideal has c generators: $I_X = (g_1, \ldots, g_c) \subseteq \mathbb{C}[z_0, \ldots, z_N]$.
Then

$$\deg(X) = \deg(g_1) \cdots \deg(g_c).$$

Complete Intersections

Suppose $X \subseteq \mathbb{P}^N$ of codimension c.

Special case: the ideal has c generators: $I_X = (g_1, \ldots, g_c) \subseteq \mathbb{C}[z_0, \ldots, z_N]$. Then

$$\deg(X) = \deg(g_1) \cdots \deg(g_c).$$

What happens in the affine setting?

Suppose $X \subseteq \mathbb{A}^N$ of codimension c, with $I_X = (g_1, \ldots, g_c) \subseteq \mathbb{C}[z_1, \ldots, z_N]$.

Homogenize the generators, to get a homogeneous ideal in $\mathbb{C}[z_0, \ldots, z_N]$ and let $Y \subseteq \mathbb{P}^N$ be the *set* (it is a projective scheme!) that this cuts out.

Certainly $\overline{X} \subseteq Y$.

If equality holds, then one gets $\deg(X) = \deg(g_1) \cdots \deg(g_c)$.

Complete Intersections: Example

Let $X \subseteq \mathbb{A}^3$ be the curve parametrized by (t, t^2, t^3). Then $I_X = (g_1, g_2)$

$$g_1 = z_1^2 - z_2$$

$$g_2 = z_1 z_2 - z_3$$

Complete Intersections: Example

Let $X \subseteq \mathbb{A}^3$ be the curve parametrized by (t, t^2, t^3). Then $I_X = (g_1, g_2)$

$$g_1 = z_1^2 - z_2$$
$$g_2 = z_1 z_2 - z_3$$

Homogenize them

$$\widehat{g}_1 = z_1^2 - z_2 z_0$$
$$\widehat{g}_2 = z_1 z_2 - z_3 z_0$$

Complete Intersections: Example

Let $X \subseteq \mathbb{A}^3$ be the curve parametrized by (t, t^2, t^3). Then $I_X = (g_1, g_2)$

$$g_1 = z_1^2 - z_2$$
$$g_2 = z_1 z_2 - z_3$$

Homogenize them

$$\widehat{g}_1 = z_1^2 - z_2 z_0$$
$$\widehat{g}_2 = z_1 z_2 - z_3 z_0$$

Let $Y = \{\widehat{g}_1 = \widehat{g}_2 = 0\} \subseteq \mathbb{P}^3$. Then

$$Y = \overline{X} \cup L$$

where $L = \{z_0 = z_1 = 0\} \subseteq \mathbb{P}^3$ is a line.

So Y has one component supported at infinity!

These components contribute to $\deg(Y)$ but not to $\deg(\overline{X})$.

Stiefel setting

For $n \geq k$, we have

$$St(k,n) = \{C \in Mat_{k\times n} : CC^T - \mathbb{I}_k = 0\}.$$

- Homogenize: $CC^T - z_0^2\mathbb{I}_k = 0$
- Look at infinity: $CC^T = 0$.

We hope the variety

$$\mathcal{Z}_\infty = \{[C] \in \mathbb{P}Mat_{k\times n} : CC^T = 0\}$$

has dimension strictly smaller than $\dim St(k,n)$.

Stiefel manifold at infinity

Fact:

$$\mathcal{Z}_\infty = \left\{ C \in Mat_{k\times n} : \mathrm{Im}\ C^T \subseteq \{q_n = 0\} \right\}.$$

We realize $\mathcal{Z}_\infty$ as a vector bundle over the *isotropic grassmannian*.

Stiefel manifold at infinity

Fact:

$$\mathcal{Z}_\infty = \big\{ C \in Mat_{k\times n} : \mathrm{Im}\ C^T \subseteq \{q_n = 0\} \big\}.$$

We realize $\mathcal{Z}_\infty$ as a vector bundle over the *isotropic grassmannian*.

Theorem: [BG'19]
If $n \geq 2k-1$, then $\dim \mathcal{Z}_\infty < \dim St(k,n)$.

Stiefel manifold at infinity

Fact:

$$\mathcal{Z}_\infty = \left\{ C \in Mat_{k\times n} : \mathrm{Im}\ C^T \subseteq \{q_n = 0\} \right\}.$$

We realize $\mathcal{Z}_\infty$ as a vector bundle over the *isotropic grassmannian*.

Theorem: [BG'19]
If $n \geq 2k-1$, then $\dim \mathcal{Z}_\infty < \dim St(k,n)$.

So the degree of the homogenization is the same as $\deg St(k,n)$.

Stiefel manifold at infinity

Fact:

$$\mathcal{Z}_\infty = \left\{ C \in Mat_{k\times n} : \mathrm{Im}\ C^T \subseteq \{q_n = 0\} \right\}.$$

We realize $\mathcal{Z}_\infty$ as a vector bundle over the *isotropic grassmannian*.

Theorem: [BG'19]
If $n \geq 2k - 1$, then $\dim \mathcal{Z}_\infty < \dim St(k, n)$.

So the degree of the homogenization is the same as $\deg St(k, n)$.

Recall:

- Ideal of $St(k, n)$ is defined by $\binom{k+1}{2}$ quadratic equations;
- $\operatorname{codim} St(k, n) = \binom{k+1}{2}$.

Stiefel manifold at infinity

Fact:

$$\mathcal{Z}_\infty = \left\{ C \in Mat_{k \times n} : \mathrm{Im}\; C^T \subseteq \{q_n = 0\} \right\}.$$

We realize $\mathcal{Z}_\infty$ as a vector bundle over the *isotropic grassmannian*.

Theorem: [BG'19]
If $n \geq 2k - 1$, then $\dim \mathcal{Z}_\infty < \dim St(k, n)$.

So the degree of the homogenization is the same as $\deg St(k, n)$.

Recall:

- Ideal of $St(k, n)$ is defined by $\binom{k+1}{2}$ quadratic equations;
- $\operatorname{codim} St(k, n) = \binom{k+1}{2}$.

We obtain the green part of the table:

$$\deg St(k, n) = 2^{\binom{k+1}{2}} \quad \textit{if} \;\; n \geq 2k - 1.$$

More on degrees of algebraic varieties

Consider

- $X \subseteq \mathbb{A}^N$ an affine variety of dimension m with ideal I_X;
- $\mathbb{C}[X] = \mathbb{C}[z_1, \ldots, z_N]/I_X$ the affine coordinate ring;
- Hilbert polynomial: $\mathrm{HP}_X(t) = \dim \mathbb{C}[X]_{\leq t}$ for $t \gg 1$.

More on degrees of algebraic varieties

Consider

- $X \subseteq \mathbb{A}^N$ an affine variety of dimension m with ideal I_X;
- $\mathbb{C}[X] = \mathbb{C}[z_1, \ldots, z_N]/I_X$ the affine coordinate ring;
- Hilbert polynomial: $\mathrm{HP}_X(t) = \dim \mathbb{C}[X]_{\leq t}$ for $t \gg 1$.

Fact:

$$\deg(X) = m! \cdot \text{ leading coefficient of } \mathrm{HP}_X;$$

More on degrees of algebraic varieties

Consider

- $X \subseteq \mathbb{A}^N$ an affine variety of dimension m with ideal I_X;
- $\mathbb{C}[X] = \mathbb{C}[z_1, \ldots, z_N]/I_X$ the affine coordinate ring;
- Hilbert polynomial: $\mathrm{HP}_X(t) = \dim \mathbb{C}[X]_{\leq t}$ for $t \gg 1$.

Fact:

$$\deg(X) = m! \cdot \text{ leading coefficient of } \mathrm{HP}_X;$$

equivalently

$$\deg(X) = m! \lim_{t \to \infty} \frac{\dim \mathbb{C}[X]_{\leq t}}{t^m}.$$

Algebraic Peter-Weyl Theorem

Intrinsic description for $\mathbb{C}[X]$ when $X = G/H$.

In our case $G = SO(n)$, $H = SO(n-k)$:

$$\mathbb{C}[St(k,n)] = \bigoplus_{\lambda} V_\lambda \otimes {V_\lambda^*}^{SO(n-k)}$$

Algebraic Peter-Weyl Theorem

Intrinsic description for $\mathbb{C}[X]$ when $X = G/H$.

In our case $G = SO(n)$, $H = SO(n-k)$:

$$\mathbb{C}[St(k,n)] = \bigoplus_{\lambda} V_\lambda \otimes {V_\lambda^*}^{SO(n-k)}$$

λ ranges over irreducible representations of $SO(n)$:

- if $n = 2s+1$, $\lambda = (\lambda_1, \dots, \lambda_s)$ with $\lambda_1 \geq \cdots \geq \lambda_s$;
- if $n = 2s$, $\lambda = (\lambda_1, \dots, \lambda_s)$ with $\lambda_1 \geq \cdots \geq \lambda_{s-1} \geq |\lambda_s|$.

Algebraic Peter-Weyl Theorem

Intrinsic description for $\mathbb{C}[X]$ when $X = G/H$.

In our case $G = SO(n)$, $H = SO(n-k)$:

$$\mathbb{C}[St(k,n)] = \bigoplus_{\lambda} V_\lambda \otimes V_\lambda^{*\,SO(n-k)}$$

λ ranges over irreducible representations of $SO(n)$:

- if $n = 2s+1$, $\lambda = (\lambda_1, \dots, \lambda_s)$ with $\lambda_1 \geq \dots \geq \lambda_s$;
- if $n = 2s$, $\lambda = (\lambda_1, \dots, \lambda_s)$ with $\lambda_1 \geq \dots \geq \lambda_{s-1} \geq |\lambda_s|$.

What about the grading?

Fact: V_λ occurs in $\mathbb{C}[X]_{\leq t}$ if and only if $|\lambda| := \lambda_1 + \dots + \lambda_{s-1} + |\lambda_s| \leq t$.

Obtain

$$\deg St(k,n) = m! \lim_{t \to \infty} \frac{\sum_{|\lambda| \leq t} \dim V_\lambda \cdot \dim V_\lambda^{SO(n-k)}}{t^m}$$

Computing invariants

- V_λ representation of $SO(n)$;

Question: What is dim $V_\lambda^{SO(n-k)}$?

Answer: Branching rules.

Computing invariants

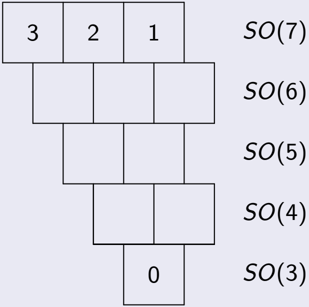
- V_λ representation of $SO(n)$;

Question: What is dim $V_\lambda^{SO(n-k)}$?

Answer: Branching rules.

Example: Say $n = 7$ with $\lambda = (3, 2, 1)$; $k = 4$.

Draw the diagram:



Computing invariants

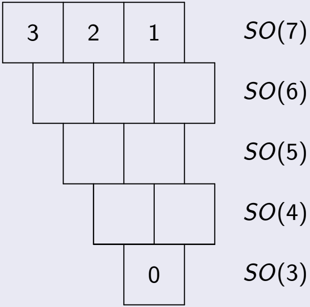
- V_λ representation of $SO(n)$;

Question: What is dim $V_\lambda^{SO(n-k)}$?

Answer: Branching rules.

Example: Say $n = 7$ with $\lambda = (3, 2, 1)$; $k = 4$.

Draw the diagram:



The number of ways to fill the boxes with "interlacing rule" is the number of invariants.

Computing invariants cont'd

Filling the diagram with interlacing rule is an affine linear constraint.

Computing invariants cont'd

Filling the diagram with interlacing rule is an affine linear constraint.

Interpret $\dim V_\lambda^{SO(n-k)}$ as the number of lattice points in a polytope, called the Gelfand-Tsetlin (GT) polytope associated to λ.

$$\dim V_\lambda^{SO(n-k)} = \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)).$$

Computing invariants cont'd

Filling the diagram with interlacing rule is an affine linear constraint.

Interpret $\dim V_\lambda^{SO(n-k)}$ as the number of lattice points in a polytope, called the Gelfand-Tsetlin (GT) polytope associated to λ.

$$\dim V_\lambda^{SO(n-k)} = \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)).$$

$$\deg St(k, n) = m! \lim_{t\to\infty} \frac{\sum_{|\lambda|\leq t} \dim V_\lambda \cdot \dim V_\lambda^{SO(n-k)}}{t^m}$$

Computing invariants cont'd

Filling the diagram with interlacing rule is an affine linear constraint.

Interpret $\dim V_\lambda^{SO(n-k)}$ as the number of lattice points in a polytope, called the Gelfand-Tsetlin (GT) polytope associated to λ.

$$\dim V_\lambda^{SO(n-k)} = \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)).$$

$$\begin{aligned}\deg St(k,n) &= m! \lim_{t\to\infty} \frac{\sum_{|\lambda|\leq t} \dim V_\lambda \cdot \dim V_\lambda^{SO(n-k)}}{t^m} \\ &= m! \lim_{t\to\infty} \sum_{|\lambda|\leq t} \frac{1}{t^m} \mathcal{L}(GT_{SO(1)}^{SO(n)}(\lambda)) \cdot \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda))\end{aligned}$$

Computing invariants cont'd

Filling the diagram with interlacing rule is an affine linear constraint.

Interpret $\dim V_\lambda^{SO(n-k)}$ as the number of lattice points in a polytope, called the Gelfand-Tsetlin (GT) polytope associated to λ.

$$\dim V_\lambda^{SO(n-k)} = \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)).$$

$$\begin{aligned}
\deg St(k,n) &= m! \lim_{t\to\infty} \frac{\sum_{|\lambda|\le t} \dim V_\lambda \cdot \dim V_\lambda^{SO(n-k)}}{t^m} \\
&= m! \lim_{t\to\infty} \sum_{|\lambda|\le t} \frac{1}{t^m} \mathcal{L}(GT_{SO(1)}^{SO(n)}(\lambda)) \cdot \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)) \\
&= m! \int_{\mathcal{W}} vol(GT_{SO(1)}^{SO(n)}(\lambda)) \cdot vol(GT_{SO(n-k)}^{SO(n)}(\lambda))
\end{aligned}$$

Computing invariants cont'd

Filling the diagram with interlacing rule is an affine linear constraint.

Interpret $\dim V_\lambda^{SO(n-k)}$ as the number of lattice points in a polytope, called the Gelfand-Tsetlin (GT) polytope associated to λ.

$$\dim V_\lambda^{SO(n-k)} = \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)).$$

$$\begin{aligned}
\deg St(k,n) &= m! \lim_{t\to\infty} \frac{\sum_{|\lambda|\le t} \dim V_\lambda \cdot \dim V_\lambda^{SO(n-k)}}{t^m} \\
&= m! \lim_{t\to\infty} \sum_{|\lambda|\le t} \frac{1}{t^m} \mathcal{L}(GT_{SO(1)}^{SO(n)}(\lambda)) \cdot \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)) \\
&= m! \int_{\mathcal{W}} vol(GT_{SO(1)}^{SO(n)}(\lambda)) \cdot vol(GT_{SO(n-k)}^{SO(n)}(\lambda)) \\
&= \cdots =
\end{aligned}$$

Computing invariants cont'd

Filling the diagram with interlacing rule is an affine linear constraint.

Interpret $\dim V_\lambda^{SO(n-k)}$ as the number of lattice points in a polytope, called the Gelfand-Tsetlin (GT) polytope associated to λ.

$$\dim V_\lambda^{SO(n-k)} = \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)).$$

$$\begin{aligned}
\deg St(k,n) &= m! \lim_{t\to\infty} \frac{\sum_{|\lambda|\le t} \dim V_\lambda \cdot \dim V_\lambda^{SO(n-k)}}{t^m} \\
&= m! \lim_{t\to\infty} \sum_{|\lambda|\le t} \frac{1}{t^m} \mathcal{L}(GT_{SO(1)}^{SO(n)}(\lambda)) \cdot \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)) \\
&= m! \int_{\mathcal{W}} vol(GT_{SO(1)}^{SO(n)}(\lambda)) \cdot vol(GT_{SO(n-k)}^{SO(n)}(\lambda)) \\
&= \cdots = \\
&= 2^k \cdot \text{ big determinant of binomial coefficients}
\end{aligned}$$

Computing invariants cont'd

Filling the diagram with interlacing rule is an affine linear constraint.

Interpret $\dim V_\lambda^{SO(n-k)}$ as the number of lattice points in a polytope, called the Gelfand-Tsetlin (GT) polytope associated to λ.

$$\dim V_\lambda^{SO(n-k)} = \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)).$$

$$\begin{aligned}
\deg St(k,n) &= m! \lim_{t\to\infty} \frac{\sum_{|\lambda|\le t} \dim V_\lambda \cdot \dim V_\lambda^{SO(n-k)}}{t^m} \\
&= m! \lim_{t\to\infty} \sum_{|\lambda|\le t} \frac{1}{t^m} \mathcal{L}(GT_{SO(1)}^{SO(n)}(\lambda)) \cdot \mathcal{L}(GT_{SO(n-k)}^{SO(n)}(\lambda)) \\
&= m! \int_{\mathcal{W}} vol(GT_{SO(1)}^{SO(n)}(\lambda)) \cdot vol(GT_{SO(n-k)}^{SO(n)}(\lambda)) \\
&= \cdots = \\
&= 2^k \cdot \text{ big determinant of binomial coefficients}
\end{aligned}$$

Gessel-Viennot: Determinants of binomial coefficients count non-intersecting lattice path configurations.

Generalize to higher degree?

Let $f \in S^d\mathbb{C}^k$.

Generalize to higher degree?

Let $f \in S^d\mathbb{C}^k$.

- Not all f are the same. Choose one: e.g., $f = x_1^d + \cdots + x_k^d$ with $R(f) = k$.

Generalize to higher degree?

Let $f \in S^d\mathbb{C}^k$.

- Not all f are the same. Choose one: e.g., $f = x_1^d + \cdots + x_k^d$ with $R(f) = k$.
- In general, $\mathcal{VSP}_r(f)$ is not even irreducible, certainly not a homogeneous space. No rich rep-theoretic structure.

Generalize to higher degree?

Let $f \in S^d\mathbb{C}^k$.

- Not all f are the same. Choose one: e.g., $f = x_1^d + \cdots + x_k^d$ with $R(f) = k$.
- In general, $\mathcal{VSP}_r(f)$ is not even irreducible, certainly not a homogeneous space. No rich rep-theoretic structure.
- We expect a stabilization result:

  If $r \gg k$, then $\deg \mathcal{VSP}_r(f) = d^{\binom{d+k-1}{d}}$.

Generalize to higher degree?

Let $f \in S^d\mathbb{C}^k$.

- Not all f are the same. Choose one: e.g., $f = x_1^d + \cdots + x_k^d$ with $R(f) = k$.
- In general, $\mathcal{VSP}_r(f)$ is not even irreducible, certainly not a homogeneous space. No rich rep-theoretic structure.
- We expect a stabilization result:

  If $r \gg k$, then $\deg \mathcal{VSP}_r(f) = d^{\binom{d+k-1}{d}}$.
- What about other notions of rank?