Seminar Linear Operators
Boundary Triplets and Self-adjoint Extensions. Positive Self-adjoint Extensions

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Introduction

Studying the constructions and properties of self-adjoint extensions of a densely defined lower semi-bounded symmetric operator T on some Hilbert space $\mathcal{H}$ is one of the most important topics in [2]. By using a boundary triplet $(\mathcal{K}, \Gamma_0, \Gamma_1)$ of the adjoint T^*, all self-adjoint extensions of T can be parametrized as restrictions $T_{\mathcal{B}}$ of T^* on

$$D(T_{\mathcal{B}}) = \{x \in D(T^*) : (\Gamma_0 x, \Gamma_1 x) \in \mathcal{B}\},$$

where $\mathcal{B}$ is a self-adjoint relation on $\mathcal{K}$, or equivalently, as restrictions T_B of T^* on

$$D(T_B) = \{x \in D(T^*) : \Gamma_0 x \in D(B) \text{ and } B\Gamma_0 x = P_B \Gamma_1 x\},$$

where B is a self-adjoint operator on some closed subspace $\mathcal{K}_B$ of $\mathcal{K}$, and P_B is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_B$.([2, Section 14.2, esp. Theorem 14.10])

Moreover, by Theorem 1.21 of Krein-Naimark, one can represent $T_{\mathcal{B}}$ and its domain by Gamma-Fields, Weyl functions, and the operator $T_0 := T_{\{0\}\oplus\mathcal{K}} \equiv T_{\mathcal{B}}|_{\mathcal{B}=\{0\}\oplus\mathcal{K}}$.

This representation leads to an interesting fact, which is the main result of Section 2 of this text: if the operator T_0 is the Friedrich extension T_F of T, one can express the form associated to an extenstion $T_{\mathcal{B}}$ in terms of the form associated to T_F and of the positive self-adjoint relation $\mathcal{B} - M(\lambda)$.

Section 3 considers the case where the greatest lower bound m_T of T is positive. In this case, we obtain the decomposition of the form of T_B in terms of the form of the Friedrich extension T_F and of B. One can think the part of T_F as the "hard"[1] one of T_B, which is "fixed", and the part of B as the "soft" one, which may be change if one changes B. This decomposition is very useful to compare positive self-adjoint extensions of T (Theorem 3.1). Considering

[1] According to [2, Remark in p.292], Krein called T_F the hard extension of T.

an extreme case of the operator B, we obtain a representation of the Krein-von Neumann extensions, which is the smallest extension among all positive self-adjoint extensions of T.

The last section introduces two examples for the Krein-von Neumann extension, where T is an differential operator and the Krein-von Neumann extension T_N is expressed as a differential operator with boundary conditions. The main tool to solve theses examples is the decomposition of T_N stated in Section 3.

The materials, which we use in this text, but do not belongs to [2, Section 14.7-8], are recalled in the preliminary, Section 1.

1 Preliminary

In this section, we recall some important knowledge about symmetric and self-adjoint operators on Hilbert spaces, which we use in next Sections.

1.1 Some basic definitions: lower semi-boundedness, self-adjoint operators and their forms, Friedrichs extension

Definition 1.1. Assume A is a self-adjoint operator on some Hilbert space $\mathcal{H}$. Let E_A denote its spectral measure by the well-known spectral theorem (for self-adjoint operators). We define the (sesquilinear) form of A by

$$D[A] := D(|A|^{1/2}) = \left\{ x \in \mathcal{H} : \int_{\mathbb{R}} |\lambda| d \langle E_A(\lambda)x, x\rangle < \infty \right\}$$
$$A[x,y] := \int_{\mathbb{R}} \lambda \langle E_A(\lambda)x, y\rangle, \qquad \text{for } x, y \in D[A].$$

Remark 1. In order to understand this definition, one should be familiar with the spectral theorem and the functional calculus. We recall only very essential facts. By the spectral theorem and functional calculus, given a self-adjoint operator A on some Hilbert space $\mathcal{H}$, we can define for a Borel function f a new operator $f(A)$ by

$$D(f(A)) = \left\{ x \in \mathcal{H} : \int_{\mathbb{R}} |f(\lambda)|^2 d \langle E_A(\lambda)x, x\rangle < \infty \right\}$$
$$\langle f(A)x, y\rangle := \int_{\mathbb{R}} f(\lambda) d \langle E_A(\lambda)x, y\rangle, \qquad \text{for } x \in D(A), y \in \mathcal{H}.$$

(see [2, eq. (5.10) and (5.11)].) Now it rises a question of whether we get the "origin" domain of A if we use the above formula for $f(\lambda) := \lambda$. The answer is YES (see the paragraph after [2, eq. (5.11)]), and therefore, the domain of A can be represent as

$$D(A) = \left\{ x \in \mathcal{H} : \int_{\mathbb{R}} |\lambda|^2 d \langle E_A(\lambda)x, x\rangle < \infty \right\}.$$

Moreover, if we choose $f(\lambda) := |\lambda|^{1/2}$, then, by the above representation of $D(A)$, we have $D(A) \subseteq D(|A|^{1/2}) \equiv D[A]$. Then, for $x, y \in D[A]$,

$$\begin{aligned} |A[x,y]| &\equiv \left| \int_{\mathbb{R}} \lambda \left\langle E_A(\lambda)x, y \right\rangle \right| \\ &\leq \left(\int_{\mathbb{R}} \lambda d \left\langle E_A(\lambda)x, x \right\rangle \right)^{1/2} \left(\int_{\mathbb{R}} \lambda d \left\langle E_A(\lambda)x, x \right\rangle \right)^{1/2} \\ &< \infty \end{aligned}$$

by [2, Lemma 4.8. (ii)]. Therefore $A[x, y]$ is well-defined for $x, y \in D[A]$.

Definition 1.2. *lower semi-bounded operators* Let $A : \mathcal{H} \supseteq D(A) \to \mathcal{H}$ be a linear operator on some Hilbert space $\mathcal{H}$. Then A is called bounded from below if there is a constant $m \in \mathbb{R}$ such that

$$\langle Ax, x \rangle \geq m \|x\|^2, \qquad \text{for } x \in D(A).$$

Further, $m_A := \sup \left\{ \langle Ax, x \rangle \|x\|^{-2} : x \in D(A), \|x\| \neq 0 \right\}$ denotes the greatest lower bound of A.

Definition 1.3. Let A be a lower semi-bounded self-adjoint operator on some Hilbert space $\mathcal{H}$. Then the form norm of A can be defined by

$$\|x\|_A := A[x] + (1 - m_A) \|x\|, \forall x \in D[A].$$

We can replace m_A by another lower bound $m \leq m_A$ in order to get an equivalent norm. See Proposition 1.6.

Proposition 1.4. *Let A be self-adjoint on $\mathcal{H}$. Then $D(A)$ is the set of $x \in D[A]$ for which there exists a vector $u \in \mathcal{H}$ such that $A[x, y] = \langle u, y \rangle$ for $y \in D[A]$. If this holds, then $u = Ax$, and hence*

$$A[x, y] = \langle Ax, y \rangle \qquad \text{for } x \in D(A), y \in D[A].$$

Proof. [2, Proposition 10.4] □

Definition 1.5. *order relations* Let A and B be symmetric operators on some Hilbert space $\mathcal{H}$. We write $A \succeq B$ if $D(A) \subseteq D(B)$ and $\langle Ax, x \rangle \geq \langle Bx, x \rangle$ for $x \in D(A)$. If A and B are self-adjoint, we write $A \geq B$ if $D[A] \subseteq D[B]$ and $A[x] \geq B[x]$ for all $x \in D[A]$.

Proposition 1.6. *Let $A \geq mI$ be a lower semi-bounded self-adjoint operator.*

1. *For $x, y \in D[A] = D((A - mI)^{1/2})$, we have*

$$A[x, y] = \left\langle (A - mI)^{1/2} x, (A - mI)^{1/2} y \right\rangle + m \langle x, y \rangle.$$

2. *$D(A)$ is dense in $D[A]$ with respect to the form norm of A.*

3. *If B is a linear operator such that $D(B) \subseteq D[A]$ and $A[x,y] = \langle Bx, y\rangle$ for $x \in D(B)$ and $y \in D(A)$, then $B \subseteq A$.*

4. *For $\lambda \leq m_A$, the norm $\left\|(A-\lambda I)^{1/2}\cdot\right\|$ is equivalent to the form norm of A, which is stronger than the norm induced from $\mathcal{H}$. This means that in the definition of the form norm (Definition 1.3), we can replace m_A by another lower bound m of A.*

Proof. We prove only the last assertion. For the others, see [2, Proposition 10.5]. To do this, we show

$$\min(1; m_A - \lambda) \|u\|_A \leq (A - \lambda I) \|u\| \leq \max(1; m_A - \lambda) \|u\|_A, \qquad \text{for } u \in D[A].$$

Let $u \in D[A]$. Then

$$\begin{aligned}(A-\lambda I)[u] &= A[u] - \lambda \|u\|^2 \\ &= (A[u] - m_A \|u\|^2) + (m_A - \lambda) \|u\|^2 \\ &\leq \max(1, m_A - \lambda) \left(A[u] - m_A \|u\|^2 + \|u\|^2\right) \\ &= \max(1, m_A - \lambda) \|u\|_A^2,\end{aligned}$$

and

$$\begin{aligned}(A-\lambda I)[u] &= (A[u] - m_A \|u\|^1) + (m_A - \lambda) \|u\|^2 \\ &\geq \min(1, m_A - \lambda) \left(A[u] - m_A \|u\|^2 + \|u\|^2\right) \\ &= \min(1, m_A - \lambda) \|u\|_A^2.\end{aligned}$$

Clearly, the form norm is stronger than the norm induced from $\mathcal{H}$, because

$$\|u\|_A^2 = A[u] - m_A \|u\|^2 + \|u\|^2 \geq \|u\|^2, \qquad \text{for } u \in D[A].$$

□

Proposition 1.7. *Let A and B be lower semi-bounded self-adjoint operators on some Hilbert space $\mathcal{H}$, and let $\lambda \in \mathbb{R}$, $\lambda < \min(m_A, m_B)$. Then $\lambda \in \rho(A) \cap \rho(B)$, and the relation $A \geq B$ holds if and only if $(B - \lambda I)^{-1} \geq (A - \lambda I)^{-1}$.*

Proof. This is [2, Corollary 10.13] □

Essentials about Friedrichs and Krein-von Neumann extensions We recall only the essential facts about the well-known Friedrichs and Krein-von Neumann extensions. Let T be a densely defined lower semi-bounded symmetric operator. We recall that the Friedrich extension of T, which is denoted by T_F, is the largest with respect to the relation $\geq$ among all lower semi-bounded self-adjoint extension of T. Moreover, the greatest lower bound of T_F is equal to that of T, i.e. $m_{T_F} = m_T$. Moreover, $D(T)$ is dense in $D(T_F)$ with respect to the form norm of T_F defined in Definition 1.3, where A is replaced by T_F.

More about this topic, see [2, 10.4]. It may be useful to write again the idea to construct T_F. One can show that the form s_T defined by $s_T[x,y] := \langle Tx, y\rangle$ for $x, y \in D(s_T) := D(T)$ is closable with the (closed!) closure $\overline{s_T}$ (see [2, Lemma 10.16], and also [2, Section 10.1] for basic facts of forms, closed and closable forms), and T_F is defined as

$$D(T_F) = \{x \in D(\overline{s_T}) : \text{ there is } u_x \in \mathcal{H} \text{ with } \overline{s_T}[x,y] = \langle u_x, y\rangle \text{ for } y \in D(\overline{s_T})\},$$

and $T_F x := u_x$, which is uniquely determined because $D(\overline{s_T}) \supseteq D(T)$ is dense in $\mathcal{H}$. Because $\overline{s_T}$ is closed, T_F is self-adjoint by the Representation Theorem of semi-bounded forms ([2, Theorem 10.17]).

The Krein-von Neumann extension T_N is the topic of [2, Chapter 13]. If T is a densely defined POSITIVE symmetric operator, then the Krein-von Neumann extension T_N of T is the smallest (with respect to $\geq$) among all positive self-adjoint extension of T on $\mathcal{H}$. In this case, we have $T_F \geq A \geq T_N$ for all positive self-adjoint extension A of T on $\mathcal{H}$.

1.2 Linear Relations on Hilbert spaces

Definition 1.8. *Linear Relations on Hilbert spaces* [2, Section 14.1]
Let $(\mathcal{H}_i, \langle\cdot,\cdot\rangle_1)$,$i = 1, 2$ be Hilbert spaces. A linear subspace $T \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$ is called a linear relation from $\mathcal{H}_1$ into $\mathcal{H}_2$. For such a linear relation, we define the domain $D(T)$, the range $\mathcal{R}(T)$, the kernel $\mathcal{N}(T)$, and the multivalued part $\mathcal{M}(T)$ by

$$\begin{aligned}
D(T) &:= \{x \in \mathcal{H}_1 : (x,y) \in T \text{ for some } y \in \mathcal{H}_2\},\\
\mathcal{R}(T) &:= \{y \in \mathcal{H}_2 : (x,y) \in T \text{ for some } x \in \mathcal{H}_1\},\\
\mathcal{N}(T) &:= \{x \in \mathcal{H}_1 : (x,0) \in T\},\\
\mathcal{M}(T) &:= \{y \in \mathcal{H}_2 : (0,y) \in T\}.
\end{aligned}$$

The closure $\overline{T}$ is the closure of the linear subspace T with respect to the product topology of $\mathcal{H}_1 \oplus \mathcal{H}_2$. If $T = \overline{T}$, then T is called closed. Further, we define the inverse T^{-1} and the adjoint T^* by

$$\begin{aligned}
T^{-1} &:= \{(y,x) : (x,y) \in T\},\\
T^* &:= \{(y,x) : \langle v, y\rangle_2 = \langle u, x\rangle_1 \qquad \text{for all } (u,v) \in T\}.
\end{aligned}$$

If T is (the graph of) a linear operator, then these notions introduced here coincides with the corresponding notions of operators, if they exist. But, different from linear operators, the adjoint and the closure of linear relations are always defined.

Moreover, if S and T be linear relations from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let $\alpha \in \mathbb{C} - \{0\}$, we define

$$\begin{aligned}
\alpha T &:= \{(x, \alpha y) : (x,y) \in T\}\\
S + T &:= \{(x, u+v) : (x,u) \in S \text{ and } (x,v) \in T\}.
\end{aligned}$$

A linear relation T on a Hilbert space $\mathcal{H}$ is called symmetric if $T \subseteq T^*$, i.e. $\langle x, v \rangle = \langle y, u \rangle$ for all $(x, y), (u, v) \in T$, and self-adjoint, if $T = T^*$.

Let T be a symmetric relation. For $\alpha \in \mathbb{R}$, we write $T \geq \alpha I$, if $\langle y, x \rangle \geq \alpha \langle x, x \rangle$ for all $(x, y) \in T$. In this case, T is called lower semi-bounded. In particular, we call T positive, if $\langle y, x \rangle \geq 0$ for $(x, y) \in T$.

By orthogonal projection, one can construct an self-adjoint operator B from a self-adjoint linear relation $\mathcal{B}$ on some Hilbert space $\mathcal{H}$, such that this correspondence $B \leftrightarrow \mathcal{B}$ is one-to-one. This is the content of the following proposition, which we do not want to prove. Here, we denote $\mathcal{S}$ the set of all self-adjoint B acting on a closed linear subspace $\mathcal{H}_B$ of $\mathcal{H}$, i.e. $B : \mathcal{H}_B \supseteq D(B) \to \mathcal{H}_B$, and P_B the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_B$.

Proposition 1.9. *There is a one-to-one correspondence between the sets of operators $B \in \mathcal{S}(\mathcal{H})$ and of self-adjoint relations $\mathcal{B}$ on $\mathcal{H}$ given by*

$$\begin{aligned}\mathcal{B} &= G(B) \oplus \left(\{0\} \oplus (\mathcal{H}_B)^{\perp}\right) \\ &\equiv \left\{(x, Bx + y) : x \in D(B), y \in (\mathcal{H}_B)^{\perp}\right\},\end{aligned}$$

where B is the operator part $\mathcal{B}_s$, and $(\mathcal{H}_B)^{\perp}$ is the multivalued part $\mathcal{M}(\mathcal{B})$ of $\mathcal{B}$. Moreover $\mathcal{B} \geq 0$ if and only if $B \geq 0$.

Proof. This is [2, Proposition 14.2]. □

It leads to

Definition 1.10. Let $\mathcal{B}$ be a self-adjoint relation and B its operator part. We define the form associated with $\mathcal{B}$ by

$$\mathcal{B}[x, x'] := B[x, x'], \qquad \text{for all } x, x' \in D[\mathcal{B}] := D[B].$$

Moreover, we have

Lemma 1.11. *Let $\mathcal{B}$ be a linear relation on some Hilbert space $\mathcal{H}$ such that $\mathcal{B}^{-1}$ is the graph of a positive self-adjoint operator on $\mathcal{H}$. Then $\mathcal{B}$ is a positive self-adjoint relation.*

Proof. This is [2, Lemma 14.3]. □

1.3 Boundary Triplets of Adjoints of Symmetric Operators

Definition 1.12. Let T be a densely defined symmetric operator on some Hilbertspace $\mathcal{H}$. Let $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$ be a Hilbert space, and $\Gamma_0 : D(T^*) \to \mathcal{K}$ and $\Gamma_1 : D(T^*) \to \mathcal{K}$ be linear mappings. Then $(\mathcal{K}, \Gamma_0, \Gamma_1)$ is a boundary triplet for the adjoint T^*, if the following conditions are satisfied:

i) $[x, y]_{T^*} := \langle T^* x, y \rangle - \langle x, T^* y \rangle = \langle \Gamma_1 x, \Gamma_0 y \rangle_{\mathcal{K}} - \langle \Gamma_0 x, \Gamma_1 y \rangle_{\mathcal{K}}$ for $x, y \in D(T^*)$,

ii) the mapping $D(T^*) \ni x \to (\Gamma_0 x, \Gamma_1 x) \in \mathcal{K} \oplus \mathcal{K}$ is surjective.

Definition 1.13. Let (K, Γ_0, Γ_1) be a boundary triple for T^* and $\mathcal{B}$ a linear relation on $\mathcal{K}$. We define the parametrized restriction[2] $T_{\mathcal{B}}$ of T^* to the domain

$$D(T_{\mathcal{B}}) := \{x \in D(T^*) : (\Gamma_0 x, \Gamma_1 x) \in \mathcal{B}\}$$

Recall that $\mathcal{S}(\mathcal{K})$ the set of self-adjoint operators B acting on a closed subspace $\mathcal{K}_B$ of $\mathcal{K}$ and P_B is the orthogonal projection onto $\mathcal{K}_{\mathcal{B}}$.

Definition 1.14. For $B \in \mathcal{S}(\mathcal{K})$, we define T_B to be the parameterized restriction of T^* to the domain

$$D(T_B) := \{x \in D(T^*) : \Gamma_0 x \in D(B) \text{ and } B\Gamma_0 x = P_B \Gamma_1 x\}.$$

Comparing to above definitions, we see that if B is the operator part of $\mathcal{B}$, then $T_B = T_{\mathcal{B}}$.

The parametrization $\{T_{\mathcal{B}}; \mathcal{B} \text{ relation on } \mathcal{K}\}$, or equivalently $\{T_B; B \in \mathcal{S}(\mathcal{K})\}$ gives us a characterization of the set of self-adjoint extension of T. This is the following result which we do not want to prove.

Theorem 1.15. *Suppose T is a densely defined symmetric operator on some Hilbert space $\mathcal{H}$, and $(\mathcal{K}, \Gamma_0, \Gamma_1)$ is a boundary triplet for T^*. For any operator S on $\mathcal{H}$, the following are equivalent:*

(i) S is a self-adjoint extension of T on $\mathcal{H}$.

(ii) There is a self-adjoint relation $\mathcal{B}$ on $\mathcal{K}$ such that $S = T_{\mathcal{B}}$, or equivalently, there is an operator $B \in \mathcal{S}(K)$ such that $S = T_B$.

Moreover, the relation $\mathcal{B}$ and the operator B are uniquely determined by S.

Proof. see [2, Theorem 14.10] □

Corollary 1.16. *If $(\mathcal{K}, \Gamma_0, \Gamma_1)$ is a boundary triplet for T^*, then there exist self-adjoint extensions T_0 and T_1 of the symmetric operator T on $\mathcal{H}$ defined by $D(T_0) = \mathcal{N}(\Gamma_0)$ and $D(T_1) = \mathcal{N}(\Gamma_1)$.*

Proof. This is [2, Corollary 14.8]. Choose $\mathcal{B}_0 = \{0\} \oplus \mathcal{K}$ and $\mathcal{B}_1 = \mathcal{K} \oplus \{0\}$, then $T_0 := T_{\mathcal{B}_0}$ and $T_1 := T_{\mathcal{B}_1}$ are self-adjoint operators satisfying $D(T_0) = \mathcal{N}(\Gamma_0)$ and $D(T_1) = \mathcal{N}(\Gamma_1)$. □

Now let T be a symmetric operator with a real regular point μ. Recall that μ is called a regular point of T if there is a $c = c_\mu > 0$ such that $\|(T - \mu I)x\| \geq c_\mu \|x\|$ for all $x \in D(T)$. Then a basic result ([2, Proposition 3.16]) implies that there exists a self-adjoint extension A of T such that μ is in the resolvent $\rho(A)$ of A. If we have a boundary triplet $(\mathcal{K}, \Gamma_0, \Gamma_1)$, we can "parameterize" A by $A = T_{\mathcal{B}}$ for a relation $\mathcal{B}$. Now it rises the question of whether we can choose a boundary triplet such that A is exactly the operator T_0 in Corollary 1.16. The answer is YES. Moreover, we have $M(\mu) = 0$ and $\gamma(\mu) = I_{\mathcal{K}} \upharpoonright \mathcal{K}$. This is the result of [2, Example 14.6, Theorem 14.12, Example 14.12].

[2] I have not found a special name for this in the literature.

Theorem 1.17. *Let T be a densely defined symmetric operator on some Hilbert space $\mathcal{H}$. Suppose that A is a fixed self-adjoint extension of T on $\mathcal{H}$ and $\mu \in \mathbb{R} \cap \rho(A)$. Set $\mathcal{N}_\mu := \mathcal{N}(T^* - \mu I)$. For $B \in \mathcal{S}(\mathcal{N}_\mu)$, let*

$$\begin{aligned} &D(T_B) := \left\{ x + R_\mu(A)(Bu + v) + u : x \in D(\overline{T}), u \in D(B), v \in \mathcal{N}_\mu \cap D(B)^\perp \right\}, \\ &T_B := T^* \upharpoonright D(T_B), \text{ that is,} \\ &T_B\left(x + R_\mu(A)(Bu+v) + u\right) := \overline{T}x + (I + \mu R_\mu(A))(Bu + v) + \mu u. \end{aligned}$$

Then the operator T_B is a self-adjoint extension of T on $\mathcal{H}$. Each self-adjoint extension of T is of the form T_B with uniquely determined $B \in \mathcal{S}(\mathcal{N}_\mu)$

Proof. [2, Example 14.6, Theorem 14.12] □

1.4 Gamma Fields and Weyl Functions

Let T be a densely defined symmetric operator on $\mathcal{H}$ and $(\mathcal{K}, \Gamma_0, \Gamma_1)$ be a boundary triplet for T^*, and T_0 the self-adjoint extension of T in Corollary 1.16 such that $D(T_0) = \mathcal{N}(\Gamma_0)$. Set $\mathcal{N}_z := \mathcal{N}(T^* - zI)$.

Lemma 1.18.

1. *Γ_0 and Γ_1 are continuous mappings of $(D(T^*), \|\cdot\|_{T^*})$ into $\mathcal{K}$.*
2. *For each $z \in \rho(T_0)$, the mapping Γ_0 is a continuous bijective mapping of the subspace $\mathcal{N}_z$ onto $\mathcal{K}$ with bounded inverse denote by $\gamma(z)$.*

Proof. This is [2, Lemma 14.13] □

It leads to

Definition 1.19. We call the map $\rho(T_0) \ni z \to \gamma(z) \in \mathbf{B}(\mathcal{K}, \mathcal{H})$ the gamma field and the map $\rho(T_0) \ni z \to M(z) \in \mathbf{B}(\mathcal{K})$ the Weyl function of the operator T_0 associated with the boundary triplet $(\mathcal{K}, \Gamma_0, \Gamma_1)$.

Proposition 1.20. Some properties of gamma fields and Weyl functions *Let $z \in \rho(T_0)$, where T_0 is defined in Corollary 1.16. Let $\gamma(z), M(z)$ be the gamma field and Weyl function as in the above definition. Then*

1. *$\mathcal{N}(\gamma(\lambda)^*) = \mathcal{N}_z^\perp$ and $\gamma(z)^*$ is a bijection of $\mathcal{N}_z$ onto $\mathcal{K}$.*
2. *For $z \in \rho(T_0)$, we have $M(z)^* = M(\overline{z})$.*

1.5 The Krein Naimark Resolvent Formula

Theorem 1.21. Krein Naimark Resolvent Formula *Let T be a densely defined symmetric operator on $\mathcal{H}$, and $(\mathcal{K}, \Gamma_0, \Gamma_1)$ a boundary triplet for T^*. Suppose* *that $\mathcal{B}$ is a closed relation on $\mathcal{K}$ and $z \in \rho(T_0)$. Then the proper extension $T_\mathcal{B}$ of T is given by*

$$D(T_\mathcal{B}) = \left\{ \begin{array}{c} f = (T_0 - zI)^{-1}(y+v) + \gamma(z)u : \\ u \in \mathcal{K}, v \in \mathcal{N}_{\overline{z}}, \\ (u, \gamma(\overline{z})^* v) \in \mathcal{B} - M(z), \\ y \in \mathcal{H} \ominus \mathcal{N}_{\overline{z}} \end{array} \right\} \tag{1}$$

$$T_\mathcal{B} f = T_B \left((T_0 - zI)^{-1}(y+v) + \gamma(z)u \right) = zf + y + v. \tag{2}$$

If $z \in \rho(T_\mathcal{B}) \cap \rho(T_0)$, the relation $\mathcal{B} - M(z)$ has an inverse $(\mathcal{B} - M(z))^{-1} \in \mathbf{B}(\mathcal{K})$, and

$$(T_B - zI)^{-1} - (T_0 - zI)^{-1} = \gamma(z)(\mathcal{B} - M(z))^{-1}\gamma(\overline{z})^*. \tag{3}$$

Proof. [2, Theorem 14.18] □

2 Boundary Triplets and Semibounded Self-adjoint Operators

In this section we assume that T is a densely defined lower semi-bounded symmetric operator and $(\mathcal{K}, \Gamma_0, \Gamma_1)$ is a boundary triplet for T^*. Recall that T_0 denotes the self-adjoint operator defined in Corollary 1.16, and $\gamma(z)$ and $M(z)$ denote the gamma field and the Weyl function of T_0.

Proposition 2.1. *Suppose that T_0 is the Friedrichs extension T_F of T. Let $\mathcal{B}$ be a self-adjoint relation on $\mathcal{K}$. If the self-adjoint operator $T_\mathcal{B}$ is lower semibounded, so is the relation B. More precisely, if $\lambda < m_T$ and $\lambda \leq m_{T_\mathcal{B}}$, then $\mathcal{B} - M(\lambda) \geq 0$.*

Proof. i) Fix $\lambda' < \lambda$. First, assume we can show that $\mathcal{B} - M(\lambda') \geq 0$, which is equivalent to

$$\langle y, x \rangle - \langle M(\lambda')x, x \rangle \geq 0, \qquad \text{for all } x \in D(\mathcal{B}).$$

Letting $\lambda' \to \lambda$, by the fact that $M(\cdot)$ is continous $\mathbf{B}(\mathcal{K})$-valued function, we obtain

$$\langle y, x \rangle - \langle M(\lambda)x, x \rangle \geq 0, \qquad \text{for all } x \in D(\mathcal{B}),$$

which means $\mathcal{B} - M(\lambda) \geq 0$.

ii) We show now, that $\mathcal{B} - M(\lambda') \geq 0$. Recall that for the Friedrichs extension, we have $m_T = m_{T_F}$. Since $\lambda' < m_T = m_{T_F} = m_{T_0}$ and $\lambda' < \lambda \leq m_{T_\mathcal{B}}$, we have $\lambda' \in \rho(T_\mathcal{B}) \cap \rho(T_0)$, by Proposition 1.7. By the resolvent formula (3) in Theorem 1.21,

$$(T_\mathcal{B} - \lambda' I)^{-1} - (T_0 - \lambda' I)^{-1} = \gamma(\lambda')(\mathcal{B} - M(\lambda'))^{-1}\gamma(\lambda')^*.$$

Since T_F is the largest lower semibounded self-adjoint extension of T, we have $T_0 = T_F \geq T_{\mathcal{B}}$, and taking the inversion (Proposition 1.7) leads to $(T_{\mathcal{B}} - \lambda' I)^{-1} \geq (T_0 - \lambda' I)^{-1}$. By the above resolvent formula, we obtain

$$\gamma(\lambda')(\mathcal{B} - M(\lambda'))^{-1}\gamma(\lambda')^* \geq 0,$$

which is equivalent to

$$\left\langle \gamma(\lambda')(\mathcal{B} - M(\lambda'))^{-1}\gamma(\lambda')^* x, x \right\rangle \geq 0,$$

and to

$$\left\langle (\mathcal{B} - M(\lambda'))^{-1}\gamma(\lambda')^* x, \gamma(\lambda')^* x \right\rangle \geq 0,$$

for all $x \in \mathcal{K}$. Recall that $\gamma(\lambda')^* \in \mathbf{B}(\mathcal{K})$ and $\gamma(\lambda')^*\mathcal{K} = \mathcal{K}$. Hence we can replace $\gamma(\lambda)^* x$ by y in the previous formula to get

$$\left\langle (\mathcal{B} - M(\lambda'))^{-1} y, y \right\rangle \geq 0, \qquad \text{for all } y \in \mathcal{K}.$$

By the inversion (using Lemma 1.11), $\mathcal{B} - M(\lambda') \geq 0$.

□

The following theorem is the main result of this section, it states that the decomposition of the form of $T_{\mathcal{B}} - \lambda$ in terms of the form of $T_F - \lambda$ and the form of the positiv linear relation $\mathcal{B} - M(\lambda)$. The main tool to prove this theorem is Theorem 1.21.

Recall that m_T is the greatest lower bound of T, as in Definition 1.2.

Theorem 2.2. *Let T be a densely defined lower semibounded symmetric operator, and let $(\mathcal{K}, \Gamma_0, \Gamma_1)$, be a boundary triplet for T^* such that the operator T_0 is the Friedrichs extension T_F of T. Let $\lambda \in \mathbb{R}, \lambda < m_T$. Suppose that $\mathcal{B}$ is a self-adjoint relation on $\mathcal{K}$ such that $\mathcal{B} - M(\lambda) \geq 0$. Then $T_{\mathcal{B}} - \lambda I$ is a positive self-adjoint operator, and*

$$\begin{aligned} D[T_{\mathcal{B}}] &= D[T_F] \dotplus \gamma(\lambda) D[\mathcal{B} - M(\lambda)] \\ (T_{\mathcal{B}} - \lambda I)[x + \gamma(\lambda)u, x' + \gamma(\lambda)u'] &= (T_F - \lambda I)[x, x'] + (\mathcal{B} - M(\lambda))[u, u'] \end{aligned} \tag{4}$$

for $x, x' \in D[T_F]$ and $u, u' \in D[\mathcal{B} - M(\lambda)]$.

Proof. The idea of this proof is to consider $D(T_{\mathcal{B}})$ instead of $D[T_{\mathcal{B}}]$. Recall that we have a representation of $D(T_{\mathcal{B}})$ by Theorem 1.21. From this, we obtain the sum

$$D(T_{\mathcal{B}}) = D(T_F) + \gamma(\lambda) D(\mathcal{B} - M(\lambda)),$$

which looks like the first equation in (4), but we have here only (). Also, by this representation, we can show the second formula in (4), but only for $x \in D(T_F)$ and $u \in D(\mathcal{B} - M(\lambda))$. The remaining can be treated by approximation. On the other hand, we see that the second formula of (4) looks like $a^2 = b^2 + c^2$, which comes from the definition of the Gamma-field.

1. In the first step, we show the "orthogonality" $(T_\mathcal{B} - \lambda I)[x, \gamma(\lambda)u] = 0$ for $x \in D[T_F]$ and $\gamma(\lambda)u \in D[T_\mathcal{B}], u \in \mathcal{K}$. By the properties of the Friedrichs extension in Subsection 1.1, there is a sequence (x_n) from $D(T)$, which converges to x in the form norm of T_F, and $T_F \geq T_\mathcal{B}$. The latter implies that $D[T_F] \subseteq D[T_\mathcal{B}]$, and that the form norm of $T_\mathcal{B}$ is weaker than the form norm of T_F. Therefore, the sequence (x_n) converges to x also in the form norm of $T_\mathcal{B}$.

   Using that $T \subset T_\mathcal{B}$ and $\gamma(\lambda)u \in \mathcal{N}_\lambda := \mathcal{N}(T^* - \lambda I)$, we derive

$$\begin{aligned}(T_\mathcal{B} - \lambda I)[x, \gamma(\lambda)u] &= \lim_{n\to\infty} (T_\mathcal{B} - \lambda I)[x_n, \gamma(\lambda)u] \\ &= \lim_{n\to\infty} \langle (T_\mathcal{B} - \lambda I)x_n, \gamma(\lambda)u\rangle \\ &= \lim_{n\to\infty} \langle (T - \lambda I)x_n, \gamma(\lambda)u\rangle = 0.\end{aligned}$$

   In the second equation, we used the fact that

$$A[x, y] = \langle Ax, y\rangle, \qquad \text{for all } x \in D(A), y \in D[A],$$

   where A is replaced by $T_\mathcal{B} - \lambda$.

2. Since $\mathcal{B}$ is self-adjoint, $M(\lambda) = M(\lambda)^* \in \mathbf{B}(\mathcal{K})$ by Proposition 1.20, and $\mathcal{B} - M(\lambda) \geq 0$ by assumption, $\mathcal{B} - M(\lambda)$ is a positive self-adjoint relation. Let C denote its operator part, which is defined in Propsition 1.9. Then C is a positive self-adjoint operator acting on some subspace $\mathcal{K}_C$ of $\mathcal{K}$, more precisely $C : \mathcal{K}_C \supseteq D(C) \to \mathcal{K}_C$, and the form associated with $\mathcal{B} - M(\lambda)$ is defined by $(\mathcal{B} - M(\lambda))[u, u'] = C[u, u']$ for $u, u' \in D[\mathcal{B} - M(\lambda)] := D(C)$.

   Clearly, $\lambda < m_T = m_{T_F}$ implies that $\lambda \in \rho(T_F)$. Hence, by Theorem 1.21, where z is replaced by $\lambda \in \mathbb{R}$ and $T_0 = T_F$, the operator $T_\mathcal{B}$ is described by

$$D(T_\mathcal{B}) = \left\{ \begin{array}{c} f = (T_F - \lambda I)^{-1}(y + v) + \gamma(\lambda)u \text{ with} \\ u \in \mathcal{K}, v \in \mathcal{N}_\lambda, \\ (u, \gamma(\lambda)^* v) \in \mathcal{B} - M(\lambda), \\ y \in H \ominus \mathcal{N}_\lambda \end{array} \right\}, \tag{5}$$

$$T_\mathcal{B} f \equiv T_\mathcal{B}\left((T_F - \lambda)^{-1}(y + v) + \gamma(\lambda)u\right) = \lambda f + y + v. \tag{6}$$

   Now let $f \in D(T_\mathcal{B})$. By the above description, there exist $v \in \mathcal{N}_\lambda$ and $y \in H \ominus \mathcal{N}_\lambda$ such that

$$\begin{aligned}&f = (T_F - \lambda I)^{-1}(y + v) + \gamma(z)u =: x + \gamma(\lambda)u, \text{ and} \\ &(u, \gamma(\lambda)^* v) \in \mathcal{B} - M(\lambda).\end{aligned}$$

   The latter implies that $u \in D(\mathcal{B} - M(\lambda)) = D(C) \subset \mathcal{K}_C$ and $Cu = P_C\gamma(\lambda)^* v$, where P_C is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_C$. By (6), we have $(T_\mathcal{B} - \lambda I)f = y + v$. Since $\gamma(\lambda)u \in \mathcal{N}_\lambda$ (recall that $\gamma(\lambda) := (\Gamma_0 \upharpoonright$

$\mathcal{N}_\lambda)^{-1})$, we have $y \perp \gamma(\lambda)u$. Using these facts, we compute

$$\begin{aligned}\langle (T_{\mathcal{B}} - \lambda I)f, f\rangle &= \langle y + v, x + \gamma(\lambda)u\rangle \\ &= \langle y + v, x\rangle + \langle v, \gamma(\lambda)u\rangle \\ &= \langle (T_F - \lambda I)x, x\rangle + \langle \gamma(\lambda)^* v, u\rangle_{\mathcal{K}} \\ &= \langle (T_F - \lambda I)x, x\rangle + \langle P_C \gamma(\lambda)^* v, u\rangle_{\mathcal{K}} \\ &= \langle (T_F - \lambda I)x, x\rangle + \langle Cu, u\rangle_{\mathcal{K}}\,.\end{aligned}$$

Therefore, since $T_F - \lambda I \geq 0$ and $C \geq 0$, it follows that $T_{\mathcal{B}} - \lambda I \geq 0$ and

$$\left\| (T_{\mathcal{B}} - \lambda I)^{1/2} f \right\|^2 = \left\| (T_F - \lambda I)^{1/2} x \right\|^2 + \left\| C^{1/2} u \right\|_{\mathcal{K}}^2. \tag{7}$$

3. Now we prove the inclusion $D[T_F] + \gamma(\lambda)D[C] \subseteq D[T_{\mathcal{B}}]$. Because T_F is the largest self-adjoint extension of T, $T_F \geq T_{\mathcal{B}}$, and so $D[T_F] \subseteq D[T_{\mathcal{B}}]$. Hence it suffices to show $\gamma(\lambda)D[C] \subseteq D[T_{\mathcal{B}}]$.

   Let $u \in D(C)$, where $D(C)$ is dense in $D[C]$ with respect to the form norm of C. By Proposition 1.20, $\gamma(\lambda)^*$ is a bijection of $\mathcal{N}_\lambda$ onto $\mathcal{K}$, so there exists a vector $v \in \mathcal{N}_\lambda$ such that $\gamma(\lambda)^* = Cu$. Since C is the operator part of $\mathcal{B} - M(\lambda)$, we have

$$(u, \gamma(\lambda)^* v) = (u, Cu) \in \mathcal{B} - M(\lambda).$$

   Hence, by choosing $y = 0$ in (5) and (6), we obtain

$$f := (T_F - \lambda I)^{-1} v + \gamma(\lambda)u =: x + \gamma(\lambda)u \in D(T_{\mathcal{B}})$$

   and $(T_{\mathcal{B}} - \lambda I)f = v$. Since

$$\begin{aligned}x &\in D(T_F) \subseteq D[T_F] \subseteq D[T_{\mathcal{B}}] \\ f &\in D(T_{\mathcal{B}}) \subseteq D[T_{\mathcal{B}}],\end{aligned}$$

   we have $\gamma(\lambda)u = f - x \in D[T_{\mathcal{B}}]$,, so $(T_{\mathcal{B}} - \lambda I)[x, \gamma(\lambda)u] = 0$ as shown in the first step. Further, $v = (T_F - \lambda I)x$, and $\gamma(\lambda)$ is bounded, so the adjoint $\gamma(\lambda)^*$ is everywhere defined. From these facts, we obtain

$$\begin{aligned}(T_{\mathcal{B}} - \lambda I)[f] &= \langle T_{\mathcal{B}} f, f\rangle \\ &= \langle v, x + \gamma(\lambda)u\rangle \\ &= \langle (T_F - \lambda I)x, x\rangle + \langle \gamma(\lambda)^* v, u\rangle_{\mathcal{K}} \\ &= (T_F - \lambda I)[x] + \langle Cu, u\rangle_{\mathcal{K}}\,, \\ (T_{\mathcal{B}} - \lambda I)[f] &= (T_{\mathcal{B}} - \lambda I)[x + \gamma(\lambda)u] \\ &= (T_{\mathcal{B}} - \lambda I)[x] + (T_{\mathcal{B}} - \lambda I)[\gamma(\lambda)u] + 2\mathrm{Re}\left\{(T_{\mathcal{B}} - \lambda I)[x, \gamma(\lambda)u]\right\} \\ &= (T_{\mathcal{B}} - \lambda I)[x] + (T_{\mathcal{B}} - \lambda I)[\gamma(\lambda)u].\end{aligned}$$

   Comparing both formulas yields $\langle Cu, u\rangle_{\mathcal{K}} = (T_{\mathcal{B}} - \lambda I)[\gamma(\lambda)u]$. Therefore,

$$\left\| C^{1/2} u \right\|_{\mathcal{K}} = \left\| (T_{\mathcal{B}} - \lambda I)^{1/2} \gamma(\lambda)u \right\|, \qquad \text{for } u \in D(C). \tag{8}$$

Now we show that the above formula can be extended to all $u \in D[C]$. Let $u \in D[C]$. Since $D(C)$ is dense in $D[C]$ with respect to the form norm of C, there exist a sequence $(u_n) \subseteq D(C)$ with $\|u_n - u\| \to 0$ and $C[u_n - u] \to 0$. By the boundedness of $\gamma(\lambda)$, we have $\gamma(\lambda)u_n \to \gamma(\lambda)u$ in $\mathcal{H}$. Now replace u by $u_n - u_m$ in (8), we see that $\big((T_{\mathcal{B}} - \lambda I)^{1/2}\gamma(\lambda)u_n\big)$ is a Cauchy sequence in $\mathcal{H}$. Because $(T_{\mathcal{B}} - \lambda I)^{1/2}$ is closed, we conclude that $\gamma(\lambda)u \in D[(T_{\mathcal{B}} - \lambda I)^{1/2}] = D[T_{\mathcal{B}}]$. Therefore $\gamma(\lambda)D[C] \subseteq D[T_{\mathcal{B}}]$.

4. We prove the converse inclusion $D[T_{\mathcal{B}}] \subseteq \gamma(\lambda)D[C]$. We use (7). Because $T_{\mathcal{B}}$,T_F and C are self-adjoint and positive, the norms

$$\left\|(T_{\mathcal{B}} - \lambda I)^{1/2}\cdot\right\|, \left\|(T_F - \lambda I)^{1/2}\cdot\right\|, \left\|C^{1/2}\cdot\right\|$$

are equivalent to the form norm of $T_{\mathcal{B}}$, T_F, and C respectively. Now let $f \in D[T_{\mathcal{B}}]$, then there is a Cauchy sequence $(f_n) \subseteq D(T_{\mathcal{B}})$ which converges to f with respect to the form norm of $T_{\mathcal{B}}$. Write $f_n = x_n + \gamma(\lambda)u_n$ with $x_n \in D(T_F)$ and $u_n \in D(C)$ as in previous steps. By (7), and the facts of equivalent norms, (x_n) is a Cauchy sequence with respect to the form norm of T_F and therefore in $\mathcal{H}$, and (u_n) is a Cauchy sequence with respect to the form norm of C and therefore in $\mathcal{K}$. Let $x \in D[T_F]$ and $u \in D[C]$ be the corresponding limits. Because $\gamma(\lambda)$ is bounded, we have $\gamma(\lambda)u_n \to \gamma(\lambda)u$ in $\mathcal{H}$, and hence $x_n + \gamma(\lambda)u_n \to x + \gamma(u)$ in $\mathcal{H}$. This means $f = x + \gamma(\lambda)u \in D[T_F] + \gamma(\lambda)D[C]$.

5. Putting things together, we have $D[T_{\mathcal{B}}] = D[T_F] + \gamma(\lambda)D[C]$. Moreover, using the Cauchy sequences in the preceeding paragraph, we can take the limit as $n \to \infty$ in (7), where f, x, u are replaced by (f_n, x_n, u_n), to conclude that this formula remains valid for $f = x + \gamma(\lambda)u \in D[T_{\mathcal{B}}]$. Since $(\mathcal{B} - M(\lambda))[u] = C[u] = \left\|C^{1/2}u\right\|_{\mathcal{K}}$ by Definition 1.10, this means that

$$(T_{\mathcal{B}} - \lambda I)[x + \gamma(\lambda)u] = (T_F - \lambda I)[x] + (\mathcal{B} - M(\lambda))[u],$$

and using polarization, we have

$$(T_{\mathcal{B}} - \lambda I)[x + \gamma(\lambda)u, x' + \gamma(\lambda)u'] = (T_F - \lambda I)[x, x'] + (\mathcal{B} - M(\lambda))[u, u'],$$

for $x, x' \in D[T_F]$ and $u, u' \in D[\mathcal{B} - M(\lambda)]$.

6. Finally, we show that $D[T_F] + \gamma(\lambda)D[C]$ is a direct sum. By the previous separation $f = x + \gamma(\lambda)$, assume $f = 0$, then the r.h.s of (7) is 0. Because the norm $\left\|(T_F - \lambda I)^{1/2}\cdot\right\|$ is equivalent to the form norm of T_F which is stronger than the norm of $\mathcal{H}$, we have $x = 0$, and so $\gamma(\lambda)u = f - x = 0$.

□

3 Positive Self-adjoint Extensions

In this section, suppose T is a densely defined symmetric operator on H with positive greatest lower bound $m_T > 0$, i.e.

$$\langle Tx, x\rangle \geq m_T \|x\|^2, \qquad \text{for all } x \in D(T).$$

Our aim is to apply the preceeding results to investigate the set of all positive self-adjoint extensions of T.

Since $0 < m_T = m_{T_F}$, we have $0 \in \rho(T_F)$, by Proposition 1.7. Hence, Theorem 1.17 applies with $\mu = 0$ and $A = T_F$. Assume that $B \in \mathcal{S}(\mathcal{N}(T^*))$, where $\mathcal{S}(\mathcal{X})$ the set of self-adjoint operators on some Hilbert space $\mathcal{X}$. Then the self-adjoint extensions of T on $\mathcal{H}$ are precisely the operators T_B defined as follows

$$D(T_B) = \left\{ \begin{array}{c} x + (T_F)^{-1}(Bu + v) + u : \\ x \in D(\overline{T}), u \in D(B), \\ v \in \mathcal{N}(T^*) \cap D(B)^{\perp} \end{array} \right\}, \qquad (9)$$
$$T_B\big(x + (T_F)^{-1}(Bu + v) + u\big) = \overline{T}x + Bu + v.$$

Let $\mathcal{S}(\mathcal{N}(T^*))_+$ denote the set of positive operators in $\mathcal{S}(\mathcal{N}(T^*))$.

Remark 2. The following Theorem 3.1 expresses the decomposition of the extension T_B into terms of T_F and B. Its proof is based on the above formulas of $D(T_B)$ and T_B, which is proved through the construction of a boundary triplet, and therefore, do not look intuitive, if one omits to read precisely its proof. This is the disadvantage of the introduction of [2]. The introduction of [1] is maybe more intuitive: one gets T_B via its forms defined by

$$D[T_B] = D[T_F] \dot{+} D[B],$$
$$T_B[y + u, y' + u'] = T_F[y, y'] + B[u, u'], \qquad \text{for } y, y' \in D[T_F], u, u' \in D[B],$$

as the formulas in Theorem 3.1 (of course, one has to show that this form is a closed quadratic form, which represents a self-adjoint operator!), and proves with this definition the representation of T_B at the beginning of this section.

Theorem 3.1. *(a) For $B \in \mathcal{S}(\mathcal{N}(T^*))$, we have $T_B \geq 0$ if and only if $B \geq 0$. In this case the greatest lower bound m_B and m_{T_B} satisfy*

$$m_T m_B (m_T + m_B)^{-1} \leq m_{T_B} \leq m_B.$$

(b) If $B \in \mathcal{S}(\mathcal{N}(T^*))_+$, then $D[T_B] = D[T_F] \dot{+} D[B]$, and

$$T_B[y + u, y' + u'] = T_F[y, y'] + B[u, u'], \qquad \textit{for } y, y' \in D[T_F], u, u' \in D[B].$$

(c) If $B_1, B_2 \in \mathcal{S}(\mathcal{N}(T^*))_+$, then $B_1 \geq B_2$ is equivalent to $T_{B_1} \geq T_{B_2}$.

Proof. i) First, suppose that $B \geq 0$. Let $f \in D(T_B)$, then f is of the form $f = y + u$ with $y = x + (T_F)^{-1}(Bu + v)$ with $x \in D(\overline{T})$, $u \in D(B)$, and $v \in \mathcal{N}(T^*) \cap D(B)^{\perp}$, and we have $T_B f = \overline{T}x + Bu + v = T_F y$. We can write this because $\overline{T} \subseteq \overline{T_F} = T_F$, so $x \in D(\overline{T}) \subseteq D(T_F)$, therefore $y \in D(T_F)$. Further, $m_T = m_{T_F}$, $T^* u = 0$, and $\langle v, u \rangle = 0$. Putting things together, we compute

$$\begin{aligned}\langle T_B f, f\rangle &= \langle T_F y, y+u\rangle \\ &= \langle T_F y, y\rangle + \left\langle \overline{T}x + Bu + v, u\right\rangle \\ &= \langle T_F y, y\rangle + \langle Bu, u\rangle \\ &\geq m_T \|y\|^2 + m_B \|u\|^2 \\ &\geq m_T m_B (m_T + m_B)^{-1} (\|y\| + \|u\|)^2 \\ &\geq m_T m_B (m_T + m_B)^{-1} \|y+u\|^2 \\ &\geq m_T m_B (m_T + m_B)^{-1} \|f\|^2 .\end{aligned}$$

In the third equation, we used $\left\langle \overline{T}x, u\right\rangle = \langle T^{*\,*}x, u\rangle = \langle x, T^* u\rangle = 0$, and in the second inequality, we used

$$\alpha a^2 + \beta b^2 \geq \alpha\beta(\alpha+\beta)^{-1}(a+b)^2, \qquad \text{for } \alpha > 0, a, b, \beta \geq 0.$$

Therefore we proved the part $B \geq 0 \Rightarrow T_B \geq 0$ and $m_T m_B (m_T + m_B)^{-1} \leq m_{T_B}$.

ii) In order to prove the remaining assertion, we apply the result stated in the paragraph before Theorem 1.17 for $A = T_F$ (as a self-adjoint extension of T) and $\mu = 0$. We stress that there exists a boundary triplet $(\mathcal{K}, \Gamma_0, \Gamma_1)$ such that A is the operator T_0 in Corollary 1.16. We have $M(0) = 0$ and $\gamma(0) = I_{\mathcal{H}}|\mathcal{K}$. Because $T_F = T_0$, we can apply Proposition 2.1 and Theorem 2.2 in the last section. To obtain b) and the part $T_B \geq 0 \Rightarrow B \geq 0$, we apply both results with $\lambda = 0$. Set $y = y' = 0$ and $u = u' \in D(B)$ in b), we have

$$\langle Bu, u\rangle = \langle T_B u, u\rangle \geq m_{T_B} \|u\|^2 ,$$

so it is obvious, that $m_{T_B} \leq m_B$.

Part c) is an immediate consequence of part b).

□

By the definition of the order relation $\geq$ (Definition 1.5) the set $\mathcal{S}(\mathcal{N}(T^*))_+$ contains a largest operator and a smallest operator. These are the following two extreme cases:

1) $D(B) = \{0\}$. By definition of $\geq$, the operator B is larger than all other operators, then B is the largest operators in $\mathcal{S}(\mathcal{N}(T^*))_+$. Then by (9), and $\overline{T} \subseteq T_F$, we have $D(T_F) \supseteq D(T_B) = D(\overline{T}) + (T_F)^{-1}\mathcal{N}(T^*)$, and $T_B(x + (T_F)^{-1}v) = \overline{T}x + v = T_F(x + (T_F)^{-1}v)$ for $x \in D(\overline{T})$ and $v \in \mathcal{N}(T^*)$. Because $T_F \geq T_B$ by the paragraph about Friedrichs and Krein-von Neumann extensions stated in Subsection 1.1, we have $D(T_F) \subseteq D(T_B)$, and so in this case $T_B = T_F$. Hence, $D(T_F) = D(\overline{T}) + (T_F)^{-1}\mathcal{N}(T^*)$.

2) $B = 0$, $D(B) = \mathcal{N}(T^*)$. By definition of $\geq$, this operator is smaller than all other operators in $\mathcal{S}(\mathcal{N}(T^*))_+$, it is the smallest element of $\mathcal{S}(\mathcal{N}(T^*))_+$.

Therefore, by Corollary 13.15, this operator T_B is the Krein-von Neumann extension T_N of T. From (9) (with $Bu = 0$ and $v = 0$) and Theorem 3.1, we deduce the following formula for the Krein-von Neumann extension:

$$\begin{aligned}
&D(T_N) = D(\overline{T}) \dotplus \mathcal{N}(T^*),\\
&T_N(x+u) = \overline{T}x \qquad \text{for } x \in D(\overline{T}), u \in \mathcal{N}(T^*),\\
&D[T_N] = D[T_F] \dotplus \mathcal{N}(T^*),\\
&T_N[y+u, y'+u'] = T_F[y,y'] \qquad \text{for } y, y' \in D[T_F], u, u' \in \mathcal{N}(T^*).
\end{aligned} \tag{10}$$

The direct sums in the first and the third equation are shown in Theorem 3.1.

Theorem 3.2. *Let T be a densely defined positive symmetric operator on $\mathcal{H}$ such that $m_T > 0$. For any positive self-adjoint operator A on $\mathcal{H}$, the following statements are equivalent:*

a) A is an extension of T.

b) There is an operator $B \in \mathcal{S}(\mathcal{N}(T^*))_+$ such that $A = T_B$.

c) $T_F \geq A \geq T_N$.

Proof. *a)* $\to$ *b)* and *b)* $\to$ *c)* are just simple applications of the previous results. To show *a)* $\to$ *b)*, we use the parameterization $A = T_B$ in Theorem 1.17 b). Because $A = T_B$ is positive, by Theorem 3.1, the operator B is also positive. Further, *b)* $\to$ *c)* holds by Theorem 3.1 c) because T_F and T_N are the two above extreme cases.

Now we show *c)* $\to$ *a)*.

i) The inequalities $T_F \geq A \geq T_N$ mean that $D[T_N] \supseteq D[A] \supseteq D[T_F]$ and $t_{T_N} \leq t_A \leq t_{T_F}$. If $y \in D[T_F]$, then $T_F[y] = T_N[y]$ by the last equation of (10), and hence $A[y] = T_F[y] = T_N[y]$, because $t_{T_N} \leq t_A \leq t_{T_F}$. Therefore, by polarization,

$$A[x,y] = T_F[x,y] \qquad \text{for } x, y \in D[T_F].$$

ii) Next, we verify that

$$A[x,u] = 0, \qquad \text{for } x \in D[T_F], u \in \mathcal{N}(T^*) \cap D[A].$$

Let $\lambda \in \mathbb{C}$. Using the last equation of (10) and $t_{T_N} \leq t_A$, we compute

$$\begin{aligned}
T_F[x] = T_N[x + \lambda u] &\leq A[x + \lambda u] = A[x + \lambda u, x + \lambda u]\\
&= A[x,x] + 2\mathrm{Re}\,(\lambda A[x,u]) + |\lambda|^2 A[u,u]\\
&= T_F[x] + 2\mathrm{Re}\,(\lambda A[x,u]) + |\lambda|^2 A[u,u]
\end{aligned}$$

Thus, $0 \leq 2\mathrm{Re}\lambda A[x,u] + |\lambda|^2 A[u,u]$ for all λ, which implies, by $\lambda \to 0$, that $A[x,u] = 0$.

iii) Finally, in order to show $T \subseteq A$, we need to show, by Proposition 1.4, that $A[x, f] = \langle Tx, f \rangle$ for all $x \in D(T)$ and $f \in D(A)$. Let $x \in D(T)$ and $f \in D(A)$. Then $f \in D[T_N] = D[T_F] \dotplus \mathcal{N}(T^*)$, so $f = y + u$ with $y \in D[T_F]$ and $u \in \mathcal{N}(T^*)$. Since $x \in D[T_F]$ and $u = f - y \in D[A]$, we have $A[x, u] = 0$ as shown in the last step. Further, $T^*u = 0$. We obtain

$$\begin{aligned} A[x, f] &= A[x, y + u] = A[x, y] \\ &= T_F[x, y] = \langle T_F x, y \rangle \\ &= \langle Tx, y \rangle = \langle Tx, f - u \rangle = \langle Tx, f \rangle . \end{aligned}$$

□

4 Some examples

In the following examples, we note that T is always closed. Our aim is to characterize the domain $D(T_N)$ of the Krein-von Neumann extension T_N as a subspace of the Sobolev space H^2 which satisfies some boundary conditions.

Example 1. Let $T := -\frac{d^2}{dx^2}$ with $D(T) := H_0^2(0, 1)$ on the Hilbert space $\mathcal{H} = L^2(0, 1)$. with the adjoint operator $T^* = d - \frac{d^2}{dx^2}$ with $D(T^*) = H^2(0, 1)$. (see [2, Example 1.4]). By the Poincare Inequality, there is a $c > 0$ with

$$\langle Tf, f \rangle = \|f'\|^2 \geq c \|f\|^2 , \qquad \text{for } f \in D(T).$$

Therefore, T has a positive lower bound. Further T is symmetric. We will show that Krein-von Neumann extension T_N of T has the domain

$$D(T_N) = \left\{ f \in H^2(0, 1) : f'(1) = f'(0) = f(1) - f(0) \right\} .$$

(Because $H^2(0, 1) \subseteq C^1([0, 1])$, each boundary value here is well-defined.) Let $f \in D(T_N) = D(T) \dotplus \mathcal{N}(T^*)$ (first formula of (10)). Here, by solving the ODE $y'' = 0$, we have $\mathcal{N}(T^*)$ is the set of functions of the form $x \to c + dx$, where c, d are constant. Therefore $f(x) = g(x) + c + dx$ for a function $g \in D(T) = H_0^1(0, 1)$. Recall that

$$H_0^1(0, 1) = \left\{ g \in H_0^1(0, 1) : g(0) = g(1) = g'(0) = g'(1) \right\} .$$

Therefore, by a little algebra, the function f satisfies the boundary condition $f'(1) = f'(0) = f(1) - f(0)$. Conversely, assume $f \in D(T^*) = H^2(0, 1)$ satisfies this boundary condition, then we set $c = f(0), d = f(1)$ and $g(x) := f(x) - f(0) - (f(1) - f(0))x$. It is easy to check that $g(0) = g(1) = g'(0) = g'(1)$. Then $g \in H_0^2(0, 1)$. Therefore, $f \in D(T) + \mathcal{N}(T^*) = D(T_N)$.

In the last example, one can solve $\mathcal{N}(T^*)$ explicitly. Therefore it is easy to find the boundary condition of T_N by the decomposition (10). In general, it is not trivial to formulate and solve the same problem for d-dimensional Dirichlet Laplacian operators as seen in the following

Example 2. *Dirichlet Laplacian on bounded domains of* $\mathbb{R}^d$ Differential Operators are not the topic of this text. We recall only essential points which we need for the example. Let $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ be bounded with smooth boundary $\partial\Omega$. Let $\mathcal{H} = L^2(\Omega)$. We define the operator $L = -\Delta$ first for smooth functions with compact supports

$$Lf = -\Delta f := -\sum_1^d \frac{\partial^2 f}{\partial x_i^2}, \qquad \text{for } f \in D(L) = C_0^\infty(\Omega).$$

By partial integration, L is symmetric. Moreover, L is densely defined. Therefore L is closable. We define the minimal operator $L_{\min}$ as the closure $\overline{L}$ of L, and the Dirichlet Laplacian as the Friedrichs extension of $L_{\min}$. Now set $T = L_{\min}$. Then T is lower semibounded (by the Poincare Inequality) and symmetric. We can show that

$$D(T) = H_0^2(\Omega) = \left\{ f \in H^2(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0 \right\},$$
$$D(T_F) = H^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega) = \left\{ f \in H^2(\Omega) : f|_{\partial\Omega} = 0 \right\},$$

where $f|_{\partial\Omega}, \frac{\partial f}{\partial \nu}|_{\partial\Omega}$ can be defined as the trace operators. (see [2, Theorem 10.19, Theorem D.7].)

In order to formulate this example, we need the following decomposition

$$D(T^*) = D(T_F) \dot{+} \mathcal{N}(T^*), \tag{11}$$

which is [2, Eq (14.19)] with A replaced by T_F and μ replaced by 0.

Let $f \in H^2(\Omega)$. Then $f \in D(T^*)$, because, by integration by part, we have

$$\langle f, (-\Delta)g \rangle = \langle (-\Delta)f, g \rangle, \qquad \text{for } g \in D(-\Delta) = C_0^\infty(\Omega),$$

hence $f \in D((-\Delta)^*) = D(\overline{(-\Delta)}^*) = D(T^*)$. Now by the above decomposition, there exists uniquely an $H(f) \in D(T^*)$ such that $f - H(f) \in D(T_F) \subseteq H^2(\Omega)$. Because $f \in H^2(\Omega)$, we have $H(f) \in H^2(\Omega)$. Moreover, by the boundary condition of $D(T_F)$, we have $f - H(f) = 0$ on $\partial\Omega$, so $f|_{\partial\Omega} = H(f)|_{\partial\Omega}$.

Now we show that $f \in D(T_N)$ if and only if $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = \frac{\partial H(f)}{\partial \nu}|_{\partial\Omega}$. Suppose $f \in D(T_N)$, then by (10) $f = g + h$ where $g \in D(T)$ and $h \in \mathcal{N}(T^*)$. Then $g \in D(T_F)$, and by the decomposition (11), we have $h = H(f)$. Since $g \in \mathcal{D}(T)$, we have $\frac{\partial g}{\partial \nu}|_{\partial\Omega} = 0$, and hence $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = \frac{\partial h}{\partial \nu}|_{\partial\Omega} = \frac{\partial H(f)}{\partial \nu}|_{\partial\Omega}$. Conversely, suppose $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = \frac{\partial H(f)}{\partial \nu}|_{\partial\Omega}$. Set $g := f - H(f)$. Then $\frac{\partial g}{\partial \nu}|_{\partial\Omega} = 0$. On the other hand, by the definition of $H(f)$ and the boundary condition of $D(T_F)$, we have $g|_{\partial\Omega} = 0$. Therefore $g \in H_0^2(\Omega) = D(T)$, and hence $f = g + h \in D(T) + \mathcal{N}(T^*) = D(T_N)$.

References

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