Technische Universität Berlin

SEMINARARBEIT ZUM FACH FUNKTIONALANALYSIS

Sectorial Forms and m-Sectorial Operators

Misagheh Khanalizadeh,

Berlin, den 21.10.2013

Contents

Introduction

This elaboration is based on the book "Unbounded Self-adjoint Operators on Hilbert Space", written by Konrad Schmüdgen. The basic item in this paper is the Representation Theorem of Sectorial forms. In the first section we will be confronted with many definitions and properties of sesquilinear forms. In the second section we introduce the operator associated with forms and prove a representation theorem for bounded coercive forms. This theorem is very important and will be applied to show the representation Theorem of sectorial forms. After these preparations we will prove in the section 3 the main Theorem of this elaboration. In the last section we will see some applications of sectorial forms and operators.

1 Bounded Coercive Sesquilinear forms

In this section we deal with sesquilinear forms. A sesquilinear form or briefly a form on a linear subspace $\mathcal{D}(\mathfrak{t})$ of a Hilbert space $\mathcal{H}$ is a mapping $\mathfrak{t}[\cdot,\cdot] : \mathcal{D}(\mathfrak{t}) \times \mathcal{D}(\mathfrak{t}) \to \mathbb{C}$ with

$$\mathfrak{t}[\alpha x + \beta y, z] = \alpha \mathfrak{t}[x, z] + \beta \mathfrak{t}[y, z]$$
$$\mathfrak{t}[z, \alpha x + \beta y] = \overline{\alpha} \mathfrak{t}[x, z] + \overline{\beta} \mathfrak{t}[y, z],$$

where α, $\beta \in \mathbb{C}$ and x, y, $z \in \mathcal{D}(\mathfrak{t})$. The quadratic form which is associated with $\mathfrak{t}$ is defined by

$$\mathfrak{t}[x] := \mathfrak{t}[x, x].$$

We say that a form $\mathfrak{t}$ is positive if $\mathfrak{t} \geq 0$, that ist, if $\mathfrak{t}[x] \geq 0$ for all $x \in \mathcal{D}(\mathfrak{t})$.

Definition 1. *i) Let $\mathcal{H}$ be a Hilbert space and $(V, \|\cdot\|_V)$ be a normed space containing the domain $\mathcal{D}(\mathfrak{t})$ A symmetric form is called lower semibounded if there exsists an $m \in \mathbb{R}$, such that*

$$\mathfrak{t}[x] \geq m\|x\|^2, \ \forall\ x \in \mathcal{D}(\mathfrak{t}).$$

We call m a lower bound of $\mathfrak{t}$ and write $\mathfrak{t} \geq m$.

ii) We say that the form $\mathfrak{t}$ is bounded no V if there is a constant $c > 0$ such that

$$|\mathfrak{t}[x, y]| \leq c\|x\|_V \|y\|_V, \ \forall\ x, y \in \mathcal{D}(\mathfrak{t}).$$

iii) We say that the $\mathfrak{t}$ is coercive on V if there is a constant $\widetilde{c} > 0$ such that

$$|\mathfrak{t}[x]| \geq \|x\|_V^2 \ \forall\ x \in \mathcal{D}(\mathfrak{t}).$$

Let m be a lower bound of $\mathfrak{t}$. Now, we define a skalar product and a corresponding norm on $\mathcal{D}(\mathfrak{t})$ as follows:

$$\langle x, y\rangle_{\mathfrak{t}} := \mathfrak{t}[x, y] + (1 - m)\langle x, y\rangle, \ x, y \in \mathcal{D}(\mathfrak{t})$$
$$\|x\|_{\mathfrak{t}} := \langle x, y\rangle_{\mathfrak{t}}^{1/2} = \Big(\mathfrak{t}[x, y] + (1 - m)\|x\|^2\Big)^{1/2}, \ x \in \mathcal{D}(\mathfrak{t}).$$

The norm $\|\cdot\|_{\mathfrak{t}}$ satisfies $\|x\|_{\mathfrak{t}} \geq \|x\|$ on $\mathcal{D}(\mathfrak{t})$. From the Cauchy-Schwarz inequality, applied to the positive form $\mathfrak{t} - m$, and the relation $\|x\|_{\mathfrak{t}} \geq \|x\|$ we easily derive that

$$|\mathfrak{t}[x, y]| \leq (1 + |m|)\|x\|_{\mathfrak{t}}\|y\|_{\mathfrak{t}}, \qquad x, y \in \mathcal{D}(\mathfrak{t}). \tag{1.1}$$

That is, $\mathfrak{t}$ is a bounded form on the pre-Hilbert space $(\mathcal{D}(\mathfrak{t}), \langle\cdot,\cdot\rangle_{\mathfrak{t}})$.

Definition 2. *Let $\mathfrak{t}$ be a lower semibounded form, then $\mathfrak{t}$ is called closed if $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\mathfrak{t}})$ is complete. The form $\mathfrak{t}$ is said to be closable if there exists a closed lower semibounded form which is an extension of $\mathfrak{t}$.*

And at last, We define the adjoint form $\mathfrak{t}^*$ by

$$t^*[x, y] = \overline{\mathfrak{t}[y, x]}, \qquad x, y \in \mathcal{D}(\mathfrak{t}^*) := \mathcal{D}(\mathfrak{t}).$$

and we define the real part and the imaginary part of $\mathfrak{t}$ as follows:

$$\text{Re}\ \mathfrak{t} = \frac{1}{2}(t + t^*), \qquad \text{Im}\ \mathfrak{t} = \frac{1}{2i}(t - t^*).$$

2 Operators associated with forms

In this section we deal with self-adjoint operators which are assiciated with forms. Selfadjoint operators are essential in functional analysis but it is not easy to work with them. Numerous types of self-adjoint operators can be constructed by means of closed forms as we will see in the representation theorem of the sectorial forms . We begin with associating a form $\mathfrak{t}_A$ with an operator A. Let A be a self-adjoint operator on $\mathcal{H}$ and let E_A be the spectral measure of A. Define

$$\mathcal{D}(\mathfrak{t}_A) := \mathcal{D}\left(|A|^{1/2}\right) = \left\{x \in \mathcal{H} : \int_{\mathbb{R}} |\lambda| d\langle E_A x, x\rangle < \infty\right\}$$
$$t_A[x,y] = A[x,y] = \int_{\mathbb{R}} |\lambda| d\langle E_A x, y\rangle, \; x, y \in \mathcal{D}(\mathfrak{t}_A).$$

The existence of the latter integral follows by [**6**], Lemma 4.8.

We call $\mathfrak{t}_A$ the form associated with the self-adjoint operator A. Note that

$$\mathcal{D}(A) = \mathcal{D}(|A|) \subseteq \mathcal{D}\left(|A|^{1/2}\right).$$

Definition 3. *Let $\mathfrak{t}$ be a densely defined form on $\mathcal{H}$. We define the operator $A_{\mathfrak{t}}$ associated with the form $\mathfrak{t}$ by $A_{\mathfrak{t}} x := u_x$ for $x \in \mathcal{D}(A_{\mathfrak{t}})$, where*

$$\mathcal{D}(A_{\mathfrak{t}}) = \{x \in \mathcal{D}(\mathfrak{t}) \;:\; \text{There exists } u_x \in \mathcal{H} \text{ such that } \mathfrak{t}[x,y] = \langle u_x, y\rangle \text{ for } y \in \mathcal{D}(\mathfrak{t})\}$$

The vector u_x is uniquely determined by x (since $\mathcal{D}(\mathfrak{t})$ is dense in $\mathcal{H}$) and the operator $A_{\mathfrak{t}}$ is therefore well defined and linear. We have $\mathcal{D}(A_{\mathfrak{t}}) \subseteq \mathcal{D}(\mathfrak{t})$, and

$$\mathfrak{t}[x,y] = \langle A_{\mathfrak{t}} x, y\rangle \; \forall x \in \mathcal{D}(A_{\mathfrak{t}}), \; y \in \mathcal{D}(\mathfrak{t}).$$

Theorem 4. *Let $\mathfrak{t}$ be a densely defined lower semibounded closed form on a Hilbert space $\mathcal{H}$. Then the operator $A_{\mathfrak{t}}$ is self-adjoint, and $\mathfrak{t}$ is equal to the form $\mathfrak{t}_{(A_{\mathfrak{t}})}$ associated with $A_{\mathfrak{t}}$.*

Proof. A proof of this theorem is given in [**6**], P. 228. □

Next, we will introduce the Lemma of Lax-Milgram and is basic for our work.

Lemma 5 (Lax-Milgram)**.** *Let $\mathfrak{t}$ be a bounded coervice form on a Hilbert space $(V, \langle\cdot,\cdot\rangle_V)$, where $\mathcal{D}(\mathfrak{t}) = V$. Then there exists an operator $B \in \mathbf{B}(V)$ with inverse $B^{-1} \in \mathbf{B}(V)$ which satisfies the following equation*

$$\mathfrak{t}[u,v] = \langle Bu, v\rangle_V \quad \text{for } u, v \in V. \tag{2.1}$$

Moreover, if c and $\widetilde{c}$ are positive constants such that

$$|\mathfrak{t}[u,v]| \le C\|u\|_V \|v\|_V \;\; \text{for } u, v \in V$$

and

$$|\mathfrak{t}[u]| \ge c\|u\|_V^2 \;\; \text{for } u, v \in V$$

hold, then

$$\widetilde{c} \le \|B\|_V \le c \;\; \text{and } c^{-1} \le \|B^{-1}\|_V \le \widetilde{c}^{-1}.$$

Proof. One can find a proof of this Lemma also in [**6**], P. 252.

□

Proposition 6. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. The following conditions are equivalent for any $\lambda \in \mathbb{C}$:*

i) $\lambda \in \rho(T)$.

ii) *There exists a constant $c_\lambda > 0$ such that $\| (T - I\lambda) x\| \geq c_\lambda \|x\|$ for all $x \in \mathcal{D}(T)$.*

iii) $\mathcal{R}(T - \lambda I) = \mathcal{H}$.
Moreover, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\lambda \in \rho(T)$.

Proof. To prove this proposition, see [**6**], Proposition 3.10. □

We say that the Hilbert $(V, \langle\cdot,\cdot\rangle_V)$ is densely and continuously embedded into the Hilbert space $\mathcal{H}$ if V is a dense linear subspace of $\mathcal{H}$ and the embedding $\mathcal{I} : V \to \mathcal{H}$ is continuous, that is, there exists a positive number a such that

$$\|x\| \leq a\|x\|_V, \quad x \in V.$$

The following important theorem helps us to prove the representation theorem for the sectorial forms.

Theorem 7. *Let $\mathcal{H}$ be a Hilbert space and let $(V, \langle\cdot,\cdot\rangle_V)$ be a Hilbert space, which is densely and continuously embedded into $\mathcal{H}$. Let $\mathfrak{t}$ be form on $\mathcal{H}$ with $\mathcal{D}(\mathfrak{t}) = V$ which is bounded coercive on the Hilbert space $(V, \langle\cdot,\cdot\rangle_V)$. Then the form $\mathfrak{t}$ can be considered as a densely defined form on $\mathcal{H}$ and we have:*

i) *The operator $A_\mathfrak{t}$ associated with $\mathfrak{t}$ is a densely defined closed operator on $\mathcal{H}$ with bounded inverse $A_\mathfrak{t}^{-1} \in \mathbf{B}(\mathcal{H})$.*

ii) *$\mathcal{D}(A_\mathfrak{t})$ is a dense linear subspace on the Hilbert space $(V, \|\cdot\|_V)$.*

Proof. *i)* We subdivide the proof into several steps to obtain a better overview.

Step 1

Consider the scalar product $\mathfrak{s} := \langle\cdot,\cdot\rangle_V$ as a form on $\mathcal{H}$. This scalar product is clearly a positive form on $\mathcal{H}$. We consider the operator $A_\mathfrak{s}$ associated with $\mathfrak{s}$ and show that $A_\mathfrak{s}^{-1/2} \in \mathbf{B}(\mathcal{H})$.
The form norm of $\mathfrak{s}$ is equivalent to the norm of V ($\|\cdot\|_\mathfrak{t} \equiv \|\cdot\|_V$) (this is for example shown in [**4**], Chapter 6) and since $(V, \langle\cdot,\cdot\rangle_V)$ is a Hilbert space, we get that $(\mathcal{D}(\mathfrak{s}), \langle\cdot,\cdot\rangle_\mathfrak{s})$ is a Hilbertspace, i. e. $\mathfrak{s}$ is closed. Since V is densely and continuously embedded into $\mathcal{H}$, the form $\mathfrak{s}$ is a densely defined lower semi-bounded form on $\mathcal{H}$ and we obtain according to Theorem 4 that the operator

$A_{\mathfrak{s}}$ is self-adjoint and $\mathfrak{s} = \mathfrak{t}_{A_{\mathfrak{s}}}$. Since $A_{\mathfrak{s}}$ is a positive self-adjoint operator, there exists a unique positive self-adjoint operartor $A_{\mathfrak{s}}^{1/2}$ with $(A_{\mathfrak{s}}^{1/2})^2 = \mathcal{H}$. It follows

$$\begin{aligned}\|y\|_V^2 = \langle y, y\rangle_V = \mathfrak{s}[y,y] &\overset{D.(3)}{=} \langle A_{\mathfrak{s}} y, y\rangle = \langle A_{\mathfrak{s}}^{1/2} A_{\mathfrak{s}}^{1/2} y, y\rangle \\ &= \langle A_{\mathfrak{s}}^{1/2} y, A_{\mathfrak{s}}^{1/2} y\rangle = \|A_{\mathfrak{s}}^{1/2} y\|^2,\end{aligned}$$

for $y \in \mathcal{D}(\mathfrak{s})$. Note again that $A_{\mathfrak{s}}^{1/2}$ is positive an thus, it holds

$$\mathcal{D}(A_{\mathfrak{s}}^{1/2}) = \mathcal{D}(|A_{\mathfrak{s}}|^{1/2}) = \mathcal{D}(\mathfrak{t}_{A_{\mathfrak{s}}}) = \mathcal{D}(\mathfrak{s}) = V.$$

Since the embedding $J : V \to \mathcal{H}$, $Jv = v$ is continuous, there exists a positive number a, such that

$$\|v\| \le a\|v\|_V, \quad \forall v \in V.$$

and therefore we get

$$a^{-1}\|y\| \le \|y\|_V = \|A_{\mathfrak{s}}^{1/2} y\| \tag{2.2}$$

for $y \in \mathcal{D}\left(A_{\mathfrak{s}}^{1/2}\right) = V$. Here, we have to be carefully: by (2.2) one can conclude that $A_{\mathfrak{s}}^{1/2}$ is injective and continuously invertible, but it is not that $A_{\mathfrak{s}}^{-1/2}$ is defined in everywhere on $\mathcal{H}$. But we can use the Proposition 6 and get that $0 \in \rho\left(A_{\mathfrak{s}}^{1/2}\right)$ and this implies that $A_{\mathfrak{s}}^{-1/2} \in \mathbf{B}(\mathcal{H})$.

Step 2

Let $B \in \mathbf{B}(V)$ be the operator for $\mathfrak{t}$ considered as a form on V given by the Lemma of Lax-Milgram. Then we have

$$\mathfrak{t}[x, y] = \langle Bx, y\rangle, \quad \forall x, y \in V.$$

Because $\mathfrak{t}$ is coercive on $(V, \langle \cdot, \cdot\rangle_V)$, there is a positive c, such that we have

$$|\mathfrak{t}[x]| \ge c\|x\|_V^2 \quad \forall x \in \mathcal{D}(\mathfrak{t}).$$

By Lemma of Lax-Milgram one get

$$\|B^{-1}\|_V \le c^{-1}.$$

Now, we have that $B^{-1} \in \mathbf{B}(V)$ and the operator $S := B^{-1} A_{\mathfrak{s}}^{-1/2}$ is well-defined.

$$\begin{aligned}\|Sx\| = \|B^{-1} A_{\mathfrak{s}}^{-1/2} x\| &\le a\|B^{-1} A_{\mathfrak{s}}^{-1/2} x\|_V \le ac^{-1}\|A_{\mathfrak{s}}^{-1/2} x\|_V \\ &= ac^{-1}\|A_{\mathfrak{s}}^{1/2} A_{\mathfrak{s}}^{-1/2} x\| = ac^{-1}\|x\|.\end{aligned}$$

This implies that $S \in \mathbf{B}(\mathcal{H})$ and hence,

$$T := S A_{\mathfrak{s}}^{-1/2} = B^{-1} A_{\mathfrak{s}}^{-1} \in \mathbf{B}(\mathcal{H}).$$

Note that $\mathcal{D}(A_{\mathfrak{s}}) \subseteq V$ and hence, $\mathcal{R}(A_{\mathfrak{s}}^{-1}) \subseteq V$. The operator T is as a conjunction of two injective operators injective and thus, T is invertible.

Step 3

Our goal is to show that $T^{-1} = A_{\mathfrak{t}}$. We use the equation

$$\begin{aligned}\mathfrak{t}[Tx, y] &= \mathfrak{t}[B^{-1}A_{\mathfrak{s}}^{-1}x, y] = \langle BB^{-1}A_{\mathfrak{s}}^{-1}x, y\rangle_V \\ &= \langle A_{\mathfrak{s}}^{-1}x, y\rangle_V = \langle A_{\mathfrak{s}}A_{\mathfrak{s}}^{-1}x, y\rangle = \langle x, y\rangle, \quad x \in \mathcal{H}, \quad y \in V = \mathcal{D}(\mathfrak{t}). \end{aligned} \tag{2.3}$$

Setting $x = T^{-1}u$ in the above formula, we obtain

$$\mathfrak{t}[u, y] = \langle T^{-1}u, y\rangle, \quad y \in \mathcal{D}(\mathfrak{t}).$$

Then we get $u \in \mathcal{D}(A_{\mathfrak{t}})$ and $T^{-1}u = A_{\mathfrak{t}}u$. This implies that $T^{-1} \subseteq A_{\mathfrak{t}}$. Since $\mathcal{D}(T) = \mathcal{H}$, the operator T is surjective. The operator $A_{\mathfrak{t}}$ is clearly injective and thus, $T^{-1} = A_{\mathfrak{t}}$. Hence, $A_{\mathfrak{t}}^{-1} = T \in \mathbf{B}(\mathcal{H})$.

ii) Let $v \in V$ be orthogonal to $\mathcal{D}(\mathcal{A}_{\mathfrak{t}}) = \mathcal{R}(T)$ in $(V, \|\cdot\|_V)$. We have to show that $v = 0$ and this implies clearly that $\mathcal{D}(\mathcal{A}_{\mathfrak{t}})$ is dense in V. We apply again the above formula (2.3) and obtain:

$$\langle x, (B^{-1})^*v\rangle = \langle A_{\mathfrak{s}}^{-1}x, (B^{-1})^*v\rangle_V = \langle B^{-1}A_{\mathfrak{s}}^{-1}x, v\rangle_V = \langle Tx, v\rangle_V = 0$$

for all $x \in \mathcal{H}$. Thus, we get $(B^*)^{-1}v = (B^{-1})^*v = 0$ and hence $v = 0$.

□

Theorem 8. *Let V and the form $\mathfrak{t}$ be as in Theorem 7. Then we have the following assertions.*

i) $(A_{\mathfrak{t}})^* = A_{\mathfrak{t}^*}$.

ii) If the embedding $J\colon V \to \mathcal{H}$ is compact, then $R_\lambda(A_{\mathfrak{t}})$ is compact for all $\lambda \in \rho(A_{\mathfrak{t}})$ and the operator $A_{\mathfrak{t}}$ has a purely discrete spectrum.

Proof. The proof is very easy and is given by [**6**], P. 254. □

Corollary 9. *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be bounded coercive forms on Hilbert spaces $(V_1, \langle\cdot,\cdot\rangle_{V_1})$ and $(V_2, \langle\cdot,\cdot\rangle_{V_2})$ respectively, which are densely and continuously embedded into the Hilbert space $\mathcal{H}$. If $A_{\mathfrak{t}_1} = A_{\mathfrak{t}_2}$, then $V_1 = V_2$ (as vector spaces) and $\mathfrak{t}_1 = \mathfrak{t}_2$.*

Proof. This proof is also very easy and is given by [**6**], P. 254. □

Corollary 10. *Let $\mathfrak{t}$ be a bounded coercive form on the Hilbert space $(V, \langle\cdot,\cdot\rangle_V)$. Then $A_{\mathfrak{t}}$ is self-adjoint iff $\mathfrak{t}$ is symmetric.*

Proof. This follows directly by applying Theorem 7 part (ii) and Corollary 9. □

3 The Representation Theorem of sectorial forms

3.1 Sectorial Forms and m-sectorial operators

Before we deal with the Representation Theorem of sectorial forms, we need some definitions and properties of sectorial forms.

Definition 11. *(numerical range)*

i) *The numerical range of a form* $\mathfrak{t}$ *is definied as*

$$\Theta(\mathfrak{t}) := \{\mathfrak{t}[x] \colon x \in \mathcal{D}(\mathfrak{t}),\ \|x\| = 1\}.$$

ii) *The numerical range of a linear operator* T *in* $\mathcal{H}$ *is defined by*

$$\Theta(T) := \{\langle Tx, x\rangle \colon x \in \mathcal{D}(\mathfrak{t}),\ \|x\| = 1\}.$$

Definition 12. *.*

i) *The form* $\mathfrak{t}$ *on* $\mathcal{H}$ *is called sectorial if there are numbers* $c \in \mathbb{R}$ *and* $\theta \in [0, \pi/2)$ *such that*

$$\Theta(\mathfrak{t}) \subseteq S_{c,\theta} := \{\lambda \in \mathbb{C} \colon |\operatorname{Im}\lambda| \le \tan\theta(\operatorname{Re}\lambda - c)\} = \{\lambda \in \mathbb{C} \colon |\arg(\lambda - c)| \le \theta)\}.$$

ii) *The operator* T *is called sectorial, if* $\Theta(T) \subseteq S_{c,\theta}$ *for some* $c \in \mathbb{R}$ *and* $\theta \in [0, \pi/2)$.

iii) *The operator* T *is said to be m-sectorial if we have again* $\Theta(T) \subseteq S_{c,\theta}$ *and if* T *is closed and* $\mathcal{R}(T - \alpha I)$ *is dense in* $\mathcal{H}$*, where* $\alpha \in \mathbb{C} \setminus S_{c,\theta}$.
We call c *a vertex and* θ *a corresponding semi-angle.*

Remark 13. $m-$*sectorial operators are maximal sectorial. (See [**6**], Proposition 3.24).*

Example 14. *Let* T *be an operator on a Hilbert space* $\mathcal{H}$ *and set*

$$t[x, y] = \langle Tx, y\rangle, \quad \mathcal{D}(\mathfrak{t}) = \mathcal{D}(T).$$

Then T *and* $\mathfrak{t}$ *have clearly the same numerical range and the form* $\mathfrak{t}$ *is sectorial iff* T *is sectorial.*

Example 15. *Let* $\mathcal{H} = \ell^2$ *and*

$$\mathfrak{t}[x, y] = \sum_{i=1}^{\infty} \alpha_i x_i \overline{y_i}, \quad \alpha_i \in \mathbb{C}$$

$\mathcal{D}(\mathfrak{t})$ *is the set of all* $x = (x_i)_{i \in \mathbb{N}} \in \mathcal{H}$*, where* $\sum_{i=1}^{\infty} |\alpha_i||x_i|^2 < \infty$*. The form* $\mathfrak{t}$ *is sectorial if and only if all the* α_i *lie in a sector* $S_{c,\theta}$.

Proof. Let $\|x\| = 1$ and $M_\alpha := \{\alpha_i \colon i \in \mathbb{N}\} \subset S_{c,\theta}$. Then

$$\langle Tx, x\rangle = \sum_{i=1}^{\infty} |\alpha_i||x_i|^2$$

lies in the convex hull of M_α. Since $S_{c,\theta}$ is convex and contains M_α it also contains the convex hull of M_α and thus $\langle Tx, x\rangle \in S_{c,\theta}$.
Now assume that there is a $i \in \mathbb{N}$ with $\alpha_i \notin S_{c,\theta}$. Then $\langle Te_i, e_i\rangle = \alpha_i \notin S_{c,\theta}$. $\square$

The above result hols also for Hilbert space L^2 :

Example 16. *Let $\mathcal{H} = L^2(K)$ and let f be a complex-valued measurable function on K and*

$$\mathfrak{t}[u, v] = \int_K f(x)u(x)\overline{v(x)}dx$$

$\mathcal{D}(\mathfrak{t})$ is the set of all $u \in \mathcal{H}$, such that $\int_K |f(x)||u(x)| < \infty$. The form $\mathfrak{t}$ is sectorial if the values of f lie in the sector sector $S_{c,\theta}$.

Remark 17. *There are various ways to define sectorial forms and m-sectorial operators. Since the sectorial forms and m-sectorial operators are the main subjects in this paper, it is convenient to compare some of these definitions. An aquivalent definition is given in [**1**]: A form $\mathfrak{t}$ on $\mathcal{H}$ is called sectorial if there exist a sector*

$$S_\theta = \left\{re^{i\alpha} \mid r \geq 0, |\alpha| \leq \theta\right\},$$

with $\theta \in [0, \pi/2)$, and a $c \in \mathbb{R}$, such that $\mathfrak{t}[x] - c\|x\|^2 \in S_\theta$ for all $x \in \mathcal{D}(\mathfrak{t})$.

An operator $T : \mathcal{D}(T) \to \mathcal{H}$ with $\mathcal{D}(T) \subset \mathcal{H}$ is called sectorial if there are a $c \in \mathbb{R}$, and $\theta \in [0, \pi/2)$, such that

$$\langle Tx, x\rangle - c\|x\|^2 \in S_\theta$$

for all $x \in D(T)$. The operator T is called m-sectorial if it is sectorial and $\lambda I - A$ is surjective for some $\lambda \in \mathbb{R}$, $\lambda < c$.

*We show the aquivalence between the definition of sectorial forms as we defined in this work and that introduced by [**1**]:*

$$\begin{aligned}
\mathfrak{t}[x] - c\|x\|^2 \in S_\theta \ \ \forall x \ &\Leftrightarrow \ \mathfrak{t}[x] - c\|x\|^2 \in S_\theta, \quad \forall x \in \mathcal{H} \setminus \{0\} \\
&\Leftrightarrow \ \mathfrak{t}\left[\frac{x}{\|x\|}\right] \|x\|^2 - c\|x\|^2 \in S_\theta, \quad \forall x \in \mathcal{H} \setminus \{0\} \\
&\Leftrightarrow \ \mathfrak{t}\left[\frac{x}{\|x\|}\right] - c \in \frac{1}{\|x\|^2} S_\theta = S_\theta, \quad \forall x \in \mathcal{H} \setminus \{0\} \\
&\Leftrightarrow \ \mathfrak{t}\left[\frac{x}{\|x\|}\right] \in S_{c,\theta}, \quad \forall x \in \mathcal{H} \setminus \{0\} \\
&\Leftrightarrow \ \mathfrak{t}[x] \in S_{c,\theta} \ \ \forall x \in \mathcal{H} : \ \|x\| = 1
\end{aligned}$$

*Another aquivalent definition is given by [**4**]. Here, an operator T is said to be m−sectorial if it is sectorial and quasi-m-accretive. The equivalence of this definition is given by applying the following Propositions 18 and 19 (for more details see [**4**], Chapter V, § 3).*

*However sometimes one has to be carefully as by [**5**]. There, the sectorial forms are per definition assumed to be densely defined and a definition is given by means of the elliptic forms. In the last section we show that elliptic forms are the same as densely defined closed sectorial forms.*

Proposition 18. *Let T be a closed operator on $\mathcal{H}$, and $\Theta(T) \subseteq S_{c,\theta}$ where $c \in \mathbb{R}$ and $\theta \in [0, \pi/2)$. Then we have the following equivalent assertions:*

i) T *is m-sectorial.*

ii) $\lambda \in \rho(T)$ *for one and hence, for all* $\lambda \in \mathbb{C} \setminus S_{c,\theta}$.

iii) $\mathcal{D}(T)$ *is dense in* $\mathcal{H}$, *and* T^* *is m-sectorial.*

Proof. A very detailed proof of this proposition can be founded in [**6**], Proposition 3.19, P. 49. □

Proposition 19. *Let T be a closed accretive operator. Then the following are equivalent:*

i) T *is m-accretive*

ii) $\lambda \in \rho(T)$ *for some, hence all,* $\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < 0$

iii) $\mathcal{D}(T)$ *is dense in* $\mathcal{H}$ *and* T^* *is* T^* *is m-accretive.*

Furthermore, if T is m-accretive, then $\sigma(T) \subset \{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda \geq 0\}$ and for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ we have

$$\|(T - \lambda I)^{-1}\| \leq |\operatorname{Re}\lambda|^{-1}. \tag{3.1}$$

Proof. This proposition can be proven using the above one. □

Before we come to our next definition, we note that if the form $\mathfrak{t}$ is sectorial then $\operatorname{Re}\mathfrak{t}$ is lower semibounded, as we can see in the soon following Proposition 24. Thus, we can now define the items cloesedness of sectorial forms and core of a sectorial form.

Definition 20 (Closed, closable, core). *A sectorial form $\mathfrak{t}$ is said to be* closed *if $\operatorname{Re}\mathfrak{t}$ is closed. It is called* closable *if there is a closed sectorial form which extends $\mathfrak{t}$. A* core *of a sectorial form $\mathfrak{t}$ is a dense linear subspace of the pre-Hilbert space* $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$.

Remark 21. *We also note that in some other references like [4] these items are defined differently, but using some propositions and theorems, one can show the equivalence of these definitions. See [4] for finding some examples of closed and closable forms and several useful theorems like the following one:*

Theorem 22. *A closable sectorial form* $\mathfrak{t}$ *with domain* $\mathcal{H}$ *is bounded.*

Proof. To prove this theorem see [**4**], Chapter 6, Theorem 1.20. □

The following theorem is not directly pertient for our work, but we introduce it to get more similar with some properties of sectorial forms.

Theorem 23. *Let* $\mathfrak{t}_1, \cdots, \mathfrak{t}_k$*,* $k \in \mathbb{N}$*, be sectorial forms in a Hilbert space* $\mathcal{H}$ *and let* $\mathfrak{t} = \mathfrak{t}_1 + \cdots + \mathfrak{t}_k$ *(with* $\mathcal{D}(\mathfrak{t}) = \mathcal{D}(\mathfrak{t}_l) \cap \cdots \cap \mathcal{D}(\mathfrak{t}_k)$*). Then* $\mathfrak{t}$ *is sectorial.*

Proof. We assume that all the t_i, $i \in \mathbb{N}$ have a vertex zero. We let θ_i be a semi-angle of $\mathfrak{t}_i$ for vertex zero. It holds:

$$\Theta(\theta_i) \subseteq S_{c,\theta} \ \Rightarrow \ |arg\lambda| \leq \theta_i < \pi/2 \ \Rightarrow \ |\arg\lambda| \leq \theta = \max\theta_i < \pi/2.$$

This implies that $\mathfrak{t}$ is sectorial with a vertex zero and semi-angle θ. □

3.2 The Representation Theorem

In this part of the paper we will prove the Representation Theorems of sectorial forms. For a bounded form which is defined everywhere on a Hilbert space $\mathcal{H}$, there is a bounded operator $T \in \mathcal{H}$ such that $\mathfrak{t}[x, y] = \langle Tx, y\rangle$. One can now generalize this assertion to an unbounded form $\mathfrak{t}$, but with the condition that the form $\mathfrak{t}$ is densely defined, sectorial and closed. The representation Theorem states that the operator T is m-sectorial. To prove this Theorem at the way of Schmüdgen we have to apply the Theorem 7 and we additionally require some assertions about the relationship between sectorial boundedness of a form $\mathfrak{t}$ and lower semiboundedness of its real part. But of course, one can prove this Theorems in other ways. A very beautiful proof is given in [**4**], Chapter 6. Theorem 2.1. But one has to take care by part (ii). To this end, see [**4**], Chapter 6, P. 314, especially the definition of a pre-Hilbert space associated with $\mathfrak{t}$ and see P. 313 for the assertion that $\mathfrak{t}$ is closed if and only if $\mathrm{Re}\,\mathfrak{t}$ is closed.

We begin with the following useful proposition.

Proposition 24. *For any form* $\mathfrak{t}$ *on* $\mathcal{H}$*, the following assertions are equivalent:*

i) $\mathfrak{t}$ *is sectorial*

ii) $\operatorname{Re}\mathfrak{t}$ *is lower semibounded and* $\operatorname{Im}\mathfrak{t}$ *is bounded on* $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$.

iii) $\operatorname{Re}\mathfrak{t}$ *is lower semibounded and* $\mathfrak{t}$ *is bounded on* $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$.

iv) There is number $c \in \mathbb{R}$ *such that* $\operatorname{Re}\mathfrak{t} \geq c$ *and* $\mathfrak{t} + 1 - c$ *is a bounded coercive for on* $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$.

Further, if $\Theta(\mathfrak{t}) \subseteq S_{c,\theta}$*, then* $\operatorname{Re}\mathfrak{t} \geq c$ *and the form* $\mathfrak{t} + 1 - c$ *is bounded and coercive on* $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$.

The proof of this Proposition is very easy, but because of the importance of this Proposition it is convience to prove it.

Proof. $(i) \to (ii)$

$$\begin{aligned}\mathfrak{t} \text{ is sectorial } &\Rightarrow \exists c \in \mathbb{R}, \theta \in [0, \pi/2) \ : \ \Theta(\mathfrak{t}) \subseteq S_{c,\theta} \\ &\Rightarrow \ \forall x \in \mathcal{D}(\mathfrak{t}) : \ |\operatorname{Im}\mathfrak{t}[x]| \leq \tan\theta(\operatorname{Re}\mathfrak{t}[x] - c\|x\|^2) \\ &\Rightarrow \ \operatorname{Re}\mathfrak{t} \geq c.\end{aligned}$$

Thus, the real part of a sectorial form is lower semibounded. Moreover, we have:

$$\|x\|^2_{\operatorname{Re}\mathfrak{t}} = \operatorname{Re}\mathfrak{t}[x] + (1-c)\|x\|^2) \geq \operatorname{Re}\mathfrak{t}[x] - c\|x\|^2.$$

This implies that

$$|\operatorname{Im}\mathfrak{t}[x]| \leq \tan\theta \|x\|^2_{\operatorname{Re}\mathfrak{t}}$$

That means, $\operatorname{Im}\mathfrak{t}$ is bounded on $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$.

$(ii) \to (iii)$ Applying the (1.1) on $\operatorname{Re}\mathfrak{t}$, we get:

$$|\operatorname{Re}\mathfrak{t}| \leq (1 + |m|)\|x\|_{\operatorname{Re}\mathfrak{t}}\|x\|_{\operatorname{Re}\mathfrak{t}}$$

This implies that $\operatorname{Re}\mathfrak{t}$ is always bounded on $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$ and using the part (ii) we conclude that $\mathfrak{t}$ is bounded on $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$.

$(iii) \to (iv)$
By (iii) there exists a $c \in \mathbb{R}$, such that $\operatorname{Re}\mathfrak{t} \geq c$. Therefor we get:

$$\|x\|^2_{\operatorname{Re}\mathfrak{t}} = (\operatorname{Re} t + 1 - c)[x] = \operatorname{Re}(\mathfrak{t} + 1 - c[x]) \leq |(\mathfrak{t} + 1 - c[x])|, \ x \in \mathcal{D}(\mathfrak{t}).$$

This shows that $t + 1 - c$ is coercive on $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$. Becase $\mathfrak{t}$ is bounded on $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$, so is $t + 1 - c$.

$(iv) \to (i)$
Choose a $c \in \mathbb{R}$, such that $\operatorname{Re}\mathfrak{t} \geq c$ and $\mathfrak{t} + 1 - c$ is bounded. Then, $\operatorname{Im}\mathfrak{t}$ is also bounded. That means, exists a $C > 0$, such that

$$|\operatorname{Im}\mathfrak{t}| \leq C\|x\|^2_{\operatorname{Re}\mathfrak{t}} = C(\operatorname{Re}\mathfrak{t}[x] + (1-c)\|x\|^2).$$

Choosing a θ, with $\tan\theta = C$, we get that $\Theta(\mathfrak{t}) \subseteq S_{c-1,\theta}$.

□

Proposition 25. *Let T be a (not necessary densely defined) sectorial operator. Then the form $\mathfrak{s}_T$ defined by $\mathfrak{s}_T[x,y] = \langle Tx, y\rangle$, $x, y \in \mathcal{D}(\mathfrak{s}_T) := \mathcal{D}(T)$, is closeable. There is a unique closed sectorial extension $\overline{\mathfrak{s}_T}$ of $\mathfrak{s}_T$ and $\Theta(T)$ is dense in $\Theta(\overline{\mathfrak{s}_T})$.*

Theorem 26 (Representation theorem for sectorial forms)**.** *Suppose that $\mathfrak{t}$ is a densely defined closed sectorial form on $\mathcal{H}$. Then we have:*

i) The operator $A_\mathfrak{t}$ is m-sectorial.

ii) $\mathcal{D}(A_\mathfrak{t})$ is a dense linear subspace on the Hilbert space $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$.

Proof. *i)* Since the form $\mathfrak{t}$ is sectorial,we can apply the Proposition 24 and find a $c \in \mathbb{R}$, such that is $\mathfrak{t} - (c-1)$ is a bounded coercive form on $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$. The closedness of the form $\mathfrak{t}$ implies that the vector space $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$ is complete. Becaus of the inequality $\|\cdot\| \leq \|\cdot\|_{\operatorname{Re}\mathfrak{t}}$ and since $\mathcal{D}(\mathfrak{t})$ is dense in $\mathcal{H}$ the Hilbert space $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$ is continuously and densely embedded into the Hilbert space $\mathcal{H}$. We can now apply the Theorem 7 and get that the operator $A_{\mathfrak{t}-(c-1)} = A_\mathfrak{t} - (c-1)\cdot I$, associated with the bounded coercive form $\mathfrak{t} - (c-1)$, possesses a bounded inverse in $\mathbf{B}(\mathcal{H})$. Therefor, it holds that $c - 1 \in \rho(A_\mathfrak{t})$ and using the Proposition 18 we obtain that $A_\mathfrak{t}$ is m-sectorial.

ii) This part follows applying the Theorem 7 ii).

□

Theorem 27. *Let $\mathfrak{t}$ be a densely defined closed sectorial form on $\mathcal{H}$. Then we have:*

i) $(A_\mathfrak{t})^* = A_{\mathfrak{t}^*}$.

ii) If the embedding of $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$ into $\mathcal{H}$ is compact, then $R_\lambda(A_\mathfrak{t})$ is compact for all $\lambda \in \rho(A_\mathfrak{t})$ and the operator $A_\mathfrak{t}$ has a purely discrete spectrum.

iii) If $\Theta(\mathfrak{t})$ is contained in a sector $S_{c,\theta}$ for $c \in \mathbb{R}$ and $\theta \in [0, \pi/2)$ then $\sigma(A_\mathfrak{t}) \subseteq S_{c,\theta}$.

iv) Let B be a linear operator on $\mathcal{H}$ such that $\mathcal{D}(B) \subseteq \mathcal{D}(\mathfrak{t})$ and let $\mathcal{D}$ be a core for $\mathfrak{t}$. If $\mathfrak{t}[x,y] = \langle Bx, y\rangle$ for all $x \in \mathcal{D}(B)$ and $y \in \mathcal{D}$, then $B \subseteq A_\mathfrak{t}$.

Proof. This theorem can be proven by applying Theorem 7. □

Corollary 28. *The map* $\mathfrak{t} \to A_{\mathfrak{t}}$ *gives a one-to-one correspondence between densely defined closed sectorial forms and* m*-sectorial operators on a Hilbert space.*

Proof. The existence and injectivity follow from Theorem 26 and Corollary 9. We shall show the surjectivity.
Let T be an m-sectorial operator, and let $\mathfrak{t}$ be the closure of the form $\mathfrak{s}_T$ from Proposition 25. The operator T and hence the form $\mathfrak{t}$ are by Proposition 18 densely defined. Because of the relation

$$\mathfrak{t}[x,y] = \mathfrak{s}_T[x,y] = \langle Tx, y\rangle, \quad x, y \in \mathcal{D}(T)$$

and since $\mathcal{D}(T)$ is a core for $\mathfrak{t} = \overline{\mathfrak{s}_T}$, we get that $T \subseteq A_{\mathfrak{t}}$. Since T and $A_{\mathfrak{t}}$ are m-sectorial and m-sectorial operators are maximal sectorial, we get $T = A_{\mathfrak{t}}$. $\square$

4 Applications of sectorial forms

Let $(V, \langle\cdot,\cdot\rangle_V)$ be a Hilbert space that is densely and continuously embedded into the Hilbert space $\mathcal{H}$.

Definition 29. *i) A form* $\mathfrak{t}$ *with domain* $\mathcal{D}(\mathfrak{t}) = V$ *is called elliptic if there exist constants* $c \in \mathbb{R}$ *and* $\alpha > 0$ *such that the abstract Garding inequality*

$$(\operatorname{Re} \mathfrak{t})\,[x] - c\|x\|^2 \geq \alpha\|x\|_V^2, \quad x \in V \tag{4.1}$$

is satisfied and $\mathfrak{t}$ *is bounded on* $\mathcal{H}$:

$$\exists\; C > 0 : |\mathfrak{t}[x,y]| \leq C\|x\|_V\|y\|_V.$$

ii) Let $j : V \to \mathcal{H}$ *be a bounded linear operator. The form* $\mathfrak{t}$ *is called* $j-$*elliptic if there exist a* $c \in \mathbb{R}$ *and* $\alpha > 0$ *such that*

$$(\operatorname{Re} \mathfrak{t})\,[x] + c\|j(x)\|^2 \geq \alpha\|x\|_V^2, \quad x \in V \tag{4.2}$$

Setting $c = 0$ *in the formula (4.2) one get the definition of coercive forms.*

The Garding inequality implies that the form $\operatorname{Re}\mathfrak{t}$ is lower semibounded with lower bound c.

Proposition 30. *Let* $\mathcal{H}$ *be a Hilbert space and* $\mathcal{D}(\mathfrak{t}) = V$. *If* $\mathfrak{t}$ *is a densely defined closed sectorial form then* $\mathfrak{t}$ *is elliptic on the Hilbert space* $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$.

Proof.

Since $\mathfrak{t}$ is sectorial, we get applying the Proposition 24 that $\mathfrak{t}$ is bounded on $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$ and $\mathfrak{t}+1-c$ is coercive on $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\operatorname{Re}\mathfrak{t}})$. We can easily see that the Garding inequality is satisfied and the assertion is shown. ☐

Proposition 31. *If* $\mathfrak{t}$ *is an elliptic form on a Hilbert space* V *then* $\mathfrak{t}$ *is sectorial.*

Proof. Let $\mathfrak{t}$ be an elliptic form on V. For a $x \in V$, we have:

$$\begin{aligned}\alpha\|x\|_V^2 &\overset{(4.1)}{\leq} \operatorname{Re}\mathfrak{t}[x] - c\|x\|^2 \leq \operatorname{Re}\mathfrak{t}[x] + (1-c)\|x\|^2 = \|x\|_{\operatorname{Re}\mathfrak{t}}^2 \\ &\leq |\mathfrak{t}[x]| + (1-c)\|x\|^2 \leq C\|x\|_V^2 + (1+|c|)\,.\end{aligned}$$

The above inequalities show that the norms $\|\cdot\|_{\operatorname{Re}\mathfrak{t}}$ and $\|\cdot\|_V$ are equivalent. Proposition 24(iii) yields that $\mathfrak{t}$ is sectorial. ☐

The above propositions illustrate the close relationship between elliptic forms and densely defined sectorial forms. Indeed, one can apply the representation theorem on j-elliptic forms:

Theorem 32. *Let* $\mathcal{H}$, V *be Hilbert spaces and* $j : V \to \mathcal{H}$ *a bounded linear operator such that* $j(V)$ *is dense in* $\mathcal{H}$. *Let* $\mathfrak{t} : V \times V \to \mathbb{C}$ *be a continuous* j*-elliptic form. Then we have:*

1. *There exists an operator* A *in* $\mathcal{H}$ *such that for all* $x, u_x \in \mathcal{H}$ *one has* $x \in \mathcal{D}(A)$ *and* $Ax = u_x$ *if and only if there exists a* $u \in V$ *such that* $j(u) = x$ *and* $\mathfrak{t}(u,v) = \langle u_x, j(y)\rangle$ *for all* $v \in V$.
2. *The operator* A *is* m*-sectorial.*

Proof. Since the proof is a bit technical, we just refer to [**1**], Theorem 2.1. $\square$

Sectorial formes are very useful tools for studying of holomorphic C_0-semigroups (also known as strongly continuous one-parameter semigroups). One can show the important theorem:

Theorem 33. *Let* $\mathfrak{t}$ *be a densely defined closed sectorial form on* $\mathcal{H}$*, such that* $\Theta(\mathfrak{t}) \subseteq S_{0,\theta}$*. The operator* $-A_{\mathfrak{t}}$ *generates of a strongly continuous contraction semigroup* $e^{-tA_{\mathfrak{t}}}$ *on* $\mathcal{H}$ *that is holomorphic in the sector* $\{z \in \mathbb{C} : |\arg z| < \pi/2 - \theta\}$*. (See also [4], Chapter IX, Theorem 1.24).*

The framework of sectorial forms has proven to be useful in dealing with the problem of the existence and uniqueness and of certain parabolic evolution equations (See [**3**]).
In conclusion, we refer to a perhaps ineresting paper [**2**]. Here, sectorial forms and especially the representation theorem of the sectorial forms are applied to examine the closure of the Creation Operator of Quantum Mechanics $A^{\dagger}$, which is an unbounded subnormal operator given by

$$\left(A^{\dagger} f\right)(x) = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right) f(x), \quad x \in \mathbb{R}, f \in \mathcal{D}\left(A^{\dagger}\right).$$

5 References

[**1**] ARENDT, W. and Elst, A.F.M. TER., *Sectorial forms and degenerate differential operators*

[**2**] ATHAVALE, A. and CHAVAN, S.,*Sectorial forms and unbounded subnormals.* Math. Proc. Camb. Phil. Soc. (2007), 143, 685.

[**3**] BAROUN, M., MANIAR, L. and SCHNAUBELT, R., *Almost periodicy of parabolic evolution equations with inhomogenuos boundary values*

[**4**] Kato, T., *Perturbation theory for linear operators.* Corrected printing of the second edition, Grundlehren der mathematischen Wissenschaften 132. Springer Verlag, Berlin etc., 1980.

[**5**] MIKLAVCIC, M., *Applied Functional Analysis and Partial Differential Equations* (World Scientific,1998).

[**6**] Schmüdgen, K., *Unbounded Self-adjoint Operators on Hilbert Space.* Dordrecht Heidelberg New York London : Springer-Verlag, 2012