A short note on piecewise constant and piecewise linear interpolation

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Abstract

The analysis of the time discretization of nonlinear evolution equations by means of the backward Euler scheme is often based on the study of piecewise constant and piecewise linear interpolation of the time discrete solution. We provide elementary proofs of two crucial properties.

Let $T > 0$ be given and let $\{N_\ell\}$ be a sequence of positive integers with $N_\ell \to \infty$ as $\ell \to \infty$. We set $\tau_\ell = T/N_\ell$ and $t_\ell^n = n\tau_\ell$ $(n = 0, 1, \ldots, N_\ell)$. Let $(H, (\cdot,\cdot), |\cdot|)$ be a separable Hilbert space.

We consider a sequence of grid functions $\{(t_\ell^n, z_\ell^n)\}_{n=0}^{N_\ell} \subset [0,T] \times H$ and the corresponding piecewise constant and piecewise linear interpolations

$$z_\ell(t) = z_\ell^n \text{ for } t \in (t_\ell^{n-1}, t_\ell^n] \ (n = 1, 2, \ldots, N_\ell) \text{ with } z_\ell(0) = z_\ell^1$$

and

$$\hat{z}_\ell(t) = \frac{t_\ell^n - t}{\tau_\ell}\, z_\ell^{n-1} + \frac{t - t_\ell^{n-1}}{\tau_\ell}\, z_\ell^n \text{ for } t \in [t_\ell^{n-1}, t_\ell^n] \ (n = 1, 2, \ldots, N_\ell)\,,$$

respectively.

Proposition 1. *Assume that*

$$K := \sup_\ell \max_{n=0,1,\ldots,N_\ell} |z_\ell^n| < \infty\,. \tag{1}$$

Then there is $z \in L^\infty(0,T;H)$ and a subsequence (still denoted by ℓ) such that

$$z_\ell \overset{*}{\rightharpoonup} z \text{ as well as } \hat{z}_\ell \overset{*}{\rightharpoonup} z \text{ in } L^\infty(0,T;H) \text{ as } \ell \to \infty\,.$$

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Proof. An immediate consequence of (1) is that $\{z_\ell\}$ and $\{\hat{z}_\ell\}$ are bounded in $L^\infty(0,T;H)$. Since $L^\infty(0,T;H)$ is the dual of the separable Banach space $L^1(0,T;H)$, it is known that there exist $z, \hat{z} \in L^\infty(0,T;H)$ and a subsequence such that

$$z_\ell \overset{*}{\rightharpoonup} z \text{ as well as } \hat{z}_\ell \overset{*}{\rightharpoonup} \hat{z} \text{ in } L^\infty(0,T;H) \text{ as } \ell \to \infty\,.$$

It remains to show $\hat{z} = z$.

The set of functions

$$w_{(a,b),h}(t) = \chi_{(a,b)}(t)h \text{ with } (a,b) \subseteq (0,T)\,,\ h \in H$$

is dense in $L^1(0,T;H)$. We thus have to show that

$$\langle z - \hat{z}, w_{(a,b),h}\rangle = 0$$

for all $(a,b) \subseteq (0,T)$ and all $h \in H$.

Since

$$\langle z - \hat{z}, w_{(a,b),h}\rangle = \langle z - z_\ell, w_{(a,b),h}\rangle + \langle z_\ell - \hat{z}_\ell, w_{(a,b),h}\rangle + \langle \hat{z}_\ell - \hat{z}, w_{(a,b),h}\rangle$$

and since the first and the third term on the right-hand side of the foregoing identity tend to zero as $\ell \to \infty$, we only have to focus on the second term.

Without loss of generality, we may assume $b - a \geq \tau_\ell$. Let $m, n \in \mathbb{N}$ be such that $t_\ell^{m-1} < a \leq t_\ell^m$ and $t_\ell^n \leq b < t_\ell^{n+1}$. Then

$$\langle z_\ell - \hat{z}_\ell, w_{(a,b),h}\rangle = \int_a^b (z_\ell(t) - \hat{z}_\ell(t), h)\, dt = \int_a^{t_\ell^m} \cdots + \int_{t_\ell^m}^{t_\ell^n} \cdots + \int_{t_\ell^n}^b \cdots\,.$$

The first and third term on the right-hand side of the foregoing identity are easily shown to converge to zero as $\ell \to \infty$ since $\{z_\ell\}$ and $\{\hat{z}_\ell\}$ are bounded in $L^\infty(0,T;H)$ and since $t_\ell^m - a \leq \tau_\ell$, $b - t_\ell^n \leq \tau_\ell$. For the remaining second term, we find

$$\begin{aligned}
\int_{t_\ell^m}^{t_\ell^n} (z_\ell(t) - \hat{z}_\ell(t), h)\, dt &= \sum_{j=m+1}^{n} \int_{t_\ell^{j-1}}^{t_\ell^j} \left(\frac{t_\ell^j - t}{\tau_\ell} (z_\ell^j - z_\ell^{j-1}), h \right) dt \\
&= \frac{\tau_\ell}{2} \sum_{j=m+1}^{n} \left(z_\ell^j - z_\ell^{j-1}, h \right) \\
&= \frac{\tau_\ell}{2} (z_\ell^n - z_\ell^m, h)
\end{aligned}$$

and thus

$$\left| \int_{t_\ell^m}^{t_\ell^n} (z_\ell(t) - \hat{z}_\ell(t), h)\, dt \right| \leq \tau_\ell K\, |h| \to 0 \text{ as } \ell \to \infty\,.$$

This proves the assertion. □

Proposition 2. *Assume, in addition to (1), that (passing to a subsequence if necessary)*

$$\hat{z}_\ell \to z \text{ in } L^2(0,T;H) \text{ as } \ell \to \infty\,. \tag{2}$$

Then also

$$z_\ell \to z \text{ in } L^2(0,T;H) \text{ as } \ell \to \infty\,.$$

Proof. Passing to a subsequence if necessary, we can assume that the assertion of Proposition 1 holds true. This shows that, in particular,

$$z_\ell \rightharpoonup z \text{ as well as } \hat{z}_\ell \rightharpoonup z \text{ in } L^2(0,T;H) \text{ as } \ell \to \infty\,. \tag{3}$$

With

$$\|z_\ell - z\|_{L^2(0,T;H)} \le \|z_\ell - \hat{z}_\ell\|_{L^2(0,T;H)} + \|\hat{z}_\ell - z\|_{L^2(0,T;H)},$$

together with (2) for the second term on the right-hand side, and

$$\begin{aligned}\|z_\ell - \hat{z}_\ell\|^2_{L^2(0,T;H)} &= (z_\ell - \hat{z}_\ell, z_\ell - \hat{z}_\ell)_{L^2(0,T;H)} \\ &= (z_\ell - \hat{z}_\ell, z_\ell + 3\hat{z}_\ell)_{L^2(0,T;H)} - 4(z_\ell - \hat{z}_\ell, \hat{z}_\ell)_{L^2(0,T;H)}\,,\end{aligned}$$

together with (2) and (3) for the second term on the right-hand side, it is sufficient to show that

$$(z_\ell - \hat{z}_\ell, z_\ell + 3\hat{z}_\ell)_{L^2(0,T;H)} \to 0 \text{ as } \ell \to \infty\,.$$

A straightforward calculation shows that

$$\begin{aligned}&(z_\ell - \hat{z}_\ell, z_\ell + 3\hat{z}_\ell)_{L^2(0,T;H)} \\ &= \sum_{j=1}^{N_\ell} \int_{t_\ell^{j-1}}^{t_\ell^j} \left(\frac{t_\ell^j - t}{\tau_\ell}(z_\ell^j - z_\ell^{j-1}), \left(1 + 3\frac{t - t_\ell^{j-1}}{\tau_\ell}\right) z_\ell^j + 3\frac{t_\ell^j - t}{\tau_\ell}\, z_\ell^{j-1} \right) dt \\ &= \tau_\ell \sum_{j=1}^{N_\ell} \left(z_\ell^j - z^{j-1}, z_\ell^j + z^{j-1} \right) = \tau_\ell \sum_{j=1}^{N_\ell} \left(|z_\ell^j|^2 - |z^{j-1}|^2 \right) \\ &= \tau_\ell \left(|z_\ell^{N_\ell}|^2 - |z^0|^2 \right).\end{aligned}$$

We thus come up with

$$\left|(z_\ell - \hat{z}_\ell, z_\ell + 3\hat{z}_\ell)_{L^2(0,T;H)}\right| \le 2\tau_\ell K^2 \to 0 \text{ as } \ell \to \infty\,.$$

This proves the assertion. □