FULL DISCRETIZATION OF SECOND-ORDER NONLINEAR EVOLUTION EQUATIONS: STRONG CONVERGENCE AND APPLICATIONS*

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Abstract. Recent results on convergence of fully discrete approximations combining the Galerkin method with the explicit-implicit Euler scheme are extended to strong convergence under additional monotonicity assumptions. It is shown that these abstract results, formulated in the setting of evolution equations, apply, for example, to the partial differential equation for vibrating membrane with nonlinear damping and to another partial differential equation that is similar to one of the equations used to describe martensitic transformations in shape-memory alloys. Numerical experiments are performed for the vibrating membrane equation with nonlinear damping which support the convergence results.

Key words. Evolution equation of second order, Time discretization, Galerkin method, Convergence, Vibrating membrane, Shape-memory alloys

Subject classifications. 47J35, 65M12, 47H05, 47G20, 74K15, 74B20

1. Introduction Nonlinear partial differential equations of second order in time are being used to describe a variety of problems in physical sciences. Explicit formulae for solutions to such partial differential equations are rare. Hence there are many methods of solving such problems numerically. This article focuses on conforming finite element methods in space and a partitioned explicit-implicit Euler method in time (generalising the well known Störmer–Verlet or leap-frog method).

Recently, Emmrich and Thalhammer [8] have demonstrated weak convergence of time discrete approximations for second-order doubly nonlinear evolution equations with damping. This has been recently extended by Emmrich and Thalhammer [9] to strong convergence of time and space discrete approximations for second-order doubly nonlinear evolution equations and weak convergence for more general second-order doubly nonlinear evolution equations.

This paper has two main aims. The first is to improve the strong convergence results in the case when stronger monotonicity of the damping term can be assumed. The second aim is to demonstrate that the abstract results apply to the vibrating membrane equation with nonlinear damping as well as to another partial differential equation which resembles one of the possible equations modelling transformations in shape-memory alloys. For the nonlinear vibrating membrane equation with nonlinear damping, numerical experiments supporting the theoretical convergence are presented. The two concrete nonlinear partial differential equations considered are:

1. Vibrating membrane equation / wave equation with a nonlinear damping term:

$$u_{tt} + |u_t|^{p-2} u_t - \Delta u = f,$$

   where u is the displacement of the membrane on some bounded domain, $p \geq 2$ and the initial displacement and initial velocity together with the right-hand

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side f are given. The earliest reference is, to the authors knowledge, Andreassi and Torelli [1]. For further references, see Section 3.

2. An equation similar to an equation for martensitic transformations in shape-memory alloys:
$$u_{tt} - \mu\Delta u_t - \operatorname{div}(\sigma(\nabla u)) + \lambda\Delta^2 u = f, \tag{1.1}$$
where $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ is given. The equation corresponds to one of the equations for martensitic transformations in shape-memory alloys and the function $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ determines the various phases in the so-called shape-memory alloy. The function σ arises from the potential $\varphi : \mathbb{R}^3 \to \mathbb{R}$, the stored energy density, via $\sigma := \varphi'$. An example of φ found in the literature is a fourth order polynomial. Hence σ should be allowed to have at least cubic growth in applications. For an overview, see Plecháč and Roubíček [18] and further references in Section 4. It is emphasised that this paper does not cover the situation where σ has cubic growth. Only linear growth is allowed. So the abstract framework of this article does not appear to be optimal for the equation modelling martensitic transformations in shape-memory alloys. Nevertheless, Roubíček [22, Chapter 11, p. 354] also considers (1.1) together with linear growth of σ.

The functional analytic formulation of the foregoing problems leads to initial value problems of the form
$$u'' + Au' + Bu = f \quad \text{in } (0,T), \quad u(0) = u_0\,, u'(0) = v_0\,, \tag{1.2}$$
where A and B are possibly nonlinear operators defined on appropriate, perhaps different, function spaces.

The functional analytic setting for (1.2) is as follows. Let $(V_A, \|\cdot\|_{V_A})$ be a reflexive and separable Banach space that is dense and continuously embedded in a Hilbert space $(H, (\cdot,\cdot), |\cdot|)$ such that $V_A \subset H \subset V_A^*$ form a Gelfand triple. Let A be the Nemytskii operator corresponding to a family of nonlinear operators $\{A(t)\}_{t\in[0,T]}$. It is assumed that $A(t)$ is composed of a principal part $A_0(t) : V_A \to V_A^*$ and a perturbation $A_1(t) : V_A \to V_A^*$ such that $A(t) = A_0(t) + A_1(t)$. The precise assumptions on A_0 and A_1 will be stated in Section 2. For now it is sufficient to say that $A_0(t)$ shall be hemicontinous, monotone and coercive (up to a shift) and satisfying a certain growth condition, uniformly in t. It is required that A_1 fulfils a certain lower semi-boundedness condition (so that $A(t)$ remains up to a shift coercive), a growth condition and local Hölder-like continuity condition.

Now consider the setting for the operator B. Let $(V_B, \|\cdot\|_B)$ be a separable Banach space such that $V_B \subset H \subset V_B^*$ again form a Gelfand triple. Let B be the Nemytskii operator corresponding to a family of operators $\{B(t)\}_{t\in[0,T]}$. Again it is assumed that $B(t)$ is composed of a principal part B_0 and a perturbation $B_1(t)$ such that $B(t) = B_0 + B_1(t)$. The principal part of $B(t)$ is assumed to be time-independent (though this is not necessary, see Lions and Strauss [15]), linear, bounded, symmetric and strongly positive. Observe that this implies that B induces an inner product on V_B thus making V_B into a Hilbert space. As such V_B is reflexive. The time dependent perturbations $B_1(t)$ are assumed to be bounded and locally continuous in a Hölder-like sense. The precise assumptions will be given in Section 2.

Furthermore let $V := V_A \cap V_B$ and assume that V is dense in both the spaces V_A and V_B. Thus we have the following scale of spaces:
$$V_A \cap V_B = V \overset{d}{\hookrightarrow} V_C \overset{d}{\hookrightarrow} H = H^* \overset{d}{\hookrightarrow} V_C^* \overset{d}{\hookrightarrow} V^* = V_A^* + V_B^*, \quad C \in \{A, B\},$$

where $\overset{d}{\hookrightarrow}$ denotes continuous and dense embedding.

Equivalently (1.2) can be written as the first-order system

$$\begin{cases} u' - v = 0 & \text{in} \quad (0,T), \quad u(0) = u_0, \\ v' + Av + Bu = f & \text{in} \quad (0,T), \quad v(0) = v_0. \end{cases} \tag{1.3}$$

In this situation, like in Lions and Strauss [15] and in Emmrich and Thalhammer [9], great care needs to be taken with the integration by parts formulae, since, in general, the second time derivative $u''(t) = v'(t)$ only takes values in $V^* = V_A^* + V_B^*$, whereas $u'(t) = v(t)$ takes values in V_A. So the duality pairing between $u''(t) = v'(t)$ and $u'(t) = v(t)$ is not defined.

Let $\{V_m\}_{m\in\mathbb{N}}$ be a Galerkin scheme for V (recall that V is the intersection of the separable spaces V_A and V_B and hence a Galerkin basis exists). For a given $m \in \mathbb{N}$ and a variable time grid

$$\mathbb{I}: 0 = t_0 < t_1 < \cdots < t_N = T, \ \tau_n = t_n - t_{n-1} \text{ for } n = 1,2,\ldots,N \in \mathbb{N}, \ \tau_{\max} := \max_{n=1,\ldots,N} \tau_n, \tag{1.4}$$

the aim is to find fully discrete approximations $\{u^n\}_{n=1}^N \subset V_m$ with $u^n \approx u(t_n)$ and $\{v^n\}_{n=1}^{N-1} \subset V_m$ with $v^n \approx v(t_n) = u'(t_n)$ such that for all $\varphi \in V_m$

$$\begin{cases} \dfrac{(u^{n+1} - u^n, \varphi)}{\tau_{n+1}} - (v^n, \varphi) = 0, \quad n = 0,1,\ldots,N-1, \\ \dfrac{(v^n - v^{n-1}, \varphi)}{\frac{\tau_{n+1}+\tau_n}{2}} + \langle A(t_n)v^n, \varphi\rangle + \langle B(t_n)u^n, \varphi\rangle = \langle f^n, \varphi\rangle, \quad n = 1,\ldots,N-1. \end{cases} \tag{1.5}$$

Here $u^0 \approx u_0$, $v^0 \approx v_0$ and $\{f^n\}_{n=1}^{N-1} \approx f$ are given. This scheme arises simply by applying the explicit Euler scheme to the first equation in (1.3) and the implicit Euler scheme to the second equation. This corresponds to the simplest partitioned Runge–Kutta method. Observe that if $A = 0$ then the scheme would correspond to the leapfrog scheme or the well known Störmer–Verlet method.

The existence of solutions to (1.2) goes back to Lions and Strauss, where in [15] they prove existence and uniqueness in the case $A_1 = B_1 = 0$, but not assuming $V_A \hookrightarrow V_B$. Existence in the case with perturbations (but not uniqueness) is proved in Emmrich and Thalhammer [9]. For the linear case, the time discretization combined with a conforming finite element method has been studied in Raviart and Thomas [20, Chapter 8]. More recently Verwer [25] studied Runge–Kutta time integration methods for the wave equation with linear damping. Colli and Favini [7] have proved the convergence of time discretization to (1.2) with constant step sizes under the much more restrictive assumptions that $V_A = V_B$, A_0 is time independent and maximal monotone and $A_1 = B_1 = 0$. This also forces V_A to be a Hilbert space.

Emmrich and Thalhammer [8] have proved weak convergence of time discretizations (1.5) under the assumption $V_A \hookrightarrow V_B$. Later this has been extended, in Emmrich and Thalhammer [9], where weak convergence has been proved in the case with no perturbations ($A_1 = B_1 = 0$), even if V_A is not embedded in V_B. Furthermore in the case of perturbations, under an additional assumption on the space H and the Galerkin scheme, or under the assumption $V_A \hookrightarrow V_B$, strong convergence (of a subsequence in case (1.2) does not have a unique solution) of the piecewise constant prolongations of the approximations $\{v^n\}_{n=0}^{N-1}$ of the first time derivative of the solution has been proved in $L^r(0,T;H)$ for any $r \in [1,\infty)$. This article extends this to strong convergence in $L^q(0,T;V_A)$, with $q\in[1,p)$ under the additional d-monotonicity assumption and to $q=p$ under a uniform monotonicity assumption on A_0. Here $p\geq 2$ comes from the coercivity assumption. Moreover, strong convergence of the piecewise constant prolongations of the approximations $\{u^n\}_{n=0}^N$ of the exact solution is shown in $L^r(0,T;V_A+V_B)$ for any $r\in[1,\infty)$.

This paper is organised as follows. In Section 2 the strong convergence results are proved. Sections 3 and 4 show that the convergence results apply to the vibrating membrane equation with nonlinear damping and the equation (1.1) respectively. Finally in Section 5 numerical results for discretising the vibrating membrane equation with nonlinear damping are presented.

2. Strong convergence In what follows, the space of Bochner integrable (for $r=\infty$ Bochner measurable and essentially bounded) abstract functions mapping $[0,T]$ into a (reflexive) Banach space X is denoted by $L^r(0,T;X)$ ($r\in[1,\infty]$) and equipped with the standard norm $\|\cdot\|_{L^r(0,T;X)}$. Let u' and u'' denote the first and second time derivative of the abstract function $u=u(t)$ in the distributional sense, respectively. Moreover, let c be a generic positive constant. For $p\in(1,\infty)$, let $p^*:=\frac{p}{p-1}$.

Let $V=V_A\cap V_B$ with norm $\|\cdot\|=\|\cdot\|_{V_A}+\|\cdot\|_{V_B}$. The space V is assumed to be dense in each of the spaces V_A and V_B. Obviously, V is also continuously embedded in each of the spaces V_A and V_B. The dual $V^*=V_A^*+V_B^*$ is equipped with the norm

$$\|f\|_*=\inf\left\{\max\left(\|f_A\|_{V_A^*},\|f_B\|_{V_B^*}\right):f=f_A+f_B \text{ with } f_A\in V_A^*, f_B\in V_B^*\right\}.$$

Observe that $V\subseteq H\subseteq V^*$ form a Gelfand triple. Besides $V=V_A\cap V_B$, we also use the space V_A+V_B equipped with the standard norm

$$\|w\|_{V_A+V_B}=\inf\{\max(\|w_A\|_{V_A},\|w_B\|_{V_B}):w=w_A+w_B \text{ with } w_A\in V_A, w_B\in V_B\}.$$

Note that $V_C\hookrightarrow V_A+V_B$ with $\|w\|_{V_A+V_B}=\|w\|_{V_C}$ if $w\in V_C$ ($C\in\{A,B\}$).

The following assumptions on the operators $A_0(t):V_A\to V_A^*$ and $B_0:V_B\to V_B^*$ will be needed to prove the convergence results. The additional monotonicity assumptions will be stated separately.

Assumption (A_0)**.** $\{A_0(t)\}_{t\in[0,T]}$ is a family of monotone and hemicontinuous operators $A_0(t):V_A\to V_A^*$ such that for all $v\in V_A$ the mapping $t\mapsto A_0(t)v:[0,T]\to V_A^*$ is continuous for almost all $t\in[0,T]$. For a suitable $p\in[2,\infty)$, there are constants $\mu_A,c>0,\lambda\geq 0$ such that for all $t\in[0,T]$ and $v\in V_A$

$$\langle A_0(t)v,v\rangle\geq\mu_A\|v\|_{V_A}^p-\lambda,\quad \|A_0(t)v\|_{V_A^*}\leq c\left(1+\|v\|_{V_A}^{p-1}\right).$$

In fact it would be sufficient to require monotonicity and coercivity for $A_0(t)+\kappa I$, with $\kappa>0$. To simplify the presentation this is omitted, noting that the additional term can be considered as a perturbation.

Assumption (B_0)**.** $B_0:V_B\to V_B^*$ is a linear, bounded, symmetric, and strongly positive operator: There are constants $\mu_B,c_B>0$ such that for all $v\in V_B$

$$\langle B_0v,v\rangle\geq\mu_B\|v\|_{V_B}^2,\quad \|B_0v\|_{V_B^*}\leq c_B\|v\|_{V_B}.$$

Due to [9, Theorem 2 and Theorem 7], the discrete problem (1.5) has a solution: if $A=A_0$, $B=B_0$ and if Assumptions (A_0) and (B_0) are satisfied and $u^0,v^0\in V_m$ and $\{f^n\}_{n=1}^{N-1}\subset V^*$ are given then (1.5) has a unique solution $\{u^n\}_{n=0}^N\subset V_m$.

Let $\{(V_{m_\ell},\mathbb{I}_\ell)\}_{\ell\in\mathbb{N}}$ be a sequence of finite dimensional spaces $V_{m_\ell}\in\{V_m\}_{m\in\mathbb{N}}$ and time grids $\mathbb{I}_\ell$ of type (1.4) fulfilling the following assumption.

Assumption $(V_m,\mathbb{I})$**.** For each $m\in\mathbb{N}$, let $c_{V_B\leftarrow V_A}(m)$ be a positive constant such that

$$\|v\|_{V_B}\le c_{V_B\leftarrow V_A}(m)\,\|v\|_{V_A}\quad\text{for all } v\in V_m\,. \tag{2.1}$$

Let $r_n:=\tau_n/\tau_{n-1}$ for $n=2,\dots,N$. The sequence $\{(V_{m_\ell},\mathbb{I}_\ell)\}_{\ell\in\mathbb{N}}$ satisfies

$$m_\ell\to\infty \text{ and } \tau_{\max}(\mathbb{I}_\ell)\to 0 \text{ as } \ell\to\infty,$$

$$\sup_{\ell\in\mathbb{N}} c_{V_B\leftarrow V_A}(m_\ell)^2\tau_{\max}(\mathbb{I}_\ell)<\min\left(1,\frac{\mu_A}{c_B}\right),\quad c_{V_B\leftarrow V_A}(m_\ell)^2\tau_{\max}(\mathbb{I}_\ell)\to 0 \text{ as } \ell\to\infty,$$

$$\sup_{\ell\in\mathbb{N}}\left(\max_{n=2,\dots,N_\ell} r_n(\mathbb{I}_\ell)\right)<\infty,\quad \inf_{\ell\in\mathbb{N}}\left(\min_{n=2,\dots,N_\ell} r_n(\mathbb{I}_\ell)\right)>0,$$

$$\sup_{\ell\in\mathbb{N}}\left(\sum_{n=3}^{N}\max\left(0,\frac{1}{r_n(\mathbb{I}_\ell)}-\frac{1}{r_{n-1}(\mathbb{I}_\ell)}\right)\right)<\infty,\quad \sup_{\ell\in\mathbb{N}}\left(\sum_{n=2}^{N}\frac{(\tau_n(\mathbb{I}_\ell)-\tau_{n-1}(\mathbb{I}_\ell))^2}{(\tau_n(\mathbb{I}_\ell)+\tau_{n-1}(\mathbb{I}_\ell))^3}\right)<\infty.$$

Relation (2.1) can always be satisfied as all norms on finite dimensional spaces are equivalent. The coupling of $c_{V_B\leftarrow V_A}(m_\ell)$ with $\tau_{\max}(\mathbb{I}_\ell)$ however does create a restriction, unless V_A is continuously embedded in V_B. The remaining assumptions are always fulfilled for equidistant time grids but can also be satisfied for variable time grids with rather large deviations from an equidistant one. With respect to the initial data it is required that the following holds.

Assumption (IC)**.** The initial values for (1.5) satisfy

$$u^0(m_\ell,\mathbb{I}_\ell),v^0(m_\ell,\mathbb{I}_\ell)\in V_{m_\ell}\ (\ell\in\mathbb{N}),\quad \sup_{\ell\in\mathbb{N}}\tau_{\max}(\mathbb{I}_\ell)\|v^0(m_\ell,\mathbb{I}_\ell)\|_{V_A}^p<\infty,$$
$$u^0(m_\ell,\mathbb{I}_\ell)\to u_0 \text{ in } V_B \text{ and } v^0(m_\ell,\mathbb{I}_\ell)\to v_0 \text{ in } H \text{ as } \ell\to\infty.$$

In order to simplify notation, v_ℓ and u_ℓ will be used in place of $v(m_\ell,\mathbb{I}_\ell)$ and $u(m_\ell,\mathbb{I}_\ell)$. Let

$$\tau_{n+1/2}:=\frac{\tau_{n+1}+\tau_n}{2},\quad t_{n+1/2}:=t_n+\frac{\tau_{n+1}}{2}.$$

Let χ_D denote the characteristic function of a set D. For the solution $\{u^n\}_{n=0}^{N_\ell}\subset V_{m_\ell}$ and $\{v^n\}_{n=1}^{N_\ell-1}\subset V_{m_\ell}$ to (1.5) corresponding to a time grid $\mathbb{I}_\ell$ define the following piecewise constant prolongations

$$u_\ell(t):=\sum_{n=1}^{N-1}u^n\chi_{(t_{n-1/2},t_{n+1/2}]}(t),\quad v_\ell(t):=\sum_{n=1}^{N-1}v^n\chi_{(t_{n-1/2},t_{n+1/2}]}(t) \tag{2.2}$$

and the piecewise linear prolongation

$$\hat v_\ell(t):=v^0\chi_{[0,t_{1/2}]}(t)+\sum_{n=1}^{N-1}\left(v^n+\frac{t-t_{n+1/2}}{\tau_{n+1/2}}(v^n-v^{n-1})\right)\chi_{(t_{n-1/2},t_{n+1/2}]}(t)$$
$$+v^{N-1}\chi_{(t_{N-1/2},t_N]}(t).$$

Note that $\hat v_\ell$ is piecewise linear and continuous in time, and thus differentiable in the weak sense. For the right-hand side, given $f\in L^{p^*}(0,T;V_A^*)$, let

$$f^n:=\frac{1}{\tau_{n+1/2}}\int_{t_{n-1/2}}^{t_{n+1/2}}f(t)dt\quad\text{and}\quad f_\ell(t):=\sum_{n=1}^{N-1}f^n\chi_{(t_{n-1/2},t_{n+1/2}]}(t).$$

Finally, let $A_{0,\ell}$ be defined as a piecewise constant approximation of A_0, i.e.,

$$A_{0,\ell}(t) := A_0(t_1)\chi_{[0,t_{1/2}]}(t) + \sum_{n=1}^{N-1} A_0(t_n)\chi_{(t_{n-1/2},t_{n+1/2}]}(t) + A_0(t_{N-1})\chi_{(t_{N-1/2},t_N]}(t).$$

Let $A_{1,\ell}$ and $B_{1,\ell}$ be defined analogously.

Theorem 1. *Let $A_1 = 0$ and $B_1 = 0$. Let Assumptions (V_m,$\mathbb{I}$), (IC), (A_0) and (B_0) be satisfied. Further assume that $f \in L^{p^*}(0,T;V_A^*)$.*

If for all $t \in [0,T]$ the operator $A(t): V_A \to V_A^*$ satisfies

$$\langle A(t)w - A(t)\overline{w}, w - \overline{w}\rangle \geq (\alpha(\|w\|_{V_A}) - \alpha(\|\overline{w}\|_{V_A}))(\|w\|_{V_A} - \|\overline{w}\|_{V_A}) \quad \forall w,\overline{w} \in V_A \tag{2.3}$$

for some monotonically increasing function $\alpha: [0,\infty) \to \mathbb{R}$, i.e., if it is d-monotone, and if V_A is uniformly convex, then v_ℓ converges towards the first time derivative $v = u'$ of the exact solution u to (1.2) strongly in $L^q(0,T;V_A)$ for any $q \in [1,p)$ as $\ell \to \infty$. Furthermore, if $A(t): V_A \to V_A^*$ satisfies for all $t \in [0,T]$

$$\langle A(t)w - A(t)\overline{w}, w - \overline{w}\rangle \geq \mu\|w - \overline{w}\|_{V_A}^p \quad \forall w,\overline{w} \in V_A, \tag{2.4}$$

for some $\mu > 0$ then v_ℓ converges towards $v = u'$ strongly in $L^p(0,T;V_A)$ as $\ell \to \infty$.

In both cases, u_ℓ converges towards u strongly in $L^r(0,T;V_A + V_B)$ for any $r \in [1,\infty)$ as $\ell \to \infty$.

Proof. Let $A_\ell(t) = A_{0,\ell}(t)$. This makes sense since $A_1 = 0$. Due to Assumption (B_0), the operator $B = B_0$ is independent of time. If $\{u^n\}_{n=0}^N \subset V_m$ and $\{v^n\}_{n=1}^{N-1} \subset V_m$ is a solution to (1.5) then the second equation of (1.5) is equivalent to

$$\langle \hat{v}_\ell'(t), \phi\rangle + \langle A_\ell(t)v_\ell(t), \phi\rangle + \langle Bu_\ell(t), \phi\rangle = \langle f_\ell(t), \phi\rangle \quad \forall \phi \in V_{m_\ell}, \tag{2.5}$$

for almost all t in $(0,T)$. Furthermore the second equation in (1.3) is equivalent to

$$\int_0^T \langle (v' + Bu)(t), w(t)\rangle dt + \int_0^T \langle A(t)v(t), w(t)\rangle dt = \int_0^T \langle f(t), w(t)\rangle dt \quad \forall w \in L^p(0,T;V_A). \tag{2.6}$$

Here $v = u'$ is the weak limit of v_ℓ and u is the unique solution to (1.2) as is shown in [9, Theorem 4].

Consider first the simpler case when $A(t): V_A \to V_A^*$ satisfies (2.4) for all $t \in [0,T]$. From (2.4) it can be seen that

$$\mu \int_0^T \|v_\ell(t) - v(t)\|_{V_A}^p dt \leq \int_0^T \langle A_\ell(t)v_\ell(t) - A_\ell(t)v(t), v_\ell(t) - v(t)\rangle dt. \tag{2.7}$$

By adding zero to the right-hand side it can be obtained that

$$\begin{aligned}
&\int_0^T \langle A_\ell(t)v_\ell(t) - A_\ell(t)v(t), v_\ell(t) - v(t)\rangle dt \\
&= \int_0^T \langle A_\ell(t)v_\ell(t) - A(t)v(t), v_\ell(t) - v(t)\rangle dt + \int_0^T \langle A(t)v(t) - A_\ell(t)v(t), v_\ell(t) - v(t)\rangle dt \\
&= \int_0^T \langle A_\ell(t)v_\ell(t), v_\ell(t)\rangle dt - \int_0^T \langle A_\ell(t)v_\ell(t), v(t)\rangle dt - \int_0^T \langle A(t)v(t), v_\ell(t) - v(t)\rangle dt \\
&\quad + \int_0^T \langle A(t)v(t) - A_\ell(t)v(t), v_\ell(t) - v(t)\rangle dt.
\end{aligned}$$

Testing with $v_\ell(t)$ in (2.5) and integrating leads to

$$\int_0^T \langle A_\ell(t)v_\ell(t),v_\ell(t)\rangle dt = \int_0^T \langle f_\ell(t),v_\ell(t)\rangle dt - \int_0^T \langle \hat{v}_\ell'(t),v_\ell(t)\rangle dt - \int_0^T \langle Bu_\ell(t),v_\ell\rangle dt.$$

Hence the above inequality can be transformed to

$$\mu\int_0^T \|v_\ell(t)-v(t)\|_{V_A}^p dt \le P_\ell^1 - P_\ell^2 + P_\ell^3,$$

where

$$\begin{aligned} P_\ell^1 &:= \int_0^T \langle f_\ell(t),v_\ell(t)\rangle dt - \int_0^T \langle A_\ell(t)v_\ell(t),v(t)\rangle dt - \int_0^T \langle A(t)v(t),v_\ell(t)-v(t)\rangle dt,\\ P_\ell^2 &:= \int_0^T \langle \hat{v}_\ell'(t),v_\ell(t)\rangle dt + \int_0^T \langle Bu_\ell(t),v_\ell(t)\rangle\, dt,\\ P_\ell^3 &:= \int_0^T \langle A(t)v(t)-A_\ell(t)v(t),v_\ell(t)-v(t)\rangle dt. \end{aligned}$$

Due to [9, proof of Theorem 4, (2.39)], it is known that for any w in $L^p(0,T;V_A)$, the sequence $A_\ell w$ converges strongly to Aw in $L^{p^*}(0,T;V_A^*)$ as $\ell\to\infty$. Also, due to the a priori estimates [9, Theorem 3], the sequence $\{v_\ell - v\}_{\ell\in\mathbb{N}}$ is bounded in $L^p(0,T;V_A)$. Hence $P_\ell^3\to 0$ as $\ell\to\infty$. Consider now P_ℓ^1. Recall that $f\in L^{p^*}(0,T;V_A^*)$ and $f_\ell\to f$ strongly in $L^{p^*}(0,T;V_A^*)$ as $\ell\to\infty$. Furthermore, due to the a priori estimates in [9, Theorem 3] $\{v_\ell\}_{\ell\in\mathbb{N}}$ is a bounded sequence in $L^p(0,T;V_A)$. Hence

$$\lim_{\ell\to\infty}\int_0^T \langle f_\ell(t),v_\ell(t)\rangle dt = \int_0^T \langle f(t),v(t)\rangle dt.$$

Due to the weak convergence of v_ℓ to v in $L^p(0,T;V_A)$, see [9, Lemma 5], we obtain

$$\lim_{\ell\to\infty}\int_0^T \langle A(t)v(t),v_\ell(t)-v(t)\rangle dt = 0. \tag{2.8}$$

Since $A_\ell v_\ell \rightharpoonup Av$ in $L^{p^*}(0,T;V_A^*)$, due to [9, proof of Theorem 4, (2.25) and (2.39)], it can be concluded that

$$\lim_{\ell\to\infty}\int_0^T \langle A_\ell(t)v_\ell(t),v(t)\rangle dt = \int_0^T \langle A(t)v(t),v(t)\rangle dt.$$

This together with (2.8) implies that

$$\lim_{\ell\to\infty} P_\ell^1 = \int_0^T \langle f(t),v(t)\rangle dt - \int_0^T \langle A(t)v(t),v(t)\rangle dt.$$

For any Bochner integrable function w, let $Kw(t) := \int_0^t w(s)ds$. Consider the second integral in P_ℓ^2. Due to [9, (2.38) in the proof of Theorem 4], it is known that

$$\int_0^T \langle B(u_\ell(t)-u_0-Kv_\ell(t)),v_\ell(t)\rangle dt \to 0 \quad \text{as} \quad \ell\to\infty.$$

In order to make a use of this, consider the identity

$$\int_0^T \langle Bu_\ell(t), v_\ell(t)\rangle dt = \int_0^T \langle B(u_\ell - u_0 - Kv_\ell)(t), v_\ell(t)\rangle dt + \int_0^T \langle B(u_0 + Kv_\ell)(t), v_\ell(t)\rangle dt.$$

Note that all the terms in the equation above are well defined, indeed $u_0 \in V_B$ implies that $Bu_0 \in V_B^*$ and $v_\ell(t) \in V_{m_\ell} \subset V \subset V_B$ implies $BKv_\ell(t) \in V_B^*$ for all t. Observe that V_B with

$$((v,w)) := \langle Bw, v\rangle, \quad \|v\|_B^2 := ((v,v)) \quad \forall v, w \in V_B$$

is a Hilbert space and $\|\cdot\|_{V_B}$ and $\|\cdot\|_B$ are equivalent norms. Thus the integration by parts formula holds for any function $v \in L^p(0,T;V_B)$ with $v' \in L^{p^*}(0,T;V_B^*)$, see, e.g., Gajewski, Gröger and Zacharias [12, Chapter 4, Theorem 1.17] or Roubíček [22, Lemma 7.3]. Hence

$$\int_0^T \langle B(u_0 + Kv_\ell(t)), v_\ell(t)\rangle dt = \int_0^T \langle B(u_0 + Kv_\ell)(t), (u_0 + Kv_\ell)'(t)\rangle dt$$
$$= \int_0^T \frac{1}{2}\frac{d}{dt}\|u_0 + Kv_\ell(t)\|_B^2 dt = \frac{1}{2}\|u_0 + Kv_\ell(T)\|_B^2 - \frac{1}{2}\|u_0\|_B^2.$$

Due to [9, proof of Theorem 4, (2.35) and (2.37)], it is known that as $\ell \to \infty$

$$u_0 + Kv_\ell(T) \rightharpoonup u_0 + Kv(T) \quad \text{in} \quad V_B.$$

Hence, using the weak lower semi-continuity of the norm, it is seen that

$$\|u_0 + Kv(T)\|_B \le \liminf_{\ell\to\infty} \|u_0 + Kv_\ell(T)\|_B.$$

Now consider the first integral in P_ℓ^2. Since $\hat{v}_\ell$ and v_ℓ are piecewise linear and piecewise constant respectively, the formula $(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$, which holds for $a, b \in \mathbb{R}$, can be used to rewrite the first integral in P_ℓ^2 as a telescoping sum. Hence

$$\int_0^T \langle \hat{v}_\ell'(t), v_\ell(t)\rangle dt \ge \frac{1}{2}|v_\ell^{N_\ell - 1}|^2 - \frac{1}{2}|v_\ell^0|^2.$$

Due to Assumption (IC), it is known that $v_\ell^0 \to v_0$ strongly in H as $\ell \to \infty$. Hence $|v_0| = \lim_{\ell\to\infty}|v_\ell^0|$. Furthermore due to [9, proof of Theorem 4, (2.24) and (2.30)] it is known that $v_\ell^{N_\ell - 1} \rightharpoonup v(T)$ in H as $\ell \to \infty$. This, together with the lower semi-continuity of the norm, implies that

$$|v(T)| \le \liminf_{\ell\to\infty} |v_\ell^{N_\ell - 1}|.$$

Hence

$$\limsup_{\ell\to\infty}(-P_\ell^2) \le -\frac{1}{2}\|u_0 + Kv(T)\|_B^2 + \frac{1}{2}\|u_0\|_B^2 - \frac{1}{2}|v(T)|^2 + \frac{1}{2}|v_0|^2.$$

Assume for now that T is a point for which [9, Lemma 6] holds. Then

$$\limsup_{\ell\to\infty}(-P_\ell^2) \le -\int_0^T \langle (v' + B(u_0 + Kv))(t), v(t)\rangle dt. \tag{2.9}$$

In fact the above step is the crucial step in the proof as only the sum $(v'+B(u_0+Kv))(t)$ is in the appropriate dual space V_A^*. That the sum is in V_A^* is known since $v=u'$ satisfies (2.6). An attempt to consider the terms separately fails as the duality pairings are not defined. The case when T is a point at which [9, Lemma 6] does not hold is resolved by a limiting argument as in [9, proof of Theorem 4]. See also Lions and Strauss [15]. Furthermore, due to (2.6) tested with v, which is known to be in $L^p(0,T;V_A)$, it can be seen that

$$0 \le \liminf_{\ell\to\infty} \mu \int_0^T \|v_\ell(t)-v(t)\|_{V_A}^p dt \le \limsup_{\ell\to\infty} \mu \int_0^T \|v_\ell(t)-v(t)\|_{V_A}^p dt$$
$$\le \int_0^T \langle f(t),v(t)\rangle dt - \int_0^T \langle A(t)v(t),v(t)\rangle dt - \int_0^T \langle (v'+Bu)(t),v(t)\rangle dt = 0,$$

This concludes the proof of the strong convergence of $\{v_\ell\}_{\ell\in\mathbb{N}}$ under the assumption that A satisfies (2.4).

Consider now what happens under the d-monotonicity assumption, i.e., when A satisfies (2.3). Since α is monotonically increasing, the following inequality can be established:

$$0 \le \int_0^T (\alpha(\|v_\ell(t)\|_{V_A}) - \alpha(\|v(t)\|_{V_A}))(\|v_\ell(t)\|_{V_A} - \|v(t)\|_{V_A}) dt$$
$$\le \int_0^T \langle A_\ell(t)v_\ell(t) - A_\ell(t)v(t), v_\ell(t)-v(t)\rangle dt.$$

The right hand side of this inequality is then equal to the right hand side of (2.7) and hence the above limiting argument can be repeated. As α is monotonically increasing, the integrand in the first integral of the above expression is always non-negative. Hence the limes superior of the first integral in the above expression can only go to zero if $\|v_\ell(t)\|_{V_A}$ converges to $\|v(t)\|_{V_A}$ almost everywhere in $(0,T)$.

Due to the a priori estimate [9, Theorem 3], the sequence $\{\|v_\ell\|_{V_A}^p\}_{\ell\in\mathbb{N}} \subset L^1(0,T)$ is bounded. Let $1 \le q < p$. Let $\theta(s) := s^{p/q}$. Then $\lim_{s\to+\infty} = \frac{\theta(s)}{s} = +\infty$ and

$$\sup_{\ell\in\mathbb{N}} \int_0^T \theta(\|v_\ell(t)\|_{V_A}^q) dt < c.$$

Hence, by the De La Vallée–Poussin theorem (see [3, Theorem 2.4.4]), the sequence $\{\|v_\ell\|_{V_A}^q\}_{\ell\in\mathbb{N}}$ is uniformly integrable. Thus by the Vitali theorem (see e.g. [6, Chapter 4] or [23])

$$\int_0^T \big|\|v_\ell(t)\|_{V_A}^q - \|v(t)\|_{V_A}^q\big| dt \to 0 \quad \text{as} \quad \ell\to\infty.$$

But this means that

$$\|v_\ell\|_{L^q(0,T;V_A)} \to \|v\|_{L^q(0,T;V_A)} \quad \text{as} \quad \ell\to\infty.$$

Here V_A is assumed to be uniformly convex. Hence $L^q(0,T;V_A)$ is uniformly convex due to [12, Theorem 1.15]. Furthermore, since v_ℓ already converges weakly towards v in $L^p(0,T;V_A)$ and hence in $L^q(0,T;V_A)$ as $q<p$, it can be concluded (see, e.g., Brézis [6, Proposition 3.32]) that in fact $v_\ell \to v$ strongly in $L^q(0,T;V_A)$ as $\ell\to\infty$.

We now come to the strong convergence of u_ℓ towards the exact solution u. Without loss of generality let $r \geq 2$. Since $u = u_0 + Kv$, we immediately find

$$\begin{aligned}
&\|u_\ell - u\|_{L^r(0,T;V_A+V_B)} \\
&\leq \|u_\ell - u_0 - Kv_\ell\|_{L^r(0,T;V_A+V_B)} + \|Kv_\ell - Kv\|_{L^r(0,T;V_A+V_B)} \\
&\leq \|u_\ell - u_0 - Kv_\ell\|_{L^2(0,T;V_B)}^{2/r} \|u_\ell - u_0 - Kv_\ell\|_{L^\infty(0,T;V_A+V_B)}^{1-2/r} + \|Kv_\ell - Kv\|_{L^r(0,T;V_A)}.
\end{aligned}$$

In view of [9, Lemma 5], we already know that

$$\|u_\ell - u_0 - Kv_\ell\|_{L^2(0,T;V_B)} \to 0$$

as $\ell \to \infty$. Moreover,

$$\|u_\ell - u_0 - Kv_\ell\|_{L^\infty(0,T;V_A+V_B)} \leq \|u_\ell - u_0\|_{L^\infty(0,T;V_B)} + \|Kv_\ell\|_{L^\infty(0,T;V_A)}$$

is uniformly bounded since $\{u_\ell\}_{\ell\in\mathbb{N}}$ is a bounded sequence in $L^\infty(0,T;V_B)$, $\{v_\ell\}_{\ell\in\mathbb{N}}$ is a bounded sequence in $L^p(0,T;V_A)$, and K is a linear bounded mapping of $L^1(0,T;V_A)$ into $L^\infty(0,T;V_A)$. Finally, we have $v_\ell \to v$ in $L^q(0,T;V_A)$ at least for $q < p$ (see the first part of the proof) and hence

$$\|Kv_\ell - Kv\|_{L^r(0,T;V_A)} \to 0$$

as $\ell \to \infty$. □

If $A = A_0 + A_1$ and $B = B_0 + B_1$ then (1.2) does not necessarily have a unique solution. From the sequence of approximations v_ℓ, a subsequence can be chosen that converges weakly to v, see [9, Theorem 12]. The techniques of Theorem 1 can still be used to prove the same strong convergence results but only for a subsequence of v_ℓ.

It is either needed that $V_A \hookrightarrow V_B$ or that H is an intermediate space of class $\underline{\mathcal{K}}_\eta(V^*, V_A)$ for some $\eta \in (0,1)$ in the sense of Lions and Peetre (see [14, 16, 24]), which just means that an interpolatory inequality holds for $V_A \hookrightarrow H \hookrightarrow V^*$. The following assumptions on $A_1(t): V_A \to V_A^*$ and $B_1(t): V_B \to V_A^*$ will be required. Notice that B_1 is indeed assumed to map into V_A^*.

Assumption (A_1)**.** $\{A_1(t)\}_{t\in[0,T]}$ is a family of operators $A_1(t): V_A \to V_A^*$ such that for all $v \in V_A$ the mapping $t \mapsto A_1(t)v : [0,T] \to V_A^*$ is continuous for almost all $t \in [0,T]$. There are constants $\varepsilon \in [0,1/4)$, $\kappa \geq 0$, $\lambda_1 \geq 0$, $c > 0$ such that for all $t \in [0,T]$ and all $v \in V_A$

$$\langle A_1(t)v, v\rangle \geq -\varepsilon\mu_A \|v\|_{V_A}^p - \kappa|v|^2 - \lambda_1, \quad \|A_1(t)v\|_{V_A^*} \leq c\left(1 + \|v\|_{V_A}^{p-1}\right).$$

Moreover, there is a constant $\delta_A \in (0, p-1]$ such that for any $R > 0$ there is a constant $\alpha_A = \alpha_A(R) > 0$ and for all $t \in [0,T]$ and all $v, w \in V_A$ with $|v|, |w| \leq R$ there holds

$$\|A_1(t)v - A_1(t)w\|_{V_A^*} \leq \alpha_A(R)\left(1 + \|v\|_{V_A}^{p-1-\delta_A} + \|w\|_{V_A}^{p-1-\delta_A}\right)|v-w|^{\delta_A/p}.$$

Assumption (B_1)**.** $\{B_1(t)\}_{t\in[0,T]}$ is a family of operators $B_1(t): V_B \to V_A^*$ such that for all $v \in V_B$ the mapping $t \mapsto B_1(t)v : [0,T] \to V_A^*$ is continuous for almost all $t \in [0,T]$. There is a constant $c > 0$ such that for all $t \in [0,T]$ and all $v \in V_B$

$$\|B_1(t)v\|_{V_A^*} \leq c\left(1 + \|v\|_{V_B}^{2(p-1)/p}\right).$$

Moreover, for any $R > 0$ there is a constant $\alpha_B = \alpha_B(R) > 0$ and for all $t \in [0,T]$ and all $v, w \in V_B$ with $\|v\|_{V_B}, \|w\|_{V_B} \leq R$ there holds

$$\|B_1(t)v - B_1(t)w\|_{V_A^*} \leq \alpha_B(R)|v-w|^{1-1/p}.$$

If $A=A_0+A_1$ and $B=B_0+B_1$ and Assumptions (A_1) and (B_1) are satisfied and if $\tau_{\max}$ is sufficiently small, then (1.5) has a solution $\{u^n\}_{n=0}^N, \{v^n\}_{n=0}^{N-1} \subset V_m$, due to [9, Theorem 7].

Theorem 2. *In addition to the assumptions of Theorem 1, let Assumptions (A_1) and (B_1) be satisfied, let $\tau_{max,\ell}$ be sufficiently small and assume V_A is compactly embedded in H. Moreover either let $H \in \underline{\mathcal{K}}_\eta(V^*, V_A)$ for some $\eta \in (0,1)$ and assume that the Galerkin scheme can be chosen in such a way that the operator norm in V of the corresponding orthogonal projection of H onto the finite dimensional subspaces is uniformly bounded or let $V_A \hookrightarrow V_B$.*

Then, passing to a subsequence if necessary, the conclusions of Theorem 1 hold even if A_1 and B_1 are different from zero.

Proof. The discrete problem now is

$$\begin{aligned}\langle \hat{v}_\ell'(t), \phi\rangle + \langle A_{0,\ell}(t) v_\ell(t), \phi\rangle + \langle A_{1,\ell}(t) v_\ell(t), \phi\rangle \\ + \langle B_0 u_\ell(t), \phi\rangle + \langle B_{1,\ell}(t) u_\ell(t), \phi\rangle = \langle f_\ell(t), \phi\rangle \quad \forall \phi \in V_{m_\ell},\end{aligned} \tag{2.10}$$

for almost all $t \in (0,T)$. Furthermore, the second equation in (1.3) is equivalent to

$$\begin{aligned}\int_0^T \langle v'(t) + B_0 u(t), w(t)\rangle dt + \int_0^T \langle A_0(t) v(t), w(t)\rangle dt + \int_0^T \langle A_1(t) v(t), w(t)\rangle dt \\ + \int_0^T \langle B_1 u(t), w(t)\rangle dt = \int_0^T \langle f(t), w(t)\rangle dt \quad \forall w \in L^p(0,T;V_A).\end{aligned} \tag{2.11}$$

Here $v=u'$ and u is a solution to (1.2), which is known to exist due to [9, Theorem 12], and $v_{\ell'}$ is a subsequence of $\{v_\ell\}_{\ell\in\mathbb{N}}$ that converges weakly to v in $L^p(0,T;V_A)$ as $\ell \to \infty$. The subsequence exists due to [9, Theorem 12]. Then the argument proceeds exactly as in the proof of Theorem 1 except that after testing (2.10) by $v_{\ell'}(t)$ and integrating from 0 to T the term $P_{\ell'}^1$ contains additionally $-\int_0^T \langle A_{1,\ell'}(t), v_{\ell'}(t)\rangle dt$. But this converges to $-\int_0^T \langle A_1(t) v(t), v(t)\rangle dt$ as $\ell' \to \infty$ due to [9, (3.15) in the proof of Theorem 12]. Furthermore $P_{\ell'}^2$ additionally contains $\int_0^T \langle B_{1,\ell'}(t) u_{\ell'}(t), v_{\ell'}(t)\rangle dt$. This converges to $\int_0^T \langle B_1(t) u(t), v(t)\rangle dt$ as $\ell' \to \infty$ due to [9, (3.16) in the proof of Theorem 12]. Finally, $P_{\ell'}^3$ contains additionally the term $\int_0^T \langle A_1(t) v(t) - A_{1,\ell'}(t) v(t), v_{\ell'}(t) - v(t)\rangle dt$. But this goes to zero as $\ell' \to \infty$ since due to Assumption (A_1) together with Lebesgue's theorem it can be shown that $A_{1,\ell'} v \to Av$ in $L^{p^*}(0,T;V_A^*)$ as $\ell' \to \infty$. Now testing (2.11) by $v \in L^p(0,T;V_A)$, the relevant limes superiors are seen to be zero and the rest of the proof is as before. □

3. Vibrating membrane with nonlinear damping The wave equation with a nonlinear damping term comes from Andreassi and Torelli [1], Fattorini [10, p. 165], and Lions [13, pp. 38ff., 62ff., 222ff.].

Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. Consider the following equation for some real number $p \geq 2$,

$$u_{tt} + |u_t|^{p-2} u_t - \Delta u = f \quad \text{in} \quad \Omega \times (0,T), \tag{3.1}$$

where $-\Delta$ denotes the Laplace operator and $|u_t| = \left|\frac{\partial u}{\partial t}(x,t)\right|$ is the absolute value of the partial derivative of u with respect to t evaluated at the point (x,t). Let

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \tag{3.2}$$

and consider the initial conditions

$$\begin{aligned} u(0,\cdot) &= u_0 \quad \text{in} \quad \Omega, \\ u_t(0,\cdot) &= v_0 \quad \text{in} \quad \Omega. \end{aligned} \tag{3.3}$$

It will be shown that piecewise constant prolongations of numerical approximations (1.5) converge strongly to the weak solutions to (3.1) by employing Theorem 1. The aim is to choose spaces $V_A \subset H \subset V_A^*$ and $V_B \subset H \subset V_B^*$ and define operators A and B such that (3.1) can be interpreted as (1.2). Let $H = L^2(\Omega)$,

$$V_A = L^p(\Omega) \overset{d}{\hookrightarrow} H = L^2(\Omega) \overset{d}{\hookrightarrow} V_A^* = L^{p^*}(\Omega)$$

and

$$V_B = H_0^1(\Omega) \overset{d}{\hookrightarrow} H = L^2(\Omega) \overset{d}{\hookrightarrow} V_B^* = H^{-1}(\Omega).$$

This gives the two required Gelfand triples and we have $V = V_A \cap V_B = L^p(\Omega) \cap H_0^1(\Omega)$. To obtain the weak formulation of (3.1) corresponding to the abstract formulation (1.2), let A and B be defined as follows:

$$\langle Av, w \rangle := \int_\Omega |v|^{p-2} v w dx \quad \forall v, w \in V_A, \tag{3.4}$$

$$\langle Bv, w \rangle := \int_\Omega \nabla v \cdot \nabla w dx \quad \forall v, w \in V_B. \tag{3.5}$$

Note that A and B are independent of time. For v in $L^p(\Omega)$ it can be seen that

$$\int_\Omega \left| |v|^{p-2} v \right|^{p^*} dx = \int_\Omega \left(|v|^{p-1} \right)^{p^*} dx = \int_\Omega |v|^p dx < \infty,$$

where, as before, $p^* = \frac{p}{p-1}$. Hence, $|v|^{p-2}v$ is in $L^{p^*}(\Omega)$.

Lemma 3. *The operator A defined by* (3.4) *satisfies Assumption* (A_0).

Proof. Since

$$\langle Av, v \rangle = \int_\Omega |v|^{p-2} v v dx = \|v\|_{V_A}^p,$$

A is coercive. Observe that for any $y, z \in \mathbb{R}$ and $p \geq 2$,

$$(|y|^{p-2} y - |z|^{p-2} z)(y - z) \geq \mu |y - z|^p,$$

with $\mu = 2^{-(p-2)}$. Hence

$$\langle Au - Av, u - v \rangle = \int_\Omega (|u|^{p-2} u - |v|^{p-2} v)(u - v) dx \geq \mu \int_\Omega |u - v|^p dx = \mu \|u - v\|_{V_A}^p.$$

Thus A satisfies (2.4) and so it is uniformly monotone. Using Hölder's inequality, we obtain the growth estimate

$$\|Av\|_{V_A^*} = \sup_{w \in V_A, w \neq 0} \frac{1}{\|w\|_{V_A}} \int_\Omega \left| |v|^{p-2} v w \right| dx \leq \left(\int_\Omega \left| |v|^{p-2} v \right|^{p^*} dx \right)^{\frac{1}{p^*}} = \|v\|_{V_A}^{p-1}.$$

It still remains to prove that the operator A is hemicontinuous. But this is an immediate consequence of the continuity of the function $y \mapsto |y|^{p-2} y$ mapping $\mathbb{R}$ into $\mathbb{R}$ and the Lebesgue dominated convergence theorem. □

Lemma 4. *The operator B defined by* (3.5) *satisfies Assumption* (B_0).

Proof. Clearly B is bounded, linear and symmetric. Furthermore $\langle Bv,v\rangle = \|v\|_{V_B}^2$. So $B: V_B \to V_B^*$ is strongly positive. Thus Assumption (B_0) is also satisfied with $\mu_B = 1$ and $c_B = 1$. □

It has been demonstrated that Assumptions (A_0) and (B_0) are satisfied. Assumption (IC) can easily be fulfilled. But since $L^p(\Omega)$ is not continuously embedded in $H_0^1(\Omega)$, fulfilling Assumption ($V_m, \mathbb{I}$) requires coupling of maximum time step size and the spatial discretization parameter. Hence the conclusions of Emmrich and Thalhammer [9, Theorem 4] apply, in particular there is a unique solution to (1.2) with A given by (3.4) and B given by (3.5). Furthermore by Theorem 1, the piecewise constant prolongations v_ℓ converge, as $\ell \to \infty$, to the first time derivative of the weak solution u to (3.1) with initial data (3.2) and boundary values (3.3) in $L^p(0,T;L^p(\Omega)) = L^p(\Omega \times (0,T))$, since A_0 satisfies the required additional monotonicity assumption. Moreover, u_ℓ converges to u in $L^r(0,T;L^p(\Omega)+H_0^1(\Omega))$ for any $r \in [1,\infty)$. Note that, e.g., in the one- and two-dimensional case $L^p(\Omega)+H_0^1(\Omega) = L^p(\Omega)$.

4. Another example Let Ω be a bounded domain in $\mathbb{R}^d$ with the boundary $\partial\Omega$ of class $\mathcal{C}^2$ or let Ω be a convex polygonal domain in $\mathbb{R}^2$. Let ν be the outward unit normal vector for the domain Ω. Let μ and λ be positive constants. We consider the equation

$$u_{tt} - \mu\Delta u_t - \operatorname{div}(\sigma(\nabla u)) + \lambda\Delta^2 u = g \quad \text{in} \quad \Omega \times (0,T) \tag{4.1}$$

together with the initial and boundary data

$$\begin{aligned} u(\cdot,0) &= u_0, \qquad & u_t(\cdot,0) &= v_0 \quad \text{in} \quad \Omega \\ u &= 0, & \nu \cdot \nabla u &= 0 \quad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{4.2}$$

We note that the equation above corresponds to a model for shape-memory alloys that has been developed and studied by Pego [17], Friesecke and McLeod [11], Ball et. al. [4], Roubíček [21], Rajagopal and Roubíček [19], Arndt, Griebel and Roubíček [2] and others. The list of references is far from complete. There are other mathematical models for transformations in shape-memory alloys which are not considered by this article, see Plecháč and Roubíček [18] and the references to other models cited therein. When modelling martensitic transformations, $\sigma: \mathbb{R}^d \to \mathbb{R}^d$ will arise as the Gâteaux derivative of a double well potential, that is itself a fourth order polynomial. Hence σ should be allowed to grow as a polynomial of third degree. As mentioned in the introduction, the framework of [9] and hence of this article does not cover this situation.

Assume that there exists $K > 0$ such that for all $y, z \in \mathbb{R}^d$

$$|\sigma(y)| \le K(1+|y|) \quad \text{and} \quad |\sigma(y) - \sigma(z)| \le K|y-z|. \tag{4.3}$$

As before, suitable choices of V_A, V_B and H need to be made and Assumptions (A_0), (B_0) and (B_1) need to be verified. Let

$$V_A = H_0^1(\Omega), \quad V_B = H_0^2(\Omega), \quad H = L^2(\Omega).$$

To obtain the weak formulation of (4.1), let $A = A_0$ (hence $A_1 = 0$) and $B = B_0 + B_1$ and define A_0, B_0 and B_1 as follows: for any $v, w \in V_A$, let

$$\langle A_0 v, w\rangle := \mu \int_\Omega \nabla v \cdot \nabla w dx. \tag{4.4}$$

For any $v,w \in V_B$, let

$$\langle B_0 v, w\rangle := \lambda \int_\Omega \Delta v \Delta w dx \tag{4.5}$$

and

$$\langle B_1 v, w\rangle := \int_\Omega \sigma(\nabla v)\cdot \nabla w dx. \tag{4.6}$$

It can be seen by standard arguments that A_0 fulfils Assumption (A_0) with $p=2$. At this point it is important to note that since the boundary of Ω is of class $\mathcal{C}^2$ or that since Ω is a convex polygonal domain in $\mathbb{R}^2$, the $H^2(\Omega)$ norm is on $H_0^2(\Omega)$ equivalent to the norm $\|\Delta\cdot\|_{L^2(\Omega)}$. Indeed, in the case when the boundary of Ω is twice continuously differentiable, this follows from the classical regularity results of Agmon, Douglis and Nirenberg. In the case of convex polygonal Ω, this is a consequence of results by Grisvard (see, e.g., Attouch et al. [3, p. 277] and the references cited therein). We therefore equip V_B with the norm $\|\cdot\|_{V_B} := \|\Delta\cdot\|_{L^2(\Omega)}$. Hence B_0 poses no difficulties as it is clearly symmetric, linear, bounded and strongly positive. Thus Assumption (B_0) is also satisfied. It only remains to verify Assumption (B_1).

Lemma 5. *The operator B_1 satisfies Assumption (B_1), provided that σ satisfies* (4.3).

Proof. The definition of B_1 and the embedding $H_0^2(\Omega)\hookrightarrow H_0^1(\Omega)$ immediately imply

$$\|B_1 v\|_{V_A^*} \le \left(\int_\Omega |\sigma(\nabla v)|^2 dx\right)^{1/2} \le c\left(1+\int_\Omega |\nabla v|^2 dx\right)^{1/2} \le c(1+\|v\|_{V_B}),$$

where assumption (4.3) has been used and where the constant c depends on K, d and Ω. Thus the first estimate of Assumption (B_1) is satisfied. Similarly, we find

$$\|B_1 v - B_1 w\|_{V_A^*} \le \left(\int_\Omega |\sigma(\nabla v)-\sigma(\nabla w)|^2 dx\right)^{1/2} \le K\left(\int_\Omega |\nabla v - \nabla w|^2 dx\right)^{1/2}.$$

Since

$$\|\nabla v\|_{L^2(\Omega)}^2 \le c\|\Delta v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \quad \forall v\in H_0^2(\Omega),$$

we obtain

$$\|B_1 v - B_1 w\|_{V_A^*} \le c\max(\|v\|_{V_B},\|w\|_{V_B})^{1/2}|v-w|^{1/2},$$

which is the second estimate in Assumption (B_1). □

It has been demonstrated that Assumptions (A_0), (B_0) and (B_1) are satisfied (and Assumption (A_1) is not needed as $A_1=0$). Assumption (IC) is easily satisfied and if Assumption ($V_m,\mathbb{I}$) is taken into account then all the conclusions of Emmrich and Thalhammer [9, Theorem 12] apply. In particular there is a weak solution to (4.1) in the sense that (1.2) holds in $L^2(0,T;V^*)$ with $A=A_0$ and $B=B_0+B_1$ given by (4.4), (4.5), (4.6) respectively and $V=V_A\cap V_B = H_0^2(\Omega)$. Furthermore, by Theorem 2, a subsequence of the piecewise constant prolongations $\{v_\ell\}_{\ell\in\mathbb{N}}$ converges strongly, as $\ell'\to\infty$, to the first time derivative of a weak solution u to (4.1) with the initial and boundary data (4.2) in $L^2(0,T;H_0^1(\Omega))$ since A_0 is strongly monotone, since $L^2(\Omega)\in\underline{\mathcal{K}}_{1/2}(H^{-1}(\Omega),H_0^1(\Omega))$, $H^{-1}(\Omega)\in\underline{\mathcal{K}}_{1/2}(H^{-2}(\Omega),L^2(\Omega))$ and thus $L^2(\Omega)\in\underline{\mathcal{K}}_{2/3}(H^{-2}(\Omega),H_0^1(\Omega))$, and if the $L^2(\Omega)$-orthogonal projection onto the Galerkin spaces is $H_0^2(\Omega)$-stable. Furthermore, the approximations u_ℓ converge strongly towards u in $L^r(0,T;H_0^1(\Omega))$ for any $r\in[1,\infty)$ as $\ell\to\infty$.

5. Numerical results In this section convergence of numerical approximations to solutions of the vibrating membrane equation with nonlinear damping (3.1) is presented. The numerical computations have been performed using the deal.II finite element library [5].

The square domain $\Omega = (-L,L)\times(-L,L)\subset\mathbb{R}^2$ is considered together with the initial conditions

$$u_0(x) = \cos(\pi x_1)\cos(\pi x_2)\chi_{\{|x|\le\frac{1}{2}\}},\quad v_0(x) = 0. \tag{5.1}$$

The finite element spaces V_m that form a Galerkin scheme for the space $V = V_A\cap V_B = L^p(\Omega)\cap H_0^1(\Omega)$ are defined as follows: the sides of the domain are subdivided into 2^m equal intervals (thus covering the domain in 2^{2m} non-overlapping squares with sides $2L/(2^m)$ long). The basis functions are the $\mathcal{Q}^1$ finite elements (functions that are continuous and piecewise linear in each of the two spatial variables). Taking four squares, of the squares into which the domain is subdivided, and forming a larger square gives the support of each of the basis functions. All the basis functions are zero on the boundary of Ω. There are $M_m = (2^m+1)^2 - 2^{m+2}$ vertices in the interior of the domain and hence M_m basis functions $\{\varphi_i\}_{i=1}^{M_m}$.

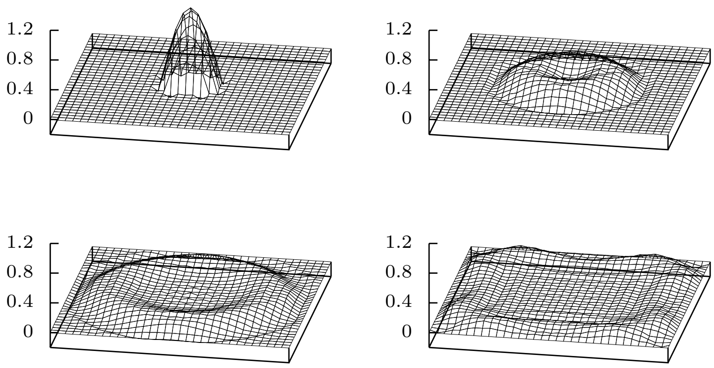

FIGURE 5.1. *Numerical solution, $p = 3$*

Constant time step τ is used. Let m be fixed (hence M is used instead of M_m to simplify notation). For $n = 0,\ldots,N-1$ let $u^n, v^n \in V^m$ be given by

$$u^n = \sum_{i=1}^{M} u_i^n \varphi_i \quad\text{and}\quad v^n = \sum_{i=1}^{M} v_i^n \varphi_i.$$

Let $\boldsymbol{u}^n := [u_1^n,\ldots,u_M^n]^T$ and $\boldsymbol{v}^n := [v_1^n,\ldots,v_M^n]^T$. The first equation in (1.5) gives

$$\frac{1}{\tau}\boldsymbol{G}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) - \boldsymbol{G}\boldsymbol{v}^n = \boldsymbol{0},\quad n = 0,1,\ldots,N-1,$$

where

$$G_{ij} = \int_\Omega \varphi_i\varphi_j\,dx,\quad\text{for}\quad i,j = 1,\ldots,M.$$

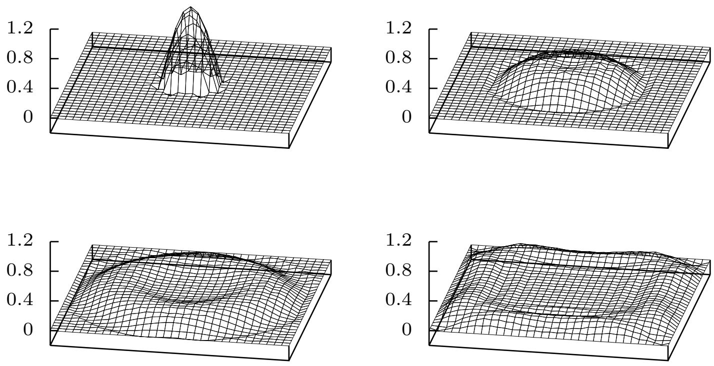

FIGURE 5.2. *Numerical solution,* $p=10$

Due to linear independence of φ_i the matrix $\boldsymbol{G}$ is invertible. This implies that

$$\boldsymbol{u}^{n+1}=\boldsymbol{v}^n+\tau\boldsymbol{u}^n,\quad n=0,1,\dots,N-1.$$

Thus at each step, $\boldsymbol{u}^{n+1}$ can easily be obtained from $\boldsymbol{v}^n$ and $\boldsymbol{u}^n$. Calculating $\boldsymbol{v}^n$ from $\boldsymbol{v}^{n-1}$ and $\boldsymbol{u}^n$ (both known) is the harder part where a nonlinear system must be solved. The second equation in (1.5) gives

$$\frac{1}{\tau}\boldsymbol{G}(\boldsymbol{v}^n-\boldsymbol{v}^{n-1})+\boldsymbol{A}(\boldsymbol{v}^n)+\boldsymbol{S}\boldsymbol{u}^{\boldsymbol{n}}=\boldsymbol{f}^n,\quad n=1,\dots,N \tag{5.2}$$

where the vectors $\boldsymbol{f}^n$ are defined by

$$\boldsymbol{f}^n:=[f_1^n,\dots,f_M^n]^T\quad\text{with}\quad f_i^n:=\int_\Omega f^n\varphi_i dx,$$

the system matrix $\boldsymbol{S}$ is

$$S_{ij}:=\int_\Omega\nabla\varphi_i\nabla\varphi_j dx,\quad\text{where}\quad i,j=1,\dots,M,$$

and the (nonlinear) vector valued function $\boldsymbol{A}:\mathbb{R}^M\to\mathbb{R}^M$ is defined as

$$\boldsymbol{A}(\boldsymbol{v}^n)=[\alpha_1(\boldsymbol{v}^n),\dots,\alpha_M(\boldsymbol{v}^n)]^T$$

with

$$\alpha_j(\boldsymbol{v}^n):=\int_\Omega\left|\sum_{i=1}^M v_i^n\varphi_i\right|^{p-2}\sum_{i=1}^M v_i^n\varphi_i\varphi_j dx.$$

Finally, let

$$\boldsymbol{F}(\boldsymbol{v}^n):=\boldsymbol{G}\boldsymbol{v}^n+\tau\boldsymbol{A}(\boldsymbol{v}^n)-\tau\boldsymbol{f}^{\boldsymbol{n}}+\tau\boldsymbol{S}\boldsymbol{u}^n-\boldsymbol{G}\boldsymbol{v}^{n-1}.$$

Then solving (5.2) is equivalent to solving $\boldsymbol{F}(\boldsymbol{v}^n) = \boldsymbol{0}$. Due to Emmrich and Thalhammer [9, Theorem 2] it is known that the system has a unique solution. The Newton method is now used iteratively, starting with an initial guess $\boldsymbol{v}^{n,0}$ (which can be chosen to be, for example, $\boldsymbol{v}^{n-1}$). Each step amounts to solving

$$\boldsymbol{J}(\boldsymbol{v}^{n,k})(\boldsymbol{v}^{n,k+1} - \boldsymbol{v}^{n,k}) = -\boldsymbol{F}(\boldsymbol{v}^{n,k}) \quad \text{for} \quad k = 0, 1, \dots, k_\epsilon \ ,$$

where $\boldsymbol{J}(\boldsymbol{v}^{n,k})$ is the Jacobian matrix of $\boldsymbol{F}(\boldsymbol{v}^{n,k})$, and where k_ϵ is such that $\boldsymbol{F}(\boldsymbol{v}^{n,k_\epsilon})$ is sufficiently close to zero in a suitable norm. Observe that while in general the Jacobian matrix $\boldsymbol{J}$ can be approximated using finite difference method, this is very computationally intensive. Here $\boldsymbol{J}(\boldsymbol{x})$ can be calculated for $\boldsymbol{x} \in \mathbb{R}^M$:

$$J_{ij}(\boldsymbol{x}) := \frac{\partial F_i}{\partial x_j} = G_{ij} + \tau \int_\Omega (p-1) \left| \sum_{k=1}^{M} x_k \varphi_k \right|^{p-2} \varphi_i \varphi_j \, dx, \quad \text{where} \quad i,j = 1, \dots, M.$$

Hence, starting with given u^0 and v^0, the numerical approximations $\{u^n\}_{n=0}^N \subset V^m$ and $\{v^n\}_{n=0}^{N-1} \subset V^m$ are obtained. Choosing a subsequence of m and N, labelled ℓ and using (2.2), the piecewise constant prolongation $v_\ell \in L^p(0,T;V_A) = L^p(\Omega \times (0,T))$ can be obtained. The time steps have to be chosen such that Assumption $(V_m, \mathbb{I})$ is satisfied. In particular we need $c_{V_B \leftarrow V_A}(m_\ell)^2 \tau_{\max}(\mathbb{I}_\ell) \to 0$ as $\ell \to \infty$, where $c_{V_B \leftarrow V_A}(m_\ell)$ comes from (2.1). In our particular case, where $V_A = L^p(\Omega)$, $V_B = H_0^1(\Omega)$ and V_m are the $\mathcal{Q}^1$ finite element spaces, the constant can be estimated as $c_{V_B \leftarrow V_A}(m) \le c2^m$. Thus we require that $2^{-2m_\ell}\tau_\ell \to 0$ as $\ell \to 0$. The convergence results of Section 3 then apply to v_ℓ and u_ℓ. Note that $V_A + V_B = L^p(\Omega)$ in the two-dimensional case considered here.

The calculations have been performed with the initial data (5.1). The numerical solutions obtained can be seen in Figures 5.1 and 5.2 for $p = 3$ and $p = 10$ respectively. The initial condition is in the top left corner of the figures. As expected, different behaviour can be observed depending on p. This is best seen by comparing the plots in top right corner of the two figures.

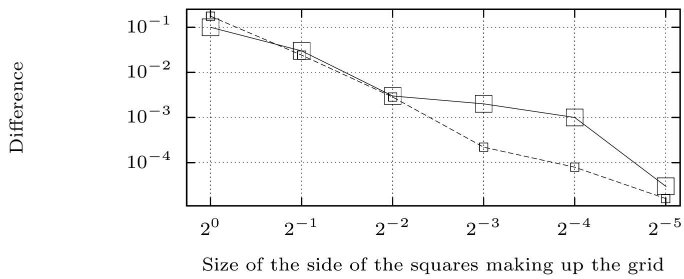

FIGURE 5.3. *Difference between the numerical solution u_ℓ calculated with a given grid and a solution on a very fine grid measured in the $L^p(\Omega \times (0,T))$ norm, calculated with $p = 10$. Dashed line and small squares correspond to calculation where τ_ℓ was chosen such that $2^{2m_\ell}\tau_\ell \to 0$ as $\ell \to 0$ is satisfied. Large squares and solid line correspond to τ_ℓ proportional to 2^{-m}.*

Convergence can also be observed. Figure 5.3 displays the difference between the numerical solution calculated on a very fine grid and the numerical solutions on coarser grids (while decreasing τ proportionally with grid refinement). The difference is measured in the $L^p(\Omega\times(0,T))$ norm. We note that convergence is observed even if the condition $2^{-2m_\ell}\tau_\ell \to 0$ as $\ell \to 0$ is not satisfied. The solid line in Figure 5.3 corresponds to τ proportional to 2^{-m} and convergence can still be observed.

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