A class of integro-differential equations arising in nonlinear elastodynamics: Existence via time discretisation‡

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Abstract. A general model for the description of, e.g., an extensible beam is studied, incorporating weak, viscous, and strong as well as Balakrishnan–Taylor damping. Convergence of a sequence of approximate solutions, resulting from a time discretisation scheme, towards a weak solution is shown. This also proves existence of a weak solution.

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1. Introduction

A variety of problems in elastodynamics require to take into account nonlinear as well as nonlocal phenomena. In this paper, we study a model involving a spatially nonlocal semilinearity and (nonlinear) damping terms for a scalar quantity (e.g., the mathematical description of the transverse motion of a viscoelastic, extensible beam in a viscous medium).

The class of equations we shall consider reads as

$$u_{tt} + \alpha\Delta^2 u - \left(\beta + \gamma\int_\Omega |\nabla u|^2 dx + \delta\left|\int_\Omega \nabla u\cdot\nabla u_t dx\right|^{q-2}\int_\Omega \nabla u\cdot\nabla u_t dx\right)\Delta u$$
$$+\zeta u + \kappa u_t - \lambda\Delta u_t + \mu\Delta^2 u_t = f \quad \text{in } \Omega\times(0,T)\,, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^d$ ($d\in\mathbb{N}$) is a bounded domain, $T>0$ the time under consideration, and $u:\overline{\Omega}\times[0,T]\to\mathbb{R}$ the unknown. Moreover, α (elasticity coefficient) and γ (extensibility coefficient) are positive parameters, whereas δ (Balakrishnan–Taylor damping coefficient), λ (viscous damping coefficient) and μ (strong damping coefficient) are nonnegative. For the real parameters β (axial force coefficient), ζ, and κ (weak damping coefficient), however, no sign conditions are imposed. The exponent q is in $[2,\infty)$. The term with δ is known to be a damping of Balakrishnan–Taylor type, which appears in the context of the control of a flexible structure (see Bass & Zes [5]).

For (1.1), we shall consider the following types of boundary conditions:

- hinged boundary described by the condition

$$u = \Delta u = 0 \quad \text{on } \partial\Omega\times(0,T)\,, \tag{1.2}$$

- clamped boundary described by the condition

$$u = n\cdot\nabla u = 0 \quad \text{on } \partial\Omega\times(0,T)\,, \tag{1.3}$$

where n denotes the outer normal. For a discussion of several types of boundary conditions, we also refer to Timoshenko & Woinowsky-Krieger [31]. Hinged boundary conditions describe, e.g., an edge of a plate that is simply supported such that the deflection and the bending moments along the edge are zero, whereas clamped means a built-in edge (see [31, p. 83]).

Note that boundary conditions prescribed for second order spatial derivatives are natural boundary conditions whereas the other conditions (Dirichlet or Neumann boundary conditions) here are essential boundary conditions since spatial derivatives of fourth order occur in the equation. One may also consider mixed boundary conditions with the above conditions on parts of $\partial\Omega$ only. For readability, we only focus on the two cases above.

We, finally, supplement the equation by initial conditions

$$u(\cdot,0) = u_0\,,\quad u_t(\cdot,0) = v_0 \quad \text{in } \Omega\,. \tag{1.4}$$

The equation (1.1) is a model for the description of vibrations of an extensible (viscoelastic) beam. The equation also arises in connection with other applications such as axial flow-induced oscillations, fluttering pipes conveying a fluid, the fluttering of a panel (see Marsden & Hughes [24]) or the dynamic buckling of imperfect viscoelastic shallow arches (see Huang & Nachbar [18]).

The first reference to (1.1) but without damping terms is Woinowsky-Krieger [32]. The main idea was to assume a nonlinear dependence of the strain on the gradient of the displacement. (For a derivation of (1.1) but with weak damping only, we also refer to Mettler & Weidenhammer [25]. For a well readable derivation of the related equations for thin plates, together with an exposition of the corresponding mathematical theory, see also Langenbach [20].) The first mathematical studies, dealing with the question of well-posedness (in the sense of weak solutions) and regularity, are due to Dickey [11] and Ball [2]. The long-term behaviour of solutions to (1.1) (with weak damping only, i.e., with $\delta = \lambda = \mu = 0$ but $\kappa > 0$) and the existence of exponential attractors has been considered in Eden & Milani [13], see also Taboada & You [28]. The case of weak damping has also been considered in Clark et al. [10] (existence and uniqueness of strong solutions in the case of moving boundary conditions). Well-posedness (in the sense of weak solutions), regularity, and asymptotic behaviour of solutions in the case of strong damping ($\mu > 0$, $\kappa \neq 0$) as well as damping of Balakrishnan–Taylor type ($\delta > 0$, $q = 2$) has been studied in Ball [3]. For the analysis of a numerical method such as a finite difference method for this equation, see also Choo & Chung [9] and the references cited therein. Damping with $\lambda > 0$ but $\mu = 0$ and damping of Balakrishnan–Taylor type ($\delta > 0$, $q \geq 2$) has been analysed in You [33], where existence and uniqueness of mild solutions as well as the existence of absorbing sets and inertial manifolds is shown. For the model equations with Balakrishnan–Taylor damping with exponent ($\delta > 0$, $q \geq 2$), see also Bass & Zes [5]. For $\delta = \lambda = 0$ but $\mu > 0$, the existence of inertial manifolds has been studied in Bianchi & Marzocchi [6]. In [17], a modification of the equations (without damping) leading to a time-delay equation has been considered in order to incorporate viscoelastic effects, see also Zaraï & Tatar [34] for a similar nonlocal-in-time generalisation.

All the afore-mentioned contributions deal with the one-dimensional case only. In Biler [7], the exponential decay in the case of weak damping has been shown for (1.1) in an abstract setting, allowing also the multi-dimensional case. A similar type of equations (including the extensibility term and including a semilinear friction term but excluding other damping terms) has been studied, in the multi-dimensional case, in Kouémou-Patcheu [19].

Our aim in this paper is a twofold: On the one hand, we wish to generalise the known existence results. On the other, we want to prove convergence of a simple temporal semi-discretisation, which is a modification of the well-known leap-frog scheme. Indeed, we prove convergence of a sequence of approximate solutions and show that the limit is a weak solution.

In Ball [2], the case $\delta = \zeta = \kappa = \lambda = \mu = 0$ has been treated in one spatial dimension for both, hinged and clamped boundary conditions. Existence of weak solutions (in the sense of Definition 2.1 below) has been proven via proving (weak and weak*) convergence of a Galerkin approximation (using monotonicity arguments). Moreover continuous dependence on the initial data and thus uniqueness has been shown. In Ball [3], the case $\zeta = \lambda = 0$ is dealt with. The Balakrishnan–Taylor damping term is now incorporated but with exponent $q = 2$ in the case of strong damping ($\mu > 0$). The Balakrishnan–Taylor damping with arbitrary exponent $q \geq 2$ in the case of viscous damping ($\lambda > 0$) with $\zeta = \kappa = \mu = 0$ has been studied in You [33], where, in particular, existence of a mild solution is shown in the case of hinged boundary conditions.

In the present work, we prove existence of weak solutions incorporating the Balakrishnan–Taylor damping with arbitrary $q \in [2, \infty)$ even if $\lambda > 0$ or if $\mu > 0$. If $\lambda = \mu = 0$, we are only able to incorporate the Balakrishnan–Taylor damping term with exponent $q = 2$.

Furthermore, we are able to prove strong convergence results, which are of interest from the numerical point of view. In contrast to Ball [2, 3], we do not investigate the question of regularity or existence of classical solutions and thus do not derive any error estimates.

The interpretation of the Balakrishnan–Taylor damping term is somewhat intrigued if $\lambda = \mu = 0$ since $u_t(\cdot, t)$ $(t \in (0, T))$ can only be shown to take values in $L^2(\Omega)$. If $\lambda > 0$ or $\mu > 0$, we obtain additional regularity of the solution such that the term is well-defined. If $\lambda = \mu = 0$, we may, at least in the case of clamped boundary conditions, carry out integration by parts and interpret the term as

$$\delta \left| \int_\Omega \Delta u \, u_t \, dx \right|^{q-2} \int_\Omega \Delta u \, u_t \, dx \, \Delta u \, .$$

Unfortunately, we are not able to prove existence if $\lambda = \mu = 0$ and $q > 2$. If $q = 2$, the Balakrishnan–Taylor damping term can be written as

$$-\frac{\delta}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 \, dx \, \Delta u \, , \tag{1.5}$$

which can be understood in the weak sense with respect to both, d/dt and Δ (see below).

For the analysis (and numerical analysis) of abstract evolution equations of second order with damping, we also refer to [4, 14, 15, 16, 22, 27, 35]. However, the equations considered there do not cover the present problem. Especially in the case $\lambda = \mu = 0$, the a priori estimates and thus the proof of convergence differ from that of Emmrich & Thalhammer [14, 15]. Moreover, the argumentation with respect to the nonlinearity here is much different from that of [14, 15] since here the damping term (except for the Balakrishnan–Taylor damping) is linear but the operator B is a nonlinear potential operator.

The outline of the paper is as follows: In Section 2, we provide the functional analytic setting for the treatment of the problem under consideration together with some preliminary results about the operators appearing. The time discretisation is then introduced in Section 3, where we also prove existence of solutions to the time discrete problem as well as a priori estimates. Finally, in Section 4, we state the main result and prove existence of a weak solution via convergence of (prolongations of) the time discrete solutions.

2. Time continuous problem and preliminaries

Let $\Omega \subset \mathbb{R}^d$ $(d \in \mathbb{N})$ be a bounded domain with sufficiently smooth boundary (see below) and let $T > 0$.

We rely upon the standard notation for Lebesgue and Sobolev spaces and spaces of continuous or continuously differentiable functions (see, e.g., Brézis [8]). For a Banach space X, we denote by $\|\cdot\|_X$ its norm. Inner product and norm in $L^2(\Omega)$, however, will be denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. By $L^p(0, T; X)$ $(p \in [1, \infty])$, we denote the usual Bochner–Lebesgue space, equipped with the standard norm. By $H^1(0, T; X)$, we denote the usual Sobolev space of functions in $L^2(0, T; X)$ which possess a distributional time derivative that also is in $L^2(0, T; X)$. We always associate with a function $u = u(x, t)$ the abstract function $u = u(t)$ via $[u(t)](x) = u(x, t)$. With this, we have in particular $L^2(0, T; L^2(\Omega)) = L^2(\Omega \times (0, T))$. By $\mathscr{AC}([0, T]; X)$,

$\mathscr{C}([0,T];X)$ and $\mathscr{C}_w([0,T];X)$, we denote the spaces of abstract functions mapping $[0,T]$ into X, which are absolutely continuous on $[0,T]$, continuous on $[0,T]$ with respect to the strong and weak topology of X, respectively. Finally, $\langle\cdot,\cdot\rangle$ denotes the duality pairing. For a number $\eta\in\mathbb{R}$, we set

$$\eta_- := \max(0,-\eta) = \begin{cases} 0 & \text{if } \eta\ge 0\,, \\ |\eta| & \text{if } \eta<0\,. \end{cases}$$

By $c>0$, we denote a generic constant.

In the case of hinged boundary conditions (1.2), let $V := H^1_0(\Omega)\cap H^2(\Omega)$. Let $\partial\Omega\in\mathscr{C}^2$ or let Ω be a convex polygonal domain in $\mathbb{R}^2$. Then $\|\Delta\cdot\|$ is equivalent to $\|\cdot\|_{H^2(\Omega)}$, and we equip V with the norm $\|\cdot\|_V = \|\Delta\cdot\|$. This follows from the classical regularity results of Agmon, Douglis and Nirenberg and of Grisvard (see, e.g., Attouch et al. [1, p. 277] and the references cited therein).

In the case of clamped boundary conditions (1.3), let $V := H^2_0(\Omega)$. We assume that $\partial\Omega\in\mathscr{C}^{0,1}$ and again equip V with the norm $\|\cdot\|_V = \|\Delta\cdot\|$.

We recall the Friedrichs inequality: there is a constant $c_F>0$ such that

$$\|u\|\le c_F\,\|\nabla u\| \quad \text{for all } u\in H^1_0(\Omega)\,. \tag{2.1}$$

Moreover, with integration by parts and the Cauchy–Schwarz inequality, there holds

$$\|\nabla u\|^2\le \|u\|\,\|\Delta u\| \quad \text{for all } u\in H^1_0(\Omega)\cap H^2(\Omega)\,, \tag{2.2}$$

which immediately implies

$$\|\nabla u\|\le c_F\,\|\Delta u\| \quad \text{for all } u\in H^1_0(\Omega)\cap H^2(\Omega)\,. \tag{2.3}$$

Since $\|u\|^2 = (u,u)\le \|u\|_{H^{-1}(\Omega)}\|\nabla u\|$ for $u\in H^1_0(\Omega)$, we also obtain

$$\|\nabla u\|^3\le \|u\|_{H^{-1}(\Omega)}\,\|\Delta u\|^2 \quad \text{for all } u\in H^1_0(\Omega)\cap H^2(\Omega)\,. \tag{2.4}$$

Identifying $L^2(\Omega)$ with its dual, we see that $V\subset L^2(\Omega)\subset V^*$ (with V^* denoting the dual of V) forms a Gelfand triple.

The weak formulation of (1.1), equipped either with hinged (1.2) or with clamped (1.3) boundary conditions and supplemented by the initial conditions (1.4) can be shown, in the standard way, to be equivalent to the abstract evolution problem

$$u'' + Au' + Bu + C(u,u') = f \;\text{ in } (0,T)\,, \tag{2.5}$$

$$u(0)=u_0\,,\; u'(0)=v_0\,, \tag{2.6}$$

where the prime denotes the time derivative in the distributional sense. This, however, requires $\delta=0$ if $\lambda=\mu=0$. The case $\delta>0$ with $q=2$ if $\lambda=\mu=0$ will be dealt with below (see Definition 2.1).

The operator A is given by

$$\langle Av,w\rangle = \int_\Omega (\kappa\, vw + \lambda\,\nabla v\cdot\nabla w + \mu\,\Delta v\,\Delta w)\,dx\,, \quad v,w\in V\,,$$

and is a linear bounded mapping of V into V^*. If $\mu=0$ then A is, indeed, a linear bounded mapping of $H^1_0(\Omega)$ into $H^{-1}(\Omega)$, and if $\lambda=\mu=0$ then A is a linear bounded mapping of $L^2(\Omega)$ into $L^2(\Omega)$. This shows that for all $v\in V$

$$\|Av\|_{V^*}\le c\begin{cases} \|v\| & \text{if } \lambda=\mu=0\,, \\ \|\nabla v\| & \text{if } \mu=0\,, \\ \|\Delta v\| & \text{otherwise.} \end{cases} \tag{2.7}$$

By standard arguments, we know that A extends to a linear bounded operator mapping $L^2(0,T;X)$ into $L^2(0,T;V^*)$ with $X = L^2(\Omega)$ if $\lambda = \mu = 0$, $X = H_0^1(\Omega)$ if $\lambda > 0$, $\mu = 0$ and $X = V$ if $\lambda > 0$, $\mu > 0$. Moreover, there holds

$$\langle Av, v\rangle = \kappa \|v\|^2 + \lambda \|\nabla v\|^2 + \mu \|\Delta v\|^2 . \tag{2.8}$$

The operator $B : V \to V^*$ is given by

$$\begin{aligned}&\langle Bu, w\rangle \\ &= \alpha\,(\Delta u, \Delta w) + \left(\beta + \gamma \|\nabla u\|^2\right)(\nabla u, \nabla w) + \zeta\,(u, w)\,, \quad u, w \in V\,.\end{aligned}$$

Occasionally, we employ the obvious decomposition $B = \alpha\, B_1 + \beta\, B_2 + \gamma\, B_3 + \zeta\, B_4$. It is clear that $B_1 : V \to V^*$, $B_2 : H_0^1(\Omega) \to H^{-1}(\Omega)$ and $B_4 : L^2(\Omega) \to L^2(\Omega)$ are linear bounded operators. With respect to $B_3 : V \to V^*$, we see that for all $u, w \in V$

$$|\langle B_3 u, w\rangle| = \|\nabla u\|^2 |(u, \Delta w)| \le \|u\|\, \|\nabla u\|^2\, \|\Delta w\|$$

such that (with (2.2))

$$\|B_3 u\|_{V^*} \le \|u\|^2\, \|\Delta u\|\,,$$

which shows that for all $u \in V$

$$\|Bu\|_{V^*} \le c\,(1 + \|u\|^2)\, \|\Delta u\|\,. \tag{2.9}$$

Moreover, $B_3 : V \to V^*$ fulfills for all $u, \overline{u}, w \in V$ the relation

$$\begin{aligned}&\langle B_3 u - B_3 \overline{u}, w\rangle \\ &= \left(\|\nabla u\|^2 - \|\nabla \overline{u}\|^2\right)(\nabla u, \nabla w) + \|\nabla \overline{u}\|^2 (\nabla(u - \overline{u}), \nabla w) \\ &= -\left(\|\nabla u\| - \|\nabla \overline{u}\|\right)\left(\|\nabla u\| + \|\nabla \overline{u}\|\right)(u, \Delta w) - \|\nabla \overline{u}\|^2 (u - \overline{u}, \Delta w)\,.\end{aligned}$$

This implies by the triangle and Cauchy–Schwarz inequality the estimate

$$\begin{aligned}&\|B_3 u - B_3 \overline{u}\|_{V^*} \\ &\le \|\nabla(u - \overline{u})\|\left(\|\nabla u\| + \|\nabla \overline{u}\|\right)\|u\| + \|\nabla \overline{u}\|^2 \|u - \overline{u}\| \\ &\le c c_F \left(\|\nabla u\|^2 + \|\nabla \overline{u}\|^2\right)\|\nabla(u - \overline{u})\| \\ &\le c c_F^4 \left(\|\Delta u\|^2 + \|\Delta \overline{u}\|^2\right)\|\Delta(u - \overline{u})\|\,,\end{aligned} \tag{2.10}$$

which shows that $B_3 : V \to V^*$ is Lipschitz continuous on bounded subsets and thus continuous. Since V is compactly embedded in $H_0^1(\Omega)$ and $L^2(\Omega)$, the above estimate also shows that $B_3 : V \to V^*$ is strongly continuous, i.e., maps weakly convergent sequences into strongly convergent sequences. With standard arguments, we may show that $B_i : V \to V^*$ $(i = 1, 2, 4)$ extends to a linear bounded mapping of $L^2(0,T;V)$ into $L^2(0,T;V^*)$, $B_3 : V \to V^*$ extends to a (nonlinear) bounded mapping of $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ into $L^2(0,T;V^*)$, and thus $B : V \to V^*$ extends to a bounded mapping of $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ into $L^2(0,T;V^*)$. Note that the Bochner measurability of the image of a Bochner measurable function under B follows from the continuity of $B : V \to V^*$.

Let us introduce the following functionals: For $w \in V$, let

$$\begin{aligned}\Phi_1(w) &= \frac{1}{2}\, \|\Delta w\|^2\,, \quad \Phi_2(w) = \frac{1}{2}\, \|\nabla w\|^2\,, \\ \Phi_3(w) &= \frac{1}{4}\, \|\nabla w\|^4\,, \quad \Phi_4(w) = \frac{1}{2}\, \|w\|^2\,.\end{aligned}$$

Note that these functionals are convex, continuous as mappings of V into $\mathbb{R}$, and thus weakly sequentially lower semicontinuous (see, e.g., Brézis [8, Corollary 3.9 on p. 61]).

Moreover, due to the compact embedding of V in $H_0^1(\Omega)$ and $L^2(\Omega)$, the functionals Φ_2, Φ_3, Φ_4 are also strongly continuous as mappings of V into $\mathbb{R}$.

It is important to observe, however, that the total functional

$$\Phi = \alpha\Phi_1 + \beta\Phi_2 + \gamma\Phi_3 + \zeta\Phi_4$$

need not be convex. Indeed, if $\beta < 0$ (which is the more interesting case) then the functional $\beta\Phi_2 + \gamma\Phi_3$ is a double-well potential in ∇u. Since $z \mapsto \frac{\beta}{2} z^2 + \frac{\gamma}{4} z^4$ $(z \in \mathbb{R})$ is bounded from below by $-\beta_-^2/(4\gamma)$, we find for all $w \in V$

$$\beta\,\Phi_2(w) + \gamma\,\Phi_3(w) \geq -\frac{\beta_-^2}{4\gamma}$$

and thus

$$\begin{aligned}\Phi(w) &\geq \frac{\alpha}{2}\,\|\Delta w\|^2 + \frac{\zeta}{2}\,\|w\|^2 - \frac{\beta_-^2}{4\gamma}\\ &= \alpha\,\Phi_1(w) + \zeta\,\Phi_4(w) - \frac{\beta_-^2}{4\gamma}\,. \end{aligned} \tag{2.11}$$

Remember here that α and γ are positive but β and ζ might be negative. Furthermore, we find with (2.2) and Young's inequality for all $w \in V$

$$\begin{aligned}\Phi_2(w) &\leq \frac{\alpha}{4|\beta|}\,\|\Delta w\|^2 + \frac{|\beta|}{4\alpha}\,\|w\|^2\\ &= \frac{\alpha}{2|\beta|}\,\Phi_1(w) + \frac{|\beta|}{2\alpha}\,\Phi_4(w) \quad \text{if } \beta < 0\,. \end{aligned} \tag{2.12}$$

Finally, the definition of the total potential, together with (2.1) and (2.2), immediately shows that for all $w \in V$

$$\Phi(w) \leq \frac{1}{2}\left(\alpha + c_F^2|\beta| + c_F^4|\zeta|\right)\|\Delta w\|^2 + \frac{\gamma}{4}\,\|\nabla w\|^4\,. \tag{2.13}$$

It can easily be shown that $B_i : V \to V^*$ is the Gâteaux derivative of the potential $\Phi_i : V \to \mathbb{R}$ $(i = 1, \dots, 4)$, i.e.,

$$\langle B_i u, w\rangle = \langle \Phi_i'(u), w\rangle = \lim_{\theta\to 0} \frac{1}{\theta}(\Phi_i(u+\theta w) - \Phi_i(u))$$

for all $u, w \in V$. Since the operators $B_i : V \to V^*$ $(i = 1, \dots, 4)$ are continuous, we find for any function $u \in \mathscr{C}^1([0,T]; V)$

$$\langle B_i u(t), u'(t)\rangle = \langle \Phi_i'(u(t)), u'(t)\rangle = \frac{d}{dt}\Phi_i(u(t)) \quad \text{in } (0,T)\,, \tag{2.14}$$

see also Gajewski et al. [16, Lemma 4.1 on p. 90]. Later we will employ a time discrete analogue of (2.14) for the total potential, which results from the following observations: For all $u, w \in V$ there holds

$$\begin{aligned}&\langle Bu, u-w\rangle = \langle \Phi'(u), u-w\rangle\\ &= \alpha\,(\Delta u, \Delta(u-w)) + \left(\beta + \gamma\,\|\nabla u\|^2\right)(\nabla u, \nabla(u-w)) + \zeta\,(u, u-w)\,. \end{aligned}$$

With the algebraic relations

$$a(a-b) = \frac{1}{2}\,a^2 - \frac{1}{2}\,b^2 + \frac{1}{2}\,(a-b)^2\,, \quad a, b \in \mathbb{R}\,, \tag{2.15}$$

and

$$a^3(a-b) = \frac{1}{4}\,a^4 - \frac{1}{4}\,b^4 + \frac{1}{12}\,(a-b)^4 + \frac{2}{3}\,\left(a + \frac{1}{2}\,b\right)^2 (a-b)^2$$
$$\geq \frac{1}{4}\,a^4 - \frac{1}{4}\,b^4 + \frac{1}{12}\,(a-b)^4\,, \quad a,b \in \mathbb{R}\,,$$

we then obtain (employing (2.12))

$$\begin{aligned}
&\langle Bu, u-w\rangle \\
&\geq \alpha\,(\Phi_1(u) - \Phi_1(w) + \Phi_1(u-w)) + \beta\,(\Phi_2(u) - \Phi_2(w) + \Phi_2(u-w)) \\
&\quad + \gamma\,\left(\Phi_3(u) - \Phi_3(w) + \frac{1}{3}\,\Phi_3(u-w)\right) + \zeta\,(\Phi_4(u) - \Phi_4(w) + \Phi_4(u-w)) \\
&\geq \Phi(u) - \Phi(w) + \Big(\alpha\,\Phi_1(u-w) + \beta\,\Phi_2(u-w) + \frac{\gamma}{3}\,\Phi_3(u-w) + \zeta\,\Phi_4(u-w)\Big) \\
&\geq \Phi(u) - \Phi(w) + \frac{\alpha}{2}\,\Phi_1(u-w) + \frac{\gamma}{3}\,\Phi_3(u-w) + \left(\zeta - \frac{\beta_-^2}{2\alpha}\right)\Phi_4(u-w)\,. \qquad (2.16)
\end{aligned}$$

With respect to the operator C, we recall the discussion on the interpretation of the Balakrishnan–Taylor damping term from the introduction. We firstly may define C, at least in the case of clamped boundary conditions, as a mapping of $V \times L^2(\Omega)$ into V^* via

$$\langle C(u,v), w\rangle = \delta\,|(\Delta u, v)|^{q-2}(\Delta u, v)\,(u, \Delta w)\,, \qquad u, w \in V,\ v \in L^2(\Omega)\,. \qquad (2.17)$$

It follows that

$$\|C(u,v)\|_{V^*} \leq \delta\,\|\Delta u\|^{q-1}\|v\|^{q-1}\|u\|\,. \qquad (2.18)$$

If the second argument is at least in $H_0^1(\Omega)$, we can define C via

$$\langle C(u,v), w\rangle = \delta\,|(\nabla u, \nabla v)|^{q-2}(\nabla u, \nabla v)\,(\nabla u, \nabla w)\,, \qquad u, w \in V,\ v \in H_0^1(\Omega)\,, \qquad (2.19)$$

and obtain (2.17) with integration by parts from (2.19).

We emphasise that in our applications later, namely in the time discrete case as well as in the continuous case with $\lambda > 0$ or $\mu > 0$, we will always have that the second argument is in $H_0^1(\Omega)$.

Commencing with (2.17), we observe that for all $u, \overline{u}, w \in V,\ v, \overline{v} \in L^2(\Omega)$

$$\begin{aligned}
&\langle C(u,v) - C(\overline{u},\overline{v}), w\rangle \\
&= \delta\,\big(|(\Delta u, v)|^{q-2}(\Delta u, v) - |(\Delta\overline{u}, \overline{v})|^{q-2}(\Delta\overline{u}, \overline{v})\big)\,(u, \Delta w) \\
&\quad + \delta\,|(\Delta\overline{u}, \overline{v})|^{q-2}(\Delta\overline{u}, \overline{v})(u - \overline{u}, \Delta w)\,.
\end{aligned}$$

With

$$\big||a|^{q-2}a - |b|^{q-2}b\big| \leq (q-1)\max(|a|,|b|)^{q-2}\,|a-b|\,, \quad a,b \in \mathbb{R}\,,$$

for $q \in [2,\infty)$ we thus come up with

$$\begin{aligned}
&\|C(u,v) - C(\overline{u},\overline{v})\|_{V^*} \\
&\leq c\,\max(\|\Delta u\|\|v\|, \|\Delta\overline{u}\|\|\overline{v}\|)^{q-2}|(\Delta u, v) - (\Delta\overline{u}, \overline{v})| \\
&\quad + c\,\|\Delta\overline{u}\|^{q-1}\|\overline{v}\|^{q-1}\|u - \overline{u}\|\,,
\end{aligned}$$

$$\begin{aligned}&\le c\,\max(\|\Delta u\|\|v\|,\|\Delta\overline{u}\|\|\overline{v}\|)^{q-2}\left(|(\Delta u, v-\overline{v})|+|(\Delta(u-\overline{u}),\overline{v})|\right)\\&\quad+c\,\|\Delta\overline{u}\|^{q-1}\|\overline{v}\|^{q-1}\|u-\overline{u}\|\\&\le c\,\max(\|\Delta u\|\|v\|,\|\Delta\overline{u}\|\|\overline{v}\|)^{q-2}\left(\|\Delta u\|\|v-\overline{v}\|+\|\nabla(u-\overline{u})\|\|\nabla\overline{v}\|\right)\\&\quad+c\,\|\Delta\overline{u}\|^{q-1}\|\overline{v}\|^{q-1}\|u-\overline{u}\|\,, \qquad (2.20)\end{aligned}$$

where, in the last step, we have again employed integration by parts which requires $\overline{v}\in H_0^1(\Omega)$. This also shows, together with

$$|(\Delta u, v)-(\Delta\overline{u},\overline{v})|\le\|\Delta u\|\,\|v-\overline{v}\|+\|\Delta(u-\overline{u})\|\,\|\overline{v}\|\,,$$

that $C: V\times L^2(\Omega)\to V^*$ is Lipschitz continuous on bounded subsets and thus continuous.

It is now clear from the continuity (ensuring that images of Bochner measurable functions are Bochner measurable) and (2.18) (ensuring integrability) that C : $V\times L^2(\Omega)\to V^*$ extends to a (nonlinear) mapping of $L^\infty(0,T;V)\times L^\infty(0,T;L^2(\Omega))$ into $L^\infty(0,T;V^*)$ that is bounded on bounded subsets. The estimate (2.20) moreover shows for a sequence $\{z_\ell\}_{\ell\in\mathbb{N}}$ that $Cz_\ell\to Cz$ in $L^1(0,T;V^*)$ as $\ell\to\infty$ if $z_\ell, z\in L^\infty(0,T;V)\times L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, $\{z_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L^\infty(0,T;V)\times L^\infty(0,T;L^2(\Omega))$ and $z_\ell\to z$ in $L^2(0,T;H_0^1(\Omega))\times L^2(\Omega\times(0,T))$ as $\ell\to\infty$ (see also the proof of Theorem 4.1 below).

Finally, there holds (recalling that $\delta\ge 0$) for all $u\in V$, $v\in H_0^1(\Omega)$

$$\langle C(u,v),v\rangle\ge 0\,. \qquad (2.21)$$

Let us now turn to the more delicate case that $\delta>0$ with $q=2$ but $\lambda=\mu=0$. The starting point is (1.5). For deriving a weak formulation, we may multiply (1.1) by a smooth function in (x,t) with compact support and carry out integration by parts for the Balakrishnan–Taylor damping term (1.5) with respect to both, the time derivative and the Laplacian. This leads to the following definition of an operator $\widetilde{C}$, which acts on time dependent functions and which replaces C in the abstract formulation (2.5): For $u\in L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;V)$, $v\in L^2(\Omega\times(0,T))$ and $w\in L^2(0,T;V)\cap H^1(0,T;L^2(\Omega))\subset\mathscr{C}([0,T];L^2(\Omega))$ with $w(0)=w(T)=0$, let

$$\langle\widetilde{C}(u,v),w\rangle:=\frac{\delta}{2}\int_0^T\|\nabla u(t)\|^2\big((v(t),\Delta w(t))+(\Delta u(t),w'(t))\big)dt\,.$$

It is easy to see that $\widetilde{C}$ is well defined as a mapping of $L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;V)\times L^2(\Omega\times(0,T))$ into the dual of $L^2(0,T;V)\cap H^1(0,T;L^2(\Omega))$. Moreover, if u is sufficiently smooth then integration by parts shows

$$\langle\widetilde{C}(u,u'),w\rangle=-\delta\int_0^T(\nabla u(t),\nabla u'(t))\,(\Delta u(t),w(t))\,dt\,,$$

which is the original term in (1.1).

We are now ready to define an appropriate notion of weak solution to the problem under consideration.

Definition 2.1 *Let $u_0\in V$, $v_0\in L^2(\Omega)$ and $f\in L^2(\Omega\times(0,T))$.*

(i) Let $\delta=0$. A function $u\in\mathscr{C}_w([0,T];V)$ with $u'\in\mathscr{C}_w([0,T];L^2(\Omega))$ and $u''\in L^2(0,T;V^*)$ is said to be a weak solution to (1.1) under the boundary conditions (1.2) or (1.3) , respectively, and subject to the initial conditions (1.4) iff (2.5) holds in the sense of $L^2(0,T;V^*)$ and (2.6) holds in the sense of V and $L^2(\Omega)$, respectively.

(ii) Let $\delta>0$ with $\lambda>0$ or $\mu>0$. In addition to the case (i), $u'\in L^2(0,T;H_0^1(\Omega))$ is required.

(iii) Let $\delta > 0$ with $q = 2$ but $\lambda = \mu = 0$. A function $u \in \mathscr{C}_w([0,T];V)$ with $u' \in \mathscr{C}_w([0,T];L^2(\Omega))$ and $u'' \in L^2(0,T;V^*)$ is then said to be a weak solution iff for all $w \in V$ and $\varphi \in \mathscr{C}_c^\infty(0,T)$

$$\langle u'' + Au + Bu + \widetilde{C}(u,u'), w\varphi\rangle = \langle f, w\varphi\rangle \tag{2.22}$$

and (2.6) holds in the sense of V and $L^2(\Omega)$, respectively.

Note that a function in $\mathscr{C}_w([0,T];V)$ is weakly measurable and thus, in view of the separability of V and the theorem of Pettis (see, e.g., Diestel & Uhl [12, Theorem 2 on p. 42]) also Bochner measurable. Moreover, by the uniform boundedness principle, functions in $\mathscr{C}_w([0,T];V)$ are norm bounded on $[0,T]$. The same argumentation shows that $\mathscr{C}_w([0,T];L^2(\Omega)) \subset L^\infty(0,T;L^2(\Omega))$.

The initial conditions (1.4) are satisfied at least in the sense of weak convergence,

$$u(t) \rightharpoonup u_0 \text{ in } V\,,\ u'(t) \rightharpoonup v_0 \text{ in } L^2(\Omega) \text{ as } t \to 0\,.$$

If $\mu > 0$ then the weak convergence can be replace by strong convergence.

With respect to (iii) in the definition above, note that

$$\begin{aligned}
&\langle u'' + Au + Bu + \widetilde{C}(u,u'), w\varphi\rangle \\
&= \int_0^T \langle u''(t) + Au'(t) + Bu(t), w\rangle\, \varphi(t)\, dt \\
&\quad + \frac{\delta}{2}\int_0^T \|\nabla u(t)\|^2 \big((u'(t),\Delta w)\,\varphi(t) + (\Delta u(t), w)\,\varphi'(t)\big)dt\,.
\end{aligned} \tag{2.23}$$

We should emphasise that even in the case $\delta > 0$ with $q = 2$ but $\lambda = \mu = 0$ the starting point for the numerical scheme is (2.5) (with the original Balarkishnan-Taylor damping term) rather than (2.22).

3. Time discrete problem: existence and a priori estimates

Let $N \in \mathbb{N}$ be given and set $\tau = T/N$, $t_n = n\tau$ $(n = 0, 1, \ldots, N)$. We look for approximations $u^n \approx u(t_n)$ to (2.5) and (2.23), respectively, given by the fully implicit scheme

$$\begin{aligned}
&\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + A\frac{u^{n+1} - u^n}{\tau} + Bu^{n+1} + C\left(u^{n+1}, \frac{u^{n+1} - u^n}{\tau}\right) = f^{n+1}\,, \\
&\qquad n = 1, 2, \ldots, N-1\,,
\end{aligned} \tag{3.1}$$

where $\{f^{n+1}\}$ is a suitable approximation of f and $u^0 \approx u_0$, $v^0 \approx v_0$ are given approximations of the initial values; we set $u^1 := u^0 + \tau v^0$.

In the sequel, we often use

$$v^n := \frac{u^{n+1} - u^n}{\tau}\,, \quad n = 0, 1, \ldots, N-1\,. \tag{3.2}$$

The above scheme can then be rewritten as

$$\begin{aligned}
&\frac{v^n - v^{n-1}}{\tau} + Av^n + Bu^{n+1} + C(u^{n+1}, v^n) = f^{n+1}\,, \\
&n = 1, 2, \ldots, N-1\,.
\end{aligned} \tag{3.3}$$

The following theorem ensures solvability of the numerical scheme.

Theorem 3.1 *Let $u^0, v^0 \in V$ (with $u^1 := u^0 + \tau v^0$) and $\{f^{n+1}\}_{n=1}^{N-1} \subset V^*$. If $\tau > 0$ is sufficiently small such that*

$$2\tau\kappa_- + \tau^2 \left(3\zeta_- + \frac{\beta_-^2}{2\alpha} \right) \le 1 \tag{3.4}$$

then there is a solution $\{u^n\}_{n=2}^{N} \subset V$ to (3.1).

Proof. We prove existence of a solution step-by-step by employing the main theorem on pseudomonotone operators. So let $u^{n-1}, u^n \in V$ be given. We then have to find $u^{n+1} \in V$. Since $u^{n+1} = u^n + \tau v^n$, this is equivalent to finding $v^n \in V$, which is a solution of the operator equation

$$Dv = f^{n+1} \quad \text{in } V^* , \tag{3.5}$$

where

$$Dv := \frac{v - v^{n-1}}{\tau} + Av + B(u^n + \tau v) + C(u^n + \tau v, v) , \quad v \in V . \tag{3.6}$$

In view of the definition and properties of the operators A, B, C studied in the previous section, it is clear that D defined by (3.6) maps V into V^* and is continuous. Moreover, the operator $D : V \to V^*$ is coercive if τ is sufficiently small. This is a consequence of the sign conditions on the parameters of the problem together with (2.15) applied to the discrete time derivative, (2.8), (2.16) (with $w = u^n$ and then dividing by τ), (2.21), and (2.11), which show for sufficiently small τ (see (3.4)) that for all $v \in V$

$$\begin{aligned}
\langle Dv, v\rangle \ge{}& \frac{1}{2\tau} \|v\|^2 - \frac{1}{2\tau} \|v^{n-1}\|^2 + \kappa \|v\|^2 + \frac{1}{\tau} \Phi(u^n + \tau v) - \frac{1}{\tau} \Phi(u^n) \\
&+ \frac{\alpha}{2\tau} \Phi_1(\tau v) + \frac{1}{\tau} \left(\zeta - \frac{\beta_-^2}{2\alpha} \right) \Phi_4(\tau v) \\
\ge{}& \frac{\alpha\tau}{4} \|\Delta v\|^2 + \frac{1}{2\tau} \left(1 - 2\tau\kappa_- - \tau^2 \left(\zeta_- + \frac{\beta_-^2}{2\alpha} \right) \right) \|v\|^2 - \frac{\zeta_-}{2\tau} \|u^n + \tau v\|^2 \\
&- \frac{1}{2\tau} \|v^{n-1}\|^2 - \frac{1}{\tau} \Phi(u^n) - \frac{\beta_-^2}{4\gamma\tau} \\
\ge{}& \frac{\alpha\tau}{4} \|\Delta v\|^2 + \frac{1}{2\tau} \left(1 - 2\tau\kappa_- - \tau^2 \left(3\zeta_- + \frac{\beta_-^2}{2\alpha} \right) \right) \|v\|^2 \\
&- \frac{\zeta_-}{\tau} \|u^n\|^2 - \frac{1}{2\tau} \|v^{n-1}\|^2 - \frac{1}{\tau} \Phi(u^n) - \frac{\beta_-^2}{4\gamma\tau} \\
\ge{}& \frac{\alpha\tau}{4} \|\Delta v\|^2 - \frac{\zeta_-}{\tau} \|u^n\|^2 - \frac{1}{2\tau} \|v^{n-1}\|^2 - \frac{1}{\tau} \Phi(u^n) - \frac{\beta_-^2}{4\gamma\tau} .
\end{aligned}$$

We can split D into the sum of two operators $D_0 : V \to V^*$ and $D_1 : V \to V^*$ defined by

$$\begin{aligned}
D_0 v &:= \frac{v - v^{n-1}}{\tau} + Av - \kappa v + \alpha\, B_1(u^n + \tau v) , \\
D_1 v &:= Dv - D_0 v , \quad v \in V .
\end{aligned}$$

Recalling the definition of A and B_1, it is obvious that $D_0 : V \to V^*$ is monotone. Furthermore, we obtain especially from (2.10) and (2.20) that for all $v, \overline{v} \in V$

$$\begin{aligned}
&\|D_1 v - D_1 \overline{v}\|_{V^*} \\
&\le c(1+\tau)\,\|v-\overline{v}\| + c\tau\left(1 + \|\nabla(u^n + \tau v)\|^2 + \|\nabla(u^n + \tau\overline{v})\|^2\right)\|\nabla(v-\overline{v})\| \\
&\qquad\qquad + c\,\max(\|\Delta(u^n+\tau v)\|\|v\|, \|\Delta(u^n+\tau\overline{v})\|\|\overline{v}\|)^{q-2} \\
&\quad \times (\|\Delta(u^n+\tau v)\|\|v-\overline{v}\| + \tau\,\|\nabla(v-\overline{v})\|\|\nabla\overline{v}\|) + c\tau\,\|\Delta(u^n+\tau\overline{v})\|^{q-1}\|\overline{v}\|^{q-1}\|v-\overline{v}\|\,.
\end{aligned}$$

Since V is compactly embedded in $H_0^1(\Omega)$ and $L^2(\Omega)$, this shows that $D_1 : V \to V^*$ is strongly continuous.

Finally, Brézis' theorem on pseudomonotone operators (see, e.g., Zeidler [35, Thm. 27A]) provides existence of a solution $v =: v^n$ to (3.5) for any $f^{n+1} \in V^*$. □

The following result is an essential prerequisite for the proof of convergence.

Theorem 3.2 *Let $u^0, v^0 \in V$ and $\{f^{n+1}\}_{n=1}^{N-1} \subset L^2(\Omega)$ and let $\{u^n\}_{n=2}^{N} \subset V$ be any solution to (3.1). If*

$$\tau < \tau_0 := \left(1 + 2\kappa_- + 3T\zeta_- + \frac{T\beta_-^2}{2\alpha}\right)^{-1} \tag{3.7}$$

then there holds for all $n = 1, 2, \dots, N-1$

$$\begin{aligned}
&\|v^n\|^2 + \sum_{j=1}^n \|v^j - v^{j-1}\|^2 + 2\tau\lambda\sum_{j=1}^n \|\nabla v^j\|^2 + 2\tau\mu\sum_{j=1}^n \|\Delta v^j\|^2 + \alpha\,\|\Delta u^{n+1}\|^2 \\
&\qquad\qquad + \frac{\alpha}{2}\sum_{j=1}^n \|\Delta(u^{j+1}-u^j)\|^2 + \frac{\gamma}{6}\sum_{j=1}^n \|\nabla(u^{j+1}-u^j)\|^4 \\
&\le c\Big(\|u^0\|^2 + \|v^0\|^2 + \|\Delta(u^0+\tau\,v^0)\|^2 + \|\nabla(u^0+\tau\,v^0)\|^4 + \tau\sum_{j=1}^n \|f^{j+1}\|^2 + 1\Big),
\end{aligned}$$

where c only depends on c_F, T, τ_0, and the coefficients of the problem. Furthermore, there holds

$$\tau\sum_{j=1}^n \left\|\frac{v^j - v^{j-1}}{\tau}\right\|_{V^*}^2 \le C\,,$$

where $C > 0$ depends on the right-hand side of the first a priori estimate and is bounded on bounded subsets.

Proof. We test (3.3) with $v^n = (u^{n+1} - u^n)/\tau$. It is important to take into account that, by construction, $v^n \in V$ $(n = 0, 1, \dots, N-1)$. With (2.15), (2.8), (2.16) and (2.21), we immediately find for $n = 1, 2, \dots, N-1$

$$\begin{aligned}
&\frac{1}{2\tau}\left(\|v^n\|^2 - \|v^{n-1}\|^2 + \|v^n - v^{n-1}\|^2\right) + \kappa\,\|v^n\|^2 + \lambda\,\|\nabla v^n\|^2 + \mu\,\|\Delta v^n\|^2 \\
&\quad + \frac{1}{\tau}\,\Phi(u^{n+1}) - \frac{1}{\tau}\,\Phi(u^n) \\
&\quad + \frac{1}{\tau}\left(\frac{\alpha}{2}\,\Phi_1(u^{n+1}-u^n) + \frac{\gamma}{3}\,\Phi_3(u^{n+1}-u^n) + \left(\zeta - \frac{\beta_-^2}{2\alpha}\right)\Phi_4(u^{n+1}-u^n)\right) \\
&\le (f^{n+1}, v^n) \le \frac{1}{2}\,\|f^{n+1}\|^2 + \frac{1}{2}\,\|v^n\|^2\,.
\end{aligned}$$

Multiplying by 2τ, summing up, and recalling the definition of the potentials leads to

$$\|v^n\|^2 + \sum_{j=1}^n \|v^j - v^{j-1}\|^2 + 2\tau\kappa \sum_{j=1}^n \|v^j\|^2 + 2\tau\lambda \sum_{j=1}^n \|\nabla v^j\|^2 + 2\tau\mu \sum_{j=1}^n \|\Delta v^j\|^2 + 2\Phi(u^{n+1})$$
$$+ \frac{\alpha}{2} \sum_{j=1}^n \|\Delta(u^{j+1} - u^j)\|^2 + \frac{\gamma}{6} \sum_{j=1}^n \|\nabla(u^{j+1} - u^j)\|^4 + \left(\zeta - \frac{\beta_-^2}{2\alpha}\right) \sum_{j=1}^n \|u^{j+1} - u^j\|^2$$
$$\le \|v^0\|^2 + 2\Phi(u^1) + \tau \sum_{j=1}^n \|f^{j+1}\|^2 + \tau \sum_{j=1}^n \|v^j\|^2 .$$

Remember here again that there is no sign condition for β, ζ, κ and that λ, μ may vanish.

With (2.11), we obtain

$$\|v^n\|^2 + \sum_{j=1}^n \|v^j - v^{j-1}\|^2 + 2\tau\lambda \sum_{j=1}^n \|\nabla v^j\|^2 + 2\tau\mu \sum_{j=1}^n \|\Delta v^j\|^2 + \alpha \|\Delta u^{n+1}\|^2$$
$$+ \frac{\alpha}{2} \sum_{j=1}^n \|\Delta(u^{j+1} - u^j)\|^2 + \frac{\gamma}{6} \sum_{j=1}^n \|\nabla(u^{j+1} - u^j)\|^4$$
$$\le \|v^0\|^2 + 2\Phi(u^1) + \tau \sum_{j=1}^n \|f^{j+1}\|^2 + \frac{\beta_-^2}{2\gamma}$$
$$+ \tau \left(1 + 2\kappa_- + \tau\left(\zeta_- + \frac{\beta_-^2}{2\alpha}\right)\right) \sum_{j=1}^n \|v^j\|^2 + \zeta_- \|u^{n+1}\|^2 .$$

For $\Phi(u^1) = \Phi(u^0 + \tau v^0)$, we employ the estimate (2.13). For the term with u^{n+1} on the right-hand side of the foregoing estimate, we use the relation

$$u^{n+1} = u^0 + \tau \sum_{j=1}^n v^j ,$$

which gives

$$\|u^{n+1}\|^2 \le 2\|u^0\|^2 + 2\tau^2 \left\| \sum_{j=1}^n v^j \right\|^2 \le 2\|u^0\|^2 + 2\tau T \sum_{j=1}^n \|v^j\|^2 .$$

Finally, a discrete Gronwall-type argument proves the first assertion.

With respect to the second estimate asserted, we obtain from (3.3) together with the definition of A, (2.9) and (2.18) (recalling that $v^n \in V$ for all $n = 0, 1, \ldots, N-1$ and thus both the interpretations of $C(\cdot,\cdot)$ coincide)

$$\left\| \frac{v^n - v^{n-1}}{\tau} \right\|_{V^*}$$
$$\le \|f^{n+1}\|_{V^*} + \|Av^n\|_{V^*} + \|Bu^{n+1}\|_{V^*} + \|C(u^{n+1}, v^n)\|_{V^*}$$
$$\le c\Big(\|f^{n+1}\| + \|v^n\| + \lambda \|\nabla v^n\| + \mu \|\Delta v^n\| + \big(1 + \|u^{n+1}\|^2\big) \|\Delta u^{n+1}\|$$
$$+ \|u^{n+1}\| \|\Delta u^{n+1}\|^{q-1} \|v^n\|^{q-1}\Big) ,$$

which yields

$$\tau \sum_{j=1}^{n} \left\| \frac{v^j - v^{j-1}}{\tau} \right\|_{V^*}^2$$
$$\leq c\Big(\tau \sum_{j=1}^{n} \|f^{j+1}\|^2 + \max_{j=1,\dots,n} \|v^j\|^2 + \tau\lambda \sum_{j=1}^{n} \|\nabla v^j\|^2 + \tau\mu \sum_{j=1}^{n} \|\Delta v^j\|^2$$
$$+ \max_{j=1,\dots,n} \|\Delta u^j\|^6 + \max_{j=1,\dots,n} \|\Delta u^j\|^{2q} \max_{j=1,\dots,n} \|v^j\|^{2(q-1)} + 1\Big).$$

This, together with the first assertion, proves the second one. □

Note that (3.7) is more restrictive than (3.4).

4. Convergence towards and existence of a weak solution

In the sequel, we consider a sequence of time grids with constant step sizes $\tau_\ell = T/N_\ell$ corresponding to a sequence $\{N_\ell\}_{\ell\in\mathbb{N}}$ of natural numbers with $N_\ell \to \infty$ as $\ell \to \infty$. We assume that

$$\sup_{\ell\in\mathbb{N}} \tau_\ell < \tau_0 \tag{4.1}$$

with τ_0 given by (3.7). For readability, we often omit writing out the dependence of a quantity on the actual time grid, e.g., we often only write t_n and u^n instead of $t_{n,\ell}$ and u^n_ℓ.

With the time discrete solutions $\{u^n\}_{n=0}^{N_\ell}$ and $\{v^n\}_{n=0}^{N_\ell-1}$ (recall here (3.2)) corresponding to the time grid with step size τ_ℓ, we associate the following prolongations on $[0,T]$:

$$u_\ell(t) := \begin{cases} 0 & \text{if } t \in [0, \tau_\ell/2]\,, \\ u^{n+1} & \text{if } t \in ((n-1/2)\tau_\ell, (n+1/2)\tau_\ell]\ (n = 1,2,\dots,N_\ell-1)\,, \\ 0 & \text{if } t \in (T-\tau_\ell/2, T]\,; \end{cases}$$

$$v_\ell(t) := \begin{cases} 0 & \text{if } t \in [0, \tau_\ell/2]\,, \\ v^{n} & \text{if } t \in ((n-1/2)\tau_\ell, (n+1/2)\tau_\ell]\ (n = 1,2,\dots,N_\ell-1)\,, \\ 0 & \text{if } t \in (T-\tau_\ell/2, T]\,; \end{cases}$$

$$\hat{v}_\ell(t) := \begin{cases} v^0 & \text{if } t \in [0, \tau_\ell/2]\,, \\ \dfrac{v^n - v^{n-1}}{\tau_\ell}(t - (n+1/2)\tau_\ell) + v^n & \text{if } t \in ((n-1/2)\tau_\ell, (n+1/2)\tau_\ell] \\ & (n = 1,2,\dots,N_\ell-1)\,, \\ v^{N_\ell-1} & \text{if } t \in (T-\tau_\ell/2, T]\,. \end{cases}$$

Whereas the functions u_ℓ, v_ℓ are piecewise constant, the function $\hat{v}_\ell$ is piecewise linear and continuous and thus differentiable in the weak sense. By construction, we have $u_\ell, v_\ell, \hat{v}_\ell \in L^\infty(0,T;V)$.

For the right-hand side $f \in L^2(\Omega\times(0,T))$, we restrict our considerations to the approximation

$$f^{n+1} := \frac{1}{\tau_\ell}\int_{(n-1/2)\tau_\ell}^{(n+1/2)\tau_\ell} f(t)\,dt\,, \quad n = 1,2,\dots,N_\ell-1\,, \tag{4.2}$$

and define

$$f_\ell(t) := \begin{cases} 0 & \text{if } t \in [0, \tau_\ell/2]\,, \\ f^{n+1} & \text{if } t \in ((n-1/2)\tau_\ell, (n+1/2)\tau_\ell]\ (n = 1,2,\dots,N_\ell-1)\,, \\ 0 & \text{if } t \in (T-\tau_\ell/2, T]\,. \end{cases}$$

For an integrable function $w = w(t)$, we introduce the antiderivative

$$(Kw)(t) := \int_0^t w(s)\,ds\,.$$

Obviously, K maps, in particular, $L^2(0,T;V)$ into $\mathscr{C}([0,T];V)$.

We recall that Y is an intermediate space of class $\underline{\mathscr{K}}_\eta(L^2(\Omega),V)$ with $\eta \in (0,1)$ in the sense of Lions and Peetre (following Lions & Peetre [21, pp. 27ff.] or, equivalently, of class J_η following Lunardi [23, pp. 27f.], see also Tartar [29, pp. 123ff.]) if $V \hookrightarrow Y \hookrightarrow L^2(\Omega)$ and

$$\|w\|_Y \le c\,\|\Delta w\|^\eta \|w\|^{1-\eta} \quad \text{for all } w \in V\,. \tag{4.3}$$

An example is given by $Y = H_0^1(\Omega)$ with (2.2).

The main result of the present paper reads as follows:

Theorem 4.1 *Let $u_0 \in V$, $v_0 \in L^2(\Omega)$ and $f \in L^2(\Omega \times (0,T))$. Then there is a weak solution u to (1.1), supplemented by (1.2) or (1.3) and by (1.4), in the sense of Definition 2.1, if $\delta = 0$ as well as if $\delta > 0$ and $\lambda > 0$ as well as if $\delta > 0$ and $\mu > 0$ as well as if $\delta > 0$ and $q = 2$ but $\lambda = \mu = 0$. In the case that $\mu > 0$, the weak solution u is in $\mathscr{C}([0,T];V)$ with $u' \in \mathscr{C}([0,T];L^2(\Omega)) \cap L^2(0,T;V)$.*

Consider the time discretisation (3.1) on a sequence of time grids with step sizes τ_ℓ ($\ell \in \mathbb{N}$, $\tau_\ell \to 0$ as $\ell \to \infty$) satisfying (4.1), with the approximation (4.2) and with approximations $\{u_\ell^0\}_{\ell\in\mathbb{N}}, \{v_\ell^0\}_{\ell\in\mathbb{N}}$ such that

$$u_\ell^0 \in V,\ u_\ell^0 \to u_0 \text{ in } V \text{ as } \ell \to \infty\,, \tag{4.4}$$

$$v_\ell^0 \in V,\ v_\ell^0 \to v_0 \text{ in } L^2(\Omega)\,,\ \tau_\ell\,\|\Delta v_\ell^0\| \to 0 \text{ as } \ell \to \infty\,. \tag{4.5}$$

If $\delta = 0$ as well as if $\delta > 0$ and $q = 2$ but $\lambda = \mu = 0$ then there is a subsequence (denoted by ℓ') such that $u_{\ell'}$ converges weakly in $L^\infty(0,T;V)$ and strongly in $L^r(0,T;Y)$ for any $r \in [1,\infty)$ and any intermediate space $Y \in \underline{\mathscr{K}}_\eta(L^2(\Omega),V)$ with $\eta \in (0,1)$ towards u as $\ell \to \infty$, $v_{\ell'}$ and $\hat{v}_{\ell'}$ converge weakly* in $L^\infty(0,T;L^2(\Omega))$ and strongly in $L^s(0,T;Z)$ for any $s \in [1,\infty)$ and any Banach space Z with $L^2(\Omega) \overset{c}{\hookrightarrow} Z \hookrightarrow V^*$ towards u', and $\hat{v}'_{\ell'}$ converges weakly in $L^2(0,T;V^*)$ towards u''.*

If $\lambda > 0$ then, in addition, $v_{\ell'}$ and $\hat{v}_{\ell'}$ converge weakly in $L^2(0,T;H_0^1(\Omega))$ and strongly in $L^2(\Omega \times (0,T))$ towards u'.

If $\mu > 0$ then, in addition, $v_{\ell'}$ and $\hat{v}_{\ell'}$ converge weakly in $L^2(0,T;V)$ and strongly in $L^2(0,T;H_0^1(\Omega))$ towards u'.

We remark that the better approximation of the initial value $v_0 \in L^2(\Omega)$ fulfilling the condition (4.5) is always possible, since V is dense in $L^2(\Omega)$. Moreover, we may choose, e.g., $Y = H_0^1(\Omega)$ and $Z = H^{-1}(\Omega)$.

The proof of the preceding theorem will be prepared by the following proposition.

Proposition 4.2 *Under the assumptions of Theorem 4.1, there is a subsequence (denoted by ℓ') and an element $u \in L^\infty(0,T;V) \cap \mathscr{C}([0,T];L^2(\Omega)) \subset \mathscr{C}_w([0,T];V)$ with $u' \in L^\infty(0,T;L^2(\Omega)) \cap \mathscr{C}([0,T];V^*) \subset \mathscr{C}_w([0,T];L^2(\Omega))$ and $u'' \in L^2(0,T;V^*)$ as well as with $u(0) = u_0$ and $u'(0) = v_0$ such that, as $\ell \to \infty$,*

$$u_{\ell'} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;V)\,,$$

$$u_{\ell'} \to u \text{ in } L^r(0,T;Y) \text{ for any } r \in [1,\infty)$$

and any intermediate space $Y \in \underline{\mathscr{K}}_\eta(L^2(\Omega),V)$ with $\eta \in (0,1)$,

$$u_\ell - u_0 - Kv_\ell \to 0 \text{ in } L^2(0,T;V)\,,$$

$$
\begin{aligned}
&v_{\ell'} \overset{*}{\rightharpoonup} u' \textit{ and } \hat v_{\ell'} \overset{*}{\rightharpoonup} u' \textit{ in } L^\infty(0,T;L^2(\Omega))\,,\\
&v_{\ell'} \to u' \textit{ and } \hat v_{\ell'} \to u' \textit{ in } L^s(0,T;Z) \textit{ for any } s\in[1,\infty)\\
&\textit{and any Banach space } Z \textit{ with } L^2(\Omega) \overset{c}{\hookrightarrow} Z \hookrightarrow V^*,\\
&v_\ell - \hat v_\ell \to 0 \textit{ in } L^2(\Omega\times(0,T))\,,\quad Kv_\ell - K\hat v_\ell \to 0 \textit{ in } \mathscr{C}([0,T];L^2(\Omega))\,,\\
&\hat v'_{\ell'} \rightharpoonup u'' \textit{ in } L^2(0,T;V^*)\,.
\end{aligned}
$$

Moreover, if $\lambda > 0$ *then there holds* $u' \in L^2(0,T;H_0^1(\Omega))$ *and*

$$
v_{\ell'}, \hat v_{\ell'} \rightharpoonup u' \textit{ in } L^2(0,T;H_0^1(\Omega))\,,\quad v_{\ell'}, \hat v_{\ell'} \to u' \textit{ in } L^2(\Omega\times(0,T))\,.
$$

If $\mu > 0$ *then* $u \in \mathscr{C}([0,T];V)$, $u' \in L^2(0,T;V)\cap\mathscr{C}([0,T];L^2(\Omega))$ *and*

$$
v_{\ell'} \rightharpoonup u' \textit{ in } L^2(0,T;V)\,,\quad v_{\ell'} \to u' \textit{ in } L^2(0,T;H_0^1(\Omega))\,.
$$

Proof. We firstly observe that the right-hand sides in the a priori estimates of Theorem 3.2 are uniformly bounded in view of (4.4), (4.5) (together with (2.3)) and since (with (4.2))

$$
\tau_\ell \sum_{j=1}^{N_\ell-1} \|f^{j+1}\|^2 \le \|f\|^2_{L^2(\Omega\times(0,T))}\,.
$$

By construction, we find

$$
\|u_\ell\|_{L^\infty(0,T;V)} = \sup_{n=2,\dots,N_\ell} \|\Delta u^n\|\,,
$$

and we obtain the boundedness of $\{u_\ell\}$ in $L^\infty(0,T;V)$ from the first estimate in Theorem 3.2. By standard arguments (corollary of the theorem of Banach–Alaoglou–Bourbaki, see Brézis [8, Corollary 3.30 on p. 76], together with the separability of $L^1(0,T;V)$), there is thus an element $u\in L^\infty(0,T;V)$ and a subsequence (denoted by ℓ') such that $u_{\ell'}$ converges weakly* in $L^\infty(0,T;V)$ towards u.

Similarly, the sequences $\{v_\ell\}$ and $\{\hat v_\ell\}$ are bounded in $L^\infty(0,T;L^2(\Omega))$. Hence, we have elements $v,\hat v\in L^\infty(0,T;L^2(\Omega))$ and can take a common subsequence of the previous subsequence (still denoted by ℓ') such that $v_{\ell'}$ and $\hat v_{\ell'}$ converge weakly* in $L^\infty(0,T;L^2(\Omega))$ towards v and $\hat v$, respectively.

Since

$$
\begin{aligned}
&\|v_\ell - \hat v_\ell\|^2_{L^2(\Omega\times(0,T))}\\
&= \int_0^{\tau_\ell/2} \|v^0\|^2\,dt + \sum_{j=1}^{N_\ell-1} \int_{(j-1/2)\tau_\ell}^{(j+1/2)\tau_\ell} \left\|\frac{v^j - v^{j-1}}{\tau_\ell}\right\|^2 (t-(j+1/2)\tau_\ell)^2\,dt\\
&\qquad + \int_{T-\tau_\ell/2}^{T} \|v^{N_\ell-1}\|^2\,dt\\
&= \frac{\tau_\ell}{2}\|v^0\|^2 + \frac{\tau_\ell}{3}\sum_{j=1}^{N_\ell-1}\|v^j - v^{j-1}\|^2 + \frac{\tau_\ell}{2}\|v^{N_\ell-1}\|^2\,,
\end{aligned}
$$

the assumptions (4.4), (4.5) together with the first a priori estimate in Theorem 3.2 show that $v_\ell - \hat v_\ell$ converges strongly in $L^2(\Omega\times(0,T))$ towards zero. Therefore, the limits v and $\hat v$ must coincide.

It immediately follows that

$$
\|Kv_\ell - K\hat v_\ell\|_{\mathscr{C}([0,T];L^2(\Omega))} \le c\,\|v_\ell - \hat v_\ell\|_{L^2(\Omega\times(0,T))} \to 0\,.
$$

Since (with the usual convention $\sum_{j=1}^{0} \equiv 0$)

$$(Kv_\ell)(t) = \begin{cases} 0 & \text{if } t \in [0, \tau_\ell/2]\,, \\ \tau_\ell \sum_{j=1}^{n-1} v^j + (t-(n-1/2)\tau_\ell)v^n & \text{if } t \in ((n-1/2)\tau_\ell, (n+1/2)\tau_\ell] \\ & (n = 1, 2, \dots, N_\ell - 1)\,, \\ \tau_\ell \sum_{j=1}^{N-1} v^j & \text{if } t \in (T - \tau_\ell/2, T] \end{cases}$$

$$= \begin{cases} 0 & \text{if } t \in [0, \tau_\ell/2]\,, \\ u^n - u^1 + (t-(n-1/2)\tau_\ell)\dfrac{u^{n+1}-u^n}{\tau_\ell} & \text{if } t \in ((n-1/2)\tau_\ell, (n+1/2)\tau_\ell] \\ & (n = 1, 2, \dots, N_\ell - 1)\,, \\ u^{N_\ell} - u^1 & \text{if } t \in (T - \tau_\ell/2, T]\,, \end{cases}$$

we find (using $u^1 - u^0 = \tau_\ell v^0$)

$$\begin{aligned} &\|u_\ell - u_0 - Kv_\ell\|^2_{L^2(0,T;V)} \\ &= \int_0^{\tau_\ell/2} \|\Delta u_0\|^2\, dt \\ &\quad + \sum_{j=1}^{N_\ell-1} \int_{(j-1/2)\tau_\ell}^{(j+1/2)\tau_\ell} \left\| \Delta \left(u^0 - u_0 + \tau_\ell v^0 - \frac{t-(j+1/2)\tau_\ell}{\tau_\ell} (u^{j+1} - u^j) \right) \right\|^2 dt \\ &\quad + \int_{T-\tau_\ell/2}^{T} \left\| \Delta \left(u^0 - u_0 + \tau_\ell v^0 - u^{N_\ell} \right) \right\|^2 dt \\ &\le c\tau_\ell \|\Delta u_0\|^2 + c \left\| \Delta \left(u^0 - u_0 + \tau_\ell v^0 \right) \right\|^2 + c\tau_\ell \sum_{j=1}^{N_\ell-1} \|\Delta(u^{j+1} - u^j)\|^2 + c\tau_\ell \|\Delta u^{N_\ell}\|^2 \,. \end{aligned}$$

The assumptions (4.4), (4.5) and the first a priori estimate in Theorem 3.2 now show that the right-hand side of the foregoing estimate tends to zero as $\ell \to \infty$.

In order to prove $u = u_0 + Kv$, we consider the relation between Kv_ℓ and Kv. Let $g \in L^1(0,T; L^2(\Omega))$. We then observe that

$$\begin{aligned} \int_0^T (g(t), (Kv_\ell)(t) - (Kv)(t))\, dt &= \int_0^T \int_0^t (g(t), v_\ell(s) - v(s))\, ds\, dt \\ = \int_0^T \int_s^T (g(t), v_\ell(s) - v(s))\, dt\, ds &= \int_0^T (G(s), v_\ell(s) - v(s))\, ds\,, \end{aligned}$$

where $G(s) := \int_s^T g(t)\, dt \in L^\infty(0,T;L^2(\Omega))$. Since $v_{\ell'}$ converges weakly* in $L^\infty(0,T;L^2(\Omega))$ towards v, the last term of the foregoing identity, passing to the subsequence if necessary, tends to zero. This shows, however, that also $Kv_{\ell'}$ converges weakly* in $L^\infty(0,T;L^2(\Omega))$ towards Kv. This, together with the weak* convergence of $u_{\ell'}$ in $L^\infty(0,T;V) \subset L^\infty(0,T;L^2(\Omega))$ towards u and the strong convergence of $u_\ell - u_0 - Kv_\ell$ in $L^2(0,T;V) \subset L^2(\Omega \times (0,T))$ towards zero, immediately shows

$$\int_0^T (g(t), u(t) - u_0 - (Kv)(t))\, dt = 0 \quad \text{for all } g \in L^1(0,T;L^2(\Omega))\,.$$

We, therefore, have $u = u_0 + Kv$. Since $v \in L^\infty(0,T;L^2(\Omega))$, this also shows that at least $u \in L^\infty(0,T;V) \cap \mathscr{AC}([0,T];L^2(\Omega))$ and that $u' = v$ almost everywhere in $(0,T)$ as well as in the weak sense. Moreover, if u is in $L^\infty(0,T;V)$ as well as in $\mathscr{C}([0,T];L^2(\Omega))$, one can easily prove (employing the reflexivity of $V \subset L^2(\Omega)$) that then $u \in \mathscr{C}_w([0,T];V)$.

With respect to the convergence of the time derivatives (in the weak sense) of $\hat{v}_\ell$, we observe that, because of the second a priori estimate in Theorem 3.2,

$$\begin{aligned}\|\hat{v}_\ell'\|^2_{L^2(0,T;V^*)} &= \sum_{j=1}^{N_\ell-1} \int_{(j-1/2)\tau_\ell}^{(j+1/2)\tau_\ell} \left\| \frac{v^j - v^{j-1}}{\tau_\ell} \right\|^2_{V^*} dt \\ &= \tau_\ell \sum_{j=1}^{N_\ell-1} \left\| \frac{v^j - v^{j-1}}{\tau_\ell} \right\|^2_{V^*}\end{aligned}$$

is uniformly bounded. By standard arguments (see Brézis [8, Theorem 3.18 on p. 69] and use the reflexivity of $L^2(0,T;V^*)$), we thus have an element $w \in L^2(0,T;V^*)$ and a subsequence (of the previous subsequence and still denoted by ℓ') such that $\hat{v}_{\ell'}'$ converges weakly in $L^2(0,T;V^*)$ towards w. The definition of the weak derivative, together with the weak* convergence of $\hat{v}_\ell$ in $L^\infty(0,T;L^2(\Omega)) \subset L^2(0,T;V^*)$ towards v, shows that $w = v' = u''$. Since then $u' = v = v_0 + Kw \in \mathscr{C}([0,T];V^*)$ but also $u' = v \in L^\infty(0,T;L^2(\Omega))$, we also get $u' \in \mathscr{C}_w([0,T];L^2(\Omega))$.

With a generalisation of the theorem of Lions–Aubin (see Roubíček [27, Lemma 7.7 on p. 194]), we conclude from the boundedness of $\{\hat{v}_\ell\}$ in $L^\infty(0,T;L^2(\Omega))$ and of $\{\hat{v}_\ell'\}$ in $L^2(0,T;V^*)$ with the strong convergence of $\hat{v}_{\ell'}$ in $L^s(0,T;Z)$ for any $s \in [1,\infty)$ and for any Banach space Z with $L^2(\Omega)$ being compactly embedded in Z and Z being continuously embedded in V^*. The limit can only be u'.

Since the difference $v_\ell - \hat{v}_\ell$ converges strongly in $L^2(\Omega \times (0,T))$ towards zero and since $\{v_\ell\}$, $\{\hat{v}_\ell\}$ are bounded in $L^\infty(0,T;L^2(\Omega))$, we also conclude with the strong convergence of $v_{\ell'}$ in $L^s(0,T;Z)$ towards u'.

It immediately follows that $Kv_{\ell'}$ (as well as $K\hat{v}_{\ell'}$) converges strongly in $\mathscr{C}([0,T];Z)$ towards $Kv = Ku' = u - u_0$. For proving the strong convergence of $u_{\ell'}$, let us take $Z = H^{-1}(\Omega)$. Since $L^2(\Omega) \in \underline{\mathscr{K}}_{1/2}(H^{-1}(\Omega), H_0^1(\Omega))$, we see that

$$\|u_\ell - u\|^2_{L^2(\Omega\times(0,T))} \le \|u_\ell - u\|_{L^2(0,T;H_0^1(\Omega))} \|u_\ell - u\|_{L^2(0,T;H^{-1}(\Omega))} ,$$

where the first factor on the right-hand side is uniformly bounded because of the boundedness of $\{u_\ell\}$ in $L^\infty(0,T;V) \hookrightarrow L^2(0,T;H_0^1(\Omega))$. For the second factor, we find

$$\begin{aligned}&\|u_\ell - u\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\le \|u_\ell - u_0 - Kv_\ell\|_{L^2(0,T;H^{-1}(\Omega))} + \|u_0 + Kv_\ell - u\|_{L^2(0,T;H^{-1}(\Omega))} ,\end{aligned}$$

where the first summand on the right-hand side vanishes as $\ell \to \infty$ since $u_\ell - u_0 - Kv_\ell$ converges strongly in $L^2(0,T;V) \hookrightarrow L^2(0,T;H^{-1}(\Omega))$ towards zero. The second summand also vanishes in the limit because of $u = u_0 + Kv$ and, passing to a subsequence if necessary, the strong convergence of $Kv_{\ell'}$ in $\mathscr{C}([0,T];Z)$ with $Z = H^{-1}(\Omega)$ towards Kv, which has just been shown.

The strong convergence of $u_{\ell'}$ in $L^2(\Omega \times (0,T))$ towards u, together with (4.3) and $V \hookrightarrow Y$, now implies (without loss of generality we consider $r(1-\eta) \ge 2$)

$$\begin{aligned}&\|u_{\ell'} - u\|^r_{L^r(0,T;Y)} \le c \int_0^T \|u_{\ell'}(t) - u(t)\|^{r(1-\eta)} \|\Delta(u_{\ell'}(t) - u(t))\|^{r\eta}\, dt \\ &\le c\|u_{\ell'} - u\|^2_{L^2(\Omega\times(0,T))} \|u_{\ell'} - u\|^{r-2}_{L^\infty(0,T;V)} ,\end{aligned}$$

where the first factor on the right-hand side tends to zero as already shown and the second one is uniformly bounded.

We now prove that $u \in \mathscr{C}_w([0,T];V)$ and $u' \in \mathscr{C}_w([0,T];L^2(\Omega))$ satisfy the initial conditions (1.4). Let $g \in L^2(\Omega)$ be arbitrary and note that $L^2(\Omega)$ is dense in V^*. We then find that

$$t \mapsto (g, u(t))\,\frac{T-t}{T} \in \mathscr{C}^1([0,T])$$

and hence with the convergence properties already shown and the construction of the prolongations

$$\begin{aligned}(g,u(0)) &= -\int_0^T \frac{d}{dt}\left((g,u(t))\,\frac{T-t}{T}\right)dt = -\int_0^T \left((g,u'(t))\,\frac{T-t}{T} - (g,u(t))\right)dt\\ &= -\lim_{\ell'\to\infty}\int_0^T \left((g,v_{\ell'}(t))\,\frac{T-t}{T} - (g,u_{\ell'}(t))\right)dt\\ &= -\lim_{\ell'\to\infty}\int_0^T \left((g,v_{\ell'}(t))\,\frac{T-t}{T} - (g,u_0+(Kv_{\ell'})(t))\right)dt\\ &= -\lim_{\ell'\to\infty}\int_0^T \frac{d}{dt}\left((g,u_0+(Kv_{\ell'})(t))\,\frac{T-t}{T}\right)dt = \lim_{\ell'\to\infty}(g,u_0+(Kv_{\ell'})(0)) = (g,u_0)\,.\end{aligned}$$

Similarly, we observe that for arbitrary $w \in V \overset{\text{dense}}{\subset} L^2(\Omega)$

$$t \mapsto (u'(t),w)\frac{T-t}{T} \in H^1(0,T)\,.$$

Using standard results on the distributional time derivative in Bochner–Lebesgue spaces (see, e.g., Temam [30, Lemma 1.1 on p. 250]), we then obtain with (4.5)

$$\begin{aligned}(u'(0),w) &= -\int_0^T \frac{d}{dt}\left((u'(t),w)\,\frac{T-t}{T}\right)dt = -\int_0^T \left(\langle u''(t),w\rangle\,\frac{T-t}{T} - (u'(t),w)\right)dt\\ &= -\lim_{\ell'\to\infty}\int_0^T \left(\langle \hat v'_{\ell'}(t),w\rangle\,\frac{T-t}{T} - (\hat v_{\ell'}(t),w)\right)dt\\ &= -\lim_{\ell'\to\infty}\int_0^T \frac{d}{dt}\left((\hat v_{\ell'}(t),w)\,\frac{T-t}{T}\right)dt = \lim_{\ell'\to\infty}(\hat v_{\ell'}(0),w) = \lim_{\ell'\to\infty}(v^0_{\ell'},w) = (v_0,w)\,.\end{aligned}$$

The proof of the additional assertions in the particular cases that λ and μ do not vanish can be carried out in a similar way: The first a priori estimate in Theorem 3.2 yields the uniform boundedness of $\|\nabla v_\ell\|_{L^2(\Omega\times(0,T))}$ if $\lambda > 0$ and of $\|\Delta v_\ell\|_{L^2(\Omega\times(0,T))}$ if $\mu > 0$. Recalling here that, by construction, $v_\ell(t) \in V$ ($t \in [0,T]$) and thus satisfies the boundary conditions (1.2) and (1.3), respectively, this shows the boundedness of $\{v_\ell\}$ in $L^2(0,T;H^1(\Omega))$ and $L^2(0,T;H^2(\Omega))$, respectively.

Moreover, since the Dirichlet trace operator is a linear bounded mapping of $H^1(\Omega)$ into $L^2(\partial\Omega)$ and thus weak-weak continuous, we find from the weak convergence of (a subsequence of) $v_\ell \in L^2(0,T;H^1_0(\Omega))$ in $L^2(0,T;H^1(\Omega))$ ($\lambda > 0$ or $\mu > 0$) that the limit u' is indeed in $L^2(0,T;H^1_0(\Omega))$. In the case that $\mu > 0$, an analogous argumentation for the Neumann trace operator shows also that u' fulfills (1.3) in the case of clamped boundary conditions such that always $u' \in L^2(0,T;V)$.

We conclude with the asserted weak convergence in $L^2(0,T;H^1_0(\Omega))$ and $L^2(0,T;V)$, respectively.

In the case $\lambda > 0$, we also have the boundedness of $\{\hat v_\ell\}$ in $L^2(0,T;H^1_0(\Omega))$ because of the first estimate in Theorem 3.2 and the assumption (4.5) since, in particular,

$$\tau_\ell\|\nabla v^0_\ell\|^2 \le \tau_\ell\|v^0_\ell\|\,\|\Delta v^0_\ell\|\,.$$

The generalised Lions–Aubin theorem (see again Roubíček [27, Lemma 7.7 on p. 194]) then yields the strong convergence (again passing to a subsequence if necessary) of $\{\hat{v}_{\ell'}\}$ towards u' in $L^2(0,T;\overline{Z})$ for any Banach space $\overline{Z}$ with $H_0^1(\Omega) \overset{c}{\hookrightarrow} \overline{Z} \hookrightarrow V^*$. In particular, we may choose $\overline{Z} = L^2(\Omega)$. Together with the strong convergence of $v_\ell - \hat{v}_\ell$ in $L^2(\Omega \times (0,T))$ towards zero, we thus find the strong convergence of $v_{\ell'}$ in $L^2(\Omega \times (0,T))$ towards u'.

The additional regularity in the case $\mu > 0$ follows from the continuous embedding of $H^1(0,T;V^*) \cap L^2(0,T;V) \ni u'$ in $\mathscr{C}([0,T];L^2(\Omega))$ and of $H^1(0,T;V) \ni u$ in $\mathscr{C}([0,T];V)$. The asserted strong convergence follows from (2.4) since

$$\|v_\ell - u'\|_{L^2(0,T;H_0^1(\Omega))} \le \|v_\ell - u'\|^{1/3}_{L^2(0,T;H^{-1}(\Omega))} \|v_\ell - u'\|^{2/3}_{L^2(0,T;V)} ,$$

and the right-hand side tends to zero as $\ell \to \infty$. □

In the case $\mu > 0$, however, we would need to assume in addition

$$\tau_\ell \|\Delta v_\ell^0\|^2 \le c$$

in order to prove the boundedness of $\{\hat{v}_\ell\}$ in $L^2(0,T;V)$.

Proof. [of Theorem 4.1] We commence with proving that the limit u of Proposition 4.2 is indeed a solution in the sense of Definition 2.1. Since we have already shown in Proposition 4.2 that u satisfies the initial conditions, it remains to show that u satisfies the differential equation. For readability, we tacitly pass to a subsequence if necessary and do not distinguish between ℓ and ℓ'.

With the definition of $u_\ell, v_\ell, \hat{v}_\ell, f_\ell$ and since $A0 = B0 = C(0,0) = 0$, the numerical scheme (3.1) (see also (3.3)) can be written in the following way as an operator-differential equation on $(0,T)$,

$$\hat{v}_\ell' + Av_\ell + Bu_\ell + C(u_\ell, v_\ell) = f_\ell \quad \text{on } (0,T)\,, \tag{4.6}$$

which holds almost everywhere on $(0,T)$ as well as in the sense of equality in $L^2(0,T;V^*)$.

As $\ell \to \infty$, Proposition 4.2 shows that $\hat{v}_\ell'$ tends to u'' in $L^2(0,T;V^*)$.

For the following, we recall that linear bounded operators are also weak-weak continuous (see, e.g., Brézis [8, Theorem 3.10 on p. 61]). Since A is a linear bounded mapping of $L^2(0,T;X)$ into $L^2(0,T;V^*)$ with $X = L^2(\Omega)$ if $\lambda = \mu = 0$, $X = H_0^1(\Omega)$ if $\lambda > 0$, $\mu = 0$ and $X = V$ if $\lambda > 0$, $\mu > 0$ (see also (2.7)), Proposition 4.2 shows that Av_ℓ converges weakly in $L^2(0,T;V^*)$ towards Au'. The same argumentation shows that $B_i u_\ell$ $(i = 1,2,4)$ converges weakly in $L^2(0,T;V^*)$ towards $B_i u$.

With respect to B_3, we recall the second inequality in (2.10), which yields

$$\begin{aligned}
&\|Bu_\ell - Bu\|_{L^2(0,T;V^*)} \\
&\le c\,\|u_\ell - u\|_{L^2(0,T;H_0^1(\Omega))} \left(\|u_\ell\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \|u\|^2_{L^\infty(0,T;H_0^1(\Omega))} \right).
\end{aligned}$$

Proposition 4.2 with $Y = H_0^1(\Omega)$ shows that the right-hand side of the foregoing estimate tends to zero as $\ell \to \infty$.

Moreover, strong convergence of f_ℓ in $L^2(\Omega \times (0,T))$ towards f can easily be shown by standard arguments.

If $\delta = 0$ (no Balakrishnan–Taylor damping), we thus obtain from (4.6) the limit equation

$$u'' + Au' + Bu = f \quad \text{in } L^2(0,T;V^*)\,,$$

which shows that the limit u of Proposition 4.2 satisfies the differential equation (1.1) in the weak sense (see Definition 2.1).

If $\delta \neq 0$ but $\lambda > 0$ or $\mu > 0$, we can employ (2.20) together with the results of Proposition 4.2. Indeed, we find with the Cauchy–Schwarz inequality

$$\begin{aligned}
&\|C(u_\ell, v_\ell) - C(u, u')\|_{L^1(0,T;V^*)} \\
&\leq c \max(\|u_\ell\|_{L^\infty(0,T;V)}\|v_\ell\|_{L^\infty(0,T;L^2(\Omega))}, \|u\|_{L^\infty(0,T;V)}\|u'\|_{L^\infty(0,T;L^2(\Omega))})^{q-2} \times \\
&\Big(\|u_\ell\|_{L^2(0,T;V)}\|v_\ell - u'\|_{L^2(\Omega\times(0,T))} + \|u_\ell - u\|_{L^2(0,T;H^1_0(\Omega))}\|u'\|_{L^2(0,T;H^1_0(\Omega))}\Big) \\
&\qquad + c\,\|u\|^{q-1}_{L^\infty(0,T;V)}\|u'\|^{q-1}_{L^\infty(0,T;L^2(\Omega))}\|u_\ell - u\|_{L^1(0,T;L^2(\Omega))}\,,
\end{aligned}$$

and the right-hand side of the foregoing estimate tends to zero as $\ell \to \infty$. Since $\{C(u_\ell, v_\ell)\}$ is bounded in $L^\infty(0,T;V^*)$, we indeed get strong convergence also in $L^2(0,T;V^*)$ and we thus obtain the limit equation

$$u'' + Au' + Bu + C(u, u') = f \quad \text{in } L^2(0,T;V^*)\,,$$

which shows that the limit u of Proposition 4.2 satisfies the differential equation (1.1) in the weak sense (see Definition 2.1).

It remains to consider the case $\delta \neq 0$ with $q = 2$ but $\lambda = \mu = 0$. So it remains to prove that for all $w \in V$ and $\varphi \in \mathscr{C}^\infty_c(0,T)$

$$\begin{aligned}
&\int_0^T \langle C(u_\ell(t), v_\ell(t)), w\rangle\,\varphi(t)\,dt \to \langle \widetilde{C}(u,u'), w\varphi\rangle \\
&= \frac{\delta}{2}\int_0^T \|\nabla u(t)\|^2\big((u'(t), \Delta w)\,\varphi(t) + (\Delta u(t), w)\,\varphi'(t)\big)dt
\end{aligned} \tag{4.7}$$

as $\ell \to \infty$, see also (2.23).

In the sequel, we set $t_{n-1/2} = (n - 1/2)\tau_\ell$ for $n \in \mathbb{Z}$. With the definition of the operator C (recall here that $q = 2$) and the prolongations, we find (applying (3.2) and (2.15))

$$\begin{aligned}
&\int_0^T \langle C(u_\ell(t), v_\ell(t)), w\rangle\,\varphi(t)\,dt = \sum_{j=1}^{N_\ell - 1}\langle C(u^{j+1}, v^j), w\rangle \int_{t_{j-1/2}}^{t_{j+1/2}} \varphi(t)\,dt \\
&= -\delta \sum_{j=1}^{N_\ell - 1}(\nabla u^{j+1}, \nabla v^j)(u^{j+1}, \Delta w) \int_{t_{j-1/2}}^{t_{j+1/2}} \varphi(t)\,dt \\
&= -\frac{\delta}{2\tau_\ell}\sum_{j=1}^{N_\ell - 1}\big(\|\nabla u^{j+1}\|^2 - \|\nabla u^j\|^2 + \|\nabla(u^{j+1} - u^j)\|^2\big)\,(u^{j+1}, \Delta w) \int_{t_{j-1/2}}^{t_{j+1/2}} \varphi(t)\,dt \\
&= -\frac{\delta}{2\tau_\ell}\sum_{j=1}^{N_\ell - 1}\|\nabla u^{j+1}\|^2(u^{j+1}, \Delta w) \int_{t_{j-1/2}}^{t_{j+1/2}} \varphi(t)\,dt \\
&\quad + \frac{\delta}{2\tau_\ell}\sum_{j=0}^{N_\ell - 2}\|\nabla u^{j+1}\|^2(u^{j+2}, \Delta w) \int_{t_{j-1/2}}^{t_{j+1/2}} \varphi(t + \tau_\ell)\,dt \\
&\quad - \frac{\delta}{2\tau_\ell}\sum_{j=1}^{N_\ell - 1}\|\nabla(u^{j+1} - u^j)\|^2(u^{j+1}, \Delta w) \int_{t_{j-1/2}}^{t_{j+1/2}} \varphi(t)\,dt =: S_{1,\ell} - S_{2,\ell} + S_{3,\ell}\,.
\end{aligned}$$

For $S_{3,\ell}$, we find with (2.2) and (3.2) that

$$|S_{3,\ell}| \leq \frac{c}{\tau_\ell}\sum_{j=1}^{N_\ell - 1}\|\tau_\ell\, v^j\|\|\Delta(u^{j+1} - u^j)\|\|u^{j+1}\|\|\Delta w\|\,\tau_\ell\|\varphi\|_{\mathscr{C}([0,T])}$$

$$\leq c\tau_\ell \sum_{j=1}^{N_\ell-1} \|\Delta(u^{j+1}-u^j)\| \max_{j=1,\ldots,N_\ell-1} \left(\|v^j\|\|u^{j+1}\|\right) \|\Delta w\| \|\varphi\|_{\mathscr{C}([0,T])}$$

$$\leq c \left(\tau_\ell \sum_{j=1}^{N_\ell-1} \|\Delta(u^{j+1}-u^j)\|^2\right)^{1/2} \max_{j=1,\ldots,N_\ell-1} \left(\|v^j\|\|u^{j+1}\|\right) \|\Delta w\| \|\varphi\|_{\mathscr{C}([0,T])} \, .$$

The first a priori estimate in Theorem 3.2 (together with (2.1) and (2.3)) now shows that the right-hand side of the foregoing estimate tends to zero as $\ell \to \infty$, which shows that $S_{3,\ell}$ vanishes as $\ell \to \infty$.

We now come back to $S_{1,\ell} - S_{2,\ell}$ and observe, after a simple but tedious calculation, that

$$\begin{aligned} & S_{1,\ell} - S_{2,\ell} \\ &= -\frac{\delta}{2\tau_\ell} \|\nabla u^{N_\ell}\|^2 (u^{N_\ell}, \Delta w) \int_{t_{N_\ell-3/2}}^{t_{N_\ell-1/2}} \varphi(t)\, dt + \frac{\delta}{2\tau_\ell} \|\nabla u^1\|^2 (u^2, \Delta w) \int_{t_{1/2}}^{t_{3/2}} \varphi(t)\, dt \\ &\quad + \frac{\delta}{2} \sum_{j=1}^{N_\ell-2} \|\nabla u^{j+1}\|^2 (v^{j+1}, \Delta w) \int_{t_{j-1/2}}^{t_{j+1/2}} \varphi(t)\, dt \\ &\quad + \frac{\delta}{2\tau_\ell} \sum_{j=1}^{N_\ell-2} \|\nabla u^{j+1}\|^2 (u^{j+2}, \Delta w) \int_{t_{j-1/2}}^{t_{j+1/2}} (\varphi(t+\tau_\ell) - \varphi(t))\, dt =: S_{4,\ell} + S_{5,\ell} + S_{6,\ell} \, . \end{aligned}$$

With the first a priori estimate in Theorem 3.2, the assumptions (4.4), (4.5) (recall that $u^1 = u^0 + \tau_\ell v^0$) and (2.2), we find

$$\begin{aligned} |S_{4,\ell}| &\leq \frac{c}{\tau_\ell} \left| \int_{t_{N_\ell-3/2}}^{t_{N_\ell-1/2}} \varphi(t)\, dt \right| \|\Delta w\| + \frac{c}{\tau_\ell} \left| \int_{t_{1/2}}^{t_{3/2}} \varphi(t)\, dt \right| \|\Delta w\| \\ &\to c\,(|\varphi(T)| + |\varphi(0)|)\, \|\Delta w\| = 0 \quad \text{as } \ell \to \infty \, . \end{aligned}$$

Let $\overline{v}_\ell : [0,T] \to V$ be piecewise constant with $\overline{v}_\ell(t) = v^{j+1}$ if $t \in (t_{j-1/2}, t_{j+1/2}]$ $(j = 1, 2, \ldots, N_\ell - 2)$ and $\overline{v}_\ell(t) = 0$ otherwise. The term $S_{5,\ell}$ can then be written as

$$S_{5,\ell} = \frac{\delta}{2} \int_0^T \|\nabla u_\ell(t)\|^2 (\overline{v}_\ell(t), \Delta w)\, \varphi(t)\, dt \, .$$

The first a priori estimate in Theorem 3.2 immediately implies the boundedness of $\{\overline{v}_\ell\}$ in $L^\infty(0,T;L^2(\Omega))$ and thus weak* convergence in $L^\infty(0,T;L^2(\Omega))$ towards an element $\overline{v}$. Since

$$\|\overline{v}_\ell - v_\ell\|^2_{L^2(\Omega\times(0,T))} = \tau_\ell \sum_{j=1}^{N_\ell-2} \|v^{j+1} - v^j\|^2 + \tau_\ell \|v^{N_\ell-1}\|^2$$

tends to zero as $\ell \to \infty$ (see again Theorem 3.2), $\overline{v}$ coincides with the weak*-limit of v_ℓ such that $\overline{v} = u'$. In view of Proposition 4.2, we also know that u_ℓ converges strongly in $L^2(0,T;H_0^1(\Omega))$ towards u as $\ell \to \infty$, which implies

$$\|\nabla u_\ell\|^2 \Delta w\, \varphi \to \|\nabla u\|^2 \Delta w\, \varphi \quad \text{in } L^1(0,T;L^2(\Omega)) \text{ as } \ell \to \infty$$

for the sequence of functions $t \mapsto \|\nabla u_\ell(t)\|^2 \Delta w\, \varphi(t)$ $(t \in [0,T])$. We, therefore, come up with

$$S_{5,\ell} \to \frac{\delta}{2} \int_0^T \|\nabla u(t)\|^2 (u'(t), \Delta w)\, \varphi(t)\, dt \quad \text{as } \ell \to \infty \, .$$

Finally, we define $\overline{u}_\ell : [0,T] \to V$ as piecewise constant with $\overline{u}_\ell(t) = u^{j+2}$ if $t \in (t_{j-1/2}, t_{j+1/2}]$ $(j = 1, 2, \dots, N_\ell - 2)$ and $\overline{u}_\ell(t) = 0$ otherwise. The term $S_{6,\ell}$ can then be written as

$$S_{6,\ell} = \frac{\delta}{2} \int_0^T \|\nabla u_\ell(t)\|^2 (\Delta \overline{u}_\ell(t), w) \, \frac{\varphi(t+\tau_\ell) - \varphi(t)}{\tau_\ell} \, dt\,.$$

With similar arguments as employed for $\overline{v}_\ell$, it is straightforward to show that $\overline{u}_\ell$ weakly* converges in $L^\infty(0,T;V)$ towards u. This, together with the strong convergence of u_ℓ in $L^2(0,T;H_0^1(\Omega))$ towards u and the strong convergence of $(\varphi(\cdot + \tau_\ell) - \varphi(t))/\tau_\ell$ in $\mathscr{C}([0,T])$ towards φ' proves that

$$S_{6,\ell} \to \frac{\delta}{2} \int_0^T \|\nabla u(t)\|^2 (\Delta u(t), w) \, \varphi'(t) \, dt \quad \text{as } \ell \to \infty\,.$$

The foregoing observations have shown that (4.7) holds true.

The asserted convergence of the numerical scheme now follows immediately from Proposition 4.2. □

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