Well-posedness of the peridynamic model with Lipschitz continuous pairwise force function

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Abstract. Peridynamics is a nonlocal theory of continuum mechanics based on a nonlocal, in general nonlinear integro-differential equation without spatial derivatives. Well-posedness of the nonlinear d-dimensional peridynamic initial value problem under the assumption of Lipschitz continuity of the pairwise force function in its second argument is shown.

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1. Introduction

Peridynamics is a nonlocal theory of continuum mechanics based on a nonlocal, in general nonlinear integro-differential equation without spatial derivatives. This is seen to be its main advantage, because the equation is therefore still meaningful in case of material failure such as cracks, which are spatial discontinuities in the deformation variable. Proposed by Silling [11] in 2000, peridynamic theory has been developed over the last decades. Especially its areas of application show a broad spectrum. For example, peridynamics is used to predict the behaviour of cracks in composites, polycrystals and nanofibre networks, see Askari et al. [3] and the references cited therein. In particular, multiscale problems can be treated, see again [3] as well as Alali and Lipton [1].

The mathematical substantiation of the peridynamic model is, however, still in its infancy. The linearized problem has shown to be well-posed in L^∞ and L^2 and in certain fractional Sobolev spaces by Emmrich and Weckner [8, 7] and by Du and Zhou [5, 12], respectively. In contrast to the linearized problem, the nonlinear problem has not been studied extensivly. A first result towards the nonlinear model can be found in Erbay, Erkip and Muslu [9] analyzing the nonlinear infinite elastic bar. Well-posedness of a one-dimensional formulation in various function spaces is shown. Here, the pairwise force function as a function of two arguments is split into a product of two functions of one argument. This product ansatz for the pairwise force function is essential in [9]. Furthermore, these functions are not integrated over the peridynamic horizon but over the full axis $\mathbb{R}$.
For a more complete overview on the peridynamic theory, we refer to Emmrich, Lehoucq and Puhst [6].

In this paper, we will show well-posedness of the nonlinear d-dimensional peridynamic initial value problem under the assumption of Lipschitz continuity of the pairwise force function in its second argument. Local Lipschitz continuity will lead to local-in-time solutions, global Lipschitz continuity will result in global-in-time solutions. Moreover, we will discuss why compactness arguments are not at hand.
This paper is organized as follows. The peridynamic model is explained in Section 2. As the nonlinear peridynamic equation of motion can be written as an operator-differential equation, in Section 3 abstract semilinear ordinary differential equations of second order are studied. To the best knowledge of the authors, these existence theorems are not provided by standard literature such as Amann [2], Deimling [4], or Ladas and Lakshmikantham [10]. In Section 4, the results of the latter section are used to prove local-in-time existence with $\mathcal{C}$ and L^∞ being the underlying function spaces and global-in-time existence with $\mathcal{C}$ and L^p $(1 \leq p \leq \infty)$.

1.1. Notation

Throughout this paper, let $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a bounded domain. The Euclidian norm will be denoted by $|\cdot|$, the inner product in $\mathbb{R}^d$ by $\cdot$. For a Banach space $(X, \|\cdot\|_X)$, the ball centred in $x_0 \in X$ with radius r is denoted by $B_X(x_0; r)$ and its closure by $\bar{B}_X(x_0; r)$. Whenever the context is clear, the norm $\|\cdot\|_X$ will be abbreviated by $\|\cdot\|$. The space of continuous functions mapping $\bar{\Omega}$ in $\mathbb{R}^d$, denoted by $\mathcal{C}(\bar{\Omega})^d$, will be equipped with the norm $\|\boldsymbol{v}\|_{\mathcal{C}(\bar{\Omega})^d} = \max_{\boldsymbol{x}\in\bar{\Omega}} |\boldsymbol{v}(\boldsymbol{x})|$. The space $L^p(\Omega)^d$ of all (equivalence classes of almost everywhere equal) Lebesgue measurable functions mapping Ω in $\mathbb{R}^d$ with finite norm is equipped with $\|\boldsymbol{v}\|_{L^p(\Omega)^d} = \left(\int_\Omega |\boldsymbol{v}(\boldsymbol{x})|^p \, d\boldsymbol{x}\right)^{1/p}$ for $1 \le p < \infty$ and with $\|\boldsymbol{v}\|_{L^\infty(\Omega)^d} = \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} |\boldsymbol{v}(\boldsymbol{x})|$ for $p = \infty$. Furthermore, we will need function spaces for functions with values in a Banach space. The space $\mathcal{C}([0,T], X)$ of continuous functions mapping $[0,T]$ into the Banach space X will be equipped with the norm $\|v\|_{\mathcal{C}([0,T],X)} = \max_{t\in[0,T]} \|v(t)\|_X$ and the space $\mathcal{C}^2([0,T], X)$ of twice continuously differentiable mappings will have the norm $\|v\|_{\mathcal{C}^2([0,T],X)} = \|v\|_{\mathcal{C}([0,T],X)} + \|v'\|_{\mathcal{C}([0,T],X)} + \|v''\|_{\mathcal{C}([0,T],X)}$. Finally, we denote by $L^p(0,T;X)$ ($1 \le p \le \infty$) the Bochner-Lebesgue space of all (equivalence classes of almost everywhere equal) Bochner measurable functions mapping $[0,T]$ into X with finite norm, equipped with

$$\|v\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|v(t)\|_X^p \, dt\right)^{1/p} & \text{for } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t\in(0,T)} \|v(t)\|_X & \text{for } p = \infty. \end{cases}$$

For a Lebesgue measurable set Ω, we denote by $\operatorname{vol}(\Omega)$ its measure. We remark that, for all $\boldsymbol{x} \in \mathbb{R}^d$, the volume of the ball centred in $\boldsymbol{x}$ with radius r is given by

$$\operatorname{vol}(B_{\mathbb{R}^d}(\boldsymbol{x}; r)) = \frac{r^d \pi^{d/2}}{\Gamma(1 + d/2)} \tag{1.1}$$

and is therefore independent of $\boldsymbol{x}$. Here Γ denotes the Gamma function. Finally, the distance of an element $x_0 \in X$ to a set $M \subseteq X$ is defined by $\operatorname{dist}(x_0, X) = \inf_{y\in M} \|x - y\|$. The distance to the empty set is defined as $\operatorname{dist}(x, \emptyset) = \infty$.
Throughout this paper, vectors in $\mathbb{R}^d$ and vectorfields are typed boldfaced.

2. Peridynamic equation of motion

Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be the bounded domain of an undeformed body and $[0,T]$ the time interval under consideration. Let $\boldsymbol{y} : \bar{\Omega} \times [0,T] \to \mathbb{R}^d$ be the deformation of the solid body. Then, for $(\boldsymbol{x}, t) \in \Omega \times (0,T)$, the nonlinear peridynamic equation of motion reads

$$\rho(\boldsymbol{x}) \partial_t^2 \boldsymbol{y}(\boldsymbol{x}, t) = \int_{\mathcal{H}(\boldsymbol{x})} \boldsymbol{f}(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}}, t) - \boldsymbol{y}(\boldsymbol{x}, t)) d\hat{\boldsymbol{x}} + \boldsymbol{b}(\boldsymbol{x}, t). \tag{2.1}$$

Here ρ is the density of the body, $\boldsymbol{b}$ describes external forces and the integration volume $\mathcal{H}(\boldsymbol{x})$ is the open ball of radius δ centred at $\boldsymbol{x}$ intersected with Ω. The radius δ is called peridynamic horizon. The integrand $\boldsymbol{f}$ is called pairwise force function and gives the force that particle $\hat{\boldsymbol{x}}$ exerts on particle $\boldsymbol{x}$. It is considered as $\mathbf{0}$ beyond the horizon, thus the integral in (2.1) can be understood as

$$\int_\Omega \tilde{\boldsymbol{f}}(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}}, t) - \boldsymbol{y}(\boldsymbol{x}, t)) d\hat{\boldsymbol{x}} \quad \text{with} \quad \tilde{\boldsymbol{f}} = \begin{cases} \boldsymbol{f}, & \hat{\boldsymbol{x}} \in \mathcal{H}(\boldsymbol{x}) \\ \mathbf{0}, & \hat{\boldsymbol{x}} \notin \mathcal{H}(\boldsymbol{x}) \end{cases}.$$

The pairwise force function $\boldsymbol{f}$ depends on the material of the body and takes into account the elastic moduli and, in the linear case, the Lamé coefficients. It is shown in Silling [11] that a direct consequence of this formulation is a Poission ratio of $\nu = 1/4$, which results, in the linear case, in the equity of both Lamé coefficients λ and μ.

In (2.1), the integral term sums up the forces that all particles within the peridynamic horizon exert on particle $\boldsymbol{x}$. These interactions are called bonds. Using the notation

$$\boldsymbol{\xi} = \hat{\boldsymbol{x}} - \boldsymbol{x} \quad \text{and} \quad \boldsymbol{\zeta} = \boldsymbol{y}(\hat{\boldsymbol{x}}, t) - \boldsymbol{y}(\boldsymbol{x}, t),$$

the pairwise force function $\boldsymbol{f}$ has to satisfy natural conditions such as Newton's principle actio et reactio and the conservation of angular momentum, namely

$$\boldsymbol{f}(-\boldsymbol{\xi}, -\boldsymbol{\zeta}) = -\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}) \quad \text{and} \quad \boldsymbol{\zeta} \times \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = \mathbf{0}. \tag{2.2}$$

A class of pairwise force functions describing isotropic materials is given by

$$\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = \phi(|\boldsymbol{\xi}|, |\boldsymbol{\zeta}|)\boldsymbol{\zeta}. \tag{2.3}$$

A first example of this form suggested in Silling [11] is the so-called bondstretch model

$$\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = c\, s(|\boldsymbol{\xi}|, |\boldsymbol{\zeta}|) \frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}, \tag{2.4}$$

where c is a constant depending on the material parameters, the dimension and the horizon, and $s(|\boldsymbol{\xi}|, |\boldsymbol{\zeta}|) = (\,|\boldsymbol{\zeta}| - |\boldsymbol{\xi}|\,)/\,|\boldsymbol{\xi}|$ is the the so-called bondstretch, which is the relative change of the Euclidian distance of the particles. Note that the bondstretch model is described by a pairwise force function, which is discontinuous in its second argument. Further examples of pairwise force functions suggested in [11] are

$$\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = c\, (|\boldsymbol{\zeta}| - |\boldsymbol{\xi}|)^2\, \boldsymbol{\zeta} \tag{2.5}$$

with another constand c depending on the material parameters, the dimension d and the horizon δ, and

$$\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = a\,(|\boldsymbol{\xi}|)\left(|\boldsymbol{\zeta}|^2 - |\boldsymbol{\xi}|^2\right)\boldsymbol{\zeta} \tag{2.6}$$

for a continuous function a, which is also depending on material parameters, the dimension d and the horizon δ. All these examples are of type (2.3) and describe materials with isotropic, microelastic behaviour.

Supplementing the peridynamic equation of motion (2.1) with inital data

$$\boldsymbol{y}(\cdot, 0) = \boldsymbol{y_0} \quad \text{and} \quad \partial_t \boldsymbol{y}(\cdot, 0) = \dot{\boldsymbol{y}}_{\mathbf{0}}, \tag{2.7}$$

the resulting initial value problem is fully described since no spatial derivatives occur and therefore, in general, no boundary conditions are neccessary.

3. Well-posedness of abstract differential equations of second order

Let $(X, \|\cdot\|)$ be a real Banach space. We discuss well-posedness of abstract initial value problems for an ordinary differential equation of second order in the Banach space X, that is

$$y''(t) = g(t, y(t)), \quad t \in (0, T), \quad y(t_0) = y_0, \; y'(t_0) = \dot{y}_0 \tag{3.1}$$

for $t_0 \in [0, T]$, $y_0, \dot{y}_0 \in X$ and some function g, whose domain will be specified below and which maps into X.

Theorem 3.1. *Let $g : [0, T] \times \bar{B}_X(y_0; r) \to X$ be continuous for some $r > 0$ and uniformly Lipschitz continuous in its second argument, i.e., there exists $L > 0$ such that for all $t \in [0, T]$ and all $v, w \in \bar{B}_X(y_0; r)$*

$$\|g(t, v) - g(t, w)\| \leq L \|v - w\| .$$

Then the initial value problem (3.1) possesses a unique solution $y \in \mathcal{C}^2(I, X)$. The interval $I \subseteq [0, T]$ is given by $I = [t_0 - a, t_0 + a] \cap [0, T]$ with

$$a = \min \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{\|\dot{y}_0\|^2}{M^2} + \frac{2r}{M}} - \frac{\|\dot{y}_0\|}{M} \right\}, \quad M = \sup_{(t,v) \in [0,T] \times \bar{B}_X(y_0; r)} \|g(t, v)\| .$$

The solution y depends continuously on the initial data.

Proof. Defining the operator $S : \mathcal{A} \to \mathcal{A}$ with $\mathcal{A} \subset \mathcal{C}([0, T], X)$ by

$$(Sv)(t) = y_0 + (t - t_0)\dot{y}_0 + \int_{t_0}^{t} (t - \tau) g(\tau, v(\tau)) d\tau, \tag{3.2}$$

the initial value problem (3.1) is equivalent to finding a fixed point $y = Sy$. Here $\mathcal{A}$ is a suitably chosen nonempty, closed subset. An application of Banach's fixed point theorem proves the first assertion. By Gronwall's lemma, the continuous dependence can be shown. □

The local-in-time solution that exists in view of the preceding theorem can be extended on some maximal existence interval. Under further assumptions on g, the behaviour of the solution can be characterized in more detail.

Theorem 3.2. *Let $J \subseteq \mathbb{R}$ be an open interval and $D \subseteq X$ be an open subset. Assume that $g : J \times D \to X$ is continuous and locally uniformly Lipschitz continuous in its second argument, i.e., for any $(t_0, y_0, \dot{y}_0) \in J \times D \times X$ there exist $a, r, L > 0$ such that for all $t \in [t_0 - a, t_0 + a]$ and all $v, w \in \bar{B}_X(y_0; r)$*

$$\|g(t, v) - g(t, w)\| \leq L \|v - w\| .$$

Then for any $(t_0, y_0, \dot{y}_0) \in J \times D \times X$ there exists a unique maximal solution of (3.1). The existence interval $J_{max} = J_{max}(t_0, y_0, \dot{y}_0) \subseteq J$ is open,

$$J_{max} = (\alpha, \beta), \quad -\infty \leq \alpha < \beta \leq \infty.$$

If g is bounded on bounded subsets of D with positive distance to the boundary of D then the boundary behaviour of the solution is characterized as follows:
There holds either $\alpha = \inf J$ or

$$\lim_{t\searrow\alpha} \min\left\{\operatorname{dist}(y(t), \partial D), \|y(t)\|^{-1}\right\} = 0$$

and similarly either $\beta = \sup J$ or

$$\lim_{t\nearrow\beta} \min\left\{\operatorname{dist}(y(t), \partial D), \|y(t)\|^{-1}\right\} = 0.$$

Proof. The existence of a maximal solution is provided by Zorn's lemma. Due to the local Lipschitz continuity, uniqueness can be shown with Gronwall's lemma. The proof of the limit behaviour of the solution is an adapted version of the proof in Amann [2, pp. 100–102] showing the same behaviour for abstract ordinary differential equations of first order. □

Replacing the local Lipschitz continuity by a global Lipschitz continuity, global existence of a solution can be proven.

Theorem 3.3. *Let $g : [0,T] \times X \to X$ be continuous and uniformly Lipschitz continuous in its second argument, i.e., there exists $L > 0$ such that for all $t \in [0,T]$ and all $v, w \in X$*

$$\|g(t,v) - g(t,w)\| \leq L \|v - w\| .$$

Then the initial value problem (3.1) has a unique solution $y \in \mathcal{C}^2([0,T], X)$, which depends continuously on the initial data.

Proof. Applying Banach's fixed point theorem to $y = Sy$, where $S : \mathcal{X} \to \mathcal{X}$ is defined by (3.2) and

$$\mathcal{X} := (\mathcal{C}([0,T], X), \|\cdot\|_{\mathcal{X}}), \quad \|v\|_{\mathcal{X}} = \max_{t\in[0,T]} e^{-\sqrt{L}|t-t_0|} \|v(t)\|$$

proves the first assertion. Note that the contracting norm $\|\cdot\|_{\mathcal{X}}$ is equivalent to the conventional one. Again, the continuous dependence can be shown with Gronwall's lemma. □

4. Well-posedness of the peridynamic equation of motion

In this section, we want to apply the theorems proven in the previous section to the initial value problem (2.1), (2.7). Identifying $\boldsymbol{y} : \bar{\Omega} \times [0,T] \to \mathbb{R}^d$ with $\tilde{\boldsymbol{y}} : [0,T] \to X$ for a function space X, which is a subset of the set of all mappings of $\bar{\Omega}$ into $\mathbb{R}^d$, by $[\tilde{\boldsymbol{y}}(t)](\boldsymbol{x}) := \boldsymbol{y}(\boldsymbol{x}, t)$ and denoting $\tilde{\boldsymbol{y}}$ by $\boldsymbol{y}$ again yields the equivalent abstract formulation

$$\boldsymbol{y}''(t) = \boldsymbol{g}(t, \boldsymbol{y}(t)), \quad \boldsymbol{y}(0) = \boldsymbol{y_0}, \ \boldsymbol{y}'(0) = \dot{\boldsymbol{y}}_{\boldsymbol{0}}. \tag{4.1}$$

Here $\boldsymbol{g}$ is defined as $\boldsymbol{g}(t, \boldsymbol{v}) := (K\boldsymbol{v} + \boldsymbol{b}(t))/\rho$, and the peridynamic integral operator K is given by

$$(K\boldsymbol{v})(\boldsymbol{x}) := \int_{\mathcal{H}(\boldsymbol{x})} \boldsymbol{f}(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{v}(\boldsymbol{x})) d\hat{\boldsymbol{x}}. \tag{4.2}$$

Note that K depends, through $\mathcal{H}$ and $\boldsymbol{f}$, on both the dimension d and the horizon δ.

Lemma 4.1. *Let $X = \mathcal{C}(\bar{\Omega})^d$. Suppose there is $r > 0$ such that the pairwise force function*

$$\boldsymbol{f} : \bar{B}_{\mathbb{R}^d}(\boldsymbol{0};\delta) \times \bar{B}_{\mathbb{R}^d}(\boldsymbol{0}; 2r + 2\,\|\boldsymbol{y_0}\|) \to \mathbb{R}^d$$

is continuous and Lipschitz continuous in its second argument, i.e., there exists a nonnegative function $L_f \in L^1(B_{\mathbb{R}^d}(\boldsymbol{0};\delta))$ such that for all $\boldsymbol{\xi} \in \bar{B}_{\mathbb{R}^d}(\boldsymbol{0};\delta)$ and all $\boldsymbol{\zeta_1}, \boldsymbol{\zeta_2} \in \bar{B}_{\mathbb{R}^d}(\boldsymbol{0}; 2r + 2\,\|\boldsymbol{y_0}\|)$ there holds

$$|\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}_1) - \boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}_2)| \le L_f(\boldsymbol{\xi})\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,. \tag{4.3}$$

Then the operator K from (4.2) is well-defined and Lipschitz continuous as a mapping of $\bar{B}_X(\boldsymbol{y_0}; r)$ into X.

Proof. In order to show that K is well-defined, we show that $K\boldsymbol{v}$ is continuous for any $\boldsymbol{v} \in \bar{B}_X(\boldsymbol{y_0}; r)$. Since $\boldsymbol{f}$ is bounded, there exists $M > 0$ such that, for all $\boldsymbol{\xi} \in \bar{B}_{\mathbb{R}^d}(\boldsymbol{0};\delta)$ and $\boldsymbol{\zeta} \in \bar{B}_{\mathbb{R}^d}(\boldsymbol{0}; 2r + 2\,\|\boldsymbol{y_0}\|)$, we have $|\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta})| \le M$. Now let $\varepsilon > 0$ be arbitrary, $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \bar{\Omega}$ with $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \tilde{\delta}$ and let $\tilde{\delta} > 0$ be chosen sufficiently small. Using the notation $\boldsymbol{\xi}_i = \hat{\boldsymbol{x}} - \boldsymbol{x}_i$ and $\boldsymbol{\zeta}_i = \boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{v}(\boldsymbol{x}_i)$ for $i \in \{1, 2\}$, we deduce

$$\begin{aligned}|(K\boldsymbol{v})(\boldsymbol{x}_1) - (K\boldsymbol{v})(\boldsymbol{x}_2)| \le &\int_{\mathcal{H}(\boldsymbol{x}_1)\cap\mathcal{H}(\boldsymbol{x}_2)} |\boldsymbol{f}(\boldsymbol{\xi}_1,\boldsymbol{\zeta}_1) - \boldsymbol{f}(\boldsymbol{\xi}_2,\boldsymbol{\zeta}_2)|\, d\hat{\boldsymbol{x}} \\ &+ \int_{\mathcal{H}(\boldsymbol{x}_1)\setminus\mathcal{H}(\boldsymbol{x}_2)} |\boldsymbol{f}(\boldsymbol{\xi}_1,\boldsymbol{\zeta}_1)|\, d\hat{\boldsymbol{x}} \\ &+ \int_{\mathcal{H}(\boldsymbol{x}_2)\setminus\mathcal{H}(\boldsymbol{x}_1)} |\boldsymbol{f}(\boldsymbol{\xi}_2,\boldsymbol{\zeta}_2)|\, d\hat{\boldsymbol{x}}.\end{aligned}$$

We denote the three integrals by I_1, I_2 and I_3. Because of

$$\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 = \boldsymbol{x}_2 - \boldsymbol{x}_1, \quad \boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2 = \boldsymbol{v}(\boldsymbol{x}_2) - \boldsymbol{v}(\boldsymbol{x}_1)$$

and the uniform continuity of $\boldsymbol{v}$ and $\boldsymbol{f}$, there exists $\tilde{\varepsilon} := \mathrm{vol}(B_{\mathbb{R}^d}(\boldsymbol{x}_1;\delta))^{-1}\varepsilon/3$ such that

$$I_1 < \tilde{\varepsilon}\,\mathrm{vol}(\mathcal{H}(\boldsymbol{x}_1) \cap \mathcal{H}(\boldsymbol{x}_2)) \le \tilde{\varepsilon}\,\mathrm{vol}(B_{\mathbb{R}^d}(\boldsymbol{x}_1;\delta)) = \frac{\varepsilon}{3}.$$

Note that the volume of the ball $B_{\mathbb{R}^d}(\boldsymbol{x}_1;\delta)$ is given by $\delta^d\pi^{d/2}/\Gamma(1 + d/2)$ (see (1.1)). Hence, it is independent of $\boldsymbol{x}_1$. Furthermore, due to

$$\begin{aligned}|\boldsymbol{\zeta}_1| &= |\boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{v}(\boldsymbol{x}_1)| \\ &\le |\boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{y_0}(\hat{\boldsymbol{x}})| + |\boldsymbol{y_0}(\hat{\boldsymbol{x}}) - \boldsymbol{y_0}(\boldsymbol{x}_1)| + |\boldsymbol{y_0}(\boldsymbol{x}_1) - \boldsymbol{v}(\boldsymbol{x}_1)| \\ &\le 2r + 2\,\|\boldsymbol{y_0}\|\,,\end{aligned}$$

the integrand is bounded by M, so we have

$$I_2 \le M\,\mathrm{vol}(\mathcal{H}(\boldsymbol{x}_1) \setminus \mathcal{H}(\boldsymbol{x}_2)) < \frac{\varepsilon}{3}.$$

Analogously, I_3 can be estimated by $\varepsilon/3$. Altogether, there holds

$$|(K\boldsymbol{v})(\boldsymbol{x}_1) - (K\boldsymbol{v})(\boldsymbol{x}_2)| < \varepsilon,$$

so $K\boldsymbol{v} \in \mathcal{C}(\bar{\Omega})^d = X$ and K is well-defined.
Now we show Lipschitz continuity of the mapping $K : \bar{B}_X(\boldsymbol{y_0}; r) \to X$. Therefore,

let $\boldsymbol{v}, \boldsymbol{w} \in \bar{B}_X(\boldsymbol{y_0}; r)$ and $\boldsymbol{x} \in \bar{\Omega}$. Since $|\boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{v}(\boldsymbol{x})| \leq 2r + 2\|\boldsymbol{y_0}\|$ it follows that $\boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{v}(\boldsymbol{x})$ and $\boldsymbol{w}(\hat{\boldsymbol{x}}) - \boldsymbol{w}(\boldsymbol{x})$ lie in the ball $\bar{B}_{\mathbb{R}^d}(\boldsymbol{0}; 2r + 2\|\boldsymbol{y_0}\|)$. Thus, by the Lipschitz continuity of $\boldsymbol{f}$, there exists $L_f \in L^1(B_{\mathbb{R}^d}(\boldsymbol{0}; \delta))$ such that

$$\begin{aligned}&|(K\boldsymbol{v})(\boldsymbol{x}) - (K\boldsymbol{w})(\boldsymbol{x})|\\ &\leq \int_{\mathcal{H}(\boldsymbol{x})} L_f(\hat{\boldsymbol{x}} - \boldsymbol{x})\Big(|\boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{w}(\hat{\boldsymbol{x}})| + |\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{w}(\boldsymbol{x})| \Big) d\hat{\boldsymbol{x}}\\ &\leq 2\|L_f\|_{L^1(B_{\mathbb{R}^d}(\boldsymbol{0};\delta))} \|\boldsymbol{v} - \boldsymbol{w}\| .\end{aligned}$$

In conclusion, there exists $L_K > 0$ such that

$$\|K\boldsymbol{v} - K\boldsymbol{w}\| \leq L_K \|\boldsymbol{v} - \boldsymbol{w}\| ,$$

which proves the assertion. □

This yields the first existence result.

Theorem 4.1. *Let $X = \mathcal{C}(\bar{\Omega})^d$ and suppose that $\boldsymbol{f}$ satisfies the assumptions of Lemma 4.1. Let $\boldsymbol{b} \in \mathcal{C}([0,T], X)$, $1/\rho \in \mathcal{C}(\bar{\Omega})$ and $\boldsymbol{y_0}, \dot{\boldsymbol{y}}_\boldsymbol{0} \in X$. Then the peridynamic abstract initial value problem (4.1) has a unique solution $\boldsymbol{y} \in \mathcal{C}^2(I, X)$ on an interval $I \subseteq [0, T]$, which depends continuously on the initial data.*

Proof. This is a direct consequence of Lemma 4.1 and Theorem 3.1. □

Two examples of pairwise force functions proposed in Silling [11], namely (2.5) and (2.6), fulfill the assumptions of this theorem.
The solution of the peridynamic abstract initial value problem (4.1) can be extended on some maximal existence time interval.

Theorem 4.2. *Let $X = \mathcal{C}(\bar{\Omega})^d$. Assume that $\boldsymbol{f} : \bar{B}_{\mathbb{R}^d}(\boldsymbol{0}; \delta) \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous, $\boldsymbol{y_0}, \dot{\boldsymbol{y}}_\boldsymbol{0} \in X$, $1/\rho \in \mathcal{C}(\bar{\Omega})$ and $\boldsymbol{b}$ is a continuous mapping from $[0, \infty)$ into X with $\sup_{t\in[0,\infty)} \|\boldsymbol{b}(t)\| < \infty$. Moreover, suppose that for all $R > 0$ the pairwise force function $\boldsymbol{f}$ restricted to the balls $\bar{B}_{\mathbb{R}^d}(\boldsymbol{0}; \delta) \times \bar{B}_{\mathbb{R}^d}(\boldsymbol{0}; R)$ is Lipschitz continuous in its second argument (in the sense of (4.3)). Then (4.1) has a unique maximal solution $\boldsymbol{y} \in \mathcal{C}^2(J_{max}, X)$ with $J_{max} = [0, T^*)$, $T^* \in (0, \infty]$. If the solution does not blow-up, then it exists globally in time with $T^* = \infty$.*

Proof. This theorem is proven by a modification of Theorem 3.2 for nonnegative time. Note that due to the assumptions, $\boldsymbol{g}$ is bounded on bounded subsets of X. □

Peridynamic theory was developed in order to be able to model cracks, which are spatial discontinuities in the deformation variable $\boldsymbol{y}$. Moving a step closer to that goal, we replace $\mathcal{C}(\bar{\Omega})^d$ by $L^\infty(\Omega)^d$.

Theorem 4.3. *Let $X = L^\infty(\Omega)^d$. Assume for some $r > 0$ that the pairwise force function*

$$\boldsymbol{f} : B_{\mathbb{R}^d}(\boldsymbol{0}; \delta) \times \bar{B}_{\mathbb{R}^d}(\boldsymbol{0}; 2r + 2\|\boldsymbol{y_0}\|) \to \mathbb{R}^d$$

is Lebesgue measurable in its first argument and Lipschitz continuous in its second argument, in the sense that there exists a nonnegative function $L_f \in L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))$ *such that for almost all* $\boldsymbol{\xi} \in B_{\mathbb{R}^d}(\mathbf{0};\delta)$ *and all* $\boldsymbol{\zeta_1}, \boldsymbol{\zeta_2} \in \bar{B}_{\mathbb{R}^d}(\mathbf{0}; 2r + 2\,\|\boldsymbol{y_0}\|)$ *there holds*

$$|\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}_1) - \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}_2)| \le L_f(\boldsymbol{\xi})\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,.$$

Moreover, suppose $\boldsymbol{y_0}, \dot{\boldsymbol{y}}_{\mathbf{0}} \in X$, $\boldsymbol{b} \in \mathcal{C}([0,T], X)$, $1/\rho \in L^\infty(\Omega)$ *and* $\boldsymbol{f}(\cdot, \mathbf{0}) \in L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))^d$. *Then the peridynamic abstract initial value problem (4.1) has a unique solution* $\boldsymbol{y} \in \mathcal{C}^2(I, X)$ *on some interval* $I \subseteq [0,T]$, *which continuously depends on the initial data.*

Proof. Because of Lemma 4.1, we have

$$\|K\boldsymbol{v}\| \le L_K\,\|\boldsymbol{v}\| + \|K\mathbf{0}\| \le L_K\,\|\boldsymbol{v}\| + \|\boldsymbol{f}(\cdot, \mathbf{0})\|_{L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))^d}\,,$$

and the function $\boldsymbol{g} : [0,T] \times \bar{B}_X(\boldsymbol{y_0}; r) \to X$ is well-defined. The continuity and the uniform Lipschitz continuity in the second argument result from Lemma 4.1 and the continuity of $\boldsymbol{b}$. By Theorem 3.1, the assertion is proven. □

The theorem above can be used to prove well-posedness for (4.1) with the example force functions stated above and discontinuous densities ρ.
Taking $X = L^p(\Omega)^d$ for $1 \le p < \infty$, we do not have almost everywhere pointwise estimates of the form

$$|\boldsymbol{y_0}(\boldsymbol{x})| \le C\,\|\boldsymbol{y_0}\|_{L^p(\Omega)^d}\,.$$

Thus, we cannot deduce uniform Lipschitz continuity of $\boldsymbol{g}$ on a ball centering $\boldsymbol{y_0}$ from Lipschitz continuity of $\boldsymbol{f}$ on a ball centering $\mathbf{0}$. Therefore, the assumptions have to be strengthened.

Lemma 4.2. *Let* $X = L^p(\Omega)^d$ *for* $1 \le p \le \infty$. *Assume that the pairwise force function* $\boldsymbol{f} : B_{\mathbb{R}^d}(\mathbf{0};\delta) \times \mathbb{R}^d \to \mathbb{R}^d$ *is measurable in its first argument and is Lipschitz continuous in its second argument, in the sense that there exists a nonnegative even function* $L_f \in L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))$ *such that for almost all* $\boldsymbol{\xi} \in B_{\mathbb{R}^d}(\mathbf{0};\delta)$ *and all* $\boldsymbol{\zeta_1}, \boldsymbol{\zeta_2} \in \mathbb{R}^d$ *there holds*

$$|\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}_1) - \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}_2)| \le L_f(\boldsymbol{\xi})\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,.$$

If $\boldsymbol{f}(\cdot, \mathbf{0}) \in L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))^d$ *then the peridynamic operator* $K : X \to X$ *is well-defined and Lipschitz continuous.*

Due to Newton's principle actio et reactio (2.2), we have

$$|\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}_1) - \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}_2)| = |\boldsymbol{f}(-\boldsymbol{\xi}, -\boldsymbol{\zeta}_1) - \boldsymbol{f}(-\boldsymbol{\xi}, -\boldsymbol{\zeta}_2)|\,,$$

so the assumption $L_f(\boldsymbol{\xi}) = L_f(-\boldsymbol{\xi})$ is no restriction.

Proof. Obviously, $K\boldsymbol{v}$ is measurable for every $\boldsymbol{v} \in L^p(\Omega)^d$. Once we have shown the Lipschitz continuity, by

$$\|K\mathbf{0}\| \le \begin{cases} (\mathrm{vol}(\Omega))^{1/p}\,\|\boldsymbol{f}(\cdot, \mathbf{0})\|_{L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))^d} & \text{if } 1 \le p < \infty\,, \\ \|\boldsymbol{f}(\cdot, \mathbf{0})\|_{L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))^d} & \text{if } p = \infty\,, \end{cases}$$

combined with

$$\|K\boldsymbol{v}\| \le \|K\boldsymbol{v} - K\mathbf{0}\| + \|K\mathbf{0}\| \le L_K \|\boldsymbol{v}\| + \|K\mathbf{0}\|,$$

we deduce that K is well-defined. For the Lipschitz continuity, we distinguish between three cases.
Case $1 < p < \infty$:
With the Hölder inequality and the conjugate exponent q defined by $\frac{1}{p} + \frac{1}{q} = 1$, we have for $\boldsymbol{v}, \boldsymbol{w} \in X$

$$\begin{aligned}
&\|K\boldsymbol{v} - K\boldsymbol{w}\|^p \\
&\le \int_\Omega \left(\int_{\mathcal{H}(\boldsymbol{x})} L_f(\hat{\boldsymbol{x}} - \boldsymbol{x}) \Big(|\boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{w}(\hat{\boldsymbol{x}})| + |\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{w}(\boldsymbol{x})| \Big) d\hat{\boldsymbol{x}} \right)^p d\boldsymbol{x} \\
&\le 2^{p-1} \|L_f\|^{p/q}_{L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))} \int_\Omega \int_{B_{\mathbb{R}^d}(\boldsymbol{x};\delta)} L_f(\hat{\boldsymbol{x}} - \boldsymbol{x}) \,|\boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{w}(\hat{\boldsymbol{x}})|^p \, d\hat{\boldsymbol{x}} d\boldsymbol{x} \\
&\quad + 2^{p-1} \|L_f\|^{p/q}_{L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))} \int_\Omega \int_{B_{\mathbb{R}^d}(\boldsymbol{x};\delta)} L_f(\hat{\boldsymbol{x}} - \boldsymbol{x}) \,|\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{w}(\boldsymbol{x})|^p \, d\hat{\boldsymbol{x}} d\boldsymbol{x}.
\end{aligned}$$

Writing the integration over the ball in the first term as an integration of the characteristic function $\chi_{[0,\delta)}(|\hat{\boldsymbol{x}} - \boldsymbol{x}|)$ over Ω, applying Fubini's theorem and the property $L(\boldsymbol{\xi}) = L(-\boldsymbol{\xi})$ yields

$$\|K\boldsymbol{v} - K\boldsymbol{w}\|^p \le 2^p \|L_f\|^p_{L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))} \|\boldsymbol{v} - \boldsymbol{w}\|^p.$$

Case $p = 1$:
Analogously to the previous case, we have

$$\begin{aligned}
&\|K\boldsymbol{v} - K\boldsymbol{w}\| \\
&\le \int_\Omega \int_{\mathcal{H}(\boldsymbol{x})} L_f(\hat{\boldsymbol{x}} - \boldsymbol{x}) \Big(|\boldsymbol{v}(\hat{\boldsymbol{x}}) - \boldsymbol{w}(\hat{\boldsymbol{x}})| + |\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{w}(\boldsymbol{x})| \Big) d\hat{\boldsymbol{x}} d\boldsymbol{x} \\
&\le 2 \|L_f\|_{L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))} \|\boldsymbol{v} - \boldsymbol{w}\|.
\end{aligned}$$

Case $p = \infty$:
This case is treated as in the proof of Lemma 4.1. We conclude that

$$\|K\boldsymbol{v} - K\boldsymbol{w}\| \le 2 \|L_f\|_{L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))} \|\boldsymbol{v} - \boldsymbol{w}\|.$$

Thus, K is Lipschitz continuous and the assertion holds. □

Now we are able to formulate the last theorem in this section proving global existence and uniqueness.

Theorem 4.4. *Let* $X = L^p(\Omega)^d$ *with* $1 \le p \le \infty$. *Assume* $\boldsymbol{b} \in \mathcal{C}([0,T], X)$, $1/\rho \in L^\infty(\Omega)$ *and* $\boldsymbol{y_0}, \dot{\boldsymbol{y}}_{\mathbf{0}} \in X$. *If* $\boldsymbol{f}$ *fulfills the assumptions of Lemma 4.2 then the peridynamic abstract initial value problem (4.1) has a unique global solution* $\boldsymbol{y} \in \mathcal{C}^2([0,T], X)$, *which depends continuously on the initial data.*

Proof. Since by Lemma 4.2, the operator K is well-defined and Lipschitz continuous,

$$\boldsymbol{g} : [0,T] \times X \to X, \quad \boldsymbol{g}(t,\boldsymbol{v}) = \frac{K\boldsymbol{v}}{\rho} + \frac{\boldsymbol{b}(t)}{\rho},$$

is well-defined, continuous in time and uniformly Lipschitz continuous in its second argument. By Theorem 3.3, the assertion is proven. □

Obviously, the linear case with $\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}) = \lambda(|\boldsymbol{\xi}|)\boldsymbol{\xi}\boldsymbol{\xi}^T(\boldsymbol{\zeta}-\boldsymbol{\xi})$ (see Emmrich and Weckner [8]) fits into this setting. Here, λ is a suitable nonnegative function as , e.g., $\lambda(r) = r^{-3}$ for $r < \delta$ (linearization of the bondstretch model). Apparently, the examples for pairwise force functions (2.5) and (2.6) are not globally Lipschitz continuous in their second arguments, and the above theorem cannot be applied. Furthermore, relying on the techniques employed in this paper, we are not able to formulate theorems handling pairwise force functions with discontinuities in the second argument such as the bondstretch model (2.4). This is open for future work.

In order to illustrate that compactness arguments (as employed, e.g., in the classical Peano theorem) are not at hand, we take a look at pairwise force functions of the form (2.3). For simplicity, the following arguments are only discussed in dimension $d = 1$. The peridynamic integral operator can then be written as a sum of two operators,

$$\begin{aligned}(Kv)(x) &= \int_{\mathcal{H}(x)} \phi(|\hat{x}-x|, |v(\hat{x})-v(x)|)(v(\hat{x})-v(x))d\hat{x} \\ &= \int_{\mathcal{H}(x)} \phi(|\hat{x}-x|, |v(\hat{x})-v(x)|)v(\hat{x})d\hat{x} \\ &\quad - \int_{\mathcal{H}(x)} \phi(|\hat{x}-x|, |v(\hat{x})-v(x)|)d\hat{x}\, v(x) \\ &=: (Av)(x) - (Bv)(x).\end{aligned}$$

If $\phi : [0,\delta] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the growth condition $|\phi(|\xi|, |\zeta|)| \leq \tilde{\phi}(|\xi|)$ with $\tilde{\phi} \in L^1(0,\delta)$ then, in view of the Arzelà-Ascoli theorem, the operator $A : \mathcal{C}(\bar{\Omega}) \to \mathcal{C}(\bar{\Omega})$ is compact. But unfortunately, the multiplication type operator $B : \mathcal{C}(\bar{\Omega}) \to \mathcal{C}(\bar{\Omega})$ is not compact in general. Indeed, let us consider the sequence of functions $v_n(x) = \cos(nx)$, $n = 1, 2, \ldots$, which is uniformly bounded. At points $x_1 = 0$, $x_2 = \pi/(2n)$ and for arbitrary $\tilde{\delta} > 0$ and $n > \pi/(2\tilde{\delta})$, we have on the one hand $|x_1 - x_2| < \tilde{\delta}$. On the other hand, we have

$$|(Bv)(x_1) - (Bv)(x_2)| = \left| \int_{-\delta}^{+\delta} \phi(|\hat{x}|, |\cos(n\hat{x}) - 1|)d\hat{x} \right|. \tag{4.4}$$

If B would be compact then, by the Arzelà-Ascoli theorem, Bv_n would be equicontinuous, which can only hold true if there does not exist an $\varepsilon > 0$ such that (4.4) is greater than ε. Thus (4.4) has to vanish, which cannot be expected in the general case. Finally, since sums of compact operators are compact, K cannot be compact in general.

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