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ORIGINAL ARTICLE

On a Multivalued Differential Equation with Nonlocality in Time

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Abstract
The initial value problem for a multivalued differential equation is studied, which is governed by the sum of a monotone, hemicontinuous, coercive operator fulfilling a certain growth condition and a Volterra integral operator in time of convolution type with exponential decay. The two operators act on different Banach spaces where one is not embedded in the other. The set-valued right-hand side is measurable and satisfies certain continuity and growth conditions. Existence of a solution is shown *via* a generalisation of the Kakutani fixed-point theorem.

Keywords Nonlinear evolution equation · Multivalued differential equation · Differential inclusion · Monotone operator · Volterra operator · Exponentially decaying memory · Existence · Kakutani fixed-point theorem

Mathematics Subject Classification (2010) 47J35 · 34G25 · 45K05 · 35R70

1 Introduction

1.1 Problem Statement and Main Result

We consider the multivalued differential equation[1]

$$\begin{aligned} v'(t) + Av(t) + (BKv)(t) &\in F(t, v(t)), \quad t \in (0, T), \\ v(0) &= v_0, \end{aligned} \tag{1}$$

[1] Also named differential inclusion by many authors. However, in order to distinguish this kind of problem from the ones containing subdifferentials or set-valued (maximal monotone) operators, we chose the name multivalued differential equation.

Dedicated to Volker Mehrmann on the occasion of his 65th birthday.

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where

$$(Kv)(t) = u_0 + \int_0^t k(t-s)v(s)\mathrm{d}s, \quad k(z) = \lambda e^{-\lambda z}. \tag{2}$$

Here, $T > 0$ defines the considered time interval, $\lambda > 0$ is a given parameter and v_0, u_0 are the given initial data of the problem.

The operator $A\colon V_A \to V_A^*$ is a monotone, hemicontinuous, coercive operator satisfying a certain growth condition, where V_A is a real, reflexive Banach space. The operator $B\colon V_B \to V_B^*$ is linear, bounded, strongly positive, and symmetric, where V_B denotes a real Hilbert space. The space V_A shall be compactly and densely embedded in a real Hilbert space H, whereas V_B shall be only continuously and densely embedded in H. The dual of H is identified with H itself, such that both V_A, H, V_A^* and V_B, H, V_B^* form a so-called Gelfand triple. However, we do not assume any relation between V_A and V_B apart from $V = V_A \cap V_B$ being separable and densely embedded in both V_A and V_B. We do not assume that V_A is embedded into V_B or the other way around. Overall, we have the scale

$$V_A \cap V_B = V \subset V_C \subset H = H^* \subset V_C^* \subset V^* = V_A^* + V_B^*, \quad C \in \{A, B\}, \tag{3}$$

of Banach and Hilbert spaces, where all embeddings are meant to be continuous and dense and the embedding $V_A \subset H$ is even meant to be compact.

The operator $F\colon [0, T] \times H \to P_{fc}(H)$ is measurable, fulfils a certain growth condition in the second argument and the graph of $v \mapsto F(t, v)$ is sequentially closed in $H \times H_w$ for almost all $t \in (0, T)$, where H_w denotes the Hilbert space H equipped with the weak topology. The set $P_{fc}(H)$ denotes the set of all nonempty, closed, and convex subsets of H.

Multivalued differential equations appear, e.g., in the formulation of optimal feedback control problems. If we consider the inclusion as an equation with a side condition on the right-hand side, i.e.,

$$\begin{aligned} v'(t) + Av(t) + (BKv)(t) &= f(t), && t \in (0, T), \\ f(t) &\in F(t, v(t)), && t \in (0, T), \\ v(0) &= v_0, \end{aligned}$$

we can consider f as the control of our system with the corresponding state v and F as the set of admissible controls, which, in the case of F depending on v, leads to a feedback control system.

Physical applications of the system we are considering in this work are, e.g., heat flow in materials with memory (see, e.g., MacCamy [25], Miller [30]) or viscoelastic fluid flow (see, e.g., Desch, Grimmer, and Schappacher [11], MacCamy [26]). Another application related to that are non-Fickian diffusion models which describe diffusion processes of a penetrant through a viscoelastic material (see, e.g., Edwards [13], Edwards and Cohen [14], Shaw and Whiteman [39]). They also appear, e.g., in mathematical biology (see, e.g., Cushing [8], Fedotov and Iomin [18], Mehrabian and Abousleiman [27]).

Due to the specific form of the kernel k given in (2), we can rewrite our system into the coupled system

$$\begin{aligned} v'(t) + Av(t) + Cu(t) &\in F(t, v(t)), \; t \in (0, T), \\ (u - Du_0)'(t) + \lambda(u - Du_0)(t) &= \lambda Dv(t), \quad t \in (0, T), \\ v(0) &= v_0, \\ u(0) &= Du_0, \end{aligned} \tag{4}$$

where C and D are suitably chosen linear operators such that $B = CD$.

Instead of the kernel $k(z) = \lambda e^{-\lambda z}$, we might also consider $k(z) = ce^{-\lambda z}$ with $c, \lambda > 0$. However, for simplicity, we will stick to the first type of kernel. Actually, this type appears naturally in many applications. In these applications, $\frac{1}{\lambda}$ is often describing a relaxation or averaged delay time. If we consider the limit $\lambda \to 0$, the system (4) decouples such that $u(t) = Du_0$, $t \in [0, T]$, is the solution of the second equation. In the case $\lambda \to \infty$, the system reduces to a single first-order equation for v without memory.

In the case of the kernel $k(z) = ce^{-\lambda z}$, the behaviour for $\lambda \to 0$ is slightly different. The limit then yields a second-order in time equation for u (see, e.g., Emmrich and Thalhammer [16]).

1.2 Literature Overview

This work is a continuation of Eikmeier, Emmrich, and Kreusler [15]. There, the single-valued instead of the multivalued differential equation is considered in the same setting concerning the spaces V_A and V_B. However, due to the structure of the proof in the present work, we additionally need the compact embedding $V_A \subset H$ and we have to assume that the right-hand side is pointwisely H-valued.

Nonlinear integro-differential equations have been considered by many authors through the years. Results on well-posedness for more general classes of nonlinear evolution equations including Volterra operators, but only in the case of Hilbert spaces $V_A = V_B$, can be found in, e.g., Gajewski, Gröger, and Zacharias [19]. In contrast to this, Crandall, Londen, and Nohel [7] study the case of a doubly nonlinear problem, where both nonlinear operators are assumed to be (possibly multivalued) maximal monotone subdifferential operators and the domain of definition of one of them has to be continuously and densely embedded in the domain of definition of the other one. For more references on nonlinear and also linear evolutionary integro-differential equations see Eikmeier, Emmrich, and Kreusler [15, Section 1.2].

Multivalued differential equations have also been studied by various authors. Basic results, also for set-valued analysis, can be found in, e.g., Aubin and Cellina [2], Aubin and Frankowska [3], or Deimling [9]. In O'Regan [32], some extensions of the results shown in Deimling [9] are presented. A semilinear multivalued differential equation with a linear, bounded, and strongly positive operator and a set-valued nonlinear operator is, e.g., considered in Beyn, Emmrich, and Rieger [4].

In particular, integro-differential equations in the multivalued case have been studied by, e.g., Papageorgiou [33–36]. The equations are considered under different assumptions with the set-valued operator appearing in the integral term. In most of the works mentioned, examples of applications in the theory of optimal control are given.

In the context of viscoelastic contact problems, evolution inclusions and hemivariational inequalities have been discussed in, e.g., Migórski [28] and Migórski, Ochal, and Sofonea [29].

The optimal feedback control of a motion of a viscoelastic fluid *via* a multivalued differential equation is, e.g., considered in Gori et al. [21] and Obukhovskiĭ, Zecca, and Zvyagin [31]. Existence of solutions for the equation are shown *via* topological degree theory.

1.3 Organisation of the Paper

The paper is organised as follows: In Section 2, we introduce the general notation and some basic results from set-valued analysis. In Section 3, we state our assumptions on the operators A, B, and F and some preliminary results concerning properties we need in the following Section 4, where we prove existence of a solution to problem (1). This is done *via* a generalisation of the Kakutani fixed-point theorem.

2 Notation

Let X be a Banach space with its dual X^*. The norm in X and the standard norm in X^* are denoted by $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$, respectively. The duality pairing between X and X^* is denoted by $\langle\cdot,\cdot\rangle$. If X is a Hilbert space, the inner product in X is denoted by $(\cdot,\cdot)$. For the intersection $X \cap Y$ of two Banach spaces X and Y, we consider the norm $\|\cdot\|_{X\cap Y} = \|\cdot\|_X + \|\cdot\|_Y$, and for the sum $X + Y$, we consider the norm

$$\|z\|_{X+Y} = \inf\{\max(\|z_X\|_X, \|z_Y\|_Y) \mid z = z_X + z_Y \text{ with } z_X \in X,\ z_Y \in Y\}.$$

Note that $(X\cap Y)^* = X^* + Y^*$ if X and Y are embedded in a locally convex space and $X\cap Y$ is dense in X and Y with respect to the norm above, see, e.g., Gajewski et al. [19, pp. 12ff.].

Now, let X be a real, reflexive, and separable Banach space and $1 \le p \le \infty$. By $L^p(0,T;X)$, we denote the usual space of Bochner measurable (sometimes also called strongly measurable), p-integrable functions equipped with the standard norm. For $1 \le p < \infty$, the duality pairing between $L^p(0,T;X)$ and its dual space $L^q(0,T;X^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$ for $p > 1$ and $q = \infty$ for $p = 1$, is also denoted by $\langle\cdot,\cdot\rangle$, and it is given by

$$\langle g, f\rangle = \int_0^T \langle g(t), f(t)\rangle \mathrm{d}t,$$

see, e.g., Diestel and Uhl [12, Theorem 1 on p. 98, Corollary 13 on p. 76, Theorem 1 on p. 79].

By $W^{1,p}(0,T;X)$, $1 \le p \le \infty$, we denote the usual space of weakly differentiable functions $u \in L^p(0,T;X)$ with $u' \in L^p(0,T;X)$, equipped with the standard norm. By $\mathscr{C}([0,T];X)$, we denote the space of functions that are continuous on $[0,T]$ with values in X, whereas $\mathscr{C}_w([0,T];X)$ denotes the space of functions that are continuous on $[0,T]$ with respect to the weak topology in X. We have the continuous embedding $W^{1,1}(0,T;X) \subset \mathscr{C}([0,T];X)$, see, e.g., Roubíček [38, Lemma 7.1]. Furthermore, a function $u \in W^{1,1}(0,T;X)$ is almost everywhere equal to a function that is absolutely continuous on $[0,T]$ with values in X, see, e.g., Brézis [5, Theorem 8.2]. We denote the set of all these functions by $\mathscr{AC}([0,T];X)$. By $\mathscr{C}^1([0,T])$, we denote the space of real-valued functions that are continuously differentiable on $[0,T]$. By c, we denote a generic positive constant.

Now, let us recall some definitions from set-valued analysis. Let (Ω,Σ) be a measurable space and let X be a complete separable metric space. By $\mathcal{L}([a,b])$ and $\mathcal{B}(X)$, we denote the Lebesgue σ-algebra on the interval $[a,b] \subset \mathbb{R}$ and the Borel σ-algebra on X, respectively. By $P_f(X)$, we denote the set of all nonempty and closed subsets $U \subset X$, and by $P_{fc}(X)$, we denote the set of all nonempty, closed, and convex subsets $U \subset X$.

For a set-valued function $F\colon \Omega \to 2^X \setminus \{\emptyset\}$, let

$$|F(\omega)| := \sup\{\|x\|_X \mid x \in F(\omega)\}, \quad \omega \in \Omega.$$

The graph of such a set-valued function is defined as

$$\operatorname{graph}(F) = \{(\omega, x) \in \Omega \times X \mid x \in F(\omega)\}.$$

A function $F\colon \Omega \to P_f(X)$ is called measurable (sometimes also called weakly measurable) if the preimage of each open set is measurable, i.e.,

$$F^{-1}(U) := \{\omega \in \Omega \mid F(\omega) \cap U \neq \emptyset\} \in \Sigma$$

for every open $U \subset X$.[2] A function $f\colon \Omega \to X$ is called measurable selection of F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$ and f is measurable. Each measurable set-valued function has a measurable selection, see, e.g., Aubin and Frankowska [3, Theorem 8.1.3].

Now, let (Ω, Σ, μ) be a complete σ-finite measure space and let X be a separable Banach space. For a set-valued function $F\colon \Omega \to P_f(X)$ and $p \in [1, \infty)$, we denote by $\mathcal{F}^p$ the set of all p-integrable selections of F, i.e.,

$$\mathcal{F}^p := \left\{f \in L^p(\Omega; X, \mu) \mid f(\omega) \in F(\omega) \text{ a.e. in } \Omega\right\},$$

where $L^p(\Omega; X, \mu)$ denotes the space of Bochner measurable, p-integrable functions with respect to μ.[3] If F is integrably bounded, i.e., there exists a nonnegative function $m \in L^p(\Omega; \mathbb{R}, \mu)$ such that $F(\omega) \subset m(\omega)B_X$ for μ-almost all $\omega \in \Omega$, where B_X denotes the unit ball in X, each measurable selection of F is in $\mathcal{F}^p$ due to Lebesgue's theorem on dominated convergence. The integral of F is defined as

$$\int_\Omega F \mathrm{d}\mu := \left\{\int_\Omega f \mathrm{d}\mu \mid f \in \mathcal{F}^1\right\}.$$

For properties of this integral, see, e.g., Aubin and Frankowska [3, Chapter 8.6].

For a set-valued function $F\colon \Omega \times X \to P_f(X)$, a function $v\colon \Omega \to X$ and $p \in [1, \infty)$, we denote by $\mathcal{F}^p(v)$ the set of all p-integrable selections of the mapping $\omega \mapsto F(\omega, v(\omega))$, i.e.,

$$\mathcal{F}^p(v) := \left\{f \in L^p(\Omega; X, \mu) \mid f(\omega) \in F(\omega, v(\omega)) \text{ a.e. in } \Omega\right\}.$$

Finally, let X, Y be Banach spaces and $\Omega \subset Y$. A set-valued function $F\colon \Omega \to 2^X \setminus \{\emptyset\}$ is called upper semicontinuous if $F^{-1}(U)$ is closed in Ω for all closed $U \subset X$.

3 Main Assumptions and Preliminary Results

Throughout this paper, let V_A be a real, reflexive Banach space and let V_B and H be real Hilbert spaces, respectively. As mentioned in Section 1, we require that $V = V_A \cap V_B$ is separable and the embeddings stated in (3) are fulfilled (with the embedding $V_A \subset H$ meant to be compact). Let also $2 \leq p < \infty$, $1 < q \leq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$.

For $A\colon V_A \to V_A^*$, we say the assumptions **(A)** are fulfilled if

i) A is monotone,
ii) A is hemicontinuous, i.e., $\theta \mapsto \langle A(u+\theta v), w\rangle$ is continuous on $[0, 1]$ for all $u, v, w \in V_A$,
iii) A fulfils a growth condition of order $p-1$, i.e., there exists $\beta_A > 0$ such that

$$\|Av\|_{V_A^*} \leq \beta_A \left(1 + \|v\|_{V_A}^{p-1}\right)$$

for all $v \in V_A$,

[2] Depending on the assumptions on (Ω, Σ) and X, there are many equivalent definitions of measurability for set-valued functions, see, e.g., Denkowski, Migórski, and Papageorgiou [10, Theorem 4.3.4].

[3] Note that in the case of a separable Banach space X, the Bochner measurability of f coincides with the Σ-$\mathcal{B}(X)$-measurability, see, e.g., Amann and Escher [1, Chapter X, Theorem 1.4], Denkowski, Migórski, and Papageorgiou [10, Corollary 3.10.5], or Papageorgiou and Winkert [37, Theorem 4.2.4]

iv) A is p-coercive, i.e., there exist $\mu_A > 0$, $c_A \geq 0$ such that

$$\langle Av, v\rangle \geq \mu_A \|v\|_{V_A}^p - c_A$$

for all $v \in V_A$.

One operator satisfying these assumptions is, e.g., the p-Laplacian $-\Delta_p = -\nabla\cdot(|\nabla|^{p-2}\nabla)$ acting between the standard Sobolev spaces $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ for a bounded Lipschitz domain Ω, see, e.g., Zeidler [40, p. 489]. For $B\colon V_B \to V_B^*$, we say the assumptions **(B)** are fulfilled if

i) B is linear,
ii) B is bounded, i.e., there exists $\beta_B > 0$ such that

$$\|Bv\|_* \leq \beta_B \|v\|$$

for all $v \in V_B$,
iii) B is strongly positive, i.e., there exists $\mu_B > 0$ such that

$$\langle Bv, v\rangle \geq \mu_B \|v\|^2$$

for all $v \in V_B$,
iv) B is symmetric.

Following these assumptions, B induces a norm $\|\cdot\|_B := \langle B\cdot,\cdot\rangle^{1/2}$ in V_B that is equivalent to $\|\cdot\|_{V_B}$. Therefore, we denote the space $L^2(0,T;(V_B,\|\cdot\|_B))$ by $L^2(0,T;B)$. An example for an operator satisfying these assumptions is the Laplacian $-\Delta$ acting between the standard Sobolev spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, again for a bounded Lipschitz domain Ω, as well as the fractional Laplacian $(-\Delta)^s$ for $\frac{1}{2} < s < 1$, acting between the standard Sobolev–Slobodeckiĭ spaces $H_0^s(\Omega)$ and $H^{-s}(\Omega)$.

Finally, we say that $F\colon [0,T]\times H \to P_{fc}(H)$ fulfils the assumptions **(F)** if

i) F is measurable,
ii) for almost all $t \in (0,T)$, the graph of the mapping $v \mapsto F(t,v)$ is sequentially closed in $H \times H_w$, where H_w denotes the Hilbert space H equipped with the weak topology,
iii) $|F(t,v)| \leq a(t) + b\|v\|_H^{2/q}$ a.e. with $a \in L^q(0,T)$, $a(t) \geq 0$ a.e. and $b > 0$.

Note that it is also possible to consider $A\colon [0,T]\times V_A \to V_A^*$ and $B\colon [0,T]\times V_B \to V_B^*$, where the mappings $t \mapsto A(t,v)$, $v \in V_A$, and $t \mapsto B(t,v)$, $v \in V_B$, are assumed to be measurable and all the assumptions above are assumed to hold uniformly in t. However, for simplicity, we will only consider the case of autonomous operators A and B.

These operators can be extended to operators defined on $L^p(0,T;V_A)$ and $L^1(0,T;V_B)$, respectively. The monotonicity and hemicontinuity of $A\colon V_A \to V_A^*$ imply demicontinuity (see, e.g., Zeidler [40, Proposition 26.4 on p. 555]). Due to the separability of V_A^*, the theorem of Pettis (see, e.g., Diestel and Uhl [12, Theorem 2 on p. 42]) then implies that A maps Bochner measurable functions $v\colon [0,T] \to V_A$ into Bochner measurable functions $Av\colon [0,T] \to V_A^*$, where $(Av)(t) = Av(t)$ for almost all $t \in (0,T)$. Due to the growth condition, we have the estimate

$$\|Av\|_{L^q(0,T;V_A^*)} \leq c\left(1 + \|v\|_{L^p(0,T;V_A)}^{p-1}\right) \tag{5}$$

for all $v \in L^p(0,T;V_A)$, i.e., A maps $L^p(0,T;V_A)$ into $L^q(0,T;V_A^*)$.

Via the same definition $(Bv)(t) = Bv(t)$ for a function $v\colon [0,T] \to V_B$, we can extend the operator $B\colon V_B \to V_B^*$ to a linear, bounded, strongly positive, and symmetric operator mapping $L^2(0,T;V_B)$ into its dual or to a linear, bounded operator mapping $L^r(0,T;V_B)$ into $L^r(0,T;V_B^*)$, $1 \le r \le \infty$, respectively.

Due to the definition (2) of the operator K, we have the following lemma.

Lemma 1 *Let X be an arbitrary Banach space, $k(z) = \lambda e^{-\lambda z}$, $\lambda > 0$, $u_0 \in X$. The operator $K\colon L^2(0,T;X) \to L^2(0,T;X)$ is well-defined, affine-linear, and bounded. The estimate*

$$\|Kv - u_0\|_{L^2(0,T;X)} \le \|k\|_{L^1(0,T)} \|v\|_{L^2(0,T;X)}$$

is satisfied for all $v \in L^2(0,T;X)$, where $\|k\|_{L^1(0,T)} = 1 - e^{-\lambda T}$. Further, the estimate

$$\|Kv - u_0\|_{\mathscr{C}([0,T];X)} \le \lambda \|v\|_{L^1(0,T;X)}$$

is satisfied for all $v \in L^1(0,T;X)$, i.e., K is also an affine-linear, bounded operator of $L^1(0,T;X)$ into $\mathscr{C}([0,T];X)$ (even $\mathscr{AC}([0,T];X)$).

The proof is based on simple calculations, therefore we omit it here. Following this lemma, we obtain the following properties of the operator BK.

Corollary 1 *Let the assumptions of Lemma 1 (with $X = V_B$) and assumption **(B)** be fulfilled. Then the operator $BK\colon L^2(0,T;V_B) \to L^2(0,T;V_B^*)$ is well-defined, affine-linear, and bounded. The same holds for $BK\colon L^1(0,T;V_B) \to \mathscr{C}([0,T];V_B^*)$.*

One crucial relation in this setting, resulting from the exponential kernel, is the following one. Let X be an arbitrary Banach space and $v \in L^1(0,T;X)$. Then we have

$$(Kv)'(t) = \lambda\big(v(t) - ((Kv)(t) - u_0)\big) \tag{6}$$

for almost all $t \in (0,T)$.

Concerning the operator F, we need a measurability result in order to be able to extract measurable selections of the multivalued mapping $t \mapsto F(t,u(t))$, where u itself is a measurable function.

Lemma 2 *Let X be a separable Banach space, let $F\colon [0,T]\times X \to P_f(X)$ be measurable and let $v\colon [0,T] \to X$ be Bochner measurable. Then the mapping $\tilde{F}_v\colon [0,T] \to P_f(X)$, $t \mapsto F(t,v(t))$, is measurable.*

Proof Let $U \subset X$ be open. Consider

$$\begin{aligned}\tilde{F}_v^{-1}(U) &= \{t \in [0,T] \mid F(t,v(t)) \cap U \neq \emptyset\}\\ &= \pi_{[0,T]}(\{(t,x) \in [0,T]\times X \mid F(t,x)\cap U \neq \emptyset, x = v(t)\})\\ &= \pi_{[0,T]}(\{(t,x) \in [0,T]\times X \mid F(t,x)\cap U \neq \emptyset\} \cap \operatorname{graph}(v)),\end{aligned}$$

where $\pi_{[0,T]}$ denotes the projection onto $[0,T]$. Since v is Bochner measurable, its graph belongs to $\mathcal{L}([0,T]) \otimes \mathcal{B}(X)$, see, e.g., Castaing and Valadier [6, Theorem III.36]. Note again that for a separable Banach space X, Bochner measurability and $\mathcal{L}([0,T])$-$\mathcal{B}(X)$-measurability are equivalent, see, e.g., Denkowski, Migórski, and Papageorgiou [10, Corollary 3.10.5]. Due to the measurability of F, the intersection space in the equation above also belongs to $\mathcal{L}([0,T]) \otimes \mathcal{B}(X)$. Since the projection maps measurable sets into measurable sets (at least in this setting, see, e.g., Castaing and Valadier [6, Theorem III.23]), we have $\tilde{F}_v^{-1}(U) \in \mathcal{L}([0,T])$, which finishes the proof. □

Finally, we need an integration-by-parts formula similar to the one provided in Roubíček [38, Lemma 7.3] for functions in the spaces $\mathcal{W}(0,T) := \{v \in L^p(0,T;V_A) \mid v' \in L^q(0,T;V^*)\}$ and $\mathscr{C}^1([0,T];V)$, respectively.

Lemma 3 *Let $v \in \mathcal{W}(0,T)$, $w \in \mathscr{C}^1([0,T];V)$. Then the integration-by-parts formula*

$$\int_{t_1}^{t_2} \big(\langle v'(t), w(t)\rangle + \langle w'(t), v(t)\rangle\big)\,\mathrm{d}t = \langle v(t_2), w(t_2)\rangle - \langle v(t_1), w(t_1)\rangle \tag{7}$$

holds for all $t_1, t_2 \in [0,T]$.

Proof Due to the density of $\mathscr{C}^1([0,T];V_A)$ in $\mathcal{W}(0,T)$ (see, e.g., Roubíček [38, Lemma 7.2]), there exists a sequence $\{v_n\} \subset \mathscr{C}^1([0,T];V_A)$ such that $v_n \to v \in \mathcal{W}(0,T)$. The formula (7) obviously holds for v_n, $n \in \mathbb{N}$, and w due to classical calculus. Therefore, we have

$$\int_{t_1}^{t_2} \big(\langle v_n'(t), w(t)\rangle + \langle w'(t), v_n(t)\rangle\big)\,\mathrm{d}t = \langle v_n(t_2), w(t_2)\rangle - \langle v_n(t_1), w(t_1)\rangle$$

for all $n \in \mathbb{N}$. The convergence $v_n \to v$ in $\mathcal{W}(0,T)$ yields

$$\int_{t_1}^{t_2} \big(\langle v_n'(t), w(t)\rangle + \langle w'(t), v_n(t)\rangle\big)\,\mathrm{d}t \to \int_{t_1}^{t_2} \big(\langle v'(t), w(t)\rangle + \langle w'(t), v(t)\rangle\big)\,\mathrm{d}t.$$

On the other hand, the continuous embedding $\mathcal{W}(0,T) \subset \mathscr{C}([0,T];V^*)$ (see, e.g., Roubíček [38, Lemma 7.1]) yields

$$\langle v_n(t), w(t)\rangle \to \langle v(t), w(t)\rangle$$

for all $t \in [0,T]$, in particular for t_1, t_2. This finishes the proof.

□

4 Existence of a Solution

Theorem 1 *Let the assumptions* **(A)**, **(B)**, *and* **(F)** *be fulfilled and let $u_0 \in V_B$, $v_0 \in H$ be given. Then there exists a solution $v \in L^p(0,T;V_A) \cap \mathscr{C}_w([0,T];H)$ to (1) with $Kv \in \mathscr{C}_w([0,T];V_B)$ and $v' \in L^q(0,T;V_A^*) + L^\infty(0,T;V_B^*)$, i.e., the initial condition is fulfilled in H and there exists $f \in \mathcal{F}^1(v)$ such that the equation*

$$v' + Av + BKv = f$$

holds in the sense of $L^q(0,T;V_A^*)$, i.e.,

$$\langle v' + BKv, \varphi\rangle + \langle Av, \varphi\rangle = \langle f, \varphi\rangle$$

for all $\varphi \in L^p(0,T;V_A)$.

Proof Following the proof of [35, Theorem 3.1], we want to apply the Kakutani fixed-point theorem, generalised by Glicksberg [20] and Fan [17] to infinite-dimensional locally convex topological vector spaces. The same fixed-point theorem has also been applied in, e.g., Kalita, Migórski, and Sofonea [22] and Kalita, Szafraniec, and Shillor [23] to prove existence of solutions to partial differential inclusions.

First, we need *a priori* estimates for the solution. Assume $v \in L^p(0,T;V_A) \cap \mathscr{C}_w([0,T];H)$ solves problem (1) with the regularity stated in the theorem. Due to

Lemma 2, there exists a measurable selection $f\colon [0,T] \to H$ of the mapping $t \mapsto F(t, v(t))$. The growth condition of F implies

$$\|f(t)\|_H \le a(t) + b\|v(t)\|_H^{2/q}$$

for almost all $t \in (0,T)$, and since $a \in L^q(0,T)$ and $v \in \mathscr{C}_w([0,T]; H)$, we have $f \in L^q(0,T;H)$. Now, test the equation

$$v' + Av + BKv = f$$

with v and integrate the resulting equation over $(0,t)$, $t \in [0,T]$, which yields

$$\int_0^t \langle v'(s) + (BKv)(s), v(s)\rangle \mathrm{d}s + \int_0^t \langle Av(s), v(s)\rangle \mathrm{d}s = \int_0^t \langle f(s), v(s)\rangle \mathrm{d}s.$$

Since we neither know $v' \in L^q(0,T;V_A^*)$ nor $BKv \in L^q(0,T;V_A^*)$, it is not possible to do integration by parts for each term separately. However, [15, Lemma 4.3] yields

$$\begin{aligned}\int_0^t \langle v'(s) + (BKv)(s), v(s)\rangle \mathrm{d}s &= \frac{1}{2}\|v(t)\|_H^2 - \frac{1}{2}\|v_0\|_H^2 + \frac{1}{2\lambda}\|(Kv)(t)\|_B^2 - \frac{1}{2\lambda}\|u_0\|_B^2 \\ &\quad - \int_0^t \langle (BKv)(s), u_0\rangle \mathrm{d}s + \int_0^t \|(Kv)(s)\|_B^2 \mathrm{d}s.\end{aligned}$$

Due to the coercivity of A and Young's inequality, we have

$$\begin{aligned}&\frac{1}{2}\|v(t)\|_H^2 + \mu_A\int_0^t \|v(s)\|_{V_A}^p \mathrm{d}s + \frac{1}{2\lambda}\|(Kv)(t)\|_B^2 + \int_0^t \|(Kv)(s)\|_B^2 \mathrm{d}s \\ &\le c_A T + \frac{1}{2}\|v_0\|_H^2 + \frac{1}{2\lambda}\|u_0\|_B^2 + \int_0^t \|f(s)\|_{V_A^*}\|v(s)\|_{V_A}\mathrm{d}s + \int_0^t \|(Kv)(s)\|_B\|u_0\|_B \mathrm{d}s \\ &\le c_A T + \frac{1}{2}\|v_0\|_H^2 + \frac{1}{2\lambda}\|u_0\|_B^2 + c\int_0^t \|f(s)\|_{V_A^*}^q \mathrm{d}s + \frac{\mu_A}{2}\int_0^t \|v(s)\|_{V_A}^p \mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \|(Kv)(s)\|_B^2 \mathrm{d}s + \frac{T}{2}\|u_0\|_B^2.\end{aligned}$$

After rearranging, the estimate on F yields

$$\begin{aligned}&\frac{1}{2}\|v(t)\|_H^2 + \frac{\mu_A}{2}\int_0^t \|v(s)\|_{V_A}^p \mathrm{d}s + \frac{1}{2\lambda}\|(Kv)(t)\|_B^2 + \frac{1}{2}\int_0^t \|(Kv)(s)\|_B^2 \mathrm{d}s \\ &\le c\left(1 + \|v_0\|_H^2 + \|u_0\|_B^2 + \int_0^t \|f(s)\|_{V_A^*}^q \mathrm{d}s\right) \\ &\le c\left(1 + \|v_0\|_H^2 + \|u_0\|_B^2 + \int_0^t \left(a(s) + b\|v(s)\|_H^{2/q}\right)^q \mathrm{d}s\right) \\ &\le c\left(1 + \|v_0\|_H^2 + \|u_0\|_B^2 + \|a\|_{L^q(0,T)}^q + \int_0^t \|v(s)\|_H^2 \mathrm{d}s\right).\end{aligned}$$

Applying Gronwall's lemma, we obtain

$$\|v(t)\|_H^2 \le M_1 \tag{8}$$

for all $t \in [0,T]$, where $M_1 > 0$ depends on the problem data. This also immediately yields

$$\int_0^t \|v(s)\|_{V_A}^2 \mathrm{d}s \le M_2 \tag{9}$$

as well as

$$\|(Kv)(t)\|_B^2 \le M_2 \tag{10}$$

for all $t \in [0, T]$, where $M_2 > 0$ also depends on the problem data.

We also need *a priori* estimates for the derivative v'. Due to the estimate (5) and the assumptions **(F)**, we have

$$\begin{aligned}
&\|v'\|_{L^q(0,T;V_A^*)+L^\infty(0,T;V_B^*)} \\
&\le \max\Big(\|Av\|_{L^q(0,T;V_A^*)} + \|f\|_{L^q(0,T;V_A^*)},\ \|BKv\|_{L^\infty(0,T;V_B^*)}\Big) \\
&\le \max\Big(c\Big(1+\|v\|^{p-1}_{L^p(0,T;V_A)}\Big) + \|a\|_{L^q(0,T)} + b\|v\|^{2/q}_{L^2(0,T;H)},\ \beta_B\|Kv\|_{L^\infty(0,T;V_B)}\Big). \quad (11)
\end{aligned}$$

The *a priori* estimates (8), (9), and (10) above yield

$$\|v'\|_{L^q(0,T;V_A^*)+L^\infty(0,T;V_B^*)} \le M_3, \tag{12}$$

where M_3 depends on the same parameters as M_1 and M_2 as well as on β_B.

Next, we define the truncation $\hat{F}$ of F *via*

$$\hat{F}(t,w) = \begin{cases} F(t,w) & \text{if } \|w\|_H \le M_1, \\ F\left(t, \frac{M_1}{\|w\|_H} w\right) & \text{if } \|w\|_H > M_1. \end{cases}$$

This set-valued function $\hat{F}$ has the same measurability and continuity properties as F: In order to prove the measurability, consider an arbitrary open subset $U \subset H$ and

$$\begin{aligned}
\hat{F}^{-1}(U) &= \big\{(t,v) \in [0,T]\times H \mid F(t, r_{M_1}(v)) \cap U \neq \emptyset\big\} \\
&= \{(t,v) \in [0,T]\times H \mid F(t,v) \cap U \neq \emptyset\} \cap \Big([0,T]\times B^H_{M_1}\Big),
\end{aligned}$$

where r_{M_1} is the radial retraction in H to the ball $B^H_{M_1}$ of radius M_1. Due to the measurability of F, the first set is an element of $\mathcal{L}([0,T]) \otimes \mathcal{B}(H)$, and since the second set is obviously an element of the same σ-algebra, $\hat{F}$ is measurable.

For proving that $\hat{F}$ fulfils the same continuity condition as F, let $N \subset [0,T]$ be the set of Lebesgue-measure 0 such that the graph of $v \mapsto F(t,v)$ is sequentially closed in $H \times H_w$ for all $t \in [0,T] \setminus N$. Now, for arbitrary $t \in [0,T]\setminus N$, consider a sequence $\{(v_n, w_n)\} \subset \operatorname{graph}(\hat{F}(t,\cdot))$ with $v_n \to v$ and $w_n \rightharpoonup w$ for some $v, w \in H$. We have to show $w \in \hat{F}(t,v)$. Since the radial retraction r_{M_1} in H is continuous, we have $r_{M_1}(v_n) \to r_{M_1}(v)$ in H. Then, $w_n \in \hat{F}(t,v_n) = F(t, r_{M_1}(v_n))$ and the continuity condition on F immediately yield $w \in F(t, r_{M_1}(v)) = \hat{F}(t,v)$.

Due to the estimate on F in the assumptions **(F)**, we have

$$|\hat{F}(t,v)| \le \hat{a}(t) := a(t) + bM_1^{2/q}$$

for almost all $t \in (0,T)$ and all $v \in H$. Now, set

$$E := \{f \in L^q(0,T;H) \mid \|f(t)\|_H \le \hat{a}(t) \text{ a.e.}\}.$$

We define the solution operator

$$G\colon E \to \overline{W}(0,T) := \big\{v \in L^p(0,T;V_A) \mid v' \in L^q(0,T;V_A^*) + L^\infty(0,T;V_B^*)\big\}$$

with $G(f) = v$, where v is the unique solution to the problem

$$\begin{aligned} v' + Av + BKv &= f, \\ v(0) &= v_0, \end{aligned} \tag{13}$$

in the sense of $L^q(0,T;V_A^*)$, which exists due to [15, Theorem 4.2, Corollary 5.2]. Note that

$$\begin{aligned} v' = f - Av - BKv &\in L^q(0,T;H) + L^q(0,T;V_A^*) + L^\infty(0,T;V_B^*) \\ &\subset L^q(0,T;V_A^*) + L^\infty(0,T;V_B^*) \end{aligned}$$

in this case. Note also that $v \in \mathscr{C}_w([0,T];H)$ and $Kv \in \mathscr{C}_w([0,T];V_B)$ by [15, Theorem 4.2]. Now, the aim is to show that G is sequentially weakly continuous.

We therefore consider a sequence $\{f_n\} \subset E$ and $f \in E$ with $f_n \rightharpoonup f$ in $L^q(0,T;H)$. Analogously to the proof of the *a priori* estimates (8), (9), (10), and (12), it can be shown that the sequence $\{v_n\}$ of corresponding solutions, i.e., $v_n = G(f_n)$, the sequence $\{v_n'\}$ of derivatives, and the sequence $\{Kv_n\}$ are bounded in $L^p(0,T;V_A) \cap L^\infty(0,T;H)$, $L^q(0,T;V_A^*) + L^\infty(0,T;V_B^*)$, and $L^\infty(0,T;V_B)$, respectively. Due to the estimate (5) on A, this implies the boundedness of the sequences $\{Av_n\}$ in $L^q(0,T;V_A^*)$, see also (11). Since $L^p(0,T;V_A)$ is a reflexive Banach space and $L^\infty(0,T;H)$, $L^q(0,T;V_A^*)$, $L^\infty(0,T;V_B)$ as well as $L^q(0,T;V_A^*) + L^\infty(0,T;V_B^*)$ are duals of separable normed spaces, there exists a subsequence (again denoted by n) and $v \in L^p(0,T;V_A) \cap L^\infty(0,T;H)$, $\hat{v} \in L^q(0,T;V_A^*) + L^\infty(0,T;V_B^*)$, $\tilde{a} \in L^q(0,T;V_A^*)$, and $u \in L^\infty(0,T;V_B)$ such that

$$\begin{aligned} v_n &\rightharpoonup v && \text{in } L^p(0,T;V_A), \\ v_n &\overset{*}{\rightharpoonup} v && \text{in } L^\infty(0,T;H), \\ v_n' &\overset{*}{\rightharpoonup} \hat{v} && \text{in } L^q(0,T;V_A^*) + L^\infty(0,T;V_B^*), \\ Av_n &\overset{*}{\rightharpoonup} \tilde{a} && \text{in } L^q(0,T;V_A^*), \\ Kv_n &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;V_B), \end{aligned}$$

as $n \to \infty$. We obviously have $\hat{v} = v'$. As $v \in L^p(0,T;V_A)$ and $v' \in L^q(0,T;V^*)$, we have $v \in \mathscr{C}([0,T];V^*)$, see, e.g., Roubíček [38, Lemma 7.1]. Together with $v \in L^\infty(0,T;H)$, this yields $v \in \mathscr{C}_w([0,T];H)$, see, e.g., Lions and Magenes [24, Chapter 3, Lemma 8.1].

In order to show that G is sequentially weakly continuous, we have to pass to the limit in the equation

$$v_n' + Av_n + BKv_n = f_n \tag{14}$$

and show that v is a solution to problem (13). First, we want to show $u = Kv$. We know that the operator $\hat{K}\colon L^2(0,T;H) \to L^2(0,T;H)$ with $\hat{K}w := Kw - u_0$ is well-defined, linear, and bounded, see Lemma 1. Thus it is weakly-weakly continuous and $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(0,T;H)$ (and therefore $v_n \rightharpoonup v$ in $L^2(0,T;H)$) implies $Kv_n - Kv = \hat{K}v_n - \hat{K}v \rightharpoonup 0$ in $L^2(0,T;H)$. This yields $u = Kv$. Due to the linearity and boundedness of $B\colon V_B \to V_B^*$ and thus its weakly*-weakly* continuity, we also have $BKv_n \overset{*}{\rightharpoonup} Bu = BKv$ in $L^\infty(0,T;V_B^*)$.

Next, let us show $v(0) = v_0$ and $v_n(T) \rightharpoonup v(T)$ in H. Due to estimate (8), the sequence $\{v_n(T)\}$ is bounded in H, so there exists $v_T \in H$ such that, up to a subsequence, $v_n(T) \rightharpoonup v_T$ in H. Now, consider $\phi \in \mathscr{C}^1([0,T])$, $w \in V$ (recall that $V = V_A \cap V_B$). Since v_n solves (14) in the sense of $L^q(0,T;V_A^*)$ and v solves

$$v' + \tilde{a} + BKv = f \tag{15}$$

in the sense of $L^q(0,T;V_A^*)$, the integration-by-parts formula in Lemma 3 and the regularity $v \in \mathscr{C}_w([0,T];H)$ yield

$$\begin{aligned}
&(v_n(T), w)\,\phi(T) - (v_n(0), w)\,\phi(0)\\
&= \int_0^T \langle v_n'(t), w\rangle \phi(t)\mathrm{d}t + \int_0^T \langle v_n(t), w\rangle \phi'(t)\mathrm{d}t\\
&= \int_0^T \langle f_n - Av_n - BKv_n, w\rangle \phi(t)\mathrm{d}t + \int_0^T \langle v_n(t), w\rangle \phi'(t)\mathrm{d}t\\
&\to \int_0^T \langle f - \tilde{a} - BKv, w\rangle \phi(t)\mathrm{d}t + \int_0^T \langle v(t), w\rangle \phi'(t)\mathrm{d}t\\
&= \int_0^T \langle v', w\rangle \phi(t)\mathrm{d}t + \int_0^T \langle v(t), w\rangle \phi'(t)\mathrm{d}t\\
&= \big(v(T), w\big)\phi(T) - (v(0), w)\,\phi(0)
\end{aligned}$$

as $n \to \infty$. Choosing $\phi(t) = 1 - \frac{t}{T}$, this yields $(v_n(0), w) \to (v(0), w)$ for all $w \in V$. Due to $v_n(0) = v_0$ for all $n \in \mathbb{N}$, we have $v(0) = v_0$. Choosing $\phi(t) = \frac{t}{T}$, we get $(v_n(T), w) \to (v(T), w)$ for all $w \in V$ and therefore also $v_T = v(T)$ in H.

Next, let us show $(Kv_n)(T) \rightharpoonup (Kv)(T)$ in V_B. Estimate (10) implies the boundedness of the sequence $\{(Kv_n)(T)\}$ in V_B, therefore there exists $u_T \in V_B$ such that, up to a subsequence, $(Kv_n)(T) \rightharpoonup u_T$ in V_B. Consider again $\phi(t) = \frac{t}{T}$, $w \in V$. Due to relation (6), we have[4]

$$\begin{aligned}
((Kv_n)(T), w) &= \int_0^T \langle (Kv_n)'(t), w\rangle \frac{t}{T}\mathrm{d}t + \int_0^T \langle (Kv_n)(t), w\rangle \frac{1}{T}\mathrm{d}t\\
&= \lambda \int_0^T \langle v_n(t) - ((Kv_n)(t) - u_0), w\rangle \frac{t}{T}\mathrm{d}t + \int_0^T \langle (Kv_n)(t), w\rangle \frac{1}{T}\mathrm{d}t\\
&\to \lambda \int_0^T \langle v(t) - ((Kv)(t) - u_0), w\rangle \frac{t}{T}\mathrm{d}t + \int_0^T \langle (Kv)(t), w\rangle \frac{1}{T}\mathrm{d}t\\
&= \int_0^T \langle (Kv)'(t), w\rangle \frac{t}{T}\mathrm{d}t + \int_0^T \langle (Kv)(t), w\rangle \frac{1}{T}\mathrm{d}t\\
&= ((Kv)(T), w)
\end{aligned}$$

as $n \to \infty$. This immediately yields $u_T = (Kv)(T)$.

In order to show that v is a solution to problem (13), it remains to show $\tilde{a} = Av$. Using the integration-by-parts formula [15, Lemma 4.3], we obtain

$$\begin{aligned}
\langle Av_n, v_n\rangle &= \langle f_n, v_n\rangle - \langle v_n' + BKv_n, v_n\rangle\\
&= \langle f_n, v_n\rangle - \frac{1}{2}\|v_n(T)\|_H^2 + \frac{1}{2}\|v_0\|_H^2 - \frac{1}{2\lambda}\|(Kv_n)(T)\|_B^2 + \frac{1}{2\lambda}\|u_0\|_B^2\\
&\quad + \int_0^T \langle (BKv_n)(s), u_0\rangle \mathrm{d}s - \|Kv_n\|_{L^2(0,T;B)}^2.
\end{aligned}$$

Since we have $v_n \rightharpoonup v$ in $\overline{W}(0,T)$ and since the embedding $\overline{W}(0,T) \subset L^p(0,T;H)$ is compact (see, e.g., Roubíček [38, Lemma 7.7]), there exists a subsequence, again denoted

[4]Note that u_0 cancels out in relation (6), so we consider $X = V_A$.

by n, such that $v_n \to v$ in $L^p(0, T; H)$. This yields $\langle f_n, v_n\rangle \to \langle f, v\rangle$.[5] We also obviously have

$$\int_0^T \langle (BKv_n)(s), u_0\rangle \mathrm{d}s \to \int_0^T \langle (BKv)(s), u_0\rangle \mathrm{d}s.$$

Due to the convergences $v_n(T) \rightharpoonup v(T)$ in H, $(Kv_n)(T) \rightharpoonup (Kv)(T)$ in V_B as well as $Kv_n \overset{*}{\rightharpoonup} Kv$ in $L^\infty(0, T; V_B)$ (and thus $Kv_n \rightharpoonup Kv$ in $L^2(0, T; V_B)$) and the lower semicontinuity of the norm, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle Av_n, v_n\rangle \;&\le\; \langle f, v\rangle - \frac{1}{2}\|v(T)\|_H^2 + \frac{1}{2}\|v_0\|_H^2 - \frac{1}{2\lambda}\|(Kv)(T)\|_B^2 + \frac{1}{2\lambda}\|u_0\|_B^2 \\ &\quad + \int_0^T \langle (BKv)(s), u_0\rangle \mathrm{d}s - \|Kv\|^2_{L^2(0,T;B)} \\ &= \langle f, v\rangle - \langle v' + BKv, v\rangle,\end{aligned}$$

using again the integration-by-parts formula [15, Lemma 4.3]. As v solves (15) in the sense of $L^q(0, T; V_A^*)$, we have

$$\limsup_{n\to\infty}\langle Av_n, v_n\rangle \le \langle \tilde{a}, v\rangle. \tag{16}$$

Now, for arbitrary $w \in L^p(0, T; V_A)$, the monotonicity of A implies

$$\begin{aligned}\langle Av_n, v_n\rangle \;&=\; \langle Av_n - Aw, v_n - w\rangle + \langle Aw, v_n - w\rangle + \langle Av_n, w\rangle \\ &\ge\; \langle Aw, v_n - w\rangle + \langle Av_n, w\rangle.\end{aligned}$$

Therefore, we obtain

$$\liminf_{n\to\infty}\langle Av_n, v_n\rangle \ge \langle Aw, v - w\rangle + \langle \tilde{a}, w\rangle$$

and, together with (16),

$$\langle Aw - \tilde{a}, v - w\rangle \le 0.$$

Choosing $w = v \pm rz$ for an arbitrary $z \in L^p(0, T; V_A)$ and $r > 0$ and using the hemicontinuity as well as the growth condition of $A\colon V_A \to V_A^*$, Lebesgue's theorem on dominated convergence yields for $r \to 0$

$$\langle Av - \tilde{a}, z\rangle = 0$$

for all $z \in L^p(0, T; V_A)$, which implies $\tilde{a} = Av$.

As the last step of this proof, consider the operator $R\colon E \to P_{fc}(E)$ with $R(f) = \mathcal{F}^1(G(f))$, where the set $\mathcal{F}^1(G(f))$ is meant with respect to the truncation $\hat{F}$ instead of F, i.e.,

$$\mathcal{F}^1(G(f)) = \left\{ f \in L^1(0, T; H) \mid f(t) \in \hat{F}(t, (G(f))(t)) \text{ a.e. in } (0, T)\right\}.$$

Following the proof of [35, Theorem 3.1], this operator is upper semicontinuous on E equipped with the weak topology. Thus, we can apply the generalisation of the Kakutani fixed-point theorem (see Glicksberg [20] and Fan [17]) to obtain the existence of $f \in E$ such that $f \in R(f) = \mathcal{F}^1(G(f))$. This implies that $v = G(f)$ solves (1) with the right-hand side $\hat{F}$. However, due to the *a priori* estimate (8), we have $\hat{F}(t, v(t)) = F(t, v(t))$ for almost all $t \in [0, T]$ which proves the assertion. □

[5] Here, we need the (in comparison to the single-valued case stronger) assumptions that the embedding $V_A \subset H$ is compact and that $f(t) \in H$ in order to identify the limit of the sequence $\{\langle f_n, v_n\rangle\}$.

5 Example

We present an example for the problem discussed in this work. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary of class $\mathscr{C}^{1,1}$ or a convex polyhedral and let $p > 6$. Consider

$$\begin{aligned} \frac{\partial}{\partial t} v(x,t) - \Delta_p v(x,t) + (\Delta^2 K v)(x,t) &\in (F(t,v))(x,t), && (x,t) \in \Omega \times (0,T), \\ v(x,t) = \Delta v(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,T), \\ v(x,0) &= v_0(x), && x \in \Omega, \end{aligned}$$

where K is given by (2) and

$$F(t,v) = \left\{ w \in L^2(\Omega) \mid \|g(v) - w\|_{L^2(\Omega)} \leq a(t) \right\} = B_{a(t)}^{L^2(\Omega)}(g(v))$$

with $g(v) = \|v\|_{L^2(\Omega)}^{1-2/p} v$ and a certain function $a \in L^q(0,T)$, $a(t) \geq 0$ a.e.

Here, we have A given by $-\Delta_p$ with $V_A = W_0^{1,p}(\Omega)$ and B given by Δ^2 with $V_B = H^2(\Omega) \cap H_0^1(\Omega)$. Obviously, V_A is not embedded in V_B. As we have $d = 3$ and $p > 6$, V_B is also not embedded in V_A. However, $V_A \cap V_B = H^2(\Omega) \cap W_0^{1,p}(\Omega)$ is separable and densely embedded in both V_A and V_B. With $H = L^2(\Omega)$, we obtain the scale (3) of Banach and Hilbert spaces. As already mentioned, the p-Laplacian fulfils the assumptions **(A)**. It is easy to see that Δ^2 fulfils the assumptions **(B)**. Thus, it remains to prove that F fulfils the assumptions **(F)**.

First, we prove the measurability of the mapping $(t,v) \mapsto d(x, F(t,v))$ for arbitrary fixed $x \in L^2(\Omega)$, which is equivalent to the measurability of F, see, e.g., Denkowski, Migórski, and Papageorgiou [10, Theorem 4.2.11]. Here, d is the distance function in $L^2(\Omega)$, i.e., $d(x,A) = \inf_{y \in A} \|x - y\|_{L^2(\Omega)}$ for $x \in L^2(\Omega)$, $A \subset L^2(\Omega)$. For $x \notin F(t,v)$, we have

$$d(x, F(t,v)) = d\left(x, B_{a(t)}^{L^2(\Omega)}(g(v))\right) = d(x, g(v)) - a(t).$$

This is obviously measurable.

To prove the continuity condition, consider $\{v_n\}, \{w_n\} \subset L^2(\Omega)$ and $v, w \in L^2(\Omega)$ with $v_n \to v$, $w_n \rightharpoonup w$, and $w_n \in F(t, v_n)$. We have to show $w \in F(t,v)$. As $w_n \in F(t,v_n)$, we have

$$\|g(v_n) - w_n\|_{L^2(\Omega)} \leq a(t).$$

The continuity of g and the lower semicontinuity of the norm yield

$$\|g(v) - w\|_{L^2(\Omega)} \leq \liminf_{n \to \infty} \|g(v_n) - w_n\|_{L^2(\Omega)} \leq a(t),$$

i.e., $w \in F(t,v)$.

Finally, we have

$$\begin{aligned} |F(t,v)| &= \sup_{w \in F(t,v)} \|w\|_{L^2(\Omega)} \\ &\leq \sup_{w \in F(t,v)} \|g(v) - w\|_{L^2(\Omega)} + \|g(v)\|_{L^2(\Omega)} \\ &\leq a(t) + \|v\|_{L^2(\Omega)}^{2-2/p}. \end{aligned}$$

As $2 - \frac{2}{p} = \frac{2}{q}$, this yields the desired growth condition on F.

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