

Existence of weak solutions to the Ericksen-Leslie model for a general class of free energies

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Funding information

Deutsche Forschungsgemeinschaft

MSC Classification: 35Q35, 35K52, 76A15

A model due to Ericksen and Leslie describing incompressible liquid crystals is studied for a general class of free energies. Global existence of weak solutions is proven via a Galerkin approximation with eigenfunctions of a strongly elliptic operator. A novelty is that the principal part of the differential operator appearing in the director equation can be nonlinear.

KEYWORDS

Ericksen-Leslie equation, existence, Galerkin approximation, liquid crystal, weak solution

1 | INTRODUCTION

Liquid crystals are fluids that exhibit anisotropic (directional depending) properties. After several reports on such materials in the second half of the 19th century (see Heinz,¹ Virchow,² and Reinitzer³), they were first named and characterized as a state of matter in between fluids and solids by Lehmann.⁴ Only in the second half of the last century liquid crystals came into the focus for many applications (see the work of Palffy-Muhoray⁵), where the liquid crystal display (see the work of Heilmeyer et al⁶) is only the most prominent one.

This article is mainly concerned with nematic liquid crystals, which is a special mesophase of liquid crystals. In this phase, the rod-like molecules are randomly distributed in space as in isotropic liquids. These molecules tend to align in a common direction, even though they are not in a rigid lattice structure as in solids (see Figure 1). The most common model describes the anisotropic properties, ie, the preferred direction of the alignment, by a unit vector field \mathbf{d} and the fluid flow by a velocity field \mathbf{v} . This model is due to Oseen⁷ and Frank⁸ in the stationary case and Ericksen^{9,10} and Leslie¹¹ in the nonstationary case. This model is indeed quite successful and has also been confirmed by experiments (see section 11.1 page 463 in the work of Beris and Edwards¹²).

In this article, we prove existence of weak solutions to the Ericksen-Leslie model under rather weak assumptions on the free energy, allowing anisotropic influences in the elastic potential, ie, that the main part of the energy depends on the alignment of the molecules such that anisotropic distortions can be described. The model under consideration reads as

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p + \nabla \cdot \left(\nabla \mathbf{d}^T \frac{\partial F}{\partial \nabla \mathbf{d}}(\mathbf{d}, \nabla \mathbf{d}) \right) - \nabla \cdot \mathbf{T}^L = \mathbf{g}, \quad (1a)$$

$$\partial_t \mathbf{d} + (\mathbf{v} \cdot \nabla) \mathbf{d} - (\nabla \mathbf{v})_{\text{skw}} \mathbf{d} + \lambda (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} + \gamma \mathbf{q} = 0, \quad (1b)$$

$$\nabla \cdot \mathbf{v} = 0. \quad (1c)$$

We recall that $\mathbf{v} : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^3$ denotes the velocity of the fluid, $\mathbf{d} : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^3$ represents the orientation of the rod-like molecules, $p : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}$ denotes the pressure, and $\mathbf{g} : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^3$ denotes an external force. Throughout this paper, let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class \mathcal{C}^2 and $T > 0$ be given.

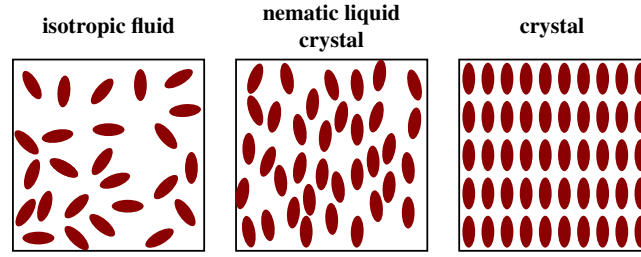


FIGURE 1 Ordering of the molecules in a nematic liquid crystal in comparison to isotropic liquids and crystals [Colour figure can be viewed at wileyonlinelibrary.com]

The Helmholtz free energy potential F , which is described rigorously in Section 3.2, is assumed to depend only on the director and its gradient, ie, $F = F(\mathbf{d}, \nabla \mathbf{d})$. The term \mathbf{q} is given by

$$\mathbf{q} := \frac{\partial F}{\partial \mathbf{d}}(\mathbf{d}, \nabla \mathbf{d}) - \nabla \cdot \frac{\partial F}{\partial \nabla \mathbf{d}}(\mathbf{d}, \nabla \mathbf{d}). \quad (2a)$$

The term $\nabla \mathbf{d}^T (\partial F / \partial \nabla \mathbf{d})$ is called Ericksen stress tensor.

The Leslie stress tensor \mathbf{T}^L , which is the dissipative part in the equation, is given by

$$\begin{aligned} \mathbf{T}^L = & \mu_1 (\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d}) \mathbf{d} \otimes \mathbf{d} + \mu_4 (\nabla \mathbf{v})_{\text{sym}} + (\mu_5 + \mu_6) (\mathbf{d} \otimes (\nabla \mathbf{v})_{\text{sym}} \mathbf{d})_{\text{sym}} \\ & + (\mu_2 + \mu_3) (\mathbf{d} \otimes \mathbf{e})_{\text{sym}} + \frac{\lambda}{\gamma} (\mathbf{d} \otimes (\nabla \mathbf{v})_{\text{sym}} \mathbf{d})_{\text{skw}} + \frac{1}{\gamma} (\mathbf{d} \otimes \mathbf{e})_{\text{skw}}, \end{aligned} \quad (2b)$$

where

$$\mathbf{e} := \partial_t \mathbf{d} + (\mathbf{v} \cdot \nabla) \mathbf{d} - (\nabla \mathbf{v})_{\text{skw}} \mathbf{d}. \quad (2c)$$

Note that \mathbf{e} is an invariant of the system with respect to translations and rotations (see the work of Leslie¹¹).

Following the work of Walkington,¹³ we have sorted the Leslie tensor into symmetric and skew symmetric parts. As established in the work of Leslie,¹¹ we have that

$$\gamma = \frac{1}{\mu_3 - \mu_2}, \quad \lambda = \gamma(\mu_6 - \mu_5). \quad (2d)$$

We emphasize that Parodi's relation

$$\mu_5 - \mu_6 + \mu_2 + \mu_3 = 0 \quad (3)$$

is often assumed to hold (see the works of Lin and Liu¹⁴ or Parodi¹⁵), which will not be the case for our existence result. Parodi's relation follows from Onsager's reciprocal relation and can be employed to obtain the stresses as derivatives of a dissipation distance (see the work of Wu et al¹⁶).

In order to ensure the dissipative character of the system, we assume that

$$\mu_1 > 0, \quad \mu_4 > 0, \quad \gamma > 0, \quad \mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3) > 0, \quad 4\gamma(\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) > (\gamma(\mu_2 + \mu_3) - \lambda)^2. \quad (4)$$

The case $\mu_1 = 0$ can be dealt with similarly but somewhat simpler.

Since we deal with a macroscopic modeling approach, the internal variables are intensive (ie, do not scale with the system size).

1.1 | Review of known results

Ericksen^{9,10} and Leslie¹¹ introduced the aforementioned system that consists of an equation for the evolution of the director, coupled with an incompressible Navier-Stokes-type equation. Originally, the equation for the director is of second order in time and includes a multiple of the second material derivative of \mathbf{d} . The coefficient, however, is proportional to the square of the size of the length of the molecules. Therefore, the term is typically negligible in a macroscopic viscous system. Moreover, $|\mathbf{d}| = 1$ is required.

The first mathematical analysis of the Ericksen-Leslie model is due to the work of Lin and Liu¹⁷ for a rather simplified model. The norm restriction $|\mathbf{d}| = 1$ is incorporated by a standard relaxation technique adding a double-well potential to the free energy. The free energy potential for this model is given by

$$F_\varepsilon(\mathbf{d}, \nabla \mathbf{d}) = \frac{k}{2} |\nabla \mathbf{d}|^2 + \frac{1}{4\varepsilon^2} (|\mathbf{d}|^2 - 1)^2, \quad k, \varepsilon > 0. \quad (5)$$

In this simplified system, the Ericksen stress appearing can be seen as a Korteweg-like stress, which is used in the modeling of miscible liquids. Lin and Liu¹⁷ proved global existence of weak solutions and local existence of strong solutions.

In the work of Lin and Liu,¹⁴ the authors generalized these results to another simplification of the original system introduced by Ericksen and Leslie that is equipped with the energy (5), neglects the second material derivative, and assumes $\lambda = 0$. With this last simplification ($\lambda = 0$), translational forces of the fluid onto the director are neglected. This enables the authors to prove a weak maximum principle, which is essential for the analysis in the works of Lin and Liu.^{14,17}

The full Ericksen-Leslie model but still neglecting the second material derivative and equipped with the Dirichlet energy and double-well potential (5) was considered by Wu et al.¹⁶ They showed the existence of weak solutions under the condition that μ_4 is large enough. Cavaterra et al.¹⁸ proved the existence of weak solutions for the same system when μ_4 is only assumed to be positive. They added a regularizing p -Laplacian to the velocity equation. Feireisl et al.¹⁹ generalized the Ericksen-Leslie model to account for nonisothermal effects by considering additionally an energy balance and an entropy inequality. They showed the global existence of weak solutions.

There are also several articles studying the local well-posedness of the Ericksen-Leslie model. Wang et al.²⁰ showed the local existence of strong solutions for the system equipped with the Dirichlet energy, where Equation (1b) for the director is replaced by

$$\mathbf{d} \times (\partial_t \mathbf{d} + (\mathbf{v} \cdot \nabla) \mathbf{d} - (\nabla \mathbf{v})_{\text{skw}} \mathbf{d} + \lambda (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} + \gamma \mathbf{q}) = 0. \quad (6)$$

Taking the equation for the director in the cross product with the director itself assures that the norm restriction is satisfied for the whole evolution. This does not need to be the case for the general Ericksen-Leslie model with energy (5).

Another approach is due to Prüss and coauthors introducing a thermodynamically consistent system²¹ and proving local existence and stability results via a semigroup approach for quasilinear equations (see the work of Hieber and Prüss²¹ and Hieber et al.²²).

The simplified model with the director equation taken in the cross-product with \mathbf{d} and equipped with the so-called Oseen-Frank energy (7) in the following was considered by Hong et al.²³ They proved the local existence of strong solutions.

For a broader overview of results concerning the analysis of liquid crystal models, we refer to Lin and Liu,²⁴ Lin and Wang,²⁵ and to Emmrich et al.²⁶ Recently, also the existence of measure-valued solutions has been established for more general models by the second author (see the work of Lasarzik²⁷) and results on the weak-strong uniqueness of weak solutions (see the work of Emmrich and Lasarzik²⁸) and of measure-valued solutions (see the work of Lasarzik²⁹) have been proved.

1.2 | Free energy potential

The free energy potential F models the inner forces and thus the influence of the molecules onto each other and on the velocity field. The focus of the present work is to generalize the global existence theory available for the Ericksen-Leslie model to a larger class of free energies, including also potentials associated to nonlinear principal parts in the director equation.

To model distortions in the material, already Leslie¹¹ suggested to consider the free energy potential due to Oseen⁷ and Frank,⁸ called the Oseen-Frank energy

$$F := k_1 (\nabla \cdot \mathbf{d})^2 + k_2 (\mathbf{d} \cdot (\nabla \times \mathbf{d}))^2 + k_3 |\mathbf{d} \times (\nabla \times \mathbf{d})|^2 + \alpha (\text{tr}(\nabla \mathbf{d}^2) - (\nabla \cdot \mathbf{d})^2). \quad (7)$$

Note that $\mathbf{d} \times (\nabla \times \mathbf{d}) = -2(\nabla \mathbf{d})_{\text{skw}} \mathbf{d}$. The last term of (7) can be expressed as the divergence of a vector field, $\nabla \cdot (\nabla \mathbf{d} \mathbf{d} - (\nabla \cdot \mathbf{d}) \mathbf{d}) = \text{tr}(\nabla \mathbf{d}^2) - (\nabla \cdot \mathbf{d})^2$, and with Gauß' formula it is already prescribed by the boundary data. With the one-constant approximation $k_1 = k_2 = k_3 = \alpha$ and employing $|\mathbf{d}| = 1$, one obtains the Dirichlet energy $F(\mathbf{d}, \nabla \mathbf{d}) = k_1 |\nabla \mathbf{d}|^2$. This gives rise to study the energy potential (5).

In the physics literature, there are several choices of free energy potentials, which are not covered by the available mathematical existence theory of generalized solutions yet. Possible electromagnetic field effects could be taken into account by considering (see section 3.2.1 in the work of de Gennes³⁰)

$$F_H(\mathbf{d}, \nabla \mathbf{d}) := F(\mathbf{d}, \nabla \mathbf{d}) - \chi_\perp |\mathbf{H}|^2 - (\chi_\parallel - \chi_\perp) (\mathbf{d} \cdot \mathbf{H})^2. \quad (8)$$

Here, \mathbf{H} denotes the magnetic field and χ_\perp and χ_\parallel are the magnetic susceptibility constants for a magnetic field parallel and perpendicular to the director, respectively. Note that the electric field can be taken into account very similar

(see section 3.3.1 in the work of de Gennes³⁰). Leslie already suggested to incorporate two additional degrees of freedom into the system, which can be achieved by considering a free energy potential of the form

$$F_A(\mathbf{d}, \nabla \mathbf{d}) := F(\mathbf{d}, \nabla \mathbf{d}) - \mathbf{d} \cdot \nabla \mathbf{d} \mathbf{b} + \frac{\bar{b}}{2} |\mathbf{d}|^2, \quad (9)$$

with $\mathbf{b} \in \mathbb{R}^3$ and $\bar{b} \in \mathbb{R}$. Furthermore, the case of the following simplified Oseen-Frank energy is not fully treated in the literature yet. For the special case $k_2 = k_3$ in (7), the Oseen-Frank energy can be simplified. Employing the norm-restriction $|\mathbf{d}| = 1$, we find (see Section 6.4)

$$F(\mathbf{d}, \nabla \mathbf{d}) = k_1 (\nabla \cdot \mathbf{d})^2 + k_2 |\nabla \times \mathbf{d}|^2,$$

with $k_1, k_2 > 0$.

It is also possible to prove the existence of weak solutions to the Ericksen-Leslie system equipped with a scaled version of the Oseen-Frank energy. This energy is given by

$$F(\mathbf{d}, \nabla \mathbf{d}) := \frac{k_1}{2} (\nabla \cdot \mathbf{d})^2 + \frac{k_2}{2} |\nabla \times \mathbf{d}|^2 + (1 + |\nabla \mathbf{d}|^2)^{-s} (1 + |\mathbf{d}|^2)^{-1} \left(\frac{k_3}{2} (\mathbf{d} \cdot \nabla \times \mathbf{d})^2 + \frac{k_4}{2} |\mathbf{d} \times \nabla \times \mathbf{d}|^2 \right), \quad (10)$$

with $k_1, k_2 > 0$ and k_3, k_4 sufficiently small and $s > 1/6$. This is a modification of the Oseen-Frank energy, taking into account the nonquadratic terms and thus anisotropic, director-depending properties of the material. The non-quadratic parts of the free energy are scaled appropriately and the energy has an anisotropic character comparable to the Oseen-Frank energy. Note that the norm restriction of the director, ie, $|\mathbf{d}| = 1$, is always incorporated by the double-well potential, as in (5). Finally, this would require the analysis of the limit for vanishing relaxation parameter $\varepsilon \rightarrow 0$ although this is not in the focus of this paper. So far, there is no global weak solvability concept (but a measure-valued one) available for the original system with norm restriction. The limit $\varepsilon \rightarrow 0$ is studied by the second author in the work of Lasarzik.²⁷

We provide the proof of existence of weak solutions to the Ericksen-Leslie equation (1a), equipped with each of the aforementioned physical relevant energies, except for the general Oseen-Frank energy (7). For the existence of measure-valued solutions to the problem with general Oseen-Frank energy, we again refer to the work of Lasarzik.²⁷

1.3 | Outline of the paper

In the present paper, we study (1a). We focus in particular on the free energy and introduce a class of free energy functions that allows us to show the global existence of weak solutions. The free energies $F = F(\mathbf{h}, \mathbf{S})$ we consider only depend on the director $\mathbf{h} = \mathbf{d}$ and its gradient $\mathbf{S} = \nabla \mathbf{d}$ in such a way that

$$\frac{\partial^2 F}{\partial \mathbf{S}^2}(\mathbf{h}, \mathbf{S}) = \Lambda + \Theta(\mathbf{h}, \mathbf{S}), \quad (11)$$

where Λ denotes a constant elliptic tensor of fourth order and $\Theta : \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3 \times 3 \times 3}$ is a sufficiently small tensor of fourth order (see Section 5.1). Moreover, F has to fulfill certain growth conditions (see (19a)). This class of free energies includes, for instance, all free energy potentials mentioned earlier except the general Oseen-Frank energy (see Section 6).

For proving existence of a solution, we employ a Galerkin method to approximate both Equations (23a) and (23b) simultaneously. This is in contrast to previous work such as Wu et al¹⁶ or Cavaterra et al,¹⁸ where the authors combined a Schauder fixed point argument with a Galerkin approximation of only the Navier-Stokes-like equation and solving the director equation in each step exactly. This method relies on existence and continuity of the solution operator to Equation (1b). To be able to use such a property, previous work had to invoke additional regularity in the approximation of the velocity field either by assuming μ_4 to be sufficiently large¹⁶ or by introducing an additional regularization.¹⁸ Due to the generalization with respect to the free energy considered in the present paper, the continuity of the solution operator to Equation (1b) is no longer at hand. Additionally, a simultaneous discretization is more suitable for a numerical approximation.

The paper is organized as follows. In Section 2, we collect some notation and important inequalities. Section 3 then contains the main result together with a detailed description of the class of free energies. The proof is then carried out in Section 4. In Section 5, we generalize the result to possible nonlinear principal parts and comment on the adaptations needed in the proof. Finally, some examples are discussed in Section 6.

2 | PRELIMINARIES

In this section, we introduce the notation and certain Gagliardo-Nirenberg-type inequalities.

2.1 | Notation

Vectors of \mathbb{R}^3 are denoted by bold small Latin letters. Matrices of $\mathbb{R}^{3 \times 3}$ are denoted by bold capital Latin letters. We also need tensors of third and fourth order, which are denoted by capital and bold capital Greek letters, respectively. Moreover, numbers are denoted by small Latin or Greek letters, and capital Latin letters are reserved for potentials.

The Euclidean inner product in \mathbb{R}^3 is denoted by a dot, $\mathbf{a} \cdot \mathbf{b} := \mathbf{a}^T \mathbf{b} = \sum_{i=1}^3 \mathbf{a}_i \mathbf{b}_i$, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The Frobenius inner product in the space $\mathbb{R}^{3 \times 3}$ of matrices is denoted by a double dot, $\mathbf{A} : \mathbf{B} := \text{tr}(\mathbf{A}^T \mathbf{B}) = \sum_{i,j=1}^3 \mathbf{A}_{ij} \mathbf{B}_{ij}$, for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3}$. We also employ the corresponding Euclidean norm with $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$, for $\mathbf{a} \in \mathbb{R}^3$, and Frobenius norm with $|\mathbf{A}|^2 = \mathbf{A} : \mathbf{A}$, for $\mathbf{A} \in \mathbb{R}^{3 \times 3}$. In addition, we define products of tensors of different order in a similar fashion. The product of a third with a second-order tensor is defined by

$$\Gamma : \mathbf{A} := \left[\sum_{j,k=1}^3 \Gamma_{ijk} \mathbf{A}_{jk} \right]_{i=1}^3, \quad \Gamma \in \mathbb{R}^{3 \times 3 \times 3}, \mathbf{A} \in \mathbb{R}^{3 \times 3}.$$

The product of a fourth order with a second-order tensor is defined by

$$\Gamma : \mathbf{A} := \left[\sum_{k,l=1}^3 \Gamma_{ijkl} \mathbf{A}_{kl} \right]_{i,j=1}^3, \quad \Gamma \in \mathbb{R}^{3 \times 3 \times 3 \times 3}, \mathbf{A} \in \mathbb{R}^{3 \times 3}.$$

The product of a fourth order with a third-order tensor is defined by

$$\Gamma : \Gamma := \left[\sum_{j,k,l=1}^3 \Gamma_{ijkl} \Gamma_{jkl} \right]_{i=1}^3, \quad \Gamma \in \mathbb{R}^{3 \times 3 \times 3 \times 3}, \Gamma \in \mathbb{R}^{3 \times 3 \times 3}.$$

The standard matrix and matrix-vector multiplication, however, is written without an extra sign for brevity

$$\mathbf{A}\mathbf{B} = \left[\sum_{j=1}^3 \mathbf{A}_{ij} \mathbf{B}_{jk} \right]_{i,k=1}^3, \quad \mathbf{A}\mathbf{a} = \left[\sum_{j=1}^3 \mathbf{A}_{ij} \mathbf{a}_j \right]_{i=1}^3, \quad \mathbf{A} \in \mathbb{R}^{3 \times 3}, \mathbf{B} \in \mathbb{R}^{3 \times 3}, \mathbf{a} \in \mathbb{R}^3.$$

The outer product is denoted by $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T = [\mathbf{a}_i \mathbf{b}_j]_{i,j=1}^3$, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Note that $\text{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$. The symmetric and skew-symmetric part of a matrix are denoted by $\mathbf{A}_{\text{sym}} := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ and $\mathbf{A}_{\text{skw}} := \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$, for $\mathbf{A} \in \mathbb{R}^{3 \times 3}$, respectively. For the Frobenius product of two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3}$, we find that

$$\mathbf{A} : \mathbf{B} = \mathbf{A} : \mathbf{B}_{\text{sym}} \text{ if } \mathbf{A}^T = \mathbf{A}, \quad \mathbf{A} : \mathbf{B} = \mathbf{A} : \mathbf{B}_{\text{skw}} \text{ if } \mathbf{A}^T = -\mathbf{A}.$$

Moreover, there holds $\mathbf{A}^T \mathbf{B} : \mathbf{C} = \mathbf{B} : \mathbf{A}\mathbf{C}$, for $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{3 \times 3}$, and $\mathbf{a} \otimes \mathbf{b} : \mathbf{A} = \mathbf{a} \cdot \mathbf{A}\mathbf{b}$, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \mathbf{A} \in \mathbb{R}^{3 \times 3}$. This implies $\mathbf{a} \otimes \mathbf{a} : \mathbf{A} = \mathbf{a} \cdot \mathbf{A}\mathbf{a} = \mathbf{a} : \mathbf{A}_{\text{sym}}\mathbf{a}$.

We use the Nabla symbol ∇ for real-valued functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, vector-valued functions $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, and matrix-valued functions $\mathbf{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$, denoting

$$\nabla f := \left[\frac{\partial f}{\partial \mathbf{x}_i} \right]_{i=1}^3, \quad \nabla \mathbf{f} := \left[\frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_j} \right]_{i,j=1}^3, \quad \nabla \mathbf{A} := \left[\frac{\partial \mathbf{A}_{ij}}{\partial \mathbf{x}_k} \right]_{i,j,k=1}^3.$$

For brevity, we write $\nabla \mathbf{f}^T$ instead of $(\nabla \mathbf{f})^T$. The divergence of a vector-valued and a matrix-valued function is defined by

$$\nabla \cdot \mathbf{f} := \sum_{i=1}^3 \frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_i} = \text{tr}(\nabla \mathbf{f}), \quad \nabla \cdot \mathbf{A} := \left[\sum_{j=1}^3 \frac{\partial \mathbf{A}_{ij}}{\partial \mathbf{x}_j} \right]_{i=1}^3.$$

The symmetric and skew-symmetric part of the gradient of a vector-valued function is denoted by $(\nabla \mathbf{f})_{\text{sym}}$ and $(\nabla \mathbf{f})_{\text{skw}}$, respectively. Note that $(\mathbf{v} \cdot \nabla) \mathbf{f} = (\nabla \mathbf{f}) \mathbf{v} = \nabla \mathbf{f} \mathbf{v}$.

We rely on the usual notation for spaces of continuous functions, ie, Lebesgue and Sobolev spaces. Spaces of vector-valued functions are emphasized by bold letters, for example $\mathbf{L}^p(\Omega) := L^p(\Omega; \mathbb{R}^3)$ and $\mathbf{W}^{k,p}(\Omega) := W^{k,p}(\Omega; \mathbb{R}^3)$. If it is clear from the context, we also use this bold notation for spaces of matrix-valued functions. For brevity, we often omit

calling the domain Ω . The standard inner product in $L^2(\Omega; \mathbb{R}^3)$ is just denoted by (\cdot, \cdot) and in $L^2(\Omega; \mathbb{R}^{3 \times 3})$ by $(\cdot : \cdot)$. In view of the Poincaré-Friedrichs inequality, the space \mathbf{H}_0^1 is equipped with the norm $\|\cdot\|_{\mathbf{H}_0^1} := \|\nabla \cdot\|_{L^2}$. We often need to work with the space $\mathbf{H}^2 \cap \mathbf{H}_0^1$ that we equip with the norm $\|\cdot\|_{\mathbf{H}^2 \cap \mathbf{H}_0^1} := \|\Delta \cdot\|_{L^2}$, which is equivalent to the standard \mathbf{H}^2 -norm (see thm. 9.15, lemma 9.17 in the work of Gilbarg and Trudinger,³¹ or p. 18f in the work of Ladyzhenskaya³²).

The space of smooth solenoidal functions with compact support is denoted by $\mathcal{C}_{c,\sigma}^\infty(\Omega; \mathbb{R}^3)$. By $\mathbf{L}_\sigma^p(\Omega)$, $\mathbf{H}_{0,\sigma}^1(\Omega)$, and $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$, we denote the closure of $\mathcal{C}_{c,\sigma}^\infty(\Omega; \mathbb{R}^3)$ with respect to the norm of $\mathbf{L}^p(\Omega)$, $\mathbf{H}^1(\Omega)$, and $\mathbf{W}^{1,p}(\Omega)$, respectively.

The dual space of a Banach space V is always denoted by V^* and equipped with the standard norm; and the duality pairing is denoted by $\langle \cdot, \cdot \rangle$. The duality pairing between $\mathbf{L}^p(\Omega)$ and $\mathbf{L}^q(\Omega)$ (with $1/p + 1/q = 1$), however, is denoted by (\cdot, \cdot) or $(\cdot : \cdot)$. The dual of \mathbf{H}_0^1 is denoted by \mathbf{H}^{-1} .

The Banach space of linear bounded operators mapping a Banach space V into itself is denoted by $\mathcal{L}(V)$ and equipped with the usual norm. For a given Banach space V , Bochner–Lebesgue spaces are denoted, as usual, by $L^p(0, T; V)$. Moreover, $W^{1,p}(0, T; V)$ denotes the Banach space of abstract functions in $L^p(0, T; V)$, whose weak time derivative exists and is again in $L^p(0, T; V)$ (see also section II.2 in the work of Diestel and Uhl³³ or section 1.5 in the work of Roubíček³⁴ for more details). We often omit the time interval $(0, T)$ and the domain Ω and just write, eg, $L^p(\mathbf{W}^{k,p})$ for brevity. By $\mathcal{A}\mathcal{C}([0, T]; V)$, $\mathcal{C}([0, T]; V)$, and $\mathcal{C}_w([0, T]; V)$, we denote the spaces of abstract functions mapping $[0, T]$ into V that are absolutely continuous, continuous, and continuous with respect to the weak topology in V , respectively.

Finally, by $c > 0$, we denote a generic positive constant.

2.2 | A few interpolation inequalities

We commence with a Gagliardo-Nirenberg-type result for time-dependent functions.

Lemma 1. *Let $p \in [2, 6]$, $q \in [6, \infty]$ and let $r, s \in [1, \infty]$ and $\theta_1, \theta_2 \in [0, 2]$ such that*

$$\frac{1}{p} = \frac{1}{2} - \frac{\theta_1}{3r}, \quad \frac{1}{q} = \frac{1}{6} - \frac{\theta_2}{3s}. \quad (12)$$

Then, there exists a constant $c > 0$ such that the inequalities

$$\|\nabla \mathbf{d}\|_{L^r(L^p)}^r \leq c \|\mathbf{d}\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)}^{\theta_1} \|\mathbf{d}\|_{L^\infty(\mathbf{H}_0^1)}^{r-\theta_1}, \quad \|\mathbf{d}\|_{L^s(L^q)}^s \leq c \|\mathbf{d}\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)}^{\theta_2} \|\mathbf{d}\|_{L^\infty(\mathbf{H}_0^1)}^{s-\theta_2}$$

hold for any $\mathbf{d} \in L^\infty(0, T; \mathbf{H}_0^1) \cap L^2(0, T; \mathbf{H}^2 \cap \mathbf{H}_0^1)$.

Proof. Since $\mathbf{H}^2 \cap \mathbf{H}_0^1$ is continuously embedded in $\mathbf{W}^{1,6}$, we immediately find with Hölder's inequality that, for $p \in [2, 6]$ and any $\mathbf{d} \in \mathbf{H}^2 \cap \mathbf{H}_0^1$,

$$\|\nabla \mathbf{d}\|_{L^p} \leq \|\nabla \mathbf{d}\|_{L^6}^{3/2-3/p} \|\nabla \mathbf{d}\|_{L^2}^{3/p-1/2} \leq c \|\mathbf{d}\|_{\mathbf{H}^2 \cap \mathbf{H}_0^1}^{3/2-3/p} \|\mathbf{d}\|_{\mathbf{H}_0^1}^{3/p-1/2}.$$

From the classical Gagliardo-Nirenberg inequality (see, eg, the works of Nirenberg,³⁵ Friedman,³⁶ or section 21.19 in the work of Zeidler³⁷), we infer that there exists $c > 0$ such that, for $q \in [6, \infty]$ and any $\mathbf{d} \in \mathbf{H}^2 \cap \mathbf{H}_0^1$,

$$\|\mathbf{d}\|_{L^q} \leq c \|\mathbf{d}\|_{\mathbf{H}^2 \cap \mathbf{H}_0^1}^{1/2-3/q} \|\mathbf{d}\|_{L^6}^{1/2+3/q} \leq c \|\mathbf{d}\|_{\mathbf{H}^2 \cap \mathbf{H}_0^1}^{1/2-3/q} \|\mathbf{d}\|_{\mathbf{H}_0^1}^{1/2+3/q},$$

the case $q = \infty$ remains true with both exponents being $1/2$.

Let us now consider $\mathbf{d} \in L^\infty(0, T; \mathbf{H}_0^1) \cap L^2(0, T; \mathbf{H}^2 \cap \mathbf{H}_0^1)$ and integrate the foregoing estimates in time. We then find

$$\|\nabla \mathbf{d}\|_{L^r(L^p)} \leq c \|\mathbf{d}\|_{L^{3r(1/2-1/p)}(\mathbf{H}^2 \cap \mathbf{H}_0^1)}^{3/2-3/p} \|\mathbf{d}\|_{L^\infty(\mathbf{H}_0^1)}^{3/p-1/2}$$

as long as $\theta_1 = 3r(1/2 - 1/p) \leq 2$ such that $L^2(0, T; \mathbf{H}^2 \cap \mathbf{H}_0^1)$ is continuously embedded in $L^{\theta_1}(0, T; \mathbf{H}^2 \cap \mathbf{H}_0^1)$. This proves the first inequality.

In the same fashion, one proves the second estimate. □

Lemma 2. *There exists a constant $c > 0$ such that, for $p \in [2, 6]$, $r \in [2, \infty]$ with*

$$\frac{1}{p} = \frac{1}{2} - \frac{2}{3r}$$

and any $\mathbf{v} \in L^\infty(0, T; \mathbf{L}^2) \cap L^2(0, T; \mathbf{H}^1)$,

$$\|\mathbf{v}\|_{L^r(L^p)}^r \leq c \|\mathbf{v}\|_{L^2(\mathbf{H}^1)}^2 \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2)}^{r-2}.$$

Proof. The proof is analogous to that of Lemma 1. \square

3 | ERICKSEN-LESLIE MODEL AND MAIN RESULT

In this section, we introduce the full system studied in this paper and its weak formulation and we state the main result.

3.1 | Governing equations

We consider the Ericksen-Leslie model (1a) for dimensionless quantities neglecting the second material derivative of the director. We focus on a rather general class of free energy functions and incorporate the restriction of the director \mathbf{d} onto unit vectors into the free energy via a classical relaxation technique, as in (5).

The free energy functional \mathcal{F} is defined by

$$\mathcal{F} : \mathbf{H}^1 \rightarrow \mathbb{R}, \quad \mathcal{F}(\mathbf{d}) := \int_{\Omega} F(\mathbf{d}, \nabla \mathbf{d}) \, dx,$$

and $\mathbf{q} = \delta \mathcal{F}(\mathbf{d}) / \delta \mathbf{d}$ (see (2a)) is its variational derivative (see section 2.1 in the work of Furihata and Matsuo³⁸).

We impose boundary and initial conditions as follows:

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega, \quad \mathbf{v}(\mathbf{x}, t) = \mathbf{0}, \quad \text{for } (\mathbf{x}, t) \in \partial\Omega \times [0, T], \quad (13a)$$

$$\mathbf{d}(\mathbf{x}, 0) = \mathbf{d}_0(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega, \quad \mathbf{d}(\mathbf{x}, t) = \mathbf{d}_1(\mathbf{x}), \quad \text{for } (\mathbf{x}, t) \in \partial\Omega \times [0, T]. \quad (13b)$$

We shall assume that $\mathbf{d}_1 = \mathbf{d}_0$ on $\partial\Omega$, which is a compatibility condition providing regularity.

3.2 | A class of free energy potentials

This section is devoted to the free energy potential that describes the inner forces between the molecules. We commence with a class of free energies that leads to a linear principal part. The more delicate case with a nonlinear principal part is dealt with in Section 5. Let us consider

$$F = F(\mathbf{h}, \mathbf{S}) \in \mathcal{C}^2(\mathbb{R}^3 \times \mathbb{R}^{3 \times 3}; \mathbb{R}), \quad (14)$$

and let us assume that

$$\frac{\partial^2 F}{\partial \mathbf{S}^2} \equiv \mathbf{\Lambda} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}, \quad (15a)$$

ie, neglecting θ in (11), where $\mathbf{\Lambda}$ satisfies the symmetry condition

$$\Lambda_{ijkl} = \Lambda_{klij}, \quad \text{for } i, j, k, l = 1, 2, 3, \quad (15b)$$

and the following strong Legendre-Hadamard (strong ellipticity) condition. There exists $\eta > 0$ such that, for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,

$$\mathbf{a} \otimes \mathbf{b} : \mathbf{\Lambda} : \mathbf{a} \otimes \mathbf{b} \geq \eta |\mathbf{a}|^2 |\mathbf{b}|^2. \quad (15c)$$

Remark 1. It is possible to generalize the assumptions on the second derivative of F with respect to \mathbf{S} . The tensor $\mathbf{\Lambda}$ can continuously depend on the spatial variable \mathbf{x} (see Remark 4). Additionally, a nonlinear term, which is sufficiently small, can be handled as a part of the second derivative of F with respect to \mathbf{S} (see Section 5).

In the course of the proof of our main result, we shall need further coercivity-type assumptions on the free energy F and its derivatives. Let us assume that there exist $\eta_1 > 0$ and $\eta_2, \eta_3 \geq 0$ such that, for all $\mathbf{h} \in \mathbb{R}^3$ and $\mathbf{S} \in \mathbb{R}^{3 \times 3}$,

$$F(\mathbf{h}, \mathbf{S}) \geq \eta_1 |\mathbf{S}|^2 - \eta_2 |\mathbf{h}|^2 - \eta_3. \quad (16)$$

For a particular free energy, such a condition may follow from (15a) together with suitable growth or nonnegativity assumptions on the lower order terms.

Later, we will have that $F = F(\mathbf{d}, \nabla \mathbf{d})$ with $\mathbf{d} = \mathbf{d}(\mathbf{x}, t)$. Under the regularity assumption (14), we may now consider (2a). With respect to \mathbf{q} , we first observe that formally

$$\begin{aligned} \mathbf{q} &= \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) - \frac{\partial^2 F}{\partial \mathbf{S}^2}(\mathbf{d}, \nabla \mathbf{d}) : \nabla(\nabla \mathbf{d})^T - \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) : \nabla \mathbf{d}^T \\ &= \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) - \nabla \cdot (\boldsymbol{\Lambda} : \nabla \mathbf{d}) - \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) : \nabla \mathbf{d}^T. \end{aligned} \quad (17)$$

For arbitrary $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, one finds

$$|\mathbf{a} - \mathbf{b} - \mathbf{c}|^2 \geq \frac{1}{2} |\mathbf{b}|^2 - 4|\mathbf{c}|^2 - 4|\mathbf{a}|^2, \quad (18)$$

and thus

$$|\mathbf{q}|^2 \geq \frac{1}{2} |\nabla \cdot (\boldsymbol{\Lambda} : \nabla \mathbf{d})|^2 - 4 \left| \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) : \nabla \mathbf{d}^T \right|^2 - 4 \left| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) \right|^2.$$

This calculation motivates the following growth conditions. There exist $C_{Sh} > 0$, $C_h > 0$ and $\gamma_1 \in [2, 10/3]$, $\gamma_2 \in [6, 10]$ such that, for all $\mathbf{h} \in \mathbb{R}^3$ and $\mathbf{S} \in \mathbb{R}^{3 \times 3}$,

$$\left| \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{h}, \mathbf{S}) \right| \leq C_{Sh} (|\mathbf{S}|^{\gamma_1/2-1} + |\mathbf{h}|^{\gamma_3} + 1) \quad (19a)$$

$$\left| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{h}, \mathbf{S}) \right| \leq C_h (|\mathbf{S}|^{\gamma_1/2} + |\mathbf{h}|^{\gamma_2/2} + 1), \quad (19b)$$

where

$$\gamma_3 := \frac{(\gamma_1 - 2)\gamma_2}{2\gamma_1}. \quad (19c)$$

This choice of exponents will allow us to derive appropriate a priori estimates. Of course, the term with $|\mathbf{S}|^{\gamma_1/2-1}$ in (19a) is superfluous for a potential fulfilling (15a) but will be essential for the analysis in Section 5.

3.3 | Existence of weak solutions

In this section, we state our main result on the existence of weak solutions. We first give a precise definition of what we mean by a weak solution. We shall work in solenoidal function spaces and thus only consider the variables velocity and director.

Let us start with a reformulation of the Ericksen stress tensor. For $\mathbf{v} \in \mathbf{H}_{0,\sigma}^1$ and $\mathbf{d} \in \mathbf{H}^2 \cap \mathbf{H}_0^1$, we find

$$\nabla((\mathbf{v} \cdot \nabla)\mathbf{d}) = \nabla \mathbf{d} \nabla \mathbf{v} + (\mathbf{v} \cdot \nabla) \nabla \mathbf{d}. \quad (20)$$

For sufficiently smooth functions $\mathbf{h} : \bar{\Omega} \rightarrow \mathbb{R}^3$, $\mathbf{S} : \bar{\Omega} \rightarrow \mathbb{R}^{3 \times 3}$, we obtain

$$\begin{aligned} (\mathbf{v} \cdot \nabla)F(\mathbf{h}, \mathbf{S}) &= \sum_{i=1}^3 \mathbf{v}_i \left(\sum_{j=1}^3 \frac{\partial F}{\partial \mathbf{h}_j}(\mathbf{h}, \mathbf{S}) \frac{\partial \mathbf{h}_j}{\partial \mathbf{x}_i} + \sum_{j,k=1}^3 \frac{\partial F}{\partial \mathbf{S}_{jk}}(\mathbf{h}, \mathbf{S}) \frac{\partial \mathbf{S}_{jk}}{\partial \mathbf{x}_i} \right) \\ &= \frac{\partial F}{\partial \mathbf{h}}(\mathbf{h}, \mathbf{S}) \cdot (\mathbf{v} \cdot \nabla)\mathbf{h} + \frac{\partial F}{\partial \mathbf{S}} : (\mathbf{v} \cdot \nabla)\mathbf{S}. \end{aligned}$$

With (1c), (2a), (20), and integration by parts, we obtain, for all $\mathbf{v} \in \mathbf{H}_{0,\sigma}^1$,

$$\begin{aligned} \left(\nabla \mathbf{d}^T \frac{\partial F}{\partial \mathbf{S}} : \nabla \mathbf{v} \right) - \langle \nabla \mathbf{d}^T \mathbf{q}, \mathbf{v} \rangle &= \left(\nabla \mathbf{d}^T \frac{\partial F}{\partial \mathbf{S}} : \nabla \mathbf{v} \right) + \left\langle \nabla \mathbf{d}^T \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}, \mathbf{v} \right\rangle - \left(\nabla \mathbf{d}^T \frac{\partial F}{\partial \mathbf{h}}, \mathbf{v} \right) \\ &= \left(\frac{\partial F}{\partial \mathbf{S}} : \nabla \mathbf{d} \nabla \mathbf{v} \right) + \left\langle \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}, (\nabla \mathbf{d}) \mathbf{v} \right\rangle - \left(\frac{\partial F}{\partial \mathbf{h}}, (\nabla \mathbf{d}) \mathbf{v} \right) \\ &= \left(\frac{\partial F}{\partial \mathbf{S}} : \nabla \mathbf{d} \nabla \mathbf{v} \right) + \left(\nabla \cdot \frac{\partial F}{\partial \mathbf{S}}, (\mathbf{v} \cdot \nabla) \mathbf{d} \right) - \left(\frac{\partial F}{\partial \mathbf{h}}, (\mathbf{v} \cdot \nabla) \mathbf{d} \right) \\ &= \left(\frac{\partial F}{\partial \mathbf{S}} : \nabla \mathbf{d} \nabla \mathbf{v} \right) - \left(\frac{\partial F}{\partial \mathbf{S}} : \nabla((\mathbf{v} \cdot \nabla)\mathbf{d}) \right) - \left(\frac{\partial F}{\partial \mathbf{h}}, (\mathbf{v} \cdot \nabla) \mathbf{d} \right) \\ &= - \left(\frac{\partial F}{\partial \mathbf{S}} : (\mathbf{v} \cdot \nabla) \nabla \mathbf{d} \right) - \left(\frac{\partial F}{\partial \mathbf{h}}, (\mathbf{v} \cdot \nabla) \mathbf{d} \right) \\ &= -(\nabla F, \mathbf{v}) = \int_{\Omega} F(\nabla \cdot \mathbf{v}) \, d\mathbf{x} = 0, \end{aligned} \quad (21)$$

where we omitted the argument $(\mathbf{d}, \nabla \mathbf{d})$ for readability.

The aforementioned identity allows us to reformulate Equation (1a) by incorporating F in a redefinition of the pressure, $p := p + F$, and by finally replacing $\nabla \cdot (\nabla \mathbf{d}^T \partial F / \partial \mathbf{S})$ by $-\nabla \mathbf{d}^T \mathbf{q}$.

Definition 1. (Weak solution)

Let $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{L}^2_\sigma \times \mathbf{H}^1_0$ and $\mathbf{g} \in L^2(0, T; (\mathbf{H}^1_{0,\sigma})^*)$. A pair (\mathbf{v}, \mathbf{d}) is said to be a solution to (1a) with (2a) and (13a) if

$$\begin{aligned} \mathbf{v} &\in L^\infty(0, T; \mathbf{L}^2_\sigma) \cap L^2(0, T; \mathbf{H}^1_{0,\sigma}) \cap W^{1,2}(0, T; (\mathbf{H}^2 \cap \mathbf{H}^1_{0,\sigma})^*), \\ \mathbf{d} &\in L^\infty(0, T; \mathbf{H}^1_0) \cap L^2(0, T; \mathbf{H}^2 \cap \mathbf{H}^1_0) \cap W^{1,4/3}(0, T; \mathbf{L}^2), \end{aligned} \quad (22)$$

if

$$\begin{aligned} & - \int_0^T (\mathbf{v}(t), \partial_t \boldsymbol{\varphi}(t)) \, dt + \int_0^T ((\mathbf{v}(t) \cdot \nabla) \mathbf{v}(t), \boldsymbol{\varphi}(t)) \, dt - \int_0^T (\nabla \mathbf{d}(t)^T \mathbf{q}(t), \boldsymbol{\varphi}(t)) \, dt \\ & + \int_0^T (\mathbf{T}^L(t) : \nabla \boldsymbol{\varphi}(t)) \, dt = \int_0^T \langle \mathbf{g}(t), \boldsymbol{\varphi}(t) \rangle \, dt, \end{aligned} \quad (23a)$$

$$\begin{aligned} & - \int_0^T (\mathbf{d}(t), \partial_t \boldsymbol{\psi}(t)) \, dt + \int_0^T ((\mathbf{v}(t) \cdot \nabla) \mathbf{d}(t), \boldsymbol{\psi}(t)) \, dt - \int_0^T ((\nabla \mathbf{v}(t))_{\text{skw}} \mathbf{d}(t), \boldsymbol{\psi}(t)) \, dt \\ & + \lambda \int_0^T ((\nabla \mathbf{v}(t))_{\text{sym}} \mathbf{d}(t), \boldsymbol{\psi}(t)) \, dt + \gamma \int_0^T (\mathbf{q}(t), \boldsymbol{\psi}(t)) \, dt = 0, \end{aligned} \quad (23b)$$

for all solenoidal $\boldsymbol{\varphi} \in \mathcal{C}_c^\infty(\Omega \times (0, T); \mathbb{R}^3)$ and all $\boldsymbol{\psi} \in C_c^\infty(\Omega \times (0, T); \mathbb{R}^3)$, and if the initial conditions are satisfied.

Remark 2. Let (\mathbf{v}, \mathbf{d}) be a solution in the sense of Definition 1. Then, $\mathbf{d} \in L^\infty(0, T; \mathbf{H}^1_0)$

$$\mathbf{d} \in W^{1,4/3}(0, T; \mathbf{L}^2) \subset \mathcal{A}\mathcal{E}([0, T]; \mathbf{L}^2) \subset \mathcal{E}_w([0, T]; \mathbf{L}^2).$$

Moreover, there holds (see ch. 3, lemma 8.1 in the work of Lions and Magenes³⁹)

$$\mathcal{E}_w([0, T]; \mathbf{L}^2) \cap L^\infty(0, T; \mathbf{H}^1_0) = \mathcal{E}_w([0, T]; \mathbf{H}^1_0),$$

such that $\mathbf{d} \in \mathcal{E}_w([0, T]; \mathbf{H}^1_0)$. Analogously, we find $\mathbf{v} \in \mathcal{E}_w([0, T]; \mathbf{L}^2_\sigma)$.

The initial conditions for the Navier-Stokes-like equation (23a) and for the director equation (23b) are thus attained in the following sense:

$$\mathbf{v}(t) \rightharpoonup \mathbf{v}_0 \text{ in } \mathbf{L}^2_\sigma, \quad \mathbf{d}(t) \rightharpoonup \mathbf{d}_0 \text{ in } \mathbf{H}^1_0 \text{ as } t \rightarrow 0. \quad (24)$$

The aforementioned notion of a weak solution can be justified as follows.

Proposition 1. *Under the regularity assumptions (22), all terms appearing in (23a) are well defined.*

Proof. Obviously, we only have to concentrate on the nonlinear terms. Let us start with Equation (23b). With Hölder's inequality, we easily find that $(\mathbf{v} \cdot \nabla) \mathbf{d} \in L^2(0, T; \mathbf{L}^{3/2})$ because

$$\|(\mathbf{v} \cdot \nabla) \mathbf{d}\|_{L^2(\mathbf{L}^{3/2})} \leq \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2)} \|\nabla \mathbf{d}\|_{L^2(\mathbf{L}^6)}$$

and because $\mathbf{H}^2 \hookrightarrow \mathbf{W}^{1,6}$. Similarly, we have

$$\|(\nabla \mathbf{v})_{\text{skw}} \mathbf{d}\|_{L^2(\mathbf{L}^{3/2})} + \|(\nabla \mathbf{v})_{\text{sym}} \mathbf{d}\|_{L^2(\mathbf{L}^{3/2})} \leq c \|\nabla \mathbf{v}\|_{L^2(\mathbf{L}^2)} \|\mathbf{d}\|_{L^\infty(\mathbf{L}^6)},$$

which shows that also $(\nabla \mathbf{v})_{\text{skw}} \mathbf{d}, (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} \in L^2(0, T; \mathbf{L}^{3/2})$. Note that $\mathbf{L}^{3/2} \hookrightarrow \mathbf{H}^{-1}$.

The term \mathbf{q} (see (2a)) is in $L^2(0, T; \mathbf{L}^2)$. Indeed, with (17), we can estimate

$$\begin{aligned} \|\mathbf{q}\|_{L^2(\mathbf{L}^2)} &\leq \left\| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) \right\|_{L^2(\mathbf{L}^2)} + \left\| \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) : \nabla \mathbf{d}^T \right\|_{L^2(\mathbf{L}^2)} + \|\nabla \cdot (\boldsymbol{\Lambda} : \nabla \mathbf{d})\|_{L^2(\mathbf{L}^2)} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Regarding the term I_1 , we see with growth condition (19b) and Lemma 1 that

$$\begin{aligned} I_1 &\leq C_h \left\| |\nabla \mathbf{d}|^{\gamma_1/2} + |\mathbf{d}|^{\gamma_2/2} + 1 \right\|_{L^2(L^2)} \\ &= C_h \left(\|\nabla \mathbf{d}\|_{L^{\gamma_1}(L^{\gamma_1})}^{\gamma_1/2} + \|\mathbf{d}\|_{L^{\gamma_2}(L^{\gamma_2})}^{\gamma_2/2} + T^{1/2} |\Omega|^{1/2} \right) \\ &\leq c \left(\|\nabla \mathbf{d}\|_{L^{10/3}(L^{10/3})}^{5/3} + \|\mathbf{d}\|_{L^{10}(L^{10})}^5 + 1 \right) \\ &\leq c \left(\|\mathbf{d}\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)} \|\mathbf{d}\|_{L^\infty(\mathbf{H}_0^1)}^{2/3} + \|\mathbf{d}\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)} \|\mathbf{d}\|_{L^\infty(\mathbf{H}_0^1)}^4 + 1 \right). \end{aligned}$$

The term I_2 is dealt with in a similar fashion. The growth condition (19a) gives

$$I_2 \leq C_{sh} \left\| |\nabla \mathbf{d}| (|\nabla \mathbf{d}|^{\gamma_1/2-1} + |\mathbf{d}|^{\gamma_3} + 1) \right\|_{L^2(L^2)}.$$

An application of Young's inequality together with definition (19c) and Lemma 1 provides that

$$\begin{aligned} I_2 &\leq c \left\| |\nabla \mathbf{d}|^{\gamma_1/2} + |\mathbf{d}|^{\gamma_2/2} + |\nabla \mathbf{d}| \right\|_{L^2(L^2)} \\ &\leq c \left(\|\nabla \mathbf{d}\|_{L^{\gamma_1}(L^{\gamma_1})}^{\gamma_1/2} + \|\mathbf{d}\|_{L^{\gamma_2}(L^{\gamma_2})}^{\gamma_2/2} + \|\nabla \mathbf{d}\|_{L^2(L^2)} \right) \\ &\leq c \left(\|\nabla \mathbf{d}\|_{L^{10/3}(L^{10/3})}^{5/3} + \|\mathbf{d}\|_{L^{10}(L^{10})}^5 + 1 \right) \\ &\leq c \left(\|\mathbf{d}\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)} \|\mathbf{d}\|_{L^\infty(\mathbf{H}_0^1)}^{2/3} + \|\mathbf{d}\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)} \|\mathbf{d}\|_{L^\infty(\mathbf{H}_0^1)}^4 + 1 \right). \end{aligned}$$

Finally, the term I_3 can be estimated by

$$I_3 \leq c \|\mathbf{d}\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)}$$

because Λ is a constant tensor (see (15a)).

Let us turn to the Navier-Stokes-like equation (23a). The convection term can be shown to be in $L^{10/7}(0, T; \mathbf{L}^{15/13})$ because

$$\|(\mathbf{v} \cdot \nabla) \mathbf{v}\|_{L^{10/7}(\mathbf{L}^{15/13})} \leq \|\mathbf{v}\|_{L^5(\mathbf{L}^{30/11})} \|\mathbf{v}\|_{L^2(\mathbf{H}_0^1)},$$

which can easily be shown with Hölder's inequality. Moreover, $\mathbf{v} \in L^5(0, T; \mathbf{L}^{30/11})$ in view of Lemma 2. Since $\mathbf{q} \in L^2(0, T; \mathbf{L}^2)$, we easily find with Hölder's inequality that

$$\|\nabla \mathbf{d}^T \mathbf{q}\|_{L^{5/4}(\mathbf{L}^{5/4})} \leq \|\nabla \mathbf{d}\|_{L^{10/3}(\mathbf{L}^{10/3})} \|\mathbf{q}\|_{L^2(\mathbf{L}^2)},$$

where the norm of $\nabla \mathbf{d}$ can be estimated as before.

It remains to estimate the Leslie stress. From definition (2b) and Hölder's inequality, it follows that

$$\begin{aligned} \|\mathbf{T}^L\|_{L^{10/9}(\mathbf{L}^{10/9})} &\leq c \left(\left\| (\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d}) \mathbf{d} \otimes \mathbf{d} \right\|_{L^{10/9}(\mathbf{L}^{10/9})} + \|\nabla \mathbf{v}\|_{L^{10/9}(\mathbf{L}^{10/9})} + \|\mathbf{d} \otimes (\nabla \mathbf{v})_{\text{sym}} \mathbf{d}\|_{L^{10/9}(\mathbf{L}^{10/9})} + \|\mathbf{d} \otimes \mathbf{e}\|_{L^{10/9}(\mathbf{L}^{10/9})} \right) \\ &\leq c \left(\|\mathbf{v}\|_{L^2(\mathbf{H}_0^1)} \|\mathbf{d}\|_{L^{10}(\mathbf{L}^{10})}^4 + \|\mathbf{v}\|_{L^2(\mathbf{H}_0^1)} + \|\mathbf{v}\|_{L^2(\mathbf{H}_0^1)} \|\mathbf{d}\|_{L^{10}(\mathbf{L}^{10})}^2 + \|\mathbf{e}\|_{L^{5/4}(\mathbf{L}^{5/4})} \|\mathbf{d}\|_{L^{10}(\mathbf{L}^{10})} \right). \end{aligned} \quad (25)$$

We can further estimate the norm of \mathbf{e} (see (2c)) by

$$\begin{aligned} \|\mathbf{e}\|_{L^{5/4}(\mathbf{L}^{5/4})} &\leq \|\partial_t \mathbf{d}\|_{L^{5/4}(\mathbf{L}^{5/4})} + \|(\mathbf{v} \cdot \nabla) \mathbf{d}\|_{L^{5/4}(\mathbf{L}^{5/4})} + \|(\nabla \mathbf{v})_{\text{skw}} \mathbf{d}\|_{L^{5/4}(\mathbf{L}^{5/4})} \\ &\leq c \|\partial_t \mathbf{d}\|_{L^{4/3}(\mathbf{L}^2)} + c \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2)} \|\mathbf{d}\|_{L^2(\mathbf{W}^{1,6})} + c \|\mathbf{v}\|_{L^2(\mathbf{H}_0^1)} \|\mathbf{d}\|_{L^\infty(\mathbf{L}^6)}. \end{aligned}$$

All this shows that $\mathbf{T}^L \in L^{10/9}(0, T; \mathbf{L}^{10/9})$. □

Our main result is as follows.

Theorem 1. (Existence of weak solutions)

Let Ω be a bounded domain of class \mathcal{C}^2 , assume (4), and let the free energy potential F fulfill the assumptions (14), (15a), (16), and (19a). For given initial data $\mathbf{v}_0 \in \mathbf{L}^2$, $\mathbf{d}_0 \in \mathbf{H}_0^1$ (such that $\mathbf{d}_1 = 0$), and the right-hand side $\mathbf{g} \in L^2(0, T; (\mathbf{H}_{0,\sigma}^1)^*)$, there exists a weak solution to the Ericksen-Leslie system (1a) with (2a) and (13a) in the sense of Definition 1.

Let us note that one may also handle a more general right-hand side $\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2$ with $\mathbf{g}_1 \in L^1(0, T; (\mathbf{L}_\sigma^2)^*)$ and $\mathbf{g}_2 \in L^2(0, T; (\mathbf{H}_{0,\sigma}^1)^*)$ (see chapter 20 in the work of Tartar⁴⁰ or the work of Simon⁴¹). For the sake of simplicity, however, we neglect \mathbf{g}_1 .

Remark 3. A nonhomogeneous boundary condition for \mathbf{d} , ie, $\mathbf{d}_1 \neq 0$, can be dealt with by a standard transformation as follows. Let $\mathbf{d}_0 \in \mathbf{H}^1(\Omega)$, $\mathbf{d}_1 \in \mathbf{H}^{3/2}(\partial\Omega)$, and assume that the trace of \mathbf{d}_0 equals \mathbf{d}_1 . As Ω is of class \mathcal{C}^2 , there exists a linear continuous extension operator $\mathcal{S} : \mathbf{H}^{3/2}(\partial\Omega) \rightarrow \mathbf{H}^2(\Omega)$ that is the right inverse of the trace operator (see thm. 8.8 in the work of Wloka⁴²). Therefore, we can replace \mathbf{d} in (1a) by $\mathbf{d} - \mathcal{S} \mathbf{d}_1$ that satisfies a homogeneous boundary condition. The regularity of \mathbf{d}_1 and thus of $\mathcal{S} \mathbf{d}_1$ is sufficient to estimate all the terms appearing in addition.

4 | GALERKIN APPROXIMATION AND PROOF OF THE MAIN RESULT

In this section, we prove the main result (Theorem 1) via convergence of a Galerkin approximation. The proof is divided into the following steps. We first (Section 4.1) introduce the Galerkin scheme and deduce local-in-time existence of a solution to the approximate problem. We then (Section 4.2) derive a priori estimates and conclude that solutions to the approximate problem exist globally in time. The crucial part is dealt with in Section 4.3, where we use the a priori estimates to extract a weakly convergent subsequence of the sequence of approximate solutions. We also prove weak convergence of a subsequence of the sequence of time derivatives of the approximate solutions. This implies strong convergence in a suitable norm and allows us to identify the initial values. Strong convergence is a prerequisite to handle the nonlinear variational derivative of the free energy. Finally, we can identify the weak limits as a solution to the director and the Navier-Stokes-like equation.

4.1 | Galerkin approximation

For the approximation of the Navier-Stokes-like equation, we follow p. 27f in the work of Temam⁴³ and use a Galerkin basis consisting of eigenfunctions $\mathbf{w}_1, \mathbf{w}_2, \dots \in \mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1$ of the Stokes operator (with homogeneous Dirichlet boundary condition). As is well known, the eigenfunctions form an orthogonal basis in \mathbf{L}_σ^2 and in $\mathbf{H}_{0,\sigma}^1$ and in $\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1$. Let $W_n = \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ ($n \in \mathbb{N}$) and let $P_n : \mathbf{L}_\sigma^2 \rightarrow W_n$ denote the \mathbf{L}_σ^2 -orthogonal projection onto W_n . The restriction of P_n on $\mathbf{H}_{0,\sigma}^1$ and $\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1$ is nothing than the $\mathbf{H}_{0,\sigma}^1$ - and $(\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1)$ -orthogonal projection onto W_n , respectively, such that

$$\|P_n\|_{\mathcal{L}(\mathbf{L}_\sigma^2)} = \|P_n\|_{\mathcal{L}(\mathbf{H}_{0,\sigma}^1)} = \|P_n\|_{\mathcal{L}(\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1)} = 1.$$

Remark that Ω is of class \mathcal{C}^2 . Hence, there exists $c > 0$ such that, for all $n \in \mathbb{N}$ and $\mathbf{v} \in \mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1$,

$$\|P_n \mathbf{v}\|_{\mathbf{H}^2} \leq c \|\mathbf{v}\|_{\mathbf{H}^2},$$

see, eg, the appendix, thm. 4.11, and lemma 4.26 in the work of Málek et al⁴⁴ together with prop. III.3.17 in the work of Boyer and Fabrie.⁴⁵

For the approximation of the director equation, we use a Galerkin basis consisting of eigenfunctions $\mathbf{z}_1, \mathbf{z}_2, \dots$ of the differential operator corresponding to the boundary value problem

$$\begin{aligned} -\nabla \cdot (\mathbf{\Lambda} : \nabla \mathbf{z}) &= \mathbf{h} \quad \text{in } \Omega, \\ \mathbf{z} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{26}$$

In view of the assumptions (15a) on $\mathbf{\Lambda}$, the aforementioned problem is a symmetric strongly elliptic system that possesses a unique weak solution $\mathbf{z} \in \mathbf{H}_0^1$ for any $\mathbf{h} \in \mathbf{H}^{-1}$ (see, eg, thm. 13.3 in the work of Chipot⁴⁶). Its solution operator is thus a compact operator in \mathbf{L}^2 . Hence, there exists an orthogonal basis of eigenfunctions $\mathbf{z}_1, \mathbf{z}_2, \dots$ in \mathbf{L}^2 .

Remark 4. If $\mathbf{\Lambda}$ depends on \mathbf{x} , the claims of the aforementioned paragraph are not true any more. The boundary-value problem (26) would not be well posed under this generalized condition. Consider a tensor $\mathbf{\Lambda} \in \mathcal{C}^{0,1}(\bar{\Omega}; \mathbb{R}^{3 \times 3 \times 3 \times 3})$, and let the strong Legendre-Hadamard condition (15c) and the symmetry condition (15b) be fulfilled uniformly in \mathbf{x} .

Under this conditions, the well posedness can be achieved by considering the boundary-value problem shifted by a multiplicative of the identity

$$\begin{aligned} -\nabla \cdot (\mathbf{\Lambda} : \nabla \mathbf{z}) + \zeta \mathbf{z} &= \mathbf{h} \quad \text{in } \Omega, \\ \mathbf{z} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (27)$$

Here, ζ is a possibly large constant. This boundary-value problem is for a sufficiently large constant ζ well posed (see, eg, prop. 13.1 and prop. 13.2 in the work of Chipot⁴⁶) and the solution operator is again compact. The existence of eigenfunctions to this system follows by the same arguments as earlier.

Consequently, the Galerkin space has to be adapted for an \mathbf{x} -dependent tensor $\mathbf{\Lambda}(\mathbf{x})$ and is the span of eigenfunctions of the solution operator to the boundary-value problem (27). However, (21) and the variational derivative have to be adapted and, in particular, additional terms with the derivative of $\mathbf{\Lambda}$ with respect to \mathbf{x} occur. It is, therefore, open whether the main result also applies to this generalization.

Moreover, the problem (26) is H^2 -regular (see, eg, thm. 6.5.6 in the work of Morrey⁴⁷ and recall that Ω is of class \mathcal{C}^2), ie, for any $\mathbf{h} \in \mathbf{L}^2$ the solution \mathbf{z} is in $\mathbf{H}^2 \cap \mathbf{H}_0^1$ and there exists a constant $c_\Lambda > 0$ such that

$$\|\mathbf{z}\|_{\mathbf{H}^2} \leq c_\Lambda \|\nabla \cdot (\mathbf{\Lambda} : \nabla \mathbf{z})\|_{\mathbf{L}^2}, \quad (28)$$

for any $\mathbf{z} \in \mathbf{H}^2 \cap \mathbf{H}_0^1$. This also shows that $\|\nabla \cdot (\mathbf{\Lambda} : \nabla \cdot)\|_{\mathbf{L}^2}$, $\|\Delta \cdot\|_{\mathbf{L}^2}$, $\|\cdot\|_{\mathbf{H}^2}$ are equivalent norms on $\mathbf{H}^2 \cap \mathbf{H}_0^1$ and that the eigenfunctions are in $\mathbf{H}^2 \cap \mathbf{H}_0^1$.

Again, the eigenfunctions form an orthogonal basis in \mathbf{L}^2 . Let $Z_n := \text{span}\{\mathbf{z}_1, \dots, \mathbf{z}_n\}$ ($n \in \mathbb{N}$) and assume $\|\mathbf{z}_i\|_{\mathbf{L}^2} = 1$ for $i = 1, 2, \dots$. Then

$$R_n : (\mathbf{H}^2 \cap \mathbf{H}_0^1)^* \longrightarrow Z_n, \quad R_n \mathbf{f} := \sum_{i=1}^n \langle \mathbf{f}, \mathbf{z}_i \rangle \mathbf{z}_i$$

is well defined and its restriction to \mathbf{L}^2 is the \mathbf{L}^2 -orthogonal projection onto Z_n such that

$$\|R_n\|_{\mathcal{L}(\mathbf{L}^2)} = 1.$$

If we equip \mathbf{H}_0^1 and $\mathbf{H}^2 \cap \mathbf{H}_0^1$ with the inner product

$$((\mathbf{\Lambda} : \nabla \mathbf{v}) : \nabla \mathbf{w}) \quad \text{and} \quad (\nabla \cdot (\mathbf{\Lambda} : \nabla \mathbf{v}), \nabla \cdot (\mathbf{\Lambda} : \nabla \mathbf{w})), \quad (29)$$

respectively, then the induced norms are equivalent to the standard norms (see also prop. 13.1 in the work of Chipot⁴⁶) and the restriction of R_n to \mathbf{H}_0^1 and $\mathbf{H}^2 \cap \mathbf{H}_0^1$ is the \mathbf{H}_0^1 - and $\mathbf{H}^2 \cap \mathbf{H}_0^1$ -orthogonal projection onto Z_n , respectively. The corresponding operator norms then equal 1 and, with respect to the standard norms, we thus find that there is a constant $c > 0$ such that, for all $n \in \mathbb{N}$,

$$\|R_n\|_{\mathcal{L}(\mathbf{H}_0^1)} \leq c, \quad \|R_n\|_{\mathcal{L}(\mathbf{H}^2 \cap \mathbf{H}_0^1)} \leq c.$$

Let $n \in \mathbb{N}$ be fixed. As usual, we consider the ansatz

$$\mathbf{v}_n(t) = \sum_{i=1}^n v_n^i(t) \mathbf{w}_i, \quad \mathbf{d}_n(t) = \sum_{i=1}^n d_n^i(t) \mathbf{z}_i.$$

Our approximation reads as

$$\begin{aligned} (\partial_t \mathbf{v}_n, \mathbf{w}) + ((\mathbf{v}_n \cdot \nabla) \mathbf{v}_n, \mathbf{w}) - (\nabla \mathbf{d}_n^T \mathbf{q}_n, \mathbf{w}) + (\mathbf{T}_n^L : \nabla \mathbf{w}) &= \langle \mathbf{g}, \mathbf{w} \rangle, \\ \mathbf{v}_n(0) &= P_n \mathbf{v}_0, \end{aligned} \quad (30a)$$

$$\begin{aligned} (\partial_t \mathbf{d}_n, \mathbf{z}) + ((\mathbf{v}_n \cdot \nabla) \mathbf{d}_n, \mathbf{z}) - ((\nabla \mathbf{v}_n)_{\text{skw}} \mathbf{d}_n, \mathbf{z}) + \lambda ((\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n, \mathbf{z}) + \gamma (\mathbf{q}_n, \mathbf{z}) &= 0, \\ \mathbf{d}_n(0) &= R_n \mathbf{d}_0, \end{aligned} \quad (30b)$$

for all $\mathbf{w} \in W_n$, and $\mathbf{z} \in Z_n$, where

$$\mathbf{q}_n := R_n \left(\frac{\delta \mathcal{F}}{\delta \mathbf{d}}(\mathbf{d}_n) \right) = R_n \left(\frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) - \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_n, \nabla \mathbf{d}_n) \right), \quad (30c)$$

and

$$\begin{aligned} \mathbf{T}_n^L := \mu_1 (\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n) \mathbf{d}_n \otimes \mathbf{d}_n + \mu_4 (\nabla \mathbf{v}_n)_{\text{sym}} - \gamma (\mu_2 + \mu_3) (\mathbf{d}_n \otimes \mathbf{q}_n)_{\text{sym}} \\ - (\mathbf{d}_n \otimes \mathbf{q}_n)_{\text{skw}} + (\mu_5 + \mu_6 - \lambda (\mu_2 + \mu_3)) (\mathbf{d}_n \otimes (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n)_{\text{sym}}. \end{aligned} \quad (30d)$$

The approximation \mathbf{T}_n^L of the Leslie stress tensor \mathbf{T}^L (see (2b)) relies upon replacing the term \mathbf{e} in which, unfortunately, the time derivative of \mathbf{d} occurs, by $\mathbf{e} = -\lambda(\nabla\mathbf{v})_{\text{sym}}\mathbf{d} - \gamma\mathbf{q}$, which follows from (1b) and (2c). Formally, this leads with (2d) to

$$\begin{aligned} \mathbf{T}^L = & \mu_1 (\mathbf{d} \cdot (\nabla\mathbf{v})_{\text{sym}}\mathbf{d}) \mathbf{d} \otimes \mathbf{d} + \mu_4 (\nabla\mathbf{v})_{\text{sym}} - \gamma(\mu_2 + \mu_3)(\mathbf{d} \otimes \mathbf{q})_{\text{sym}} \\ & - (\mathbf{d} \otimes \mathbf{q})_{\text{skw}} + (\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) (\mathbf{d} \otimes (\nabla\mathbf{v})_{\text{sym}}\mathbf{d})_{\text{sym}}. \end{aligned} \quad (31)$$

It is standard to prove the existence locally in time of solutions to the approximate problem in the sense of Carathéodory, ie, of solutions that are absolutely continuous with respect to time (see, eg, chapter I, thm. 5.2 in the work of Hale⁴⁸). Of course, the existence interval may depend on n . Global-in-time existence on $[0, T]$, however, follows later directly from suitable a priori estimates.

4.2 | Energy inequality and a priori estimates

In what follows, we derive an energy inequality and appropriate a priori estimates for the approximate solutions.

Proposition 2. *Let the assumptions of Theorem 1 be fulfilled and let $\{(\mathbf{v}_n, \mathbf{d}_n)\}$ be a solution to (30a). Then, there holds for almost all t*

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{v}_n\|_{L^2}^2 + \mathcal{F}(\mathbf{d}_n) \right) + \mu_1 \|\mathbf{d}_n \cdot (\nabla\mathbf{v}_n)_{\text{sym}}\mathbf{d}_n\|_{L^2}^2 + \mu_4 \|(\nabla\mathbf{v}_n)_{\text{sym}}\|_{L^2}^2 \\ & + (\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \|(\nabla\mathbf{v}_n)_{\text{sym}}\mathbf{d}_n\|_{L^2}^2 + \gamma \|\mathbf{q}_n\|_{L^2}^2 \\ & = \langle \mathbf{g}, \mathbf{v}_n \rangle + (\gamma(\mu_2 + \mu_3) - \lambda) (\mathbf{q}_n, (\nabla\mathbf{v}_n)_{\text{sym}}\mathbf{d}_n). \end{aligned} \quad (32)$$

Remark 5. If Parodi's relation (3) is assumed to hold for the constants appearing (see also (2d)), then the constant in front of the last term on the right-hand side of (32) vanishes.

Proof of Proposition. (Proof of Proposition 2)

We recall that the approximate solution $\{(\mathbf{v}_n, \mathbf{d}_n)\}$ is absolutely continuous in time. In order to derive the estimate asserted, we test (30a) with \mathbf{v}_n , (30b) with \mathbf{q}_n and add both equations. This leads to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{v}_n\|_{L^2}^2 + (\partial_t \mathbf{d}_n, \mathbf{q}_n) - \langle \nabla \mathbf{d}_n^T \mathbf{q}_n, \mathbf{v}_n \rangle + ((\mathbf{v}_n \cdot \nabla) \mathbf{d}_n, \mathbf{q}_n) \\ & + (\mathbf{T}_n^L : \nabla \mathbf{v}_n) - ((\nabla \mathbf{v}_n)_{\text{skw}} \mathbf{d}_n, \mathbf{q}_n) + \gamma \|\mathbf{q}_n\|_{L^2}^2 \\ & = \langle \mathbf{g}, \mathbf{v}_n \rangle - \lambda ((\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n, \mathbf{q}_n). \end{aligned}$$

Here, we have employed that the convection term vanishes because \mathbf{v}_n is solenoidal. Moreover, the projection R_n maps L^2 into Z_n , which ensures that \mathbf{q}_n takes values in Z_n .

A straightforward calculation shows that

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(\mathbf{d}_n) &= \frac{d}{dt} \int_{\Omega} F(\mathbf{d}_n, \nabla \mathbf{d}_n) dx = \int_{\Omega} \partial_t F(\mathbf{d}_n, \nabla \mathbf{d}_n) dx \\ &= \int_{\Omega} \left(\frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) \cdot \partial_t \mathbf{d}_n + \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_n, \nabla \mathbf{d}_n) : \partial_t \nabla \mathbf{d}_n \right) dx \\ &= \int_{\Omega} \partial_t \mathbf{d}_n \cdot \left(\frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) - \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_n, \nabla \mathbf{d}_n) \right) dx \\ &= \int_{\Omega} \partial_t \mathbf{d}_n \cdot R_n \left(\frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) - \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_n, \nabla \mathbf{d}_n) \right) dx = (\partial_t \mathbf{d}_n, \mathbf{q}_n). \end{aligned} \quad (33)$$

In the last but one step, we used that R_n is the L^2 -orthogonal projection onto Z_n .

For the term with the Leslie stress tensor, we find with rules recapitulated in Section 2.1 that

$$\begin{aligned} \mathbf{T}_n^L : \nabla \mathbf{v}_n &= \mu_1 (\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n)^2 + \mu_4 |(\nabla \mathbf{v}_n)_{\text{sym}}|^2 + (\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) |(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n|^2 \\ & - \gamma(\mu_2 + \mu_3) \mathbf{q}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n + \mathbf{q}_n \cdot (\nabla \mathbf{v}_n)_{\text{skw}} \mathbf{d}_n. \end{aligned}$$

The assertion now follows from putting all together. \square

An essential step in our analysis is the following energy inequality that is an adaptation of lemma 1 in the work of Lin and Liu¹⁴ to the general free energy functions considered here. In order to ensure the dissipative character of the system, the constants appearing are supposed to fulfill the constraints (4).

Corollary 1. *Under the assumptions of Theorem 1, there exist $\alpha, \beta, c > 0$ such that, for any solution $\{\mathbf{v}_n, \mathbf{d}_n\}$ to (30a), the energy inequality*

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{v}_n\|_{L^2}^2 + \mathcal{F}(\mathbf{d}_n) \right) + \mu_1 \|\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2}^2 \\ + \frac{\mu_4}{2} \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{L^2}^2 + \alpha \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2}^2 + \beta \|\mathbf{q}_n\|_{L^2}^2 \leq c \|\mathbf{g}\|_{(\mathbf{H}_{0,\sigma}^1)^*}^2 \end{aligned} \quad (34)$$

holds for almost all t .

Proof. The assertion is an immediate consequence of Proposition 2.

We first observe with Korn's first inequality (see, eg, thm. 10.1 in the work of McLean⁴⁹) and Young's inequality that

$$\begin{aligned} \langle \mathbf{g}, \mathbf{v}_n \rangle &\leq \|\mathbf{g}\|_{(\mathbf{H}_{0,\sigma}^1)^*} \|\mathbf{v}_n\|_{\mathbf{H}_{0,\sigma}^1} \leq c \|\mathbf{g}\|_{(\mathbf{H}_{0,\sigma}^1)^*} \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{L^2} \\ &\leq \frac{c^2}{2\mu_4} \|\mathbf{g}\|_{(\mathbf{H}_{0,\sigma}^1)^*}^2 + \frac{\mu_4}{2} \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{L^2}^2. \end{aligned}$$

In a second step, we find with (4) that there is $\delta \in (0, 1)$ such that

$$(\gamma(\mu_2 + \mu_3) - \lambda)^2 \leq 4\delta^2 \gamma(\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)),$$

and thus

$$\begin{aligned} (\gamma(\mu_2 + \mu_3) - \lambda) \langle \mathbf{q}_n, (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n \rangle &\leq |\gamma(\mu_2 + \mu_3) - \lambda| \|\mathbf{q}_n\|_{L^2} \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2} \\ &\leq 2\delta \sqrt{\gamma(\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3))} \|\mathbf{q}_n\|_{L^2} \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2} \\ &\leq \delta \gamma \|\mathbf{q}_n\|_{L^2}^2 + \delta(\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2}^2. \end{aligned}$$

Taking α, β appropriately proves the assertion. □

Corollary 2. (A priori estimates I)

Under the assumptions of Theorem 1, there holds

$$\begin{aligned} \frac{1}{2} \|\mathbf{v}_n\|_{L^\infty(L^2)}^2 + \sup_t \mathcal{F}(\mathbf{d}_n(t)) + \mu_1 \|\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)}^2 + \frac{\mu_4}{2} \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{L^2(L^2)}^2 + \alpha \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)}^2 + \beta \|\mathbf{q}_n\|_{L^2(L^2)}^2 \\ \leq c \left(\|\mathbf{v}_0\|_{L^2}^2 + \|\mathbf{g}\|_{L^2((\mathbf{H}_{0,\sigma}^1)^*)}^2 + \|\mathbf{d}_0\|_{\mathbf{H}_0^1}^6 + 1 \right) \end{aligned} \quad (35)$$

on the time interval of existence.

Proof. Integrating (34) with respect to time implies

$$\begin{aligned} \frac{1}{2} \|\mathbf{v}_n\|_{L^\infty(L^2)}^2 + \sup_t \mathcal{F}(\mathbf{d}_n(t)) + \mu_1 \|\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)}^2 + \frac{\mu_4}{2} \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{L^2(L^2)}^2 + \alpha \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)}^2 + \beta \|\mathbf{q}_n\|_{L^2(L^2)}^2 \\ \leq c \left(\|\mathbf{v}_0\|_{L^2}^2 + \mathcal{F}(R_n \mathbf{d}_0) + \|\mathbf{g}\|_{L^2((\mathbf{H}_{0,\sigma}^1)^*)}^2 \right) \end{aligned}$$

because $\mathbf{v}_n(0) = P_n \mathbf{v}_0$ and $\mathbf{d}_n(0) = R_n \mathbf{d}_0$. Thus, it remains to estimate $\mathcal{F}(R_n \mathbf{d}_0)$ independently of n .

In view of the smoothness of the free energy potential F , we find with the fundamental theorem of calculus

$$\begin{aligned} F(R_n \mathbf{d}_0, \nabla R_n \mathbf{d}_0) &= F(\mathbf{d}_0, \nabla \mathbf{d}_0) + \int_0^1 \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_0^n(s), \nabla \mathbf{d}_0^n(s)) \, ds \cdot (R_n \mathbf{d}_0 - \mathbf{d}_0) \\ &\quad + \int_0^1 \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_0^n(s), \nabla \mathbf{d}_0^n(s)) \, ds : \nabla (R_n \mathbf{d}_0 - \mathbf{d}_0), \end{aligned}$$

where $\mathbf{d}_0^n(s) := sR_n\mathbf{d}_0 + (1-s)\mathbf{d}_0$ ($s \in [0, 1]$). Applying the fundamental theorem of calculus again to the second term gives

$$\begin{aligned} F(R_n\mathbf{d}_0, \nabla R_n\mathbf{d}_0) &= F(\mathbf{d}_0, \nabla\mathbf{d}_0) + \int_0^1 \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_0^n(s), \nabla\mathbf{d}_0^n(s)) \, ds \cdot (R_n\mathbf{d}_0 - \mathbf{d}_0) \\ &\quad + \int_0^1 \frac{\partial F}{\partial \mathbf{S}}(0, 0) \, ds : \nabla(R_n\mathbf{d}_0 - \mathbf{d}_0) \\ &\quad + \int_0^1 \int_0^1 \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\tau\mathbf{d}_0^n(s), \tau\nabla\mathbf{d}_0^n(s)) \, d\tau \cdot \mathbf{d}_0^n(s) \, ds : \nabla(R_n\mathbf{d}_0 - \mathbf{d}_0) \\ &\quad + \int_0^1 \int_0^1 \frac{\partial^2 F}{\partial \mathbf{S}^2}(\tau\mathbf{d}_0^n(s), \tau\nabla\mathbf{d}_0^n(s)) \, d\tau : \nabla\mathbf{d}_0^n(s) \, ds : \nabla(R_n\mathbf{d}_0 - \mathbf{d}_0). \end{aligned}$$

Integrating over Ω gives

$$\begin{aligned} \mathcal{F}(R_n\mathbf{d}_0) &= \mathcal{F}(\mathbf{d}_0) + \int_0^1 \left(\frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_0^n(s), \nabla\mathbf{d}_0^n(s)), R_n\mathbf{d}_0 - \mathbf{d}_0 \right) \, ds \\ &\quad + \int_0^1 \left(\frac{\partial F}{\partial \mathbf{S}}(0, 0); \nabla(R_n\mathbf{d}_0 - \mathbf{d}_0) \right) \, ds \\ &\quad + \int_0^1 \int_0^1 \left(\frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\tau\mathbf{d}_0^n(s), \tau\nabla\mathbf{d}_0^n(s)) \cdot \mathbf{d}_0^n(s); \nabla(R_n\mathbf{d}_0 - \mathbf{d}_0) \right) \, d\tau \, ds \\ &\quad + \int_0^1 \int_0^1 \left(\frac{\partial^2 F}{\partial \mathbf{S}^2}(\tau\mathbf{d}_0^n(s), \tau\nabla\mathbf{d}_0^n(s)) : \nabla\mathbf{d}_0^n(s); \nabla(R_n\mathbf{d}_0 - \mathbf{d}_0) \right) \, d\tau \, ds \\ &= \mathcal{F}(\mathbf{d}_0) + I_1 + I_2 + I_3 + I_4, \end{aligned}$$

such that

$$|\mathcal{F}(R_n\mathbf{d}_0) - \mathcal{F}(\mathbf{d}_0)| \leq |I_1| + |I_2| + |I_3| + |I_4|.$$

Let us consider the term I_1 . Invoking the growth conditions (19b) and Young's inequality yields

$$\begin{aligned} |I_1| &\leq \int_{\Omega} \int_0^1 \left| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_0^n(s), \nabla\mathbf{d}_0^n(s)) \right| \, ds |R_n\mathbf{d}_0 - \mathbf{d}_0| \, dx \\ &\leq C_h \int_{\Omega} \int_0^1 \left(|\nabla\mathbf{d}_0^n(s)|^{\gamma_1/2} + |\mathbf{d}_0^n(s)|^{\gamma_2/2} + 1 \right) \, ds |R_n\mathbf{d}_0 - \mathbf{d}_0| \, dx. \end{aligned}$$

Using Young's and Hölder's inequality and Sobolev's embedding theorem thus leads to

$$\begin{aligned} |I_1| &\leq c \int_{\Omega} \int_0^1 \left(|\nabla\mathbf{d}_0^n(s)|^{5/3} + |\mathbf{d}_0^n(s)|^5 + 1 \right) \, ds |R_n\mathbf{d}_0 - \mathbf{d}_0| \, dx \\ &\leq c \int_{\Omega} \left(|\nabla R_n\mathbf{d}_0|^{5/3} + |R_n\mathbf{d}_0|^5 + |\nabla\mathbf{d}_0|^{5/3} + |\mathbf{d}_0|^5 + 1 \right) |R_n\mathbf{d}_0 - \mathbf{d}_0| \, dx \\ &\leq c \left(\|\nabla R_n\mathbf{d}_0\|_{L^2}^2 + \|R_n\mathbf{d}_0\|_{L^6}^6 + \|\nabla\mathbf{d}_0\|_{L^2}^2 + \|\mathbf{d}_0\|_{L^6}^6 + 1 \right)^{5/6} \|R_n\mathbf{d}_0 - \mathbf{d}_0\|_{L^6} \\ &\leq c \left(\|R_n\mathbf{d}_0\|_{H_0^1}^6 + \|\mathbf{d}_0\|_{H_0^1}^6 + 1 \right)^{5/6} \|R_n\mathbf{d}_0 - \mathbf{d}_0\|_{H_0^1}. \end{aligned}$$

Due to the continuity of the derivative of F with respect to \mathbf{S} , the term I_2 can be estimated by

$$|I_2| \leq \left| \frac{\partial F}{\partial \mathbf{S}}(0, 0) \right| \|\nabla(R_n\mathbf{d}_0 - \mathbf{d}_0)\|_{L^1} \leq c \|R_n\mathbf{d}_0 - \mathbf{d}_0\|_{H_0^1}.$$

For the term I_3 , we get with (19a) and estimating $\partial_t au$ by 1 that

$$\begin{aligned} |I_3| &\leq \int_{\Omega} \int_0^1 \left| \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}_0^n(s), \nabla\mathbf{d}_0^n(s)) \right| |\mathbf{d}_0^n(s)| \, ds |\nabla(R_n\mathbf{d}_0 - \mathbf{d}_0)| \, dx \\ &\leq C_{sh} \int_{\Omega} \int_0^1 \left((|\nabla\mathbf{d}_0^n(s)|^{\gamma_1/2-1} + |\mathbf{d}_0^n(s)|^{\gamma_3} + 1) |\mathbf{d}_0^n(s)| \right) \, ds |\nabla(R_n\mathbf{d}_0 - \mathbf{d}_0)| \, dx. \end{aligned}$$

Since γ_1 can be estimated by $10/3$ and γ_3 by 2 , we obtain

$$|I_3| \leq c \int_{\Omega} \int_0^1 \left(|\nabla\mathbf{d}_0^n(s)|^{2/3} |\mathbf{d}_0^n(s)| + |\mathbf{d}_0^n(s)|^3 + 1 \right) \, ds |\nabla(R_n\mathbf{d}_0 - \mathbf{d}_0)| \, dx.$$

An application of Young's and Hölder's inequality gives

$$\begin{aligned} |I_3| &\leq c \int_{\Omega} \int_0^1 \left(|\nabla \mathbf{d}_0^n(s)|^1 + |\mathbf{d}_0^n(s)|^3 + 1 \right) ds |\nabla(R_n \mathbf{d}_0 - \mathbf{d}_0)| dx \\ &\leq c \left(\|R_n \mathbf{d}_0\|_{\mathbf{H}_0^1}^2 + \|R_n \mathbf{d}_0\|_{L^6}^6 + \|\mathbf{d}_0\|_{\mathbf{H}_0^1}^2 + \|\mathbf{d}_0\|_{L^6}^6 + 1 \right)^{1/2} \|R_n \mathbf{d}_0 - \mathbf{d}_0\|_{\mathbf{H}_0^1} \\ &\leq c \left(\|R_n \mathbf{d}_0\|_{\mathbf{H}_0^1}^6 + \|\mathbf{d}_0\|_{\mathbf{H}_0^1}^6 + 1 \right)^{1/2} \|R_n \mathbf{d}_0 - \mathbf{d}_0\|_{\mathbf{H}_0^1}. \end{aligned}$$

For the term I_4 , we observe with (15a) that

$$|I_4| \leq c \|\mathbf{d}_0^n(s)\|_{\mathbf{H}_0^1} \|R_n \mathbf{d}_0 - \mathbf{d}_0\|_{\mathbf{H}_0^1} \leq c \left(\|R_n \mathbf{d}_0\|_{\mathbf{H}_0^1} + \|\mathbf{d}_0\|_{\mathbf{H}_0^1} \right) \|R_n \mathbf{d}_0 - \mathbf{d}_0\|_{\mathbf{H}_0^1}.$$

Finally, we come up with

$$|\mathcal{F}(R_n \mathbf{d}_0) - \mathcal{F}(\mathbf{d}_0)| \leq c \left(\|R_n \mathbf{d}_0\|_{\mathbf{H}_0^1}^5 + \|\mathbf{d}_0\|_{\mathbf{H}_0^1}^5 + 1 \right) \|R_n \mathbf{d}_0 - \mathbf{d}_0\|_{\mathbf{H}_0^1}.$$

With similar estimates as before, one can show that

$$|\mathcal{F}(\mathbf{d}_0)| \leq c \left(\|\mathbf{d}_0\|_{\mathbf{H}_0^1}^6 + 1 \right),$$

if $\mathbf{d}_0 \in \mathbf{H}_0^1$. Due to the boundedness of the projection R_n as an operator in \mathbf{H}_0^1 , we see that $\mathcal{F}(R_n \mathbf{d}_0)$ is bounded (independently of n) as long as $\mathbf{d}_0 \in \mathbf{H}_0^1$. This proves the assertion. \square

Remark 6. The proof of Corollary 2 also shows that $\mathcal{F}(R_n \mathbf{d}_0)$ converges to $\mathcal{F}(\mathbf{d}_0)$. The proof of Corollary 2 is done in such a way that it is still valid for the modified assumptions of Section 5.

Due to the strong coercivity assumptions on the free energy potential F and its derivatives, we are able to deduce a priori estimates in spaces with rather strong norms.

Corollary 3. (A priori estimates II)

Under the assumptions of Theorem 1, there is a constant $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\|\mathbf{v}_n\|_{L^\infty(L^2)}^2 + \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}_0^1)}^2 + \|\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)}^2 + \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{L^2(L^2)}^2 + \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)}^2 + \|\Delta \mathbf{d}_n\|_{L^2(L^2)}^2 \leq C. \quad (36)$$

Proof. The coercivity condition (16) implies

$$\mathcal{F}(\mathbf{d}_n(t)) = \int_{\Omega} F(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)) dx \geq \eta_1 \|\nabla \mathbf{d}_n(t)\|_{L^2}^2 - \eta_2 \|\mathbf{d}_n(t)\|_{L^2}^2 - \eta_3 |\Omega|. \quad (37)$$

The director equation (30b) tested with \mathbf{d}_n gives

$$(\partial_t \mathbf{d}_n, \mathbf{d}_n) + ((\mathbf{v}_n \cdot \nabla) \mathbf{d}_n, \mathbf{d}_n) - ((\nabla \mathbf{v}_n)_{\text{skw}} \mathbf{d}_n, \mathbf{d}_n) + \lambda ((\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n, \mathbf{d}_n) + \gamma(\mathbf{q}_n, \mathbf{d}_n) = 0.$$

Integration in time and using $2(\partial_t \mathbf{d}_n, \mathbf{d}_n) = \partial_t \|\mathbf{d}_n\|_{L^2}^2$, the vanishing divergence of \mathbf{v}_n and the skew-symmetry of $(\nabla \mathbf{v}_n)_{\text{skw}}$ shows that

$$\frac{1}{2} \|\mathbf{d}_n(t)\|_{L^2}^2 + \int_0^t \left(\lambda ((\mathbf{v}_n)_{\text{sym}} \mathbf{d}_n, \mathbf{d}_n) + \gamma(\mathbf{q}_n, \mathbf{d}_n) \right) ds = \frac{1}{2} \|\mathbf{d}_0\|_{L^2}^2.$$

The norm of the director can thus be estimated using Hölder's and Young's inequality and, in the second step Poincaré's inequality,

$$\begin{aligned} \|\mathbf{d}_n(t)\|_{L^2}^2 &\leq \int_0^t \left(\frac{\alpha}{2\eta_2} \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2}^2 + \frac{\beta}{2\eta_2} \|\mathbf{q}_n\|_{L^2}^2 + c \|\mathbf{d}_n\|_{L^2}^2 \right) ds + \|\mathbf{d}_0\|_{L^2}^2 \\ &\leq \int_0^t \left(\frac{\alpha}{2\eta_2} \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2}^2 + \frac{\beta}{2\eta_2} \|\mathbf{q}_n\|_{L^2}^2 + c \|\nabla \mathbf{d}_n\|_{L^2}^2 \right) ds + \|\mathbf{d}_0\|_{\mathbf{H}_0^1}^2. \end{aligned}$$

Applying this estimate together with (37) and (34) gives

$$\begin{aligned} & \frac{1}{2} \|\mathbf{v}_n(t)\|_{L^2}^2 + \eta_1 \|\nabla \mathbf{d}_n(t)\|_{L^2}^2 + \mu_1 \int_0^t \|\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2}^2 \, ds \\ & \quad + \frac{1}{2} \int_0^t \left(\mu_4 \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{L^2}^2 + \alpha \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2}^2 + \beta \|\mathbf{q}_n\|_{L^2}^2 \right) \, ds \\ & \leq \frac{1}{2} \|\mathbf{v}_0\|_{L^2}^2 + c \left(\|\mathbf{g}\|_{L^2((\mathbf{H}_{0,\sigma}^1)^*)}^2 + \|\mathbf{d}_0\|_{\mathbf{H}_0^1}^6 + 1 \right) \\ & \quad + \eta_2 c \int_0^t \|\nabla \mathbf{d}_n\|_{L^2}^2 \, ds + \eta_2 \|\mathbf{d}_0\|_{\mathbf{H}_0^1}^2 + \eta_3 |\Omega|. \end{aligned}$$

The lemma of Gronwall and taking the supremum over all $t \in [0, T]$ shows the estimate

$$\begin{aligned} & \frac{1}{2} \|\mathbf{v}_n\|_{L^\infty(L^2)}^2 + \eta_1 \|\nabla \mathbf{d}_n\|_{L^\infty(L^2)}^2 + \mu_1 \|\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)}^2 \\ & \quad + \frac{\mu_4}{2} \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{L^2(L^2)}^2 + \frac{\alpha}{2} \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)}^2 + \frac{\beta}{2} \|\mathbf{q}_n\|_{L^2(L^2)}^2 \\ & \leq c \left(\|\mathbf{v}_0\|_{L^2}^2 + \mathcal{F}(\mathbf{d}_0) + \|\mathbf{g}\|_{L^2((\mathbf{H}_{0,\sigma}^1)^*)}^2 + \|\mathbf{d}_0\|_{\mathbf{H}^1}^6 + 1 \right) e^{cT} =: K, \end{aligned} \quad (38)$$

and thus, the boundedness of the director in the $L^\infty(0, T; \mathbf{H}^1)$ norm.

Moreover, (30c), (17), and (18) leads to

$$|\mathbf{q}_n|^2 \geq \frac{1}{2} |R_n \nabla \cdot (\mathbf{A} : \nabla \mathbf{d}_n)|^2 - 4 \left| R_n \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) : \nabla \mathbf{d}_n^T \right|^2 - 4 \left| R_n \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) \right|^2.$$

The orthogonality properties of R_n then imply

$$\|\mathbf{q}_n\|_{L^2}^2 \geq \frac{1}{2} \|\nabla \cdot (\mathbf{A} : \nabla \mathbf{d}_n)\|_{L^2}^2 - 4 \left\| \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) : \nabla \mathbf{d}_n^T \right\|_{L^2}^2 - 4 \left\| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) \right\|_{L^2}^2.$$

The estimates (28), (19a), and (19b) show that there are constants $c_1, c > 0$ such that, for all t ,

$$\|\mathbf{q}_n\|_{L^2}^2 \geq c_1 \|\Delta \mathbf{d}_n\|_{L^2}^2 - c \left(\|\nabla \mathbf{d}_n\|_{L^{\gamma_1}}^{\gamma_1} + \|\mathbf{d}_n\|^{\gamma_3} \|\nabla \mathbf{d}_n\|_{L^2}^2 + \|\nabla \mathbf{d}_n\|_{L^2}^2 + \|\mathbf{d}_n\|_{L^{\gamma_2}}^{\gamma_2} + 1 \right). \quad (39)$$

With (19c), an application of Young's inequality (recall that $\gamma_3 = 0$ if $\gamma_1 = 2$),

$$\begin{aligned} \|\mathbf{d}_n\|^{\gamma_3} \|\nabla \mathbf{d}_n\|_{L^2}^2 & \leq \frac{2}{\gamma_1} \|\nabla \mathbf{d}_n\|_{L^{\gamma_1}}^{\gamma_1} + \frac{\gamma_1 - 2}{\gamma_1} \|\mathbf{d}_n\|_{L^{2\gamma_3\gamma_1/(\gamma_1-2)}}^{2\gamma_3\gamma_1/(\gamma_1-2)} \\ & = \frac{2}{\gamma_1} \|\nabla \mathbf{d}_n\|_{L^{\gamma_1}}^{\gamma_1} + \frac{\gamma_1 - 2}{\gamma_1} \|\mathbf{d}_n\|_{L^{\gamma_2}}^{\gamma_2}, \end{aligned}$$

and

$$\|\nabla \mathbf{d}_n\|_{L^2}^2 \leq c \|\nabla \mathbf{d}_n\|_{L^{\gamma_1}}^2 \leq 1 + c \|\nabla \mathbf{d}_n\|_{L^{\gamma_1}}^{\gamma_1},$$

we come up with

$$\|\mathbf{q}_n\|_{L^2}^2 \geq c_1 \|\Delta \mathbf{d}_n\|_{L^2}^2 - c \left(\|\nabla \mathbf{d}_n\|_{L^{\gamma_1}}^{\gamma_1} + \|\mathbf{d}_n\|_{L^{\gamma_2}}^{\gamma_2} + 1 \right),$$

and thus with

$$\|\mathbf{q}_n\|_{L^2(L^2)}^2 \geq c_1 \|\Delta \mathbf{d}_n\|_{L^2(L^2)}^2 - c \left(\|\nabla \mathbf{d}_n\|_{L^{\gamma_1}(L^{\gamma_1})}^{\gamma_1} + \|\mathbf{d}_n\|_{L^{\gamma_2}(L^{\gamma_2})}^{\gamma_2} + 1 \right).$$

Lemma 1 yields

$$\begin{aligned} \|\nabla \mathbf{d}_n\|_{L^{\gamma_1}(L^{\gamma_1})}^{\gamma_1} & \leq c \|\Delta \mathbf{d}_n\|_{L^2(L^2)}^{\theta_1} \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}_0^1)}^{\gamma_1 - \theta_1}, \quad \theta_1 = \frac{3}{2}(\gamma_1 - 2), \\ \|\mathbf{d}_n\|_{L^{\gamma_2}(L^{\gamma_2})}^{\gamma_2} & \leq c \|\Delta \mathbf{d}_n\|_{L^2(L^2)}^{\theta_2} \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}_0^1)}^{\gamma_2 - \theta_2}, \quad \theta_2 = \frac{1}{2}(\gamma_2 - 6). \end{aligned}$$

Young's inequality now leads to

$$\|\mathbf{q}_n\|_{L^2(L^2)}^2 \geq \frac{c_1}{2} \|\Delta \mathbf{d}_n\|_{L^2(L^2)}^2 - c \left(\|\mathbf{d}_n\|_{L^\infty(\mathbf{H}_0^1)}^p + 1 \right),$$

with

$$p = 2 \max \left(\frac{\gamma_1 - \theta_1}{2 - \theta_1}, \frac{\gamma_2 - \theta_2}{2 - \theta_2} \right) = 2 \max \left(\frac{6 - \gamma_1}{10 - 3\gamma_1}, \frac{6 + \gamma_2}{10 - \gamma_2} \right) < \infty.$$

Because of (38), we already know that $\eta_1 \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}_0^1)}^2$ is bounded by K . Hence, (35) leads to

$$\begin{aligned} & \frac{1}{2} \|\mathbf{v}_n\|_{L^\infty(\mathbf{L}^2)}^2 + \eta_1 \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}_0^1)}^2 + \mu_1 \|\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(\mathbf{L}^2)}^2 + \frac{\mu_4}{2} \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{L^2(\mathbf{L}^2)}^2 \\ & + \frac{\alpha}{2} \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(\mathbf{L}^2)}^2 + \frac{c_1 \beta}{4} \|\Delta \mathbf{d}_n\|_{L^2(\mathbf{L}^2)}^2 \leq K + c\beta \left(\left(\frac{K}{\eta_1} \right)^{\frac{p}{2}} + 1 \right), \end{aligned}$$

which proves the assertion. \square

The aforementioned a priori estimates ensure that the approximate solutions exist on $[0, T]$ (see, again, chapter I, thm. 5.2 in the work of Hale⁴⁸).

We are now going to estimate the time derivative of \mathbf{d}_n and \mathbf{v}_n in appropriate norms.

Proposition 3. *Under the assumptions of Theorem 1, there is a constant $C > 0$ such that, for all $n \in \mathbb{N}$,*

$$\|\partial_t \mathbf{v}_n\|_{L^2((\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1)^*)} \leq C. \quad (40)$$

Proof. Recall that P_n is the $(\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1)$ -orthogonal projection onto W_n . We thus find with (30a), for all $t \in [0, T]$ and all $\boldsymbol{\varphi} \in \mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1$,

$$\begin{aligned} |\langle \partial_t \mathbf{v}_n, \boldsymbol{\varphi} \rangle| &= |(\partial_t \mathbf{v}_n, P_n \boldsymbol{\varphi})| = \left| \langle \mathbf{g}, P_n \boldsymbol{\varphi} \rangle - ((\mathbf{v}_n \cdot \nabla) \mathbf{v}_n, P_n \boldsymbol{\varphi}) + (\nabla \mathbf{d}_n^T \mathbf{q}_n, P_n \boldsymbol{\varphi}) - (\mathbf{T}_n^L : \nabla P_n \boldsymbol{\varphi}) \right| \\ &\leq \|\mathbf{g}\|_{(\mathbf{H}_{0,\sigma}^1)^*} \|P_n \boldsymbol{\varphi}\|_{\mathbf{H}_{0,\sigma}^1} + \|(\mathbf{v}_n \cdot \nabla) \mathbf{v}_n\|_{L^1} \|P_n \boldsymbol{\varphi}\|_{L^\infty} \\ &\quad + \|\nabla \mathbf{d}_n^T \mathbf{q}_n\|_{L^1} \|P_n \boldsymbol{\varphi}\|_{L^\infty} + \|\mathbf{T}_n^L\|_{L^{6/5}} \|\nabla P_n \boldsymbol{\varphi}\|_{L^6}. \end{aligned}$$

Since \mathbf{H}^2 is continuously embedded in \mathbf{H}^1 , L^∞ , and $\mathbf{W}^{1,6}$, we find

$$\|\partial_t \mathbf{v}_n\|_{(\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1)^*} \leq c \left(\|\mathbf{g}\|_{(\mathbf{H}_{0,\sigma}^1)^*} + \|(\mathbf{v}_n \cdot \nabla) \mathbf{v}_n\|_{L^1} + \|\nabla \mathbf{d}_n^T \mathbf{q}_n\|_{L^1} + \|\mathbf{T}_n^L\|_{L^{6/5}} \right),$$

and thus

$$\|\partial_t \mathbf{v}_n\|_{L^2((\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1)^*)} \leq c \left(\|\mathbf{g}\|_{L^2((\mathbf{H}_{0,\sigma}^1)^*)} + \|(\mathbf{v}_n \cdot \nabla) \mathbf{v}_n\|_{L^2(L^1)} + \|\nabla \mathbf{d}_n^T \mathbf{q}_n\|_{L^2(L^1)} + \|\mathbf{T}_n^L\|_{L^2(L^{6/5})} \right).$$

With Hölder's inequality and Lemma 2, we see that

$$\|(\mathbf{v}_n \cdot \nabla) \mathbf{v}_n\|_{L^2(L^1)} \leq \|\mathbf{v}_n\|_{L^\infty(L^2)} \|\nabla \mathbf{v}_n\|_{L^2(L^2)} \quad \text{and} \quad \|\nabla \mathbf{d}_n^T \mathbf{q}_n\|_{L^2(L^1)} \leq \|\nabla \mathbf{d}_n\|_{L^\infty(L^2)} \|\mathbf{q}_n\|_{L^2(L^2)}.$$

In view of (36), (38), and Korn's inequality, the terms on the right-hand sides of the foregoing estimates are bounded.

Finally, we observe with (30d) the estimate

$$\begin{aligned} \|\mathbf{T}_n^L\|_{L^2(L^{6/5})} &\leq c \left(\|(\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n) \mathbf{d}_n \otimes \mathbf{d}_n\|_{L^2(L^{6/5})} + \|\nabla \mathbf{v}_n\|_{L^2(L^{6/5})} \right. \\ &\quad \left. + \|\mathbf{d}_n \otimes \mathbf{q}_n\|_{L^2(L^{6/5})} + \|\mathbf{d}_n \otimes (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^{6/5})} \right), \end{aligned}$$

where (again with Hölder's inequality)

$$\begin{aligned} \|(\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n) \mathbf{d}_n \otimes \mathbf{d}_n\|_{L^2(L^{6/5})} &\leq \|\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)} \|\mathbf{d}_n\|_{L^\infty(L^6)}, \\ \|\mathbf{d}_n \otimes \mathbf{q}_n\|_{L^2(L^{6/5})} &\leq c \|\mathbf{d}_n \otimes \mathbf{q}_n\|_{L^2(L^{3/2})} \leq c \|\mathbf{d}_n\|_{L^\infty(L^6)} \|\mathbf{q}_n\|_{L^2(L^2)}, \\ \|\mathbf{d}_n \otimes (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^{6/5})} &\leq c \|\mathbf{d}_n \otimes (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^{3/2})} \\ &\leq c \|\mathbf{d}_n\|_{L^\infty(L^6)} \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^2(L^2)}, \end{aligned}$$

which proves the assertion because of (36) and (38). \square

Proposition 4. Under the assumptions of Theorem 1, there is a constant $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\|\partial_t \mathbf{d}_n\|_{L^{4/3}(\mathbf{L}^2)} \leq C. \quad (41)$$

Proof. Recall that R_n is the \mathbf{L}^2 -orthogonal projection onto Z_n . We thus find with (30b), for all $t \in [0, T]$,

$$\begin{aligned} \|\partial_t \mathbf{d}_n\|_{\mathbf{L}^2} &= \sup_{\|\boldsymbol{\psi}\|_{\mathbf{L}^2} \leq 1} |(\partial_t \mathbf{d}_n, \boldsymbol{\psi})| = \sup_{\|\boldsymbol{\psi}\|_{\mathbf{L}^2} \leq 1} |(\partial_t \mathbf{d}_n, R_n \boldsymbol{\psi})| \\ &\leq \sup_{\|\boldsymbol{\psi}\|_{\mathbf{L}^2} \leq 1} \left\| -(\mathbf{v}_n \cdot \nabla) \mathbf{d}_n + (\nabla \mathbf{v}_n)_{\text{skw}} \mathbf{d}_n - \lambda (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n - \gamma \mathbf{q}_n \right\|_{\mathbf{L}^2} \|R_n \boldsymbol{\psi}\|_{\mathbf{L}^2} \\ &\leq \|(\mathbf{v}_n \cdot \nabla) \mathbf{d}_n\|_{\mathbf{L}^2} + \|(\nabla \mathbf{v}_n)_{\text{skw}} \mathbf{d}_n\|_{\mathbf{L}^2} + |\lambda| \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{\mathbf{L}^2} + \gamma \|\mathbf{q}_n\|_{\mathbf{L}^2}. \end{aligned} \quad (42)$$

In view of (38), we see that

$$\|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{L^{4/3}(\mathbf{L}^2)} \text{ and } \|\mathbf{q}_n\|_{L^{4/3}(\mathbf{L}^2)}$$

are bounded. It remains to consider the first two terms on the right-hand side of (42). With Hölder's inequality and Lemma 1, we find

$$\|(\mathbf{v}_n \cdot \nabla) \mathbf{d}_n\|_{L^{4/3}(\mathbf{L}^2)} \leq \|\mathbf{v}_n\|_{L^2(\mathbf{L}^6)} \|\nabla \mathbf{d}_n\|_{L^4(\mathbf{L}^3)} \leq c \|\mathbf{v}_n\|_{L^2(\mathbf{H}^1_0)} \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}^1_0)}^{1/2} \|\mathbf{d}_n\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}^1_0)}^{1/2}.$$

Note that, employing Korn's inequality, all terms on the right-hand side are bounded in view of (36). Similarly, we find that

$$\|(\nabla \mathbf{v}_n)_{\text{skw}} \mathbf{d}_n\|_{L^{4/3}(\mathbf{L}^2)} \leq 2 \|\mathbf{v}_n\|_{L^2(\mathbf{H}^1_0)} \|\mathbf{d}_n\|_{L^4(\mathbf{L}^\infty)} \leq c \|\mathbf{v}_n\|_{L^2(\mathbf{H}^1_0)} \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}^1_0)}^{1/2} \|\mathbf{d}_n\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}^1_0)}^{1/2}$$

is bounded. \square

4.3 | Convergence of the approximate solutions

In what follows, we consider a sequence of approximate solutions as $n \rightarrow \infty$. The a priori estimates for the approximate solutions imply then the following results.

Corollary 4. Under the assumptions of Theorem 1, there is a subsequence (not relabeled) of the sequence of solutions to the approximate problem (30a) such that

$$\mathbf{v}_n \overset{*}{\rightharpoonup} \mathbf{v} \quad \text{in } L^\infty(0, T; \mathbf{L}^2_\sigma), \quad (43a)$$

$$\mathbf{v}_n \rightharpoonup \mathbf{v} \quad \text{in } L^2(0, T; \mathbf{H}^1_{0,\sigma}), \quad (43b)$$

$$\mathbf{q}_n \rightharpoonup \bar{\mathbf{q}} \quad \text{in } L^2(0, T; \mathbf{L}^2), \quad (43c)$$

$$\mathbf{d}_n \overset{*}{\rightharpoonup} \mathbf{d} \quad \text{in } L^\infty(0, T; \mathbf{H}^1_0), \quad (43d)$$

$$\mathbf{d}_n \rightharpoonup \mathbf{d} \quad \text{in } L^2(0, T; \mathbf{H}^2 \cap \mathbf{H}^1_0), \quad (43e)$$

$$\partial_t \mathbf{v}_n \rightharpoonup \partial_t \mathbf{v} \quad \text{in } L^2\left(0, T; (\mathbf{H}^2 \cap \mathbf{H}^1_{0,\sigma})^*\right), \quad (43f)$$

$$\partial_t \mathbf{d}_n \rightharpoonup \partial_t \mathbf{d} \quad \text{in } L^{4/3}(0, T; \mathbf{L}^2), \quad (43g)$$

$$\mathbf{v}_n \rightarrow \mathbf{v} \quad \text{in } L^p(0, T; \mathbf{L}^2_\sigma) \text{ for any } p \in [1, \infty), \quad (43h)$$

$$\mathbf{d}_n \rightarrow \mathbf{d} \quad \text{in } L^q(0, T; L^q) \text{ for any } q \in [1, 10), \quad (43i)$$

$$\mathbf{d}_n \rightarrow \mathbf{d} \quad \text{in } L^r(0, T; \mathbf{W}^{1,r}) \text{ for any } r \in [1, 10/3). \quad (43j)$$

Proof. The existence of weakly and weakly* convergent subsequences immediately follows, by standard arguments, from the a priori estimates (36) and (38) together with Korn's inequality and from (40) and (41) together with the definition of the weak time derivative. The strong convergence follows from the Lions-Aubin compactness lemma (see théorème 1.5.2 in the work of Lions⁵⁰). With respect to \mathbf{v}_n , we observe that $\mathbf{H}_{0,\sigma}^1$ is compactly embedded in \mathbf{L}_σ^2 , which implies strong convergence in $L^2(0, T; \mathbf{L}_\sigma^2)$ and together with the boundedness in $L^\infty(0, T; \mathbf{L}_\sigma^2)$ and in $L^p(0, T; \mathbf{L}_\sigma^2)$ for any $p \in [1, \infty)$.

With respect to \mathbf{d}_n , we observe that $\mathbf{H}^2 \cap \mathbf{H}_0^1$ is compactly embedded in \mathbf{H}_0^1 , which implies strong convergence in $L^2(0, T; \mathbf{H}_0^1)$ and together with the boundedness in $L^\infty(0, T; \mathbf{H}_0^1)$ also in $L^p(0, T; \mathbf{H}_0^1)$ and thus in $L^p(0, T; \mathbf{L}^6)$ for any $p \in [1, \infty)$. Moreover, \mathbf{d}_n is also bounded in $L^{10}(0, T; \mathbf{L}^{10})$. Indeed, with Lemma 1, we find

$$\|\mathbf{d}_n\|_{L^{10}(\mathbf{L}^{10})} \leq c \|\mathbf{d}_n\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)}^{1/5} \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}_0^1)}^{4/5}.$$

This implies (43i). Finally, \mathbf{d}_n is also bounded in $L^{10/3}(0, T; \mathbf{W}^{1,10/3})$ since

$$\|\mathbf{d}_n\|_{L^{10/3}(\mathbf{W}^{1,10/3})} \leq c \|\mathbf{d}_n\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)}^{3/5} \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}^1)}^{2/5}$$

because of Lemma 1. This proves (43j). \square

Corollary 5. *Under the assumptions of Theorem 1, the limits \mathbf{v} and \mathbf{d} from Corollary 4 satisfy*

$$\mathbf{v}(0) = \mathbf{v}_0 \text{ and } \mathbf{d}(0) = \mathbf{d}_0.$$

Proof. Due to the convergence results (43f) for the time derivative of the velocity field, we get

$\mathbf{v} \in W^{1,2}(0, T; (\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1)^*) \hookrightarrow \mathcal{C}([0, T]; (\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1)^*)$ and, together with (43f), that, for any $\mathbf{w} \in \mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1$ and $\omega(t) = (T - t)/T$,

$$\langle \mathbf{v}_n(0) - \mathbf{v}(0), \mathbf{w} \rangle = - \int_0^T (\langle \mathbf{v}_n(t) - \mathbf{v}(t), \mathbf{w}\omega'(t) \rangle + \langle \partial_t \mathbf{v}_n(t) - \partial_t \mathbf{v}(t), \mathbf{w}\omega(t) \rangle) dt \rightarrow 0,$$

which shows that

$$\mathbf{v}_n(0) \rightharpoonup \mathbf{v}(0) \text{ in } (\mathbf{H}^2 \cap \mathbf{H}_{0,\sigma}^1)^*.$$

Moreover, we know that

$$\mathbf{v}_n(0) = P_n \mathbf{v}_0 \rightarrow \mathbf{v}_0 \text{ in } \mathbf{L}_\sigma^2,$$

which proves $\mathbf{v}(0) = \mathbf{v}_0$.

Analogously, one proves that $\mathbf{d}(0) = \mathbf{d}_0$. \square

With the following proposition, we identify the limit $\bar{\mathbf{q}}$ in (43c).

Proposition 5. *Under the assumptions of Theorem 1, the limit $\bar{\mathbf{q}}$ in (43c) is given by $\bar{\mathbf{q}} = \mathbf{q}$, where \mathbf{q} is given by (2a).*

Proof. In what follows, we do not relabel the subsequence that exists in view of Corollary 4. We already know (43c) and wish to establish now the weak convergence of \mathbf{q}_n to \mathbf{q} in $L^2(0, T; \mathbf{H}^{-1})$. Recalling that R_n is the L^2 -orthogonal projection onto Z_n , we find, for all $\boldsymbol{\psi} \in L^2(0, T; \mathbf{H}_0^1)$, that

$$\begin{aligned} & \int_0^T \langle \mathbf{q}_n(t) - \mathbf{q}(t), \boldsymbol{\psi}(t) \rangle dt \\ &= \int_0^T \left\langle \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)) - \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)), R_n \boldsymbol{\psi}(t) \right\rangle dt \\ & \quad - \int_0^T \left\langle \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}(t), \nabla \mathbf{d}(t)) - \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}(t), \nabla \mathbf{d}(t)), \boldsymbol{\psi}(t) \right\rangle dt \\ &= \int_0^T \left\langle \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)) - \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)), R_n \boldsymbol{\psi}(t) - \boldsymbol{\psi}(t) \right\rangle dt \\ & \quad + \int_0^T \left\langle \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)) - \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}(t), \nabla \mathbf{d}(t)), \boldsymbol{\psi}(t) \right\rangle dt \\ & \quad - \int_0^T \left\langle \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)) - \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}(t), \nabla \mathbf{d}(t)), \boldsymbol{\psi}(t) \right\rangle dt =: I_{1,n} + I_{2,n} + I_{3,n}. \end{aligned} \quad (44)$$

Regarding the term $I_{1,n}$, we note that

$$\begin{aligned} & \left\| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)) - \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)) \right\|_{L^2(\mathbf{L}^2)} \\ & \leq c \left(\|\mathbf{d}_n\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)} \|\mathbf{d}_n\|_{L^\infty(\mathbf{H}_0^1)}^4 + \|\mathbf{d}_n\|_{L^2(\mathbf{H}^2 \cap \mathbf{H}_0^1)} + 1 \right), \end{aligned}$$

which can be shown as in the proof of Proposition 1. This, together with (36), shows the boundedness of the term earlier. Moreover, R_n is the L^2 -orthogonal projection onto Z_n such that, for all $\boldsymbol{\psi} \in L^2(0, T; \mathbf{L}^2)$,

$$\lim_{n \rightarrow \infty} \|R_n \boldsymbol{\psi} - \boldsymbol{\psi}\|_{L^2(\mathbf{L}^2)} = 0.$$

This shows that $I_{1,n}$ converges to 0 as $n \rightarrow \infty$.

Let us now consider the term $I_{2,n}$. Because of (43i) and (43j), we observe that (passing to a subsequence if necessary)

$$\mathbf{d}_n(\mathbf{x}, t) \rightarrow \mathbf{d}(\mathbf{x}, t), \quad \nabla \mathbf{d}_n(\mathbf{x}, t) \rightarrow \nabla \mathbf{d}(\mathbf{x}, t), \quad (45)$$

for almost all $(\mathbf{x}, t) \in \Omega \times (0, T)$. Moreover, $|\mathbf{d}_n(\mathbf{x}, t)|$ is dominated by a function in $L^q(0, T; L^q)$ ($q \in [1, 10)$) and $|\nabla \mathbf{d}_n(\mathbf{x}, t)|$ is dominated by a function in $L^r(0, T; L^r)$ ($r \in [1, 10/3)$). The growth condition (19b) then shows that

$$\left| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n(\mathbf{x}, t), \nabla \mathbf{d}_n(\mathbf{x}, t)) \right| \leq C_h (|\nabla \mathbf{d}_n(\mathbf{x}, t)|^{\gamma_1/2} + |\mathbf{d}_n(\mathbf{x}, t)|^{\gamma_2/2} + 1)$$

is dominated by a function in $L^2(0, T; \mathbf{L}^2)$. With the continuity of $\frac{\partial F}{\partial \mathbf{h}}$ and Lebesgue's theorem on dominated convergence, we thus find that $I_{2,n}$ converges to 0 as $n \rightarrow \infty$.

For the term $I_{3,n}$, we find (see (17)) that

$$\begin{aligned} I_{3,n} &= \int_0^T \langle (\boldsymbol{\Lambda} : \nabla(\mathbf{d}_n(t) - \mathbf{d}(t))) : \nabla \boldsymbol{\psi}(t) \rangle dt \\ &\quad - \int_0^T \left\langle \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)) : \nabla \mathbf{d}_n(t)^T - \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}(t), \nabla \mathbf{d}(t)) : \nabla \mathbf{d}(t)^T, \boldsymbol{\psi}(t) \right\rangle dt. \end{aligned}$$

The first term on the right-hand side, which is linear, converges to 0 because of (43e). The second term can be dealt with similarly as $I_{2,n}$. In particular, (19a) together with Young's inequality provides that

$$\begin{aligned} \left| \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}_n(\mathbf{x}, t), \nabla \mathbf{d}_n(\mathbf{x}, t)) : \nabla \mathbf{d}_n(\mathbf{x}, t)^T \right| &\leq C_{sh} (|\nabla \mathbf{d}_n(\mathbf{x}, t)|^{\gamma_1/2-1} + |\mathbf{d}_n(\mathbf{x}, t)|^{\gamma_3} + 1) |\nabla \mathbf{d}_n(\mathbf{x}, t)| \\ &\leq c (|\nabla \mathbf{d}_n(\mathbf{x}, t)|^{\gamma_1/2} + |\mathbf{d}_n(\mathbf{x}, t)|^{\gamma_2/2} + 1) \end{aligned}$$

is dominated by a function in $L^2(0, T; \mathbf{L}^2)$. □

We are now ready to prove the main result.

Proof of Theorem. (Proof of Theorem 1)

It only remains to prove that the limit (\mathbf{v}, \mathbf{d}) from Corollary 4 satisfies the original problem in the sense of Definition 1. This is shown by passing to the limit in the approximate problem (30a).

Let us start with the approximation (30a) of the Navier-Stokes-like equation. In view of Corollary 4, we already know that the term incorporating the time derivative converges. Moreover, we find with the (43h) convergence of the convection term such that, for all solenoidal $\boldsymbol{\varphi} \in \mathcal{C}_c^\infty(\Omega \times (0, T); \mathbb{R}^3)$,

$$\int_0^T ((\mathbf{v}_n(t) \cdot \nabla) \mathbf{v}_n(t), \boldsymbol{\varphi}(t)) dt \rightarrow \int_0^T ((\mathbf{v}(t) \cdot \nabla) \mathbf{v}(t), \boldsymbol{\varphi}(t)) dt.$$

With Proposition 5, (43c), and (43i), we find that

$$\int_0^T \langle \nabla \mathbf{d}_n^T(t) \mathbf{q}_n(t), \boldsymbol{\varphi}(t) \rangle dt \rightarrow \int_0^T \langle \nabla \mathbf{d}^T(t) \mathbf{q}(t), \boldsymbol{\varphi}(t) \rangle dt.$$

With respect to the term incorporating the Leslie tensor, we only focus on the first term, that is, the least regular one. With (43b) and (43i), we find that

$$\begin{aligned} & \int_0^T \langle (\mathbf{d}_n(t) \cdot (\nabla \mathbf{v}_n(t))_{\text{sym}} \mathbf{d}_n(t)) \mathbf{d}_n(t) \otimes \mathbf{d}_n(t) : \nabla \boldsymbol{\varphi}(t) \rangle dt \\ & \rightarrow \int_0^T \langle (\mathbf{d}(t) \cdot (\nabla \mathbf{v}(t))_{\text{sym}} \mathbf{d}(t)) \mathbf{d}(t) \otimes \mathbf{d}(t) : \nabla \boldsymbol{\varphi}(t) \rangle dt. \end{aligned}$$

This, together with similar observations for the other terms, shows that

$$\int_0^T (\mathbf{T}_n^L(t) : \nabla \boldsymbol{\varphi}(t)) dt \rightarrow \int_0^T (\mathbf{T}^L(t) : \nabla \boldsymbol{\varphi}(t)) dt,$$

where \mathbf{T}^L is given by (31), which is equivalent to (2b).

Regarding the approximation (30b) of the director equation, we observe the convergence of the term incorporating the time derivative because of (43g). With (43h) and (43i), we obtain, for all $\boldsymbol{\psi} \in \mathcal{C}_c^\infty(\Omega \times (0, T); \mathbb{R}^3)$,

$$\int_0^T ((\mathbf{v}_n(t) \cdot \nabla) \mathbf{d}_n(t), \boldsymbol{\psi}(t)) dt \rightarrow \int_0^T ((\mathbf{v}(t) \cdot \nabla) \mathbf{d}(t), \boldsymbol{\psi}(t)) dt.$$

With (43b) and (43i), we find that

$$\int_0^T ((\nabla \mathbf{v}_n(t))_{\text{skw}} \mathbf{d}_n(t), \boldsymbol{\psi}(t)) dt \rightarrow \int_0^T ((\nabla \mathbf{v}(t))_{\text{skw}} \mathbf{d}(t), \boldsymbol{\psi}(t)) dt$$

and

$$\int_0^T ((\nabla \mathbf{v}_n(t))_{\text{sym}} \mathbf{d}_n(t), \boldsymbol{\psi}(t)) dt \rightarrow \int_0^T ((\nabla \mathbf{v}(t))_{\text{sym}} \mathbf{d}(t), \boldsymbol{\psi}(t)) dt.$$

For the term with \mathbf{q}_n , we employ (43c) together with Proposition 5.

All this shows that the limit (\mathbf{v}, \mathbf{d}) of the approximate solutions satisfies the original equations. Moreover, Corollary 5 shows that the initial conditions are also fulfilled. \square

5 | NONLINEAR PRINCIPAL PART

In this section, we study the more general case with $\boldsymbol{\Theta} \neq 0$ and prove the main result, as stated in Theorem 1 but with (15a) replaced by (11).

5.1 | Assumptions on the second derivative

To handle more general free energies with a possible nonlinear principal part, we consider now a more general model. More specifically, the result of Theorem 1 remains true if the assumption (15a) is replaced by (11), where $\boldsymbol{\Theta} \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}^{3 \times 3}; \mathbb{R}^{3 \times 3 \times 3 \times 3})$ is sufficiently small such that

$$|\boldsymbol{\Theta}(\mathbf{h}, \mathbf{S})| \leq c_{\boldsymbol{\Theta}} \quad \text{with} \quad c_{\boldsymbol{\Theta}} = c_{\Lambda} / (16c_{\mathbf{H}^2}), \quad (46)$$

for all $\mathbf{h} \in \mathbb{R}^3$ and $\mathbf{S} \in \mathbb{R}^{3 \times 3}$. Here, c_{Λ} denotes the constant of estimate (28) and $c_{\mathbf{H}^2}$ denotes the constant of the equivalent norm estimate of $\|\cdot\|_{\mathbf{H}^2} \leq c_{\mathbf{H}^2} \|\cdot\|_{\mathbf{H}^2 \cap \mathbf{H}_0^1} = c_{\mathbf{H}^2} \|\Delta \cdot\|_{L^2}$ (see also Section 2.1). The proof of Theorem 1 is similar under this modified assumption. We are commenting only on the necessary changes.

Theorem 2. *Let Ω be a bounded domain of class \mathcal{C}^2 , assume (4), and let the free energy potential F fulfill the assumptions (11) with (46), (14), (16), and (19a). For given initial data $\mathbf{v}_0 \in \mathbf{L}_\sigma^2$, $\mathbf{d}_0 \in \mathbf{H}_0^1$ (such that $\mathbf{d}_1 = 0$) and right-hand side $\mathbf{g} \in L^2(0, T; (\mathbf{H}_{0,\sigma}^1)^*)$, there exists a weak solution to the Ericksen-Leslie system (1a) with (2a) and (13a) in the sense of Definition 1.*

5.2 | Estimate of the variational derivative

Under assumption (11), the variational derivative (compare with Equation (17)) becomes

$$\mathbf{q} = -\nabla \cdot (\boldsymbol{\Lambda} : \nabla \mathbf{d}) - \boldsymbol{\Theta}(\mathbf{d}, \nabla \mathbf{d}) : \nabla(\nabla \mathbf{d})^T - \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) : (\nabla \mathbf{d})^T + \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}).$$

For arbitrary $\mathbf{a}_i \in \mathbb{R}^3$, $i \in \{1, 2, 3, 4\}$, one finds that

$$\begin{aligned} \left| \sum_{i=1}^4 \mathbf{a}_i \right|^2 &= |\mathbf{a}_1|^2 + |\mathbf{a}_2|^2 + |\mathbf{a}_3|^2 + |\mathbf{a}_4|^2 + 2\mathbf{a}_1 \cdot (\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4) + 2\mathbf{a}_2 \cdot (\mathbf{a}_3 + \mathbf{a}_4) + 2\mathbf{a}_3 \cdot \mathbf{a}_4 \\ &\geq \frac{1}{2} |\mathbf{a}_1|^2 + |\mathbf{a}_2|^2 + |\mathbf{a}_3|^2 + |\mathbf{a}_4|^2 - 2|\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4|^2 - |\mathbf{a}_2|^2 - |\mathbf{a}_3 + \mathbf{a}_4|^2 - |\mathbf{a}_3|^2 - |\mathbf{a}_4|^2 \\ &\geq \frac{1}{2} |\mathbf{a}_1|^2 - 4|\mathbf{a}_2|^2 - 10(|\mathbf{a}_3|^2 + |\mathbf{a}_4|^2), \end{aligned}$$

and thus

$$\begin{aligned} |\mathbf{q}|^2 &\geq \frac{1}{2} |\nabla \cdot (\boldsymbol{\Lambda} : \nabla \mathbf{d})|^2 - 4|\boldsymbol{\Theta}(\mathbf{d}, \nabla \mathbf{d}) : \nabla(\nabla \mathbf{d})^T|^2 \\ &\quad - 10 \left| \frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) : (\nabla \mathbf{d})^T \right|^2 - 10 \left| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) \right|^2. \end{aligned}$$

Inserting this estimate into Corollary 3 yields

$$|\mathbf{q}_n|^2 \geq \frac{1}{2} |R_n \nabla \cdot (\boldsymbol{\Lambda} : \nabla \mathbf{d}_n)|^2 - 4 \left| R_n (\boldsymbol{\Theta}(\mathbf{d}_n, \nabla \mathbf{d}_n) : \nabla(\nabla \mathbf{d}_n)^T) \right|^2 \quad (47)$$

$$- 10 \left| R_n \left(\frac{\partial^2 F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) : (\nabla \mathbf{d}_n)^T \right) \right|^2 - 10 \left| R_n \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}_n, \nabla \mathbf{d}_n) \right|^2. \quad (48)$$

Again, the orthogonality property of R_n and the estimate (28) yields

$$\begin{aligned} \frac{1}{2} \|R_n \nabla \cdot (\boldsymbol{\Lambda} : \nabla \mathbf{d}_n)\|_{L^2}^2 - 4 \|R_n (\boldsymbol{\Theta}(\mathbf{d}_n, \nabla \mathbf{d}_n) : \nabla(\nabla \mathbf{d}_n)^T)\|_{L^2}^2 &\geq \\ \frac{c_\Lambda}{2} \|\Delta \mathbf{d}_n\|_{L^2}^2 - 4 \|\boldsymbol{\Theta}(\mathbf{d}_n, \nabla \mathbf{d}_n) : \nabla(\nabla \mathbf{d}_n)^T\|_{L^2}^2. & \quad (49) \end{aligned}$$

The assumptions on $\boldsymbol{\Theta}$ (see Section 5.1) guarantee that

$$\|\boldsymbol{\Theta}(\mathbf{d}_n, \nabla \mathbf{d}_n) : \nabla(\nabla \mathbf{d}_n)^T\|_{L^2} \leq \|\boldsymbol{\Theta}(\mathbf{d}_n, \nabla \mathbf{d}_n)\|_{L^\infty} \|\mathbf{d}_n\|_{H^2} \leq c_{\boldsymbol{\Theta}} c_{H^2} \|\Delta \mathbf{d}_n\|_{L^2} \leq \frac{c_\Lambda}{16} \|\Delta \mathbf{d}_n\|_{L^2}.$$

Inserting this into (49) then shows that (48), together with (19b), yields the estimate (39). Now, we can proceed as in the proof of Corollary (3).

5.3 | Convergence of the nonlinear part

The only other change in the proof of Theorem 1 under the new assumptions (11) is needed in the limiting procedure in the nonlinear terms in Proposition 5. We consider the term $I_{3,n}$ of Equation (44). For this term, we find with the fundamental theorem of calculus that

$$\begin{aligned} I_{3,n} &= \int_0^T \left(\frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}_n(t), \nabla \mathbf{d}_n(t)) - \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}(t), \nabla \mathbf{d}(t), \nabla \boldsymbol{\psi}(t)) \right) dt \\ &= \int_0^T \left(\int_0^1 \frac{\partial F}{\partial \mathbf{S} \partial \mathbf{h}}(\mathbf{d}_s^n(t), \nabla \mathbf{d}_s^n(t)) \cdot (\mathbf{d}_n(t) - \mathbf{d}(t)) ds : \nabla \boldsymbol{\psi}(t) \right) dt \\ &\quad + \int_0^T \left(\int_0^1 \boldsymbol{\Theta}(\mathbf{d}_s^n(t), \nabla \mathbf{d}_s^n(t)) : (\nabla \mathbf{d}_n(t) - \nabla \mathbf{d}(t)) ds : \nabla \boldsymbol{\psi}(t) \right) dt \\ &\quad + \int_0^T ((\boldsymbol{\Lambda} : (\nabla \mathbf{d}_n(t) - \nabla \mathbf{d}(t))) : \nabla \boldsymbol{\psi}(t)) dt = J_{1,n} + J_{2,n} + J_{3,n}, \end{aligned}$$

where $\mathbf{d}_s^n(t) := s\mathbf{d}_n(t) + (1-s)\mathbf{d}(t)$. For the term $J_{1,n}$, the growth condition (19a), Young's inequality, and (19c) show that

$$\begin{aligned} & \left| \frac{\partial^2 F}{\partial S \partial \mathbf{h}} (\mathbf{d}_s^n(\mathbf{x}, t), \nabla \mathbf{d}_s^n(\mathbf{x}, t)) \cdot (\mathbf{d}_n(\mathbf{x}, t) - \mathbf{d}(\mathbf{x}, t)) \right| \\ & \leq C_{sh} \left(|\nabla \mathbf{d}_s^n(\mathbf{x}, t)|^{(\gamma_1/2-1)} + |\mathbf{d}_s^n(\mathbf{x}, t)|^{\gamma_3} + 1 \right) (|\mathbf{d}_n(\mathbf{x}, t)| + |\mathbf{d}(\mathbf{x}, t)|) \\ & \leq c \left(|\nabla \mathbf{d}_s^n(\mathbf{x}, t)|^{\gamma_1/2} + |\mathbf{d}_s^n(\mathbf{x}, t)|^{\gamma_2/2} + |\mathbf{d}_n(\mathbf{x}, t)|^{\gamma_1/2} + |\mathbf{d}(\mathbf{x}, t)|^{\gamma_1/2} + 1 \right) \\ & \leq c \left(|\nabla \mathbf{d}_n(\mathbf{x}, t)|^{\gamma_1/2} + |\mathbf{d}_n(\mathbf{x}, t)|^{\gamma_2/2} + |\nabla \mathbf{d}(\mathbf{x}, t)|^{\gamma_1/2} + |\mathbf{d}(\mathbf{x}, t)|^{\gamma_2/2} + 1 \right). \end{aligned}$$

Thus, the aforementioned function is dominated by a function in $L^2(0, T; \mathbf{L}^2)$ (see also (45)) and note that $\gamma_2 < 10$. The continuity of $\frac{\partial F}{\partial S \partial \mathbf{h}}$ and the point wise convergence (45) imply $|J_{1,n}| \rightarrow 0$ as $n \rightarrow \infty$ in view of Lebesgue's theorem on dominated convergence.

For the terms $J_{2,n}$ and $J_{3,n}$, we observe that those terms can be estimated due to the assumptions on Θ by

$$|J_{2,n} + J_{3,n}| \leq (c_\Theta + |\Lambda|) \|\nabla \mathbf{d}_n - \nabla \mathbf{d}\|_{L^2(\mathbf{L}^2)} \|\nabla \psi\|_{L^2(\mathbf{L}^2)}.$$

The strong convergence (43j) with $r = 2$ shows that $|J_{2,n} + J_{3,n}| \rightarrow 0$, for $n \rightarrow \infty$.

6 | SOME EXAMPLES

In this section, we collect different physical relevant examples of free energy potentials of the Ericksen-Leslie model that fulfill the assumptions of Section 4.2.

6.1 | Dirichlet energy with Ginzburg-Landau penalization

System (1a)-(2a) with the assumptions on the free energy (14) and (19a) is especially a generalization of the models considered by Lin and Liu¹⁴ and by Cavaterra et al.¹⁸ Therefore, the free energy considered in the aforementioned works^{14,18} fits into our setting, ie, the free energy function (5) with $k, \varepsilon > 0$ fulfills the hypothesis on the free energy potential in Section 3.2 with Λ being a multiple of the identity mapping $\mathbb{R}^{3 \times 3}$ into $\mathbb{R}^{3 \times 3}$ and $\gamma_1 = 2, \gamma_2 = 6$.

6.2 | Electromagnetic field effects

If the influence of an electromagnetic field is taken into account, which is essential because the desirable anisotropic effects are controlled in such a way, the function (8) is needed (see section 3.2.1 in the work of de Gennes³⁰). If the magnetic field is bounded such that $|\mathbf{H}(\mathbf{x})| \leq C$ for all $\mathbf{x} \in \Omega$, the energy (8) fits into our model. Especially, it fulfills the assumptions (16) and (19b). Similarly to the magnetic field, a free energy can be introduced modeling the influence of an electric field \mathbf{E} (see section 3.2.1 in the work of de Gennes³⁰). The anisotropic magnetic susceptibility constants $\chi_\perp, \chi_\parallel$ have then to be replaced by electric susceptibility constants $\varepsilon_\perp, \varepsilon_\parallel$ such that the associated potential becomes

$$F_E(\mathbf{d}, \nabla \mathbf{d}) = F(\mathbf{d}, \nabla \mathbf{d}) - \frac{\varepsilon_\perp}{8\pi} |\mathbf{E}|^2 - \frac{(\varepsilon_\parallel - \varepsilon_\perp)}{8\pi} (\mathbf{d} \cdot \mathbf{E})^2.$$

Under similar assumptions, ie, $|\mathbf{E}(\mathbf{x})| \leq C$ for all $\mathbf{x} \in \Omega$, the potential fits into our model.

As the assumption (19b) suggests, parts of the free energy, which only depend on the director \mathbf{d} and not on its gradient, are incorporated in our setting as long as they are continuously differentiable, bounded from below and of polynomial growth of a degree strictly less than 6.

6.3 | Additional degrees of freedom

Leslie recognized in a previous work¹¹ that the system (1a) possesses two additional degrees of freedom. He proposed to alter the system (1a) by adding $\mathbf{d} \otimes \mathbf{b}$ to the derivative of the free energy potential with respect to the gradient of the director and by adding $\bar{b}\mathbf{d} - \nabla \mathbf{d}\mathbf{b}$ to the derivative of the free energy with respect to the director with certain constant vector $\mathbf{b} \in \mathbb{R}^d$ and constant scalar $\bar{b} \in \mathbb{R}$.

These two changes can be introduced into the system by replacing the free energy F by a new function F_A defined by (9).

With this function, we find that

$$\frac{\partial F_A}{\partial \mathbf{S}}(\mathbf{d}, \nabla \mathbf{d}) = \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}, \nabla \mathbf{d}) - \mathbf{d} \otimes \mathbf{b}$$

and

$$\frac{\partial F_A}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) = \frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) - \nabla \mathbf{d} \mathbf{b} + \bar{\mathbf{b}} \mathbf{d},$$

as proposed by Leslie.¹¹ What remains is to check that the system (1a) is not altered somewhere else. The only other term depending on the free energy is the Ericksen stress. To calculate the altered Ericksen stress, tested with a solenoidal, sufficiently smooth function \mathbf{w} , we use the calculation (21) and obtain

$$\begin{aligned} \left\langle \nabla \mathbf{d}^T \frac{\partial F_A}{\partial \mathbf{S}} : \nabla \mathbf{w} \right\rangle &= \left\langle \nabla \mathbf{d}^T \mathbf{q}_A, \mathbf{w} \right\rangle = \left\langle \nabla \mathbf{d}^T \left(\frac{\partial F_A}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) - \nabla \cdot \frac{\partial F_A}{\partial \mathbf{S}}(\mathbf{d}, \nabla \mathbf{d}) \right), \mathbf{w} \right\rangle \\ &= \left\langle \nabla \mathbf{d}^T \left(\frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) - \nabla \mathbf{d} \mathbf{b} + \bar{\mathbf{b}} \mathbf{d} - \nabla \cdot \left(\frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}, \nabla \mathbf{d}) - \mathbf{d} \otimes \mathbf{b} \right) \right), \mathbf{w} \right\rangle \\ &= \left\langle \nabla \mathbf{d}^T \left(\frac{\partial F}{\partial \mathbf{h}}(\mathbf{d}, \nabla \mathbf{d}) - \nabla \cdot \frac{\partial F}{\partial \mathbf{S}}(\mathbf{d}, \nabla \mathbf{d}) + \bar{\mathbf{b}} \mathbf{d} \right), \mathbf{w} \right\rangle \\ &= \left\langle \nabla \mathbf{d}^T \mathbf{q}, \mathbf{w} \right\rangle + \frac{\bar{\mathbf{b}}}{2} (\nabla |\mathbf{d}|^2, \mathbf{w}) = \left\langle \nabla \mathbf{d}^T \mathbf{q}, \mathbf{w} \right\rangle = \left\langle \nabla \mathbf{d}^T \frac{\partial F}{\partial \mathbf{S}} : \nabla \mathbf{w} \right\rangle. \end{aligned}$$

The last but one equation holds because \mathbf{w} is a solenoidal function. We thus see that the system is only changed by the new free energy potential (9) as proposed by Leslie. The model with these additional degrees of freedom thus fits into our framework.

6.4 | Simplified Oseen-Frank energy

The Oseen-Frank energy (7) fits only into our setting for a particular choice of the constants appearing. Since

$$(\mathbf{d} \cdot (\nabla \times \mathbf{d}))^2 + |\mathbf{d} \times (\nabla \times \mathbf{d})|^2 = |\mathbf{d}|^2 |\nabla \times \mathbf{d}|^2$$

and since in the classical Ericksen-Leslie model the norm of the director is supposed to be one, it is convenient to consider the following reformulation of the Oseen-Frank energy for the case $k_2 = k_3$:

$$F_{\{k_2=k_3\}}(\mathbf{d}, \nabla \mathbf{d}) = k_1 (\nabla \cdot \mathbf{d})^2 + k_2 |\nabla \times \mathbf{d}|^2 + \alpha (\text{tr}(\nabla \mathbf{d}^2) - (\nabla \cdot \mathbf{d})^2).$$

For $k_1, k_2 > 0$, this energy fulfills the assumptions of Section 3.2, ie, a simple but tedious calculation shows that

$$\begin{aligned} |\nabla \mathbf{d}|^2 &= \nabla \mathbf{d} : \Lambda^0 : \nabla \mathbf{d} \quad \text{with } \Lambda_{ijkl}^0 = \delta_{ik} \delta_{jl} \\ (\nabla \cdot \mathbf{d})^2 &= \nabla \mathbf{d} : \Lambda^1 : \nabla \mathbf{d} \quad \text{with } \Lambda_{ijkl}^1 = \delta_{ij} \delta_{kl} \\ \text{tr}(\nabla \mathbf{d}^2) &= \nabla \mathbf{d} : \Lambda^2 : \nabla \mathbf{d} \quad \text{with } \Lambda_{ijkl}^2 = \delta_{il} \delta_{jk} \\ |\nabla \times \mathbf{d}|^2 &= \nabla \mathbf{d} : \Lambda^3 : \nabla \mathbf{d} \quad \text{with } \Lambda^3 = \Lambda^0 - \Lambda^2, \end{aligned} \tag{50}$$

where Λ^0 is the identity mapping $\mathbb{R}^{3 \times 3}$ into $\mathbb{R}^{3 \times 3}$. It follows that

$$F_{\{k_2=k_3\}}(\mathbf{d}, \nabla \mathbf{d}) = \frac{1}{2} \nabla \mathbf{d} : \Lambda : \nabla \mathbf{d} \quad \text{with } \Lambda = 2k_2 \Lambda^0 + 2(k_1 - \alpha) \Lambda^1 - 2(k_2 - \alpha) \Lambda^2.$$

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, we find

$$\mathbf{a} \otimes \mathbf{b} : \Lambda : \mathbf{a} \otimes \mathbf{b} = 2k_2 |\mathbf{a}|^2 |\mathbf{b}|^2 + 2(k_1 - k_2) (\mathbf{a} \cdot \mathbf{b})^2,$$

and the strong Legendre-Hadamard condition is fulfilled for any $\alpha \in \mathbb{R}$ and any $k_1, k_2 > 0$.

6.5 | Scaled Oseen-Frank energy

It does not seem to be possible to include the general Oseen-Frank energy in the presented setting, but to include its anisotropic character. We consider an energy, where the nonquadratic terms are scaled appropriately. The energy is given by (10) with sufficiently small constants k_3 and k_4 .

The associated tensor of fourth-order Λ is given by

$$\Lambda = k_1 \Lambda^1 + \min\{k_2, k_3\} (\Lambda^0 - \Lambda^2),$$

where Λ^i for $i = 0, 1, 2$ is defined in (50). Let c_Λ be the associated coercivity constant of the estimate (28). A careful calculation and estimate of the partial derivatives of the free energy potential (10) show that they fulfill the assumptions of Section 3.2 and Section 5 for $s > \frac{1}{6}$, and $k_3 + k_4$ is sufficiently small. The condition on s follows from the property (19a) and the smallness condition from property (11) and the estimate on Θ . If in addition $s \leq \frac{1}{4}$, it can be seen by roughly estimating the second partial derivative of F with respect to \mathbf{S} that the constants k_3 and k_4 have to be chosen such that

$$53(k_3 + k_4) \leq \frac{c_\Lambda}{c_H^2}.$$

ACKNOWLEDGEMENT

This work was funded by the DFG through CRC 901 *Control of self-organizing nonlinear systems: Theoretical methods and concepts of application*. (Project A8)

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How to cite this article: Emmrich E, Lasarzik R. Existence of weak solutions to the Ericksen-Leslie model for a general class of free energies. *Math Meth Appl Sci.* 2018;41:6492-6518. <https://doi.org/10.1002/mma.5172>