Analysis of operator differential-algebraic equations arising in fluid dynamics.
Part I. The finite dimensional case

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Abstract

Existence and uniqueness of solutions to initial value problems for a class of abstract differential-algebraic equations (DAEs) is shown. The class of equations cover, in particular, the spatially semi-discretized Stokes and Oseen problem describing the motion of an incompressible or nearly incompressible Newtonian fluid. Moreover, we derive explicit solution formulas.

Keywords: Differential-algebraic equation, strangeness-index, existence, uniqueness, consistency, Duhamel's principle, Stokes equation, Oseen equation

AMS(MOS) subject classification: 34A09, 34G10, 35D05, 76D07, 34H05, 65M99

1 Introduction

In this paper we study the solvability of operator equations of the form

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} + \begin{bmatrix} A & B \\ -D^T & C \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

on a time interval $[0, T]$ with linear operators M, A, B, C, D defined on appropriate Hilbert spaces and with appropriate right-hand side functions f, g. Here, the time derivative is usually understood in the distributional sense.

We are interested in solutions to initial value problems with initial condition

$$\begin{bmatrix} v(0) \\ p(0) \end{bmatrix} = \begin{bmatrix} v_0 \\ p_0 \end{bmatrix} \tag{1.2}$$

where, if the solution does not exist in the classical sense at $t = 0$ then the initial condition is viewed in a generalized sense, see [15, 19].

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Since the coefficient matrix of the time derivative is singular, (1.1) is called *operator differential-algebraic equation (DAE) or abstract DAE* [26].

Operator DAEs of the form (1.1) (with $M = I, B = D, C = 0$) arise in the functional analytic formulation of the initial value problem for the *Stokes* as well as for the *linearized Navier-Stokes* or *Oseen* equations [3, 24, 25] in which v and p, denoting velocity and pressure, respectively, then are abstract functions mapping the time interval into appropriate spatial function spaces.

The linearized Navier-Stokes equations, describing the motion of an incompressible or nearly incompressible Newtonian fluid, have the form

$$\begin{aligned} \partial_t v - \nu \Delta v + (v_\infty \cdot \nabla) v + (v \cdot \nabla) v_\infty + \nabla p = f \qquad & \text{in } \Omega \times (0,T) \\ \nabla^T v = 0 \qquad & \text{in } \Omega \times (0,T). \end{aligned} \tag{1.3}$$

They arise from a linearization of the *Navier-Stokes* equations

$$\begin{aligned} \partial_t v - \nu \Delta v + (v \cdot \nabla) v + \nabla p = f \qquad & \text{in } \Omega \times (0,T) \\ \nabla^T v = 0 \qquad & \text{in } \Omega \times (0,T) \end{aligned}$$

around a prescribed vector field v_∞. In what follows, we restrict to the case that v_∞ is independent of time. Note that if v_∞ is also independent of space then the term $(v \cdot \nabla) v_\infty$ does not appear, and the equations are then called Oseen equations (see [23]). The equations have to be supplemented by suitable initial and boundary conditions.

Operator DAEs of the form (1.1) also arise when the Oseen system is semi-discretized in space via the method of lines [1, 21] using e.g. a finite element discretization in space [16, 21] and a fixed point iteration to resolve the nonlinearity. Due to the convection term, in general, in the fixed point iteration the resulting coefficient matrix A is nonsymmetric. Furthermore, if the corresponding finite element spaces do not fulfill the discrete Babuška-Brezzi condition [3, 10, 21, 22], a stabilization is needed which then leads to an additional term in (1.3). Also quasi-compressible fluid flow [18] can be described by an additional term in (1.3).

Differential-algebraic (operator) equations are currently the standard modeling concept in many applications such as circuit simulation, multibody dynamics, and chemical process engineering, see [1, 2, 8, 11, 12, 13, 15, 20] and the references therein. They have a particular advantage for the treatment of multi-physics models arising from modern automatic modeling tools such as [7, 17] and as we have described, they arise in computational fluid dynamics in the special form of the linear operator DAE (1.1).

In this paper we carry out the analysis for the finite dimensional case. In particular, we study existence and uniqueness of (1.1). We review the classical theory for linear DAEs and apply this theory to the specially structured system given by (1.1). Moreover, we present explicit solution formulas.

2 A review of differential-algebraic equations

In this section, as a basis for the operator case, we recall some well-known results on the solvability of the initial value problem for a system of DAEs. Moreover, we provide an explicit representation of the solution, which is a generalization of Duhamel's principle, and discuss the relation between the consistency (compatibility) of the data and the index of the DAE. We follow [15] in style and notation.

For initial value problems associated with linear ordinary differential equations

$$\dot{x} + \mathcal{A}x = f, \quad x(0) = x^0,$$

with $\mathcal{A} \in \mathbb{R}^{n,n}$ and $f \in C([0,T];\mathbb{R}^n)$ (where $C([0,T];\mathbb{R}^n)$ denotes the space of continuous functions from $[0,T]$ to $\mathbb{R}^n$) one has the well-known solution formula (Duhamel's principle) [6]

$$x(t) = \mathrm{e}^{-t\mathcal{A}}x^0 + \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\, ds \tag{2.1}$$

obtained by variation of constants. The extension of this formula to initial value problems for DAEs of the form

$$\mathcal{E}\dot{x} + \mathcal{A}x = f, \tag{2.2}$$

with $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$, sufficiently smooth right-hand side f, and initial conditions

$$x(0) = x^0 \in \mathbb{R}^n \tag{2.3}$$

is also well-known, see e.g. [4, 15].

To describe this, we need the following results.

Theorem 1 (Weierstraß canonical form) *[9]. Let $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$ and suppose that the pair $(\mathcal{E}, \mathcal{A})$ is* regular, *i.e.,* $\det(\lambda\mathcal{E} + \mathcal{A})$ *does not vanish identically for all $\lambda \in \mathbb{C}$. Then, there exist nonsingular matrices $P, Q \in \mathbb{R}^{n,n}$ such that*

$$(P\mathcal{E}Q, P\mathcal{A}Q) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{2.4}$$

where J is a matrix in real Jordan canonical form, N is a nilpotent matrix also in Jordan canonical form and I denotes an identity matrix of appropriate size. Moreover, it is allowed that one or the other block is not present.

If in Theorem 1 the index of nilpotency of N is ν, then we say that the pair $(\mathcal{E}, \mathcal{A})$ has *(differentiation) index* ν and denote this by $\nu = \operatorname{ind}(\mathcal{E}, \mathcal{A})$. For a matrix $\mathcal{E} \in \mathbb{R}^{n,n}$ we set $\operatorname{ind}\mathcal{E} = \operatorname{ind}(\mathcal{E}, I)$. We have $\operatorname{ind}\mathcal{E} = 0$ if and only if $\mathcal{E}$ is nonsingular.

The explicit solution formulas require the *Drazin inverse of a matrix.*

Definition 2 *Let $\mathcal{E} \in \mathbb{R}^{n,n}$ have* $\operatorname{ind}\mathcal{E} = \nu$. *A matrix $X \in \mathbb{R}^{n,n}$ satisfying*

$$\begin{array}{l} (a)\ \mathcal{E}X = X\mathcal{E}, \\ (b)\ X\mathcal{E}X = X, \\ (c)\ X\mathcal{E}^{\nu+1} = \mathcal{E}^{\nu} \end{array} \tag{2.5}$$

is called a Drazin inverse *of $\mathcal{E}$.*

We recall some well-known facts about the Drazin inverse, see [5].

Theorem 3 *Let $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$.*

1. *$\mathcal{E}$ has one and only one Drazin inverse $\mathcal{E}^D$.*

2. *If $\mathcal{E}$ is nonsingular then $\mathcal{E}^D = \mathcal{E}^{-1}$.*

3. If $\mathcal{E}\mathcal{A} = \mathcal{A}\mathcal{E}$ then

$$\begin{array}{l} (a)\ \mathcal{E}\mathcal{A}^D = \mathcal{A}^D\mathcal{E}, \\ (b)\ \mathcal{E}^D\mathcal{A} = \mathcal{A}\mathcal{E}^D, \\ (c)\ \mathcal{E}^D\mathcal{A}^D = \mathcal{A}^D\mathcal{E}^D. \end{array} \tag{2.6}$$

To present the explicit solution representations, we first assume that in (2.2) the coefficient matrices $\mathcal{E}$ and $\mathcal{A}$ commute, i.e., that

$$\mathcal{E}\mathcal{A} = \mathcal{A}\mathcal{E}. \tag{2.7}$$

Then we have the following solution formula, see [4, 15].

Theorem 4 *Let $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$ form a regular pair satisfying (2.7). Furthermore, let $f \in C^\nu([0,T];\mathbb{R}^n)$ with $\nu = \operatorname{ind}(\mathcal{E},\mathcal{A})$. Then every solution $x \in C^1([0,T];\mathbb{R}^n)$ of (2.2) has the form*

$$\begin{aligned} x(t) = {} & \mathrm{e}^{-t\mathcal{E}^{\mathcal{D}}\mathcal{A}}\mathcal{E}^D\mathcal{E}q + \int_0^t \mathrm{e}^{-(t-s)\mathcal{E}^D\mathcal{A}}\mathcal{E}^D f(s)\,ds \\ & + \big(I - \mathcal{E}^D\mathcal{E}\big)\sum_{i=0}^{\nu-1}(-\mathcal{E}\mathcal{A}^D)^i\mathcal{A}^D f^{(i)}(t) \end{aligned} \tag{2.8}$$

for some $q \in \mathbb{R}^n$.

Evaluating the solution formula at $t = 0$ one immediately gets consistency conditions for initial values.

Corollary 5 *Let the assumptions of Theorem 4 hold. The initial value problem consisting of (2.2) and (2.3) has a solution $x \in C^1([0,T];\mathbb{R}^n)$ if and only if there exists a vector $q \in \mathbb{R}^n$ with*

$$x^0 = \mathcal{E}^D\mathcal{E}q + (I - \mathcal{E}^D\mathcal{E})\sum_{i=0}^{\nu-1}(-\mathcal{E}\mathcal{A}^D)^i\mathcal{A}^D f^{(i)}(0). \tag{2.9}$$

If this is the case, then for every such q the solution is unique.

Remark 6 Corollary 5 gives consistency conditions for classical continuously differentiable solutions. By going over to weaker smoothness requirements for the solutions, also these consistency conditions may be partially weakened, see [15, 19].

The commutativity requirement (2.7) is not really a restriction, since if $(\mathcal{E},\mathcal{A})$ is regular and $\hat{\lambda} \in \mathbb{R}$ is chosen such that $\hat{\lambda}\mathcal{E} + \mathcal{A}$ is nonsingular, then

$$\hat{\mathcal{E}} = (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}\mathcal{E}, \quad \hat{\mathcal{A}} = (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}\mathcal{A} \tag{2.10}$$

commute. Since the factor $(\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}$ represent a simple scaling of (2.2) from the left by a nonsingular matrix, results analogous to Theorem 4 and Corollary 5 hold for the general case by setting

$$\mathcal{E} \leftarrow (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}\mathcal{E}, \quad \mathcal{A} \leftarrow (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}\mathcal{A}, \quad f \leftarrow (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}f \tag{2.11}$$

in (2.8) and (2.9). It should also be noted that none of the solution formulas depends on the choice of the value $\hat{\lambda}$, see [15].

3 Well-posedness and explicit solution for DAEs of type (1.1)

In this section we specialize the results of the previous section to the specially structured finite dimensional version of the operator DAE (1.1). The associated matrix pair is then

$$(\mathcal{E},\mathcal{A}) = \left(\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ -D^T & C \end{bmatrix} \right). \tag{3.1}$$

We assume that the pair arises from a reasonable discretization that leads to a regular pair with the mass matrix M and the matrix $\mathcal{A}$ being invertible. The latter condition, which will be satisfied in practice, is not really necessary for the analysis, but if this is not the case, then the presentation becomes rather technical. Since $\mathcal{A}$ is nonsingular it follows that $\operatorname{ind}(\mathcal{E}) = \operatorname{ind}(\mathcal{E},\mathcal{A})$ as well.

Typically, the matrix C in (3.1) is singular or even 0 depending on the discretization. If C were invertible then we would immediately have that $\operatorname{ind}(\mathcal{E},\mathcal{A}) = 1$, see e.g. [15]. We also assume that B and D have full column rank. For the latter condition it is usually necessary to remove the freedom in the pressure by an extra condition or a factorization of the underlying function space [10, 14]. For C singular, let P_1 and P_2 be matrices, such that their columns span the nullspace of C and C^T, respectively. It is another reasonable assumption that $P_2^T D^T B P_1$ is square and nonsingular, [27]. Under this assumption we have that $\operatorname{ind}(\mathcal{E},\mathcal{A}) = 2$, see e.g. [15]. This holds for example in the particular case that $D = B$ has full column rank and $C = 0$, that we will study below. We thus restrict our considerations to the case $\operatorname{ind}(\mathcal{E}) = \operatorname{ind}(\mathcal{E},\mathcal{A}) \in \{1,2\}$.

To apply the explicit solution formula to (1.1), we first need to pick a value $\hat{\lambda}$ such that $\hat{\lambda}\mathcal{E} + \mathcal{A}$ is invertible. Under the given assumptions, it is sufficient to pick $\hat{\lambda} \in \mathbb{R}$ such that $\hat{\lambda}M + A$ is nonsingular, which means that $\hat{\lambda}$ is not an eigenvalue of the (discretized Laplace) operator A.

Introducing the Schur complement $S := C + D^T(\hat{\lambda}M + A)^{-1}B$, we obtain

$$\hat{\mathcal{E}} = \begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix} := \begin{bmatrix} \hat{\lambda}M + A & B \\ -D^T & C \end{bmatrix}^{-1} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \tag{3.2}$$

and

$$\hat{\mathcal{A}} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix} := \begin{bmatrix} \hat{\lambda}M + A & B \\ -D^T & C \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ -D^T & C \end{bmatrix}, \tag{3.3}$$

with the following formula for the block inverse

$$(\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1} = \begin{bmatrix} \hat{\lambda}M + A & B \\ -D^T & C \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (\hat{\lambda}M + A)^{-1} - (\hat{\lambda}M + A)^{-1}BS^{-1}D^T(\hat{\lambda}M + A)^{-1} & -(\hat{\lambda}M + A)^{-1}BS^{-1} \\ S^{-1}D^T(\hat{\lambda}M + A)^{-1} & S^{-1} \end{bmatrix},$$

and thus

$$\begin{aligned} E_{11} &= (\hat{\lambda}M + A)^{-1}M - (\hat{\lambda}M + A)^{-1}BS^{-1}D^T(\hat{\lambda}M + A)^{-1}M \\ &= [I - (\hat{\lambda}M + A)^{-1}BS^{-1}D^T](\hat{\lambda}M + A)^{-1}M, \\ E_{21} &= S^{-1}D^T(\hat{\lambda}M + A)^{-1}M, \\ A_{11} &= (\hat{\lambda}M + A)^{-1}\left(A + BS^{-1}D^T[I - (\hat{\lambda}M + A)^{-1}A]\right), \\ A_{21} &= -S^{-1}D^T[I - (\hat{\lambda}M + A)^{-1}A]. \end{aligned}$$

Note that since both $\mathcal{E}, \mathcal{A}$ are block lower triangular and commute, also the blocks E_{11} and A_{11} commute. Note further that the state vector $[v^T, p^T]^T$ remains unchanged by this operation, while the inhomogeneity transforms to

$$\begin{aligned}
\begin{bmatrix} \hat{f} \\ \hat{g} \end{bmatrix} &:= (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \hat{\lambda}M + A & B \\ -D^T & C \end{bmatrix}^{-1} \begin{bmatrix} f \\ g \end{bmatrix} \\
&= \begin{bmatrix} E_{11}M^{-1} & V_1 \\ E_{21}M^{-1} & V_2 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}
\end{aligned} \tag{3.4}$$

with $V_1 = -(\hat{\lambda}M + A)^{-1}BS^{-1}$ and $V_2 = S^{-1}$.

We will now determine the Drazin inverse

$$\hat{\mathcal{E}}^D = X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix},$$

of $\hat{\mathcal{E}}$ and assume that it is partitioned analogous to $\hat{\mathcal{E}}$.

Since we have assumed that M is invertible, we automatically have that $\operatorname{rank} \hat{\mathcal{E}} = \operatorname{rank} \mathcal{E} = \operatorname{rank} M$. By (2.5)(a), we have that

$$\begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} X_{12} = 0,$$

which thus implies $X_{12} = 0$. From (2.5)(b) we then obtain immediately that $X_{22} = 0$ and thus $X_{11}E_{11}X_{11} = X_{11}$. We now make use of (2.5)(c) and use the fact that for $j \geq 1$ we have

$$\begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}^j = \begin{bmatrix} E_{11}^j & 0 \\ E_{21}E_{11}^{j-1} & 0 \end{bmatrix}.$$

Therefore, since $\nu = \operatorname{ind} \hat{\mathcal{E}} \geq 1$ ($\mathcal{E}$ and thus $\hat{\mathcal{E}}$ is singular) we have the following equations for X_{11} and X_{21}.

$$\begin{array}{ll}
(a) & E_{11}X_{11} = X_{11}E_{11}, \\
(b) & X_{11} = X_{11}E_{11}X_{11}, \\
(c) & X_{11}E_{11}^{\nu+1} = E_{11}^{\nu}, \\
(d) & X_{21}E_{11}^{\nu+1} = E_{21}E_{11}^{\nu-1}, \\
(e) & E_{21}X_{11} = X_{21}E_{11}, \\
(f) & X_{21}E_{11}X_{11} = X_{21}.
\end{array} \tag{3.5}$$

From (3.5) (e)–(f), we have immediately that $X_{21} = E_{21}X_{11}^2$ and thus it is sufficient to determine X_{11}. For this we need information on the index of E_{11}, which is given by the following lemma.

Lemma 7 *Consider the pair (3.1) and the transformed pair $(\hat{\mathcal{E}}, \hat{\mathcal{A}})$ of (3.2)–(3.3).*

Then either

$$\operatorname{ind}(\hat{\mathcal{E}}) = \operatorname{ind}(\hat{\mathcal{E}}, \hat{\mathcal{A}}) = \operatorname{ind}(E_{11}, A_{11})$$

or

$$\operatorname{ind}(\hat{\mathcal{E}}) = \operatorname{ind}(\hat{\mathcal{E}}, \hat{\mathcal{A}}) = \operatorname{ind}(E_{11}, A_{11}) + 1.$$

Proof. Since $\hat{\mathcal{A}}$ and hence also A_{11} is invertible, we have that $\operatorname{ind}\hat{\mathcal{E}} = \operatorname{ind}(\hat{\mathcal{E}}, \hat{\mathcal{A}})$ and $\operatorname{ind} E_{11} = \operatorname{ind}(E_{11}, A_{11})$. Using that

$$(\hat{\mathcal{E}}, \hat{\mathcal{A}}) = \left(\begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix} \right),$$

and that the Weierstrass canonical form (see Theorem 1) of the regular pair (E_{11}, A_{11}) is given by

$$\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$$

with a nilpotent matrix N of nilpotency index $\operatorname{ind} E_{11}$, by simple algebraic manipulations we obtain that

$$\left(\begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix} \right)$$

is equivalent to

$$\left(\begin{bmatrix} I & 0 & 0 \\ 0 & N & 0 \\ 0 & \widetilde{E}_{32} & 0 \end{bmatrix}, \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right).$$

Since

$$\begin{bmatrix} N & 0 \\ \widetilde{E}_{32} & 0 \end{bmatrix}^j = \begin{bmatrix} N^j & 0 \\ \widetilde{E}_{32} N^{j-1} & 0 \end{bmatrix},$$

the nilpotency index of

$$\begin{bmatrix} N & 0 \\ \widetilde{E}_{32} & 0 \end{bmatrix}$$

and hence the index of the pair (E_{11}, A_{11}) can increase at most by 1. $\square$

Since for a reasonable discretization $\operatorname{ind}(\mathcal{E}) = \operatorname{ind}(\mathcal{E}, \mathcal{A}) \in \{1, 2\}$, we can assume that $\operatorname{ind}(E_{11}) \leq 2$ and typically it will be 1 or 2.

We obtain the following explicit formulas for $\mathcal{E}^D$ in terms of E_{11}^D.

Lemma 8 *Consider the pair (3.1) with* $\operatorname{ind}(\mathcal{E}, \mathcal{A}) \leq 2$ *and the transformed pair* $(\hat{\mathcal{E}}, \hat{\mathcal{A}})$ *of (3.2–3.3).*

Then we have the following formulas:

$$\begin{aligned}
&(i) \qquad \hat{\mathcal{E}}^D = \begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}^D = \begin{bmatrix} E_{11}^D & 0 \\ E_{21}(E_{11}^D)^2 & 0 \end{bmatrix}, \\
&(ii) \qquad \hat{\mathcal{A}}^D = \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}, \\
&(iii) \qquad \hat{\mathcal{E}}^D \hat{\mathcal{A}} = \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}, \\
&(iv) \qquad \hat{\mathcal{E}}^D \hat{\mathcal{E}} = \begin{bmatrix} E_{11}^D E_{11} & 0 \\ E_{21} E_{11}^D & 0 \end{bmatrix}, \\
&(v) \qquad I - \hat{\mathcal{E}}^D \hat{\mathcal{E}} = \begin{bmatrix} I - E_{11}^D E_{11} & 0 \\ -E_{21} E_{11}^D & I \end{bmatrix} \\
&(vi) \qquad \hat{\mathcal{E}} \hat{\mathcal{A}}^D = \begin{bmatrix} E_{11} A_{11}^{-1} & 0 \\ E_{21} A_{11}^{-1} & 0 \end{bmatrix}.
\end{aligned} \tag{3.6}$$

Proof. (i) Let $\alpha = \mathrm{ind}(E_{11})$, then by Lemma 7 we have that $\alpha = \nu$ or $\alpha = \nu - 1$. If $\alpha = \nu$ then by (3.5) (a)–(c) it follows that $X_{11} = E_{11}^D$ and then $X_{21} = E_{21}X_{11}^2 = E_{21}(E_{11}^D)^2$.

If $\alpha = \nu - 1$ then choosing $X_{11} = E_{11}^D$ and $X_{21} = E_{21}(E_{11}^D)^2$ the relations (3.5) (a)–(b) hold automatically and from $X_{11}E_{11}^{\alpha+1} = E_{11}^{\alpha}$, then (3.5) (c) follows by multiplying with E_{11} from the right.

Equations (3.5) (d)–(f) read as

$$\begin{aligned} &E_{21}(E_{11}^D)^2 E_{11}^{\nu+1} = E_{21}E_{11}^{\nu-1}\\ &E_{21}E_{11}^D = E_{21}(E_{11}^D)^2 E_{11}\\ &E_{21}(E_{11}^D)^2 E_{11}E_{11}^D = E_{21}(E_{11}^D)^2. \end{aligned}$$

Since $1 \le \nu = \mathrm{ind}(\hat{\mathcal{E}}) \le 2$, and since E_{11}^D and E_{11} commute, these equations are satisfied.

The assertion then follows by the uniqueness of the Drazin inverse. The other parts follow trivially. □

Note that whenever $\nu = \mathrm{ind}(\mathcal{E}, \mathcal{A}) \le 2$ it follows from (3.5) (d) that

$$E_{21}E_{11}^D E_{11}^{\nu} = E_{21}E_{11}^{\nu-1}, \tag{3.7}$$

i.e., in the range of E_{21}, the matrix E_{11} behaves like it is of index $\nu - 1$.

An immediate consequence is the following result on the well-posedness and explicit representation of the solution.

Theorem 9 *Consider the differential-algebraic equation corresponding to (3.1) with an invertible mass matrix M, an invertible matrix $\mathcal{A}$ and sufficiently smooth inhomogeneities $\hat{f}, \hat{g}$ as in (3.4). Let $\nu = \mathrm{ind}(\mathcal{E}, \mathcal{A}) \in \{1, 2\}$.*

Then for any consistent initial condition $\begin{bmatrix} v(0) \\ p(0) \end{bmatrix} = \begin{bmatrix} v_0 \\ p_0 \end{bmatrix}$, there exists a unique classical solution to the initial value problem (1.1)–(1.2) given by

$$\begin{aligned} \begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \exp(-tE_{11}^D A_{11})\widetilde{q} \\ E_{21}E_{11}^D \exp(-tE_{11}^D A_{11})\widetilde{q} \end{bmatrix} \\ &+ \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^D A_{11})E_{11}^D \hat{f}(s) \\ E_{21}E_{11}^D \exp(-(t-s)E_{11}^D A_{11})E_{11}^D \hat{f}(s) \end{bmatrix} ds \\ &+ \begin{bmatrix} (I - E_{11}^D E_{11})A_{11}^{-1}\hat{f}(t) \\ (-E_{21}E_{11}^D A_{11}^{-1} - A_{21}A_{11}^{-1})\hat{f}(t) + \hat{g}(t) \end{bmatrix} \\ &+ \sum_{i=1}^{\nu-1}(-1)^i \begin{bmatrix} (I - E_{11}^D E_{11})(E_{11}A_{11}^{-1})^i A_{11}^{-1}\hat{f}^{(i)}(t) \\ \left(-E_{21}E_{11}^D (E_{11}A_{11}^{-1})^i A_{11}^{-1} + E_{21}A_{11}^{-1}(E_{11}A_{11}^{-1})^{i-1}A_{11}^{-1}\right)\hat{f}^{(i)}(t) \end{bmatrix}, \end{aligned} \tag{3.8}$$

where $\widetilde{q}$ is a constant vector. An initial condition is consistent if the linear system

$$\begin{aligned} &\begin{bmatrix} I \\ E_{21}E_{11}^D \end{bmatrix} \widetilde{q} = \begin{bmatrix} v_0 \\ p_0 \end{bmatrix} \\ &- \begin{bmatrix} I - E_{11}E_{11}^D & 0 \\ -E_{21}E_{11}^D & I \end{bmatrix} \sum_{i=0}^{\nu-1}(-1)^i \begin{bmatrix} E_{11}A_{11}^{-1} & 0 \\ E_{21}A_{11}^{-1} & 0 \end{bmatrix}^i \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{f}^{(i)}(0) \\ \hat{g}^{(i)}(0) \end{bmatrix} \end{aligned} \tag{3.9}$$

has a solution.

Proof. According to (2.9) let

$$\begin{bmatrix} \widetilde{q} \\ \widetilde{r} \end{bmatrix} := \hat{\mathcal{E}}^D \hat{\mathcal{E}} q.$$

Applying Theorem 4, we obtain the following explicit solution of (1.1):

$$\begin{aligned}
\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \exp\left(-t \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}\right) \begin{bmatrix} \widetilde{q} \\ \widetilde{r} \end{bmatrix} \\
&+ \int_0^t \exp\left(-(t-s) \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}\right) \begin{bmatrix} E_{11}^D & 0 \\ E_{21}(E_{11}^D)^2 & 0 \end{bmatrix} \begin{bmatrix} \hat{f}(s) \\ \hat{g}(s) \end{bmatrix} ds \\
&+ \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} E_{11}^D E_{11} & 0 \\ E_{21} E_{11}^D & 0 \end{bmatrix} \right) \sum_{i=0}^{\nu-1} \left(- \begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \right)^i \times \\
&\qquad \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{f}^{(i)}(t) \\ \hat{g}^{(i)}(t) \end{bmatrix} \\
&= \exp\left(-t \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}\right) \begin{bmatrix} \widetilde{q} \\ \widetilde{r} \end{bmatrix} \qquad (3.10) \\
&+ \int_0^t \exp\left(-(t-s) \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}\right) \begin{bmatrix} E_{11}^D \hat{f}(s) \\ E_{21}(E_{11}^D)^2 \hat{f}(s) \end{bmatrix} ds \\
&+ \begin{bmatrix} (I - E_{11}^D E_{11}) A_{11}^{-1} \hat{f}(t) \\ (-E_{21}E_{11}^D A_{11}^{-1} - A_{21}A_{11}^{-1}) \hat{f}(t) + \hat{g}(t) \end{bmatrix} \\
&+ \sum_{i=1}^{\nu-1} (-1)^i \begin{bmatrix} (I - E_{11}^D E_{11})(E_{11}A_{11}^{-1})^i A_{11}^{-1} \hat{f}^{(i)}(t) \\ \left(-E_{21}E_{11}^D (E_{11}A_{11}^{-1})^i A_{11}^{-1} + E_{21}A_{11}^{-1}(E_{11}A_{11}^{-1})^{i-1} A_{11}^{-1}\right) \hat{f}^{(i)}(t) \end{bmatrix}.
\end{aligned}$$

Since with $Z = -E_{11}^D A_{11}$, $Y = E_{21}E_{11}^D$, we have

$$\begin{aligned}
\exp\left(t \begin{bmatrix} Z & 0 \\ YZ & 0 \end{bmatrix}\right) &= \sum_{i=0}^{\infty} \frac{1}{i!} \left(t \begin{bmatrix} Z & 0 \\ YZ & 0 \end{bmatrix}\right)^i = \sum_{i=0}^{\infty} \frac{t^i}{i!} \begin{bmatrix} Z^i & 0 \\ YZ^i & 0 \end{bmatrix} \\
&= \begin{bmatrix} \exp(tZ) & 0 \\ Y\exp(tZ) & 0 \end{bmatrix},
\end{aligned}$$

we can simplify the exponential functions and since we have from the commutativity of $\hat{\mathcal{E}}$ and $\hat{\mathcal{A}}$ that also E_{11}^D, E_{11}, and A_{11} commute, the result follows. The consistency condition follows by inserting $t = 0$. ∎

We will present more details about these consistency conditions in the special cases below.

4 Some special cases

In the explicit solution formula we may consider several simplifying cases.

If E_{11} is invertible, which by Lemma 7 can only happen if $\nu = 1$, then $I - E_{11}^D E_{11} = 0$, and we obtain

$$\begin{aligned}\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \exp(-tE_{11}^{-1}A_{11})\widetilde{q} \\ E_{21}E_{11}^{-1}\exp(-tE_{11}^{-1}A_{11})\widetilde{q} \end{bmatrix} \\ &+ \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^{-1}A_{11})E_{11}^{-1}\hat{f}(s) \\ E_{21}E_{11}^{-1}\exp(-(t-s)E_{11}^{-1}A_{11})E_{11}^{-1}\hat{f}(s) \end{bmatrix} ds \\ &+ \begin{bmatrix} 0 \\ (-E_{21}E_{11}^{-1}A_{11}^{-1} - A_{21}A_{11}^{-1})\hat{f}(t) + \hat{g}(t) \end{bmatrix}. \end{aligned} \tag{4.1}$$

In this case from the first equation we obtain $\widetilde{q} = v^0$ and thus the consistency condition

$$p^0 = E_{21}E_{11}^{-1}v^0 + (-E_{21}E_{11}^{-1}A_{11}^{-1} - A_{21}A_{11}^{-1})\hat{f}(0) + \hat{g}(0),$$

thus for a given v^0 then p^0 is fixed in an easy way, but we could also fix p^0 and then both equations together give a consistency condition for v^0.

If we have (as will typically be the case) that $\nu = 2$ and $\mathrm{ind}(E_{11}) = 1$, then by (2.5)(c) we have that $E_{11}^D E_{11}^2 = E_{11}$ and hence the formulas simplify to

$$\begin{aligned}\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \exp(-tE_{11}^{D}A_{11})\widetilde{q} \\ E_{21}E_{11}^{D}\exp(-tE_{11}^{D}A_{11})\widetilde{q} \end{bmatrix} \\ &+ \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^{D}A_{11})E_{11}^{D}\hat{f}(s) \\ E_{21}E_{11}^{D}\exp(-(t-s)E_{11}^{D}A_{11})E_{11}^{D}\hat{f}(s) \end{bmatrix} ds \\ &+ \begin{bmatrix} (I - E_{11}^D E_{11})A_{11}^{-1}\hat{f}(t) \\ (-E_{21}E_{11}^{D}A_{11}^{-1} - A_{21}A_{11}^{-1})\hat{f}(t) + \hat{g}(t) \end{bmatrix} \\ &- \begin{bmatrix} 0 \\ \left(-E_{21}E_{11}^D E_{11}A_{11}^{-2} + E_{21}A_{11}^{-2}\right)\dot{\hat{f}}(t) \end{bmatrix}. \end{aligned} \tag{4.2}$$

This again gives an algebraic relationship between v^0 and p^0 and again by choosing v^0 we obtain

$$\widetilde{q} = v^0 - (I - E_{11}^D E_{11})A_{11}^{-1}\hat{f}(0)$$

and this then fixes p^0 uniquely. We could also again fix p^0 and then both equations together give a consistency condition for v^0.

Looking in detail at the last term and using (3.4), which implies that

$$\dot{\hat{f}} = E_{11}M^{-1}\dot{f} + V_1\dot{g},$$

we see by (2.5)(c) that the factor of $\dot{f}$ in (4.2) vanishes, while the factor of $\dot{g}$ may not be zero if E_{11} is not invertible.

If, however, there is no inhomogeneity g then it follows that whenever E_{11} is of index 1, then the last term vanishes, thus despite the fact that $\nu = 2$, no derivative of f occurs, i.e., the system behaves somewhat like a system with $\nu = 1$.

Note that the same argument holds also for systems of this form with $\nu > 2$ and $\mathrm{ind}(E_{11}) = \nu - 1$, because also then the coefficient of the highest derivative $f^{(\nu-1)}$ is zero.

Let us now consider the even more special case that $M = I$, A is invertible, $B = D$ has full column rank and $C = 0$ and that we choose $\hat{\lambda} = 0$. In this case the Schur complement is given by $S = B^T A^{-1} B$, and we have $E_{11} = A^{-1} - A^{-1}BS^{-1}B^TA^{-1}$, $E_{21} = S^{-1}B^TA^{-1}$,

$A_{11} = I$, $A_{21} = 0$. Then it is well known that $\nu = 2$, and we obtain for $\operatorname{ind} E_{11} = 1$ the solution

$$
\begin{aligned}
\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \exp(-tE_{11}^D)\widetilde{q} \\ E_{21}E_{11}^D \exp(-tE_{11}^D)\widetilde{q} \end{bmatrix} \\
&+ \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^D)E_{11}^D(E_{11}f(s) - A^{-1}BS^{-1}g(s)) \\ E_{21}E_{11}^D \exp(-(t-s)E_{11}^D)E_{11}^D(E_{11}f(s) - A^{-1}BS^{-1}g(s)) \end{bmatrix} ds \\
&+ \begin{bmatrix} -(I - E_{11}^D E_{11})A^{-1}BS^{-1}g(t) \\ E_{21}(I - E_{11}^D E_{11})f(t) + (I + E_{21}E_{11}^D A^{-1}B)S^{-1}g(t)) \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ E_{21}(I - E_{11}^D E_{11})A^{-1}BS^{-1}\dot{g}(t) \end{bmatrix},
\end{aligned} \tag{4.3}
$$

where we have used again that $(E_{11}^D)^2 E_{11} = E_{11}^D$, $E_{11}^D E_{11}^3 = E_{11}^2$, as well as $E_{21}E_{11}^D E_{11}^2 = E_{21}E_{11}$. In this case it is very easy to determine consistent initial conditions. Rather than going through the solution formula, the second equation of (1.1) immediately gives the consistency condition

$$0 = B^T v^0 + g(0)\,. \tag{4.4}$$

This is exactly the consistency condition that will also appear in the infinite dimensional case, while directly in the system no consistency condition for p^0 arises. However, a differentiation with respect to t of the second equation of (1.1) and insertion of the first equation gives

$$B^T Bp(t) = -B^T Av(t) + B^T f(t) + \dot{g}(t), \tag{4.5}$$

which corresponds to the Poisson problem for the pressure that is typically used to solve for the pressure or to do pressure correction.

Evaluating (4.5) at $t = 0$ we get the consistency system for the initial values

$$\begin{bmatrix} B^T A & B^T B \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} v^0 \\ p^0 \end{bmatrix} = \begin{bmatrix} B^T f(0) + \dot{g}(0) \\ g(0) \end{bmatrix}. \tag{4.6}$$

Note that the invertibility of $B^T B$ allows to solve for p^0 in terms of v^0.

The initial condition is indeed consistent (see Theorem 9) if we can find a vector $\widetilde{q}$ such that (4.3) is satisfied at $t = 0$. By taking

$$\widetilde{q} = v^0 + (I - E_{11}^D E_{11})A^{-1}BS^{-1}g(0)\,, \tag{4.7}$$

the first equation of (4.3) at $t = 0$ is automatically satisfied. Inserting (4.7) into the second equation of (4.3) at $t = 0$ and employing (4.4) as well as (4.5) at $t = 0$, a straightforward calculation shows that also this second equation is fulfilled.

This, finally, proves that (4.6) is a sufficient as well as necessary condition for the solvability, in the classical sense, of the initial value problem under consideration.

As already noted, in general we do not know the index of E_{11}, however, if the discretization is such that M, A, C are symmetric and $D = B$, then we have the following Lemma.

Lemma 10 *Consider the coefficient matrices in (1.1) and the transformed coefficients in (3.2) and (3.3). If A, C, M are symmetric, if $D = B$, and if $\hat{\lambda}$ is chosen so that $W = \hat{\lambda}M + A$ is positive (or negative) definite, then* $\operatorname{ind}(E_{11}) \leq 1$.

Proof. Under the given assumptions we have that

$$E_{11} = W^{-1} - W^{-1}BS^{-1}B^TW^{-1}$$

If W is positive definite then let $Z = W^{-1/2}W^{-1/2}$, where $W^{-1/2}$ denotes the positive definite square root of W^{-1}. Then it follows that

$$E_{11} = W^{-1/2}Z^{-1/2}[Z - Z^{1/2}W^{-1/2}BS^{-1}B^TW^{-1/2}Z^{1/2}]Z^{1/2}W^{1/2},$$

i.e. E_{11} is similar to a symmetric matrix, which has index less than or equal to 1 and hence $\mathrm{ind}(E_{11}) \leq 1$. If W is negative definite then the same proof follows by replacing W by $-W$. □

We can directly apply Lemma 10 if A is definite and if we choose $\hat{\lambda} = 0$. Then we obtain the simplified formulas

$$\begin{aligned} A_{11} &= I,\ A_{21} = 0, \\ E_{11} &= I - A^{-1}BS^{-1}B^TA^{-1}M, \\ E_{21} &= S^{-1}B^TA^{-1}M. \end{aligned} \tag{4.8}$$

Let us assume that A is positive definite, the same result follows if A is negative definite by replacing A with $-A$.

Let $A = LL^T$ be the Cholesky decomposition (we could also take the positive square root $L = A^{1/2}$), let $\widetilde{B} = L^{-1}B$ and let

$$\begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q^T\widetilde{B}$$

be a QR decomposition, with Q orthogonal and R_1 upper triangular. By the assumption that B has full column rank it follows that R_1 is invertible. We then have

$$\begin{aligned} E_{11} &= L^{-T}Q\left(I - \begin{bmatrix} R_1 \\ 0 \end{bmatrix}(C + R_1^TR_1)^{-1}\begin{bmatrix} R_1^T & 0 \end{bmatrix}\right)Q^TL^{-1}M \\ &= L^{-T}Q\begin{bmatrix} Z_1 & 0 \\ 0 & I \end{bmatrix}Q^TL^{-1}M, \end{aligned} \tag{4.9}$$

with $Z_1 = I - R_1(C + R_1^TR_1)^{-1}R_1^T$. (In the special case that $C = 0$, we then have $Z_1 = 0$). In the same way it follows that

$$E_{21} = R_1^{-1}\begin{bmatrix} I - Z_1 & 0 \end{bmatrix}Q^TL^{-1}M. \tag{4.10}$$

Checking the conditions for E_{11}^D, we immediately have that

$$\begin{aligned} E_{11}^D &= M^{-1}LQ\begin{bmatrix} Z_1^D & 0 \\ 0 & I \end{bmatrix}Q^TL^T, \\ E_{11}E_{11}^D &= L^{-T}Q\begin{bmatrix} Z_1Z_1^D & 0 \\ 0 & I \end{bmatrix}Q^TL^T, \\ E_{21}E_{11}^D &= R_1^{-1}\begin{bmatrix} (I - Z_1)Z_1^D & 0 \end{bmatrix}Q^TL^T, \end{aligned}$$

and we can insert this in (4.2) and obtain with $\widetilde{q} = v^0$ that

$$\begin{aligned}
\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \exp(-tE_{11}^D)\widetilde{q} \\ E_{21}E_{11}^D \exp(-tE_{11}^D)\widetilde{q} \end{bmatrix} \\
&+ \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^D)E_{11}^D(E_{11}M^{-1}f(s) - A^{-1}BS^{-1}g(s))\, ds \\ E_{21}E_{11}^D \exp(-(t-s)E_{11}^D)E_{11}^D(E_{11}M^{-1}f(s) - A^{-1}BS^{-1}g(s)) \end{bmatrix} ds \\
&+ \begin{bmatrix} (I - E_{11}^D E_{11})(E_{11}M^{-1}f(t) - A^{-1}BS^{-1}g(t)) \\ E_{21}(I - E_{11}^D E_{11})M^{-1}f(t) + (S^{-1} + E_{21}E_{11}^D A^{-1}BS^{-1})g(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ E_{21}\left(I - E_{11}^D E_{11}\right) A^{-1}BS^{-1}\dot{g}(t) \end{bmatrix}.
\end{aligned} \tag{4.11}$$

A further special case arises when $C = 0$. In this case the formulas become even more simple, since then we have

$$\begin{aligned}
E_{11} &= L^{-T}Q \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q^T L^{-1}M, \\
E_{11}^D &= M^{-1}LQ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q^T L^T, \\
E_{21} &= R_1^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q^T L^{-1}M,
\end{aligned}$$

and from this we see immediately that $E_{21}E_{11}^D = 0$ and hence

$$\begin{aligned}
\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \exp(-tE_{11}^D)\widetilde{q} \\ 0 \end{bmatrix} \\
&+ \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^D)E_{11}^D(E_{11}M^{-1}f(s) - A^{-1}BS^{-1}g(s)) \\ 0 \end{bmatrix} ds \\
&+ \begin{bmatrix} (I - E_{11}^D E_{11})(E_{11}M^{-1}f(t) - A^{-1}BS^{-1}g(t)) \\ E_{21}M^{-1}f(t) + S^{-1}g(t) + E_{21}A^{-1}BS^{-1}\dot{g}(t) \end{bmatrix}.
\end{aligned} \tag{4.12}$$

In particular we have an explicit formula for the pressure as

$$p(t) = S^{-1}B^T A^{-1}f(t) + S^{-1}g(t) + S^{-1}B^T A^{-1}MA^{-1}BS^{-1}\dot{g}(t). \tag{4.13}$$

As a summary of the findings, we see that derivatives of the inhomogeneity f do not occur, whenever we have $\operatorname{ind}(\mathcal{E}, \mathcal{A}) \leq 2$ but an inhomogeneity g will have to be at least differentiable.

5 Conclusion

We have presented the analysis as well as explicit solution formulas for differential-algebraic equations arising from the semi-discretization of linearized Navier-Stokes and Oseen equations. The infinite dimensional case will be studied in the forthcoming Part II.

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