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On the Evolutionary Fractional p-Laplacian

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In this work, existence results on nonlinear first order as well as doubly nonlinear second-order evolution equations involving the fractional p-Laplacian are presented. The proofs do not exploit any monotonicity assumption but rely on a compactness argument in combination with regularity of the Galerkin scheme and the nonlocal character of the operator.

1 Introduction

In [14], existence of weak solutions to the nonlinear peridynamic initial value problem is shown. Peridynamics is a nonlocal elasticity theory based on an integro-differential equation without spatial derivatives (see, e.g., [13, 30]). The authors remark in [14] that the existence result also applies to the weak formulation of

$$u_{tt}(x,t) - \int_{\Omega} \frac{|u(y,t)-u(x,t)|^{p-2}}{|y-x|^{d+\sigma p}}(u(y,t)-u(x,t))\,\mathrm{d}y = f(x,t), \quad (x,t)\in \Omega\times(0,T), \tag{1.1}$$

supplemented with initial conditions, where $\sigma\in(0,1)$, $p\in[2,\infty)$, and $\Omega\subset\mathbb{R}^d$ is a bounded Lipschitz domain. Since the assumptions on the peridynamic operator acting on u are fairly general (in particular, the operator is not assumed to be monotone), the method of proof relies on compactness arguments combined with the nonlocal structure of the operator instead of monotonicity arguments. The goal of this work is to apply the latter method to various nonlinear nonlocal evolution equations. In the first part, we present an alternative existence proof (the solvability of this problem is well

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investigated, see, e.g., the references given in Section 1.1) for

$$u' + K^{\sigma,p}u = f,$$

where u' denotes the weak time derivative of the abstract function $u\colon [0,T] \to W^{\sigma,p}(\Omega)^d$ and $K^{\sigma,p}$ is the Nemytskii operator associated to the nonlinear form $k^{\sigma,p}\colon W^{\sigma,p}(\Omega)^d \times W^{\sigma,p}(\Omega)^d \to \mathbb{R}$,

$$k^{\sigma,p}(u,v) := \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{d+\sigma p}}(u(x)-u(y))\cdot(v(x)-v(y))\,\mathrm{d}x\,\mathrm{d}y, \tag{1.2}$$

given by

$$\langle K^{\sigma,p}u, v\rangle = \int_0^T k^{\sigma,p}(u(t),v(t))\,\mathrm{d}t. \tag{1.3}$$

This is indeed the operator to be investigated as a result of the weak formulation of (1.1), which follows from nonlocal integration-by-parts (see, e.g., [14]). Thus, $K^{\sigma,p}$ is (one possible) nonlocal (fractional) version of the well-known p-Laplace operator.

In the second part, we prove existence of solutions to the second-order doubly nonlinear evolution problem

$$u'' + K^{\sigma,p}u' + K^{\gamma,q}u = f.$$

Here, $K^{\gamma,q}$ is given in a similar manner as above, that is, in (1.2) the parameter $\sigma \in (0,1)$ is replaced by $\gamma \in (0,1)$ and $p \in [2,\infty)$ by $q \in [2,\infty)$. Results on vector-valued doubly nonlinear evolution equations are already obtained (see the references cited in Section 1.1), however, none of these apply to the setting given in this paper. To the best knowledge of the author, the result in Theorem 4.1 providing existence for the doubly nonlinear evolution equation of second order is new. The proofs of both the first- and second-order evolution problem are based on the method discovered in [14] which combines compactness in a slightly larger space with the given nonlocal structure of the operator and the regularity of the Galerkin scheme.

1.1 **Literature**

The fractional Laplacian as a nonlocal generalization of the Laplace operator has been studied in classical monographs such as [24, 31] as well as in very recent articles such as [8, 9, 21] and many others. Usually, singular kernels are considered in contrast to [2] and the references cited therein, where nonlocal generalizations of the Laplacian (both linear and nonlinear) are examined with smooth kernels. Further, in most papers, the fractional Laplacian is represented as an integral operator over the whole $\mathbb{R}^d$. In this work, we focus on nonlocal operators acting on bounded domains which correspond to the regional fractional Laplacian (see [21]) and can be interpreted as a nonlocal version of the Laplacian equipped with Neumann boundary conditions (see [2]). Next to the integral representation of the fractional Laplacian used in the latter references, it is also possible to define the fractional Laplacian via Fourier transform. However, this approach is restricted to $p=2$. The nonlocal generalization of the p-Laplacian, see (1.1), is hence the nonlinear pendant to the fractional regional Laplacian mentioned above and appears, for instance, as a type of nonlinear diffusion, see [34]. Due to the strong singularity of the kernel, the operators involved in the setting used in this paper are based on Sobolev–Slobodetskii spaces. Applications of equations with operators based on these spaces are listed in the introduction of [12] and range from obstacle problems over finance to water waves and material science.

Abstract evolution problems of first order are part of many text books and monographs such as [20, Kapitel VI; 29, Chapter 8; 35, Chapter IV; 36, Chapter 30]. Second-order evolution problems in form of abstract operator equations are studied, for example, in the monographs [3, Chapter V; 20, Kapitel VII; 25; 26, Chapitre 3.8; 29, Chapter 11; 35, Chapter V; 36, Chapter 33] as well as in the works [11, 14, 15, 17, 19, 23, 28] (this list is far from being comprehensive). Usually, at least one of the two operators involved is assumed to be linear or Lipschitz continuous. Compared with first-order equations, these problems are more involved and often it is not clear whether a weak solution exists (one way out of this dilemma is to weaken the concept of solutions and to study Young-measure-valued solutions). Nevertheless, existence theory for doubly nonlinear evolution problems (i.e., both operators involved are nonlinear) is studied, for instance, in [7, 16, 18]. However, in these results, monotonicity assumptions are used. Further, doubly nonlinear evolution equations are treated in the monograph [29, Chapter 11.3] and the references cited therein.

1.2 Notation

In this work, $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. We rely upon the usual notation of Lebesgue spaces (its norm is denoted by $\|\cdot\|_{0,p}$ and the $L^2(\Omega)^d$ inner product by $(\cdot,\cdot)$, whereas $L^2(\Omega)^d$ denotes the space $L^2(\Omega;\mathbb{R}^d)$) and Sobolev–Slobodetskii spaces, that is, for $\sigma \in (0,1)$ and $p \in [1,\infty)$ the Banach space $W^{\sigma,p}(\Omega)^d$ consists of all elements $u \in L^p(\Omega)^d$ with bounded Slobodetskii seminorm $|u|_{\sigma,p} < \infty$, given by

$$|u|_{\sigma,p} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

Here, $|\cdot|$ denotes an appropriate norm on $\mathbb{R}^d$. The Banach space $W^{\sigma,p}(\Omega)^d$ is equipped with the norm

$$\|\cdot\|_{\sigma,p} = \left(\|\cdot\|_{0,p}^p + |\cdot|_{\sigma,p}^p\right)^{1/p}.$$

Moreover, the standard notation for Bochner–Lebesgue spaces $L^p(0,T;X)$ and Bochner–Sobolev spaces $W^{k,p}(0,T;X)$ is applied. As usual, X^* is the dual of the Banach space X and the conjugate exponent to $p \in (1,\infty)$ is denoted by $p^* = p/(p-1)$. In particular, we denote the norm of the dual $(W^{\sigma,p}(\Omega)^d)^*$ by $\|\cdot\|_{(\sigma,p)^*}$. Finally, $\mathcal{C}([0,T];X)$ and $\mathcal{AC}([0,T];X)$ are the spaces of continuous and absolutely continuous functions mapping $[0,T]$ into X and $\mathcal{C}_w([0,T];X)$ is the space of demicontinuous functions (i.e., continuous with respect to the weak topology in X).

For Banach spaces X and Y, we write $X \hookrightarrow Y$ when X is continuously embedded in Y. By $X \overset{d}{\hookrightarrow} Y$ and $X \overset{c,d}{\hookrightarrow} Y$, we denote continuous and dense as well as compact and dense embeddings, respectively. In the sequel, we will deal with intersections and sums of Banach spaces. Following, for example, [20, p. 12] for Banach spaces X and Y, which are continuously embedded in another Banach space Z, the intersection $X \cap Y$ forms with $\|\cdot\|_{X\cap Y} = \|\cdot\|_X + \|\cdot\|_Y$ again a Banach space. If $X \cap Y$ is dense in X as well as in Y, then its dual is determined as the sum of their duals, $(X \cap Y)^* = X^* + Y^*$, equipped with the norm

$$\|g\|_{X^*+Y^*} = \inf_{\substack{g=g_X+g_Y \\ g_X\in X^*, g_Y\in Y^*}} \max\{\|g_X\|_{X^*}, \|g_Y\|_{Y^*}\}.$$

The duality pairing is given by

$$\langle g, v\rangle_{X^*+Y^*, X\cap Y} = \langle g_X, v\rangle_{X^*,X} + \langle g_Y, v\rangle_{Y^*,Y}, \quad g = g_X + g_Y.$$

In the following, $\sigma, \gamma \in (0,1)$ and $p, q \in [2,\infty)$ as well as $T > 0$ are fixed and $c > 0$ is a generic constant.

1.3 Outline

In Section 2, properties of the operators induced by the fractional p-Laplacian are summarized. Moreover, results on the nonlinear form $k^{\sigma,p}$ given by (1.2) as well as on Sobolev–Slobodetskii spaces are presented, which are needed to prove both the first- and second-order result. In Section 3, the first-order evolution problem is considered. Existence and uniqueness of a solution is shown. Subsequently, in Section 4, an existence result for the doubly nonlinear evolution problem is provided.

2 Preliminary Results

In this section, we state properties of the nonlinear form given by (1.2) and its associated Nemytskii operator given by (1.3). The following results are well known or easily to obtain by standard arguments.

Proposition 2.1. The nonlinear form $k^{\sigma,p}: W^{\sigma,p}(\Omega)^d \times W^{\sigma,p}(\Omega)^d \to \mathbb{R}$ given by (1.2) is well defined, bounded, continuous in its first and second argument, and monotone; there holds for all $u, v \in W^{\sigma,p}(\Omega)^d$

$$|k^{\sigma,p}(u, v)| \leq \tfrac{1}{2}|u|_{\sigma,p}^{p-1}|v|_{\sigma,p}, \tag{2.1}$$

$$k^{\sigma,p}(u, u-v) - k^{\sigma,p}(v, u-v) \geq 0. \tag{2.2}$$

Moreover, there holds for $u \in W^{\sigma,p}(\Omega)^d$

$$k^{\sigma,p}(u, u) = \tfrac{1}{2}|u|_{\sigma,p}^{p}. \tag{2.3}$$

The potential $\Phi^{\sigma,p}: W^{\sigma,p}(\Omega)^d \to \mathbb{R}$ given by

$$\Phi^{\sigma,p}(u) = \frac{1}{2p}|u|_{\sigma,p}^{p}$$

is well defined, bounded, nonnegative, and has the Gâteaux derivative

$$\langle (\Phi^{\sigma,p})'(u), v\rangle = k^{\sigma,p}(u, v), \quad u, v \in W^{\sigma,p}(\Omega)^d.$$

□

Furthermore, the form $k^{\sigma,p}$ induces a Nemytskii operator which inherits the properties given in the latter proposition, that is, the nonlinear operator $K^{\sigma,p}$ given by (1.3) maps $L^p(0,T; W^{\sigma,p}(\Omega)^d)$ into its dual $L^{p^*}(0,T; (W^{\sigma,p}(\Omega)^d)^*)$, is bounded, demicontinuous, and monotone. In particular, for $u \in L^p(0,T; W^{\sigma,p}(\Omega)^d)$ there holds

$$\langle K^{\sigma,p}u, u\rangle = \frac{1}{2}\int_0^T |u(t)|_{\sigma,p}^p \,\mathrm{d}t.$$

The main key to both existence proofs presented in the sequel is the following lemma providing a continuity result.

Lemma 2.2. For any $0 < \eta < \min\left\{\frac{1-\sigma}{p-1}, \sigma\right\}$, the form $k^{\sigma,p}$ given by (1.2) is also well defined on $W^{\sigma-\eta,p}(\Omega)^d \times W^{\sigma+\eta(p-1),p}(\Omega)^d$, bounded, and continuous in both of its arguments. □

This result is proved in [14, Proposition 4.2]. Due to its central role in this paper, we give the proof here again.

Proof. Observe that due to the assumption on η, the order for both Sobolev–Slobodetskii spaces is between 0 and 1,

$$0<\sigma+\eta(p-1)<1 \quad \text{as well as} \quad 0<\sigma-\eta<1.$$

For $u\in W^{\sigma-\eta,p}(\Omega)^d$ and $v\in W^{\sigma+\eta(p-1),p}(\Omega)^d$, there holds with Hölder's inequality

$$\begin{aligned}|k^{\sigma,p}(u,v)| &\leq \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{(d+(\sigma-\eta)p)/p^*}}\frac{|v(x)-v(y)|}{|x-y|^{(d+(\sigma+\eta(p-1))p)/p}}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq \frac{1}{2}|u|^{p-1}_{\sigma-\eta,p}|v|_{\sigma+\eta(p-1),p}. \end{aligned} \tag{2.4}$$

Therefore, the form is well defined and bounded. By the linearity and boundedness in its second argument, the form is continuous in its second argument. To prove continuity in its first argument, we take for fixed $v\in W^{\sigma+\eta(p-1),p}(\Omega)^d$ a sequence $\{u_\ell\}\subset W^{\sigma-\eta,p}(\Omega)^d$ converging strongly to u in $W^{\sigma-\eta,p}(\Omega)^d$. In particular, we then have $u_\ell\to u$ in $L^p(\Omega)^d$ as well as convergence of the integrands of the Slobodetskii seminorm in $L^1(\Omega\times\Omega)$. Thus, there exists a subsequence (not relabeled) and a function $h\in L^1(\Omega\times\Omega)$ such that

$$u_\ell(x)\to u(x) \quad \text{a.e. on } \Omega, \qquad \frac{|u(x)-u(y)|^p}{|x-y|^{d+(\sigma-\eta)p}}\leq h(x,y) \quad \text{a.e. on } \Omega\times\Omega.$$

By Lebesgue's theorem on dominated convergence, we find $k^{\sigma,p}(u_\ell,v)\to k^{\sigma,p}(u_\ell,v)$ for the subsequence. A standard contradiction argument yields convergence of the whole sequence. ■

A final observation beforehand is the following known fact about Sobolev–Slobodetskii spaces.

Lemma 2.3. For bounded domains Ω, the norm

$$|||\cdot||| := \|\cdot\|_{0,1}+|\cdot|_{\sigma,p}$$

is equivalent to $\|\cdot\|_{\sigma,p}$ on $W^{\sigma,p}(\Omega)^d$. □

Proof. We first observe that there holds for $u\in W^{\sigma,p}(\Omega)^d$ and its integral mean $\bar{u}=\mathrm{vol}(\Omega)^{-1}\int_\Omega u(x)\,\mathrm{d}x$

$$\|u-\bar{u}\|^p_{0,p}\leq \frac{1}{\mathrm{vol}(\Omega)}\int_\Omega\int_\Omega |u(x)-u(y)|^p\,\mathrm{d}x\,\mathrm{d}y \leq \frac{\mathrm{diam}(\Omega)^{d+\sigma p}}{\mathrm{vol}(\Omega)}|u|^p_{\sigma,p}.$$

Hence, by triangle inequality and $\|\bar{u}\|_{0,p}\leq c\|u\|_{0,1}$ we get that

$$\|u\|_{0,p}\leq c\,(\|u\|_{0,1}+|u|_{\sigma,p}).$$

On the one hand, we obtain

$$\|u\|_{\sigma,p}^{p} = \|u\|_{0,p}^{p} + |u|_{\sigma,p}^{p} \le c\,(\|u\|_{0,1}^{p} + |u|_{\sigma,p}^{p})$$

and hence, there exists $c > 0$ such that

$$\|u\|_{\sigma,p} \le c|||u|||.$$

On the other hand, by Hölder's inequality there follows $|||u||| \le c\|u\|_{\sigma,p}$. ∎

3 First-Order Evolution Problem

In this section, we prove existence and uniqueness of the before-mentioned first-order problem. Of course, the following result is well known and can be proved by standard results such as given in the references of Section 1.1. However, we give an alternative proof which uses the monotonicity only for uniqueness. The following existence result is based on a particular Galerkin approximation of the underlying function space $W^{\sigma,p}(\Omega)^d$ combined with compactness methods and the nonlocal structure of $K^{\sigma,p}$.

Theorem 3.1. For $u_0 \in L^2(\Omega)^d$, $f \in L^1(0,T;L^2(\Omega)^d)$, and $g \in L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)$, there exists a function

$$u \in \mathcal{C}([0,T];L^2(\Omega)^d) \cap L^p(0,T;W^{\sigma,p}(\Omega)^d),$$
$$u' \in L^1(0,T;L^2(\Omega)^d) + L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)$$

such that there holds $u(0) = u_0$ in $L^2(\Omega)^d$ and

$$u' + K^{\sigma,p}u = f + g \quad \text{in } L^1(0,T;L^2(\Omega)^d) + L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*).$$

Moreover, for every $t \in [0,T]$ there holds the energy equality

$$\|u(t)\|_{0,2}^2 + \int_0^t |u(s)|_{\sigma,p}^p \,\mathrm{d}s = \|u_0\|_{0,2}^2 + 2\int_0^t (f(s),u(s))\,\mathrm{d}s + 2\int_0^t \langle g(s),u(s)\rangle\,\mathrm{d}s.$$ □

Proof. We use a Galerkin discretization and therefore divide the proof into its naturally arising steps.

Existence of a discrete solution. We construct a special Galerkin scheme that satisfies the smoothness as well as the stability we require for this proof. Let $r > 0$ be such that

$$W^{r,2}(\Omega) \overset{d}{\hookrightarrow} W^{1,p}(\Omega),$$

that is, $r \geq d(1/2 - 1/p) + 1$. Moreover, let $(\cdot,\cdot)_r$ denote the $W^{r,2}(\Omega)^d$ inner product. Then the solution operator $T : L^2(\Omega)^d \to L^2(\Omega)^d$, $f \mapsto u$, to the problem of finding $u \in W^{r,2}(\Omega)^d$ to a given $f \in L^2(\Omega)^d$ such that

$$(u, v)_r = (f, v) \quad \text{for all } v \in W^{r,2}(\Omega)^d,$$

is self-adjoint, compact, nonnegative, and the kernel of T is equal to $\{0\}$. Hence, by [6, Theorem 6.11] and arguments similar to [6, Theorem 9.31] there exists an orthonormal basis $\{e_\ell\} \subset W^{r,2}(\Omega)^d$ of $L^2(\Omega)^d$ consisting of eigenfunctions of T,

$$Te_\ell = \mu_\ell e_\ell, \quad \ell = 1, 2, \dots,$$

where $\mu_\ell > 0$ are the corresponding eigenvalues with $\mu_\ell \to 0$. Then $\{\sqrt{\mu_\ell} e_\ell\}$ is an orthonormal basis of $W^{r,2}(\Omega)^d$ and by the density of the embedding, $\{\psi_\ell\}$ given by $\psi_\ell := \sqrt{\mu_\ell} e_\ell$ is a Galerkin basis of $W^{\sigma,p}(\Omega)^d$. Hence, $\{V_\ell\} \subset W^{\sigma,p}(\Omega)^d$ defined through $V_\ell := \operatorname{span}\{\psi_1, \dots, \psi_\ell\}$ is a Galerkin scheme of $W^{\sigma,p}(\Omega)^d$, that is,

$$\overline{\bigcup_{\ell \in \mathbb{N}} V_\ell} = W^{\sigma,p}(\Omega)^d.$$

The reason we choose this specific scheme is two-fold. First, we will exploit that the Galerkin space has a certain regularity, that is, $V_\ell \subset W^{\sigma+\eta(p-1),p}(\Omega)^d$ for all $\ell \in \mathbb{N}$ and $0 < \eta < \min\left\{\frac{1-\sigma}{p-1}, \sigma\right\}$ (cf. Lemma 2.2). Second, we are going to use the $W^{r,2}(\Omega)^d$-stability of the family of $L^2(\Omega)^d$-orthogonal projections $P_\ell : L^2(\Omega)^d \to V_\ell$, that is,

$$\sup_{v \in W^{r,2}(\Omega)^d \setminus \{0\}} \frac{\|P_\ell v\|_{r,2}}{\|v\|_{r,2}} \leq 1. \tag{3.1}$$

Indeed, the $L^2(\Omega)^d$-orthogonal projection $P_\ell : L^2(\Omega)^d \to V_\ell$ given by $P_\ell v = \sum_{j=1}^{\ell} (v, e_j) e_j$ is also $W^{r,2}(\Omega)^d$-orthogonal, because it is constructed from eigenfunctions. Therefore, (3.1) follows. Note that for $v \in L^2(\Omega)^d$, there holds

$$(P_\ell v, v_\ell) = (v, v_\ell) \quad \text{for all } v_\ell \in V_\ell. \tag{3.2}$$

The stability will yield a uniform estimate for the derivatives of Galerkin solutions in the very large space $(W^{r,2}(\Omega)^d)^*$, which—similarly to [25, Chapitre 1, Section 12.3]—allows the application of the Lions–Aubin lemma (see, e.g., [25, Chapitre 1, Théorème 5.1] or [29, Lemma 7.7]). Note that for any $0 < \eta < \min\left\{\frac{1-\sigma}{p-1}, \sigma\right\}$, we have the scale of Banach spaces

$$V_\ell \subset W^{r,2}(\Omega)^d \overset{c,d}{\hookrightarrow} W^{\sigma+\eta(p-1),p}(\Omega)^d \overset{c,d}{\hookrightarrow} W^{\sigma,p}(\Omega)^d \overset{c,d}{\hookrightarrow} W^{\sigma-\eta,p}(\Omega)^d \overset{c,d}{\hookrightarrow} L^2(\Omega)^d$$

and

$$L^2(\Omega)^d \overset{c,d}{\hookrightarrow} (W^{\sigma,p}(\Omega)^d)^* \overset{c,d}{\hookrightarrow} (W^{r,2}(\Omega)^d)^*.$$

By standard arguments (existence of a maximal solution in the sense of Carathéodory, see [22, Theorem 5.2]), there exists a maximal solution $u_\ell \in W^{1,1}(I; V_\ell)$ to the discretized problem of finding $u_\ell : [0, T] \to V_\ell$ such that for all $v_\ell \in V_\ell$ there holds

$$(u_\ell'(t), v_\ell) + k^{\sigma,p}(u_\ell(t), v_\ell) = (f(t), v_\ell) + \langle g(t), v_\ell\rangle \quad \text{a.e. on } (0, T), \quad u_\ell(0) = u_\ell^0. \tag{3.3}$$

Here, the initial values $u_\ell^0 \in V_\ell$ are chosen such that there holds $u_\ell^0 \to u_0$ in $L^2(\Omega)^d$. By applying the a priori estimate proved in the sequel to $t \in I \subseteq (0, T)$, arguments such as given in [1, Corollary 7.7] yield that $I = (0, T)$.

A priori estimate and convergent subsequences. Using (3.3) with $v_\ell = u_\ell(t) \in V_\ell$ yields with (2.3) for almost every $t \in (0, T)$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\ell(t)\|_{0,2}^2 + \frac{1}{2}|u_\ell(t)|_{\sigma,p}^p \le \|f(t)\|_{0,2}\|u_\ell(t)\|_{0,2} + \|g(t)\|_{(\sigma,p)^*}\|u_\ell(t)\|_{\sigma,p}.$$

By Lemma 2.3, there exists $c > 0$ such that

$$\|u_\ell(t)\|_{\sigma,p} \le c(\|u_\ell(t)\|_{0,1} + |u_\ell(t)|_{\sigma,p}) \le c(\|u_\ell(t)\|_{0,2} + |u_\ell(t)|_{\sigma,p}).$$

Therefore, we obtain with Young's inequality and $c_1, c_2 > 0$

$$\|g(t)\|_{(\sigma,p)^*}\|u_\ell(t)\|_{\sigma,p} \le c_1\|g(t)\|_{(\sigma,p)^*}\|u_\ell(t)\|_{0,2} + c_2\|g(t)\|_{(\sigma,p)^*}^{p^*} + \frac{1}{2p}|u_\ell(t)|_{\sigma,p}^p,$$

which then yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\ell(t)\|_{0,2}^2 + \frac{1}{2p^*}|u_\ell(t)|_{\sigma,p}^p \le (\|f(t)\|_{0,2} + c_1\|g(t)\|_{(\sigma,p)^*})\|u_\ell(t)\|_{0,2} + c_2\|g(t)\|_{(\sigma,p)^*}^{p^*},$$

and, after integrating and multiplying by 2, becomes

$$\begin{aligned}&\|u_\ell(t)\|_{0,2}^2 + \frac{1}{p^*}\int_0^t |u_\ell(s)|_{\sigma,p}^p\,\mathrm{d}s \\ &\quad\le 2\int_0^t (\|f(s)\|_{0,2} + c_1\|g(s)\|_{(\sigma,p)^*})\|u_\ell(s)\|_{0,2}\,\mathrm{d}s + 2c_2\int_0^t \|g(s)\|_{(\sigma,p)^*}^{p^*}\,\mathrm{d}s + \|u_\ell^0\|_{0,2}^2.\end{aligned} \tag{3.4}$$

Since $u_\ell \in W^{1,1}(0, T; V_\ell) \subset \mathcal{AC}([0, T]; V_\ell)$, all terms of the latter inequality are continuous and therefore, (3.4) holds for every $t \in [0, T]$. Moreover, there exists $\bar{t} \in [0, T]$ such that $\|u_\ell(\bar{t})\|_{0,2} = \max_{t\in[0,T]}\|u_\ell(t)\|_{0,2}$. Forgetting about the integral on the left-hand side of (3.4), we are in the situation to solve the quadratic inequality

$$\begin{aligned}\|u_\ell(\bar{t})\|_{0,2}^2 \le\ & 2(\|f\|_{L^1(0,T;L^2(\Omega)^d)} + c_1\|g\|_{L^1(0,T;(W^{\sigma,p}(\Omega)^d)^*)})\|u_\ell(\bar{t})\|_{0,2} \\ &+ \|u_\ell^0\|_{0,2}^2 + 2c_2\|g\|_{L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)}^{p^*}.\end{aligned}$$

This yields

$$\|u_\ell(\bar{t})\|_{0,2} \leq c\big(\|f\|_{L^1(0,T;L^2(\Omega)^d)} + \|g\|_{L^1(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|g\|^{p^*/2}_{L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|u^0_\ell\|_{0,2}\big).$$

Since u^0_ℓ converges to u_0 in $L^2(\Omega)^d$, the right-hand side of the latter inequality is bounded. Consequently, the right-hand side of (3.4) is bounded and we have for all $t \in [0,T]$

$$\|u_\ell(t)\|^2_{0,2} + \frac{1}{p^*}\int_0^t |u_\ell(s)|^p_{\sigma,p}\,\mathrm{d}s \leq c\big(\|f\|^2_{L^1(0,T;L^2(\Omega)^d)} + \|g\|^2_{L^1(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|g\|^{p^*}_{L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|u^0_\ell\|^2_{0,2}\big).$$

Thus, in view of Lemma 2.3 there exists $u \in L^p(0,T;W^{\sigma,p}(\Omega)^d) \cap L^\infty(0,T;L^2(\Omega)^d)$ and a subsequence of $\{u_\ell\}$, which will not be relabeled, such that

$$u_\ell \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;L^2(\Omega)^d), \tag{3.5}$$

$$u_\ell \rightharpoonup u \quad \text{in } L^p(0,T;W^{\sigma,p}(\Omega)^d). \tag{3.6}$$

Moreover, by (3.2) there holds

$$\begin{aligned}\|u'_\ell(t)\|_{(r,2)^*} &= \sup_{v\in W^{r,2}(\Omega)^d\setminus\{0\}} \frac{|(u'_\ell(t),v)|}{\|v\|_{r,2}} \\ &= \sup_{v\in W^{r,2}(\Omega)^d\setminus\{0\}} \frac{|(u'_\ell(t),P_\ell v)|}{\|v\|_{r,2}} \\ &\leq \sup_{v\in W^{r,2}(\Omega)^d\setminus\{0\}} (\|f(t)\|_{(\sigma,p)^*} + \|g(t)\|_{(\sigma,p)^*} + \|k^{\sigma,p}(u_\ell(t),\cdot)\|_{(\sigma,p)^*})\frac{\|P_\ell v\|_{\sigma,p}}{\|v\|_{r,2}} \\ &\leq c\left(\|f(t)\|_{0,2} + \|g(t)\|_{(\sigma,p)^*} + \frac{1}{2}|u_\ell(t)|^{p-1}_{\sigma,p}\right),\end{aligned}$$

since $W^{r,2}(\Omega)^d \hookrightarrow W^{\sigma,p}(\Omega)^d$ and since the family of projections is $W^{r,2}(\Omega)^d$-stable, see (3.1). Therefore, we have

$$\|u'_\ell\|_{L^1(0,T;(W^{r,2}(\Omega)^d)^*)} \leq c\big(\|f\|_{L^1(0,T;L^2(\Omega)^d)} + \|g\|_{L^1(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|u_\ell\|^{p-1}_{L^{p-1}(0,T;W^{\sigma,p}(\Omega)^d)}\big),$$

which yields uniform boundedness of the first derivatives. By the Lions–Aubin lemma [29, Lemma 7.7], there holds for $0 < \eta < \min\left\{\frac{1-\sigma}{p-1},\sigma\right\}$

$$L^p(0,T;W^{\sigma,p}(\Omega)^d) \cap W^{1,1}(0,T;(W^{r,2}(\Omega)^d)^*) \overset{c}{\hookrightarrow} L^p(0,T;W^{\sigma-\eta,p}(\Omega)^d).$$

Thus, we also have

$$u_\ell \to u \quad \text{in } L^p(0,T;W^{\sigma-\eta,p}(\Omega)^d),$$

which immediately yields (for a subsequence, not relabeled) pointwise almost everywhere convergence and the existence of $h \in L^1(0,T)$ such that

$$u_\ell(t) \to u(t) \quad \text{in } W^{\sigma-\eta,p}(\Omega)^d, \quad |u_\ell(t)|^p_{\sigma-\eta,p} \leq h(t), \quad \text{a.e. on } (0,T). \tag{3.7}$$

Passage to the limit. There holds

$$-\int_0^T (u_\ell(t), v_m)\phi'(t)\,\mathrm{d}t + \int_0^T k^{\sigma,p}(u_\ell(t), v_m)\phi(t)\,\mathrm{d}t = \int_0^T (f(t), v_m)\phi(t)\,\mathrm{d}t + \int_0^T \langle g(t), v_m\rangle \phi(t)\,\mathrm{d}t$$

for all $\phi \in \mathcal{C}_c^\infty(0,T)$ and $v_m \in V_m$ with $m \in \mathbb{N}$, $m \leq \ell$. Due to (3.5), the first term converges for $\ell \to \infty$ to $-\int_0^T (u(t), v_m)\phi'(t)\,\mathrm{d}t$. Moreover, due to the pointwise convergence (3.7), $V_m \subset W^{\sigma+\eta,p}(\Omega)^d$, and Lemma 2.2, there holds for almost all $t \in (0,T)$

$$k^{\sigma,p}(u_\ell(t), v_m) \to k^{\sigma,p}(u(t), v_m).$$

By the boundedness of $k^{\sigma,p}$ from Lemma 2.2 and Lebesgue's theorem on dominated convergence in combination with the $h \in L^1(0,T)$ from (3.7), we find

$$\int_0^T k^{\sigma,p}(u_\ell(t), v_m)\phi(t)\,\mathrm{d}t \to \int_0^T k^{\sigma,p}(u(t), v_m)\phi(t)\,\mathrm{d}t.$$

Therefore, by the limited completeness of the Galerkin scheme we obtain

$$-\int_0^T (u(t), v)\phi'(t)\,\mathrm{d}t + \int_0^T k^{\sigma,p}(u(t), v)\phi(t)\,\mathrm{d}t = \int_0^T (f(t), v)\phi(t)\,\mathrm{d}t + \int_0^T \langle g(t), v\rangle \phi(t)\,\mathrm{d}t$$

for all $\phi \in \mathcal{C}_c^\infty(0,T)$ and $v \in V$. Hence, u has a weak derivative and by the density of $\mathcal{C}_c^\infty(0,T) \otimes W^{\sigma,p}(\Omega)^d$ in $L^p(0,T;W^{\sigma,p}(\Omega)^d)$ and the weak* density of $\mathcal{C}_c^\infty(0,T) \otimes L^2(\Omega)^d$ in $L^\infty(0,T;L^2(\Omega)^d)$ we arrive at

$$u' + K^{\sigma,p}u = f + g \quad \text{in } L^1(0,T;L^2(\Omega)^d) + L^{p*}(0,T;(W^{\sigma,p}(\Omega)^d)^*).$$

It remains to show that $u(0) = u_0$. Observe that since $u \in L^p(0,T;W^{\sigma,p}(\Omega)^d)$ with $u' \in L^1(0,T;L^2(\Omega)^d) + L^{p*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)$ there holds $u \in \mathcal{C}([0,T];L^2(\Omega)^d)$ (this follows from arguments similar to those given in [32, Chapter 20]). On the one hand, since $u_\ell \in W^{1,1}(0,T;V_\ell) \subset \mathcal{AC}([0,T];V_\ell)$ we find for all $v_m \in V_m$

$$\begin{aligned}-(u_\ell^0, v_m) &= \int_0^T \left[(u_\ell(t), v_m)\frac{T-t}{T}\right]' \mathrm{d}t \\ &= \int_0^T ((f(t), v_m) + \langle g(t), v_m\rangle - k^{\sigma,p}(u_\ell(t), v_m))\frac{T-t}{T}\,\mathrm{d}t - \frac{1}{T}\int_0^T (u_\ell(t), v_m)\,\mathrm{d}t\end{aligned}$$

and on the other hand, since $u \in W^{1,1}(0,T;(W^{\sigma,p}(\Omega)^d)^*) \subset \mathcal{AC}([0,T];(W^{\sigma,p}(\Omega)^d)^*)$ there holds for all $v_m \in V_m$

$$-(u(0),v_m) = \int_0^T \left[(u(t),v_m)\frac{T-t}{T}\right]' \mathrm{d}t$$
$$= \int_0^T ((f(t),v_m) + \langle g(t),v_m\rangle - k^{\sigma,p}(u(t),v_m))\frac{T-t}{T}\,\mathrm{d}t - \frac{1}{T}\int_0^T (u(t),v_m)\,\mathrm{d}t.$$

By the convergences deduced above both right-hand sides coincide for $\ell \to \infty$ and by passing to the limit $m \to \infty$, we obtain $u_0 = u(0)$ in $L^2(\Omega)^d$.

Energy balance. Since we are allowed to test the equation with the solution, there holds for almost every $t \in (0,T)$

$$\langle u'(t),u(t)\rangle + k^{\sigma,p}(u(t),u(t)) = (f(t),u(t)) + \langle g(t),u(t)\rangle.$$

Applying the chain rule to the first term, which is valid due to arguments similar to those given in [32, Chapter 20], and using (2.3) for the second term yields

$$\|u(t)\|_{0,2}^2 + \int_0^t |u(s)|_{\sigma,p}^p\,\mathrm{d}s = \|u_0\|_{0,2}^2 + 2\int_0^t (f(s),u(s))\,\mathrm{d}s + 2\int_0^t \langle g(s),u(s)\rangle\,\mathrm{d}s. \qquad \blacksquare$$

Observe that the proof did not make use of property (2.2). However, if we exploit monotonicity of the nonlinear form, we get uniqueness of the solution.

Theorem 3.2. Under the assumptions of Theorem 3.1, the solution is unique. $\square$

Proof. Assume that we have two solutions u_1 and u_2, while u_0 and f are fixed. Then there holds almost everywhere on $(0,T)$

$$\langle u_1'(t) - u_2'(t), v\rangle + k^{\sigma,p}(u_1(t),v) - k^{\sigma,p}(u_2(t),v) = 0 \quad \text{for all } v \in W^{\sigma,p}(\Omega)^d.$$

Due to the choice of function spaces, we are allowed to test with $v = u_1(t) - u_2(t)$, which yields with (2.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_1(t) - u_2(t)\|_{0,2}^2 \le 0.$$

Since $u_1(0) = u_2(0)$, the solutions coincide (note that $u_1, u_2 \in \mathcal{C}(0,T;L^2(\Omega)^d)$). $\blacksquare$

We close this section with a remark on a numerically more suited choice of the Galerkin scheme.

Remark 3.3. From a numerical point of view, the choice of eigenfunctions as a Galerkin basis is disadvantageous. Even though we proved their existence, eigenfunctions in general are not given explicitly and it is not known how to construct them. A more suited Galerkin scheme would be based on finite elements. This Galerkin scheme still has to fulfill the smoothness condition (i.e., $V_\ell \subset W^{\sigma+\eta(p-1),p}(\Omega)^d$, which is usually no problem) and the stability condition on the family of $L^2(\Omega)^d$-orthogonal projections. Under the additional assumption that $\Omega \subset \mathbb{R}^d$ is polygonal, in [5] it is shown that the $L^2(\Omega)$-projection onto conforming finite elements (polynomial on each simplex) is stable both with respect to the $L^p(\Omega)$-norm as well as to the $W^{1,p}(\Omega)$-norm (for the 1D and 2D case this result is also proved in [10]). Hence, by the fundamental property of interpolation theory (see, e.g., [27, Theorem B.2] or also [4]), the projection is also stable on $W^{\sigma,p}(\Omega) = [L^p(\Omega), W^{1,p}(\Omega)]_{\sigma,p}$ (cf. [33, Chapters 34 and 36]). Furthermore, for the Galerkin scheme based on these finite elements the smoothness property is directly fulfilled. The drawback, however, is that this result is not known to be true yet for general Lipschitz domains. Because of that, we chose to work with eigenfunctions. □

4 Second-Order Evolution Problem

In this section, we develop a setting close to that given in [15]. For this purpose, we consider the space

$$V := W^{\sigma,p}(\Omega)^d \cap W^{\gamma,q}(\Omega)^d.$$

Of course, if $p = q$ and $\sigma \leq \gamma$ or the other way around, then one space is contained in the other. However, we would like to treat the setting in full generality and do not consider inclusions. As explained in Section 1, the dual then is determined as the sum of the duals of the Sobolev–Slobodetskii spaces,

$$V^* = (W^{\sigma,p}(\Omega)^d)^* + (W^{\gamma,q}(\Omega)^d)^*,$$

with norms $\|\cdot\|_V = \|\cdot\|_{\sigma,p} + \|\cdot\|_{\gamma,q}$ and

$$\|g\|_{V^*} = \inf_{\substack{g = g_\sigma + g_\gamma \\ g_\sigma \in (W^{\sigma,p}(\Omega)^d)^* \\ g_\gamma \in (W^{\gamma,q}(\Omega)^d)^*}} \max\{\|g_\sigma\|_{(\sigma,p)^*}, \|g_\gamma\|_{(\gamma,q)^*}\}.$$

We have the scale of Banach spaces

$$V \overset{d}{\hookrightarrow} \begin{Bmatrix} W^{\sigma,p}(\Omega)^d \\ W^{\gamma,q}(\Omega)^d \end{Bmatrix} \overset{c,d}{\hookrightarrow} L^2(\Omega)^d \overset{c,d}{\hookrightarrow} \begin{Bmatrix} (W^{\sigma,p}(\Omega)^d)^* \\ (W^{\gamma,q}(\Omega)^d)^* \end{Bmatrix} \overset{d}{\hookrightarrow} V^*.$$

Moreover, we focus on the operators $K^{\sigma,p}: L^p(0,T;W^{\sigma,p}(\Omega)^d) \to L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)$ and $K^{\gamma,q}: L^q(0,T;W^{\gamma,q}(\Omega)^d) \to L^{q^*}(0,T;(W^{\gamma,q}(\Omega)^d)^*)$ given by (1.2) and (1.3), which share all the properties given in Section 2. Note that in particular there holds $K^{\gamma,q}: L^\infty(0,T;W^{\gamma,q}(\Omega)^d) \to L^\infty(0,T;(W^{\gamma,q}(\Omega)^d)^*)$.

Theorem 4.1. For initial values $u_0 \in W^{\gamma,q}(\Omega)^d$, $v_0 \in L^2(\Omega)^d$ and right-hand sides $f \in L^1(0,T;L^2(\Omega)^d)$, $g \in L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)$, there exists a function

$$
\begin{aligned}
u &\in \mathcal{C}_w([0,T];W^{\gamma,q}(\Omega)^d),\\
u' &\in L^p(0,T;W^{\sigma,p}(\Omega)^d) \cap \mathcal{C}_w([0,T];L^2(\Omega)^d),\\
u'' &\in L^1(0,T;L^2(\Omega)^d) + L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*) + L^\infty(0,T;(W^{\gamma,q}(\Omega)^d)^*)
\end{aligned}
$$

with $u(0) = u_0$ and $u'(0) = v_0$ that solves the equation

$$
u'' + K^{\sigma,p}u' + K^{\gamma,q}u = f + g \quad \text{in } L^1(0,T;L^2(\Omega)^d) + L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*).
$$

Moreover, the solution u satisfies for almost every $t \in (0,T)$ the energy estimate

$$
\begin{aligned}
&\|u'(t)\|_{0,2}^2 + \int_0^t |u'(s)|_{\sigma,p}^p \,\mathrm{d}s + \frac{1}{q}|u(t)|_{\gamma,q}^q\\
&\quad\le 2\int_0^t (f(s),u'(s))\,\mathrm{d}s + 2\int_0^t \langle g(s),u'(s)\rangle\,\mathrm{d}s + \|v_0\|_{0,2}^2 + \frac{1}{q}|u_0|_{\gamma,q}^q.
\end{aligned}
$$

□

Proof. Again, we prove this theorem via a Galerkin discretization.

Existence of a discrete solution. Let $\{\psi_\ell\}$ be given as in the proof of Theorem 3.1. Then $\{V_\ell\} \subset V$ with $V_\ell = \operatorname{span}\{\psi_1,\dots,\psi_\ell\}$ is also a Galerkin scheme for V, that is,

$$
\overline{\bigcup_{\ell\in\mathbb{N}} V_\ell} = V.
$$

In particular, we now have for η and ω chosen appropriately later in the proof that

$$
V_\ell \subset W^{r,2}(\Omega)^d \overset{d}{\hookrightarrow} W^{1,p}(\Omega)^d \overset{c,d}{\hookrightarrow} W^{\sigma+\eta(p-1),p}(\Omega)^d \cap W^{\gamma+\omega(q-1),q}(\Omega)^d \overset{c,d}{\hookrightarrow} V
$$

and

$$
V \overset{c,d}{\hookrightarrow} W^{\sigma-\eta,p}(\Omega)^d \cap W^{\gamma-\omega,q}(\Omega)^d \overset{c,d}{\hookrightarrow} L^2(\Omega)^d
$$

as well as

$$
L^2(\Omega)^d \overset{c,d}{\hookrightarrow} V^* \overset{c,d}{\hookrightarrow} (W^{r,2}(\Omega)^d)^*
$$

and the family of $L^2(\Omega)^d$-orthogonal projections $P_\ell: L^2(\Omega)^d \to V_\ell$ is stable with respect to the $W^{r,2}(\Omega)^d$-norm.

The discretized problem then consists of finding a function $u_\ell : [0,T] \to V_\ell$ such that for all $v_\ell \in V_\ell$ there holds

$$(u_\ell''(t), v_\ell) + k^{\sigma,p}(u_\ell'(t), v_\ell) + k^{\gamma,q}(u_\ell(t), v_\ell) = (f(t), v_\ell) + \langle g(t), v_\ell\rangle, \quad t \in (0,T), \tag{4.1}$$

with $u_\ell(0) = u_\ell^0 \in V_\ell$ and $u_\ell'(0) = v_\ell^0 \in V_\ell$, for which we assume that $u_\ell^0 \to u_0$ in $W^{\gamma,q}(\Omega)^d$ and $v_\ell^0 \to v_0$ in $L^2(\Omega)^d$. By standard arguments together with the subsequent a priori estimate, there exists a solution $u_\ell \in W^{2,1}(0,T;V_\ell)$ to the discretized problem.

A priori estimate and convergent subsequences. We test (4.1) with $v_\ell = u_\ell'(t)$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\ell'(t)\|_{0,2}^2 + k^{\sigma,p}(u_\ell'(t), u_\ell'(t)) + k^{\gamma,q}(u_\ell(t), u_\ell'(t)) = (f(t), u_\ell'(t)) + \langle g(t), u_\ell'(t)\rangle.$$

Applying Proposition 2.1 there holds $k^{\gamma,q}(u_\ell(t), u_\ell'(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi^{\gamma,q}(u_\ell(t))$ as well as $k^{\sigma,p}(u_\ell'(t), u_\ell'(t)) = \frac{1}{2}|u_\ell'(t)|_{\sigma,p}^p$. It follows

$$\begin{aligned}&\|u_\ell'(t)\|_{0,2}^2 + \int_0^t |u_\ell'(s)|_{\sigma,p}^p \,\mathrm{d}s + \frac{1}{q}|u_\ell(t)|_{\gamma,q}^q \\ &\quad = 2\int_0^t (f(s), u_\ell'(s))\,\mathrm{d}s + 2\int_0^t \langle g(s), u_\ell'(s)\rangle\,\mathrm{d}s + \|v_\ell^0\|_{0,2}^2 + \frac{1}{q}|u_\ell^0|_{\gamma,q}^q,\end{aligned} \tag{4.2}$$

and hence

$$\begin{aligned}&\|u_\ell'(t)\|_{0,2}^2 + \int_0^t |u_\ell'(s)|_{\sigma,p}^p \,\mathrm{d}s + \frac{1}{q}|u_\ell(t)|_{\gamma,q}^q \\ &\quad \le 2\int_0^t \|f(s)\|_{0,2}\|u_\ell'(s)\|_{0,2}\,\mathrm{d}s + 2\int_0^t \|g(s)\|_{(\sigma,p)^*}\|u_\ell'(s)\|_{\sigma,p}\,\mathrm{d}s + \|v_\ell^0\|_{0,2}^2 + \frac{1}{q}|u_\ell^0|_{\gamma,q}^q.\end{aligned}$$

By Lemma 2.3 and Hölder's inequality, there exists $c > 0$ such that

$$\|u_\ell'(s)\|_{\sigma,p} \le c\,(\|u_\ell'(s)\|_{0,2} + |u_\ell'(s)|_{\sigma,p}).$$

Therefore, we find with Young's inequality and $c_1, c_2 > 0$

$$\|g(s)\|_{(\sigma,p)^*}\|u_\ell'(s)\|_{\sigma,p} \le c_1\|g(s)\|_{(\sigma,p)^*}\|u_\ell'(s)\|_{0,2} + c_2\|g(s)\|_{(\sigma,p)^*}^{p^*} + \frac{1}{2p}|u_\ell'(s)|_{\sigma,p}^p,$$

which then yields

$$\begin{aligned}\|u_\ell'(t)\|_{0,2}^2 + \frac{1}{p^*}\int_0^t |u_\ell'(s)|_{\sigma,p}^p\,\mathrm{d}s + \frac{1}{q}|u_\ell(t)|_{\gamma,q}^q &\le 2\int_0^t (\|f(s)\|_{0,2} + c_1\|g(s)\|_{(\sigma,p)^*})\|u_\ell'(s)\|_{0,2}\,\mathrm{d}s \\ &\quad + 2c_2\int_0^t \|g(s)\|_{(\sigma,p)^*}^{p^*}\,\mathrm{d}s + \|v_\ell^0\|_{0,2}^2 + \frac{1}{q}|u_\ell^0|_{\gamma,q}^q.\end{aligned} \tag{4.3}$$

Since $u_\ell \in W^{2,1}(0,T;V_\ell)$ all terms in the latter inequality are continuous and hence, it holds for every $t \in [0,T]$. In particular, there exists $\bar{t} \in [0,T]$ such that $\|u_\ell'(\bar{t})\|_{0,2} = \max_{t\in[0,T]} \|u_\ell'(t)\|_{0,2}$. Similarly to the first-order setting, we obtain the quadratic inequality

$$\begin{aligned}\|u_\ell'(\bar{t})\|_{0,2}^2 \le 2(\|f\|_{L^1(0,T;L^2(\Omega)^d)} + c_1\|g\|_{L^1(0,T;(W^{\sigma,p}(\Omega)^d)^*)})\|u_\ell'(\bar{t})\|_{0,2}\\ + 2c_2\|g\|^{p^*}_{L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|v_\ell^0\|_{0,2}^2 + \frac{1}{q}|u_\ell^0|^q_{\gamma,q},\end{aligned}$$

which immediately yields

$$\|u_\ell'(\bar{t})\|_{0,2} \le c\big(\|f\|_{L^1(0,T;L^2(\Omega)^d)} + \|g\|_{L^1(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|g\|^{p^*/2}_{L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|v_\ell^0\|_{0,2} + |u_\ell^0|^{q/2}_{\gamma,q}\big).$$

Combining this estimate with (4.3) yields for all $t \in [0,T]$

$$\begin{aligned}&\|u_\ell'(t)\|_{0,2}^2 + \frac{1}{p^*}\int_0^t |u_\ell'(s)|^p_{\sigma,p}\,\mathrm{d}s + \frac{1}{q}|u_\ell(t)|^q_{\gamma,q}\\ &\quad\le c\big(\|f\|^2_{L^1(0,T;L^2(\Omega)^d)} + \|g\|^2_{L^1(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|g\|^{p^*}_{L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*)} + \|v_\ell^0\|_{0,2}^2 + |u_\ell^0|^q_{\gamma,q}\big). \quad (4.4)\end{aligned}$$

The right-hand side of this inequality is bounded, because v_ℓ^0 converges to v_0 in $L^2(\Omega)^d$ and u_ℓ^0 converges to u_0 in $W^{\gamma,q}(\Omega)^d$. Note that due to the boundedness of $u_\ell'(t)$ with respect to $|\cdot|_{\sigma,p}$ as well as $\|\cdot\|_{0,2}$, we obtain boundedness with respect to the full norm $\|\cdot\|_{\sigma,p}$ (see Lemma 2.3). Moreover, (4.4) immediately implies boundedness of $u_\ell(t)$ in $L^2(\Omega)^d$ since

$$\|u_\ell\|_{L^\infty(0,T;L^2(\Omega)^d)} \le \|u_\ell^0\|_{0,2} + T\|u_\ell'\|_{L^\infty(0,T;L^2(\Omega)^d)},$$

and the sequence of initial values is bounded. Therefore, the bound of the seminorm $|u_\ell(t)|_{\gamma,q}$ implies boundedness of the full norm $\|u_\ell(t)\|_{\gamma,q}$. In view of all these bounds, there exist elements u, w_0, w_σ and a subsequence, which will not be relabeled, such that

$$u_\ell \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;W^{\gamma,q}(\Omega)^d), \quad (4.5)$$

$$u_\ell' \overset{*}{\rightharpoonup} w_0 \quad \text{in } L^\infty(0,T;L^2(\Omega)^d), \quad (4.6)$$

$$u_\ell' \rightharpoonup w_\sigma \quad \text{in } L^p(0,T;W^{\sigma,p}(\Omega)^d). \quad (4.7)$$

It is clear that $u' = w_0 = w_\sigma$. In order to pass to the limit in the next step, strong convergence is needed. Similarly, to the first-order setting, the strong convergence we use takes place in a larger space. Now fix $0 < \omega < \min\left\{\frac{1-\gamma}{q-1}, \gamma\right\}$. Then by the Lions–Aubin lemma [29, Lemma 7.7], there holds

$$L^\infty(0,T;W^{\gamma,q}(\Omega)^d) \cap W^{1,\infty}(0,T;L^2(\Omega)^d) \overset{c}{\hookrightarrow} L^q(0,T;W^{\gamma-\omega,q}(\Omega)^d).$$

Thus, by the a priori estimates we gain

$$u_\ell \to u \quad \text{in } L^q(0,T;W^{\gamma-\omega,q}(\Omega)^d),$$

and therefore

$$u_\ell(t) \to u(t) \quad \text{in } W^{\gamma-\omega,q}(\Omega)^d, \quad \text{a.e. on } (0,T). \tag{4.8}$$

Moreover, due to the $W^{r,2}(\Omega)^d$-stability of the orthogonal projection $P_\ell : L^2(\Omega)^d \to V_\ell$ together with its property $(u_\ell''(t), v - P_\ell v) = 0$ as well as with the embedding $W^{r,2}(\Omega)^d \hookrightarrow V$ we find

$$\begin{aligned}
\|u_\ell''(t)\|_{(r,2)^*} &= \sup_{v\in W^{r,2}(\Omega)^d\setminus\{0\}} \frac{|(u_\ell''(t), v)|}{\|v\|_{r,2}} \\
&= \sup_{v\in W^{r,2}(\Omega)^d\setminus\{0\}} \frac{|(u_\ell''(t), P_\ell v)|}{\|v\|_{r,2}} \\
&\le \sup_{v\in W^{r,2}(\Omega)^d\setminus\{0\}} (\|f(t)\|_{V^*} + \|g(t)\|_{V^*} + \|k^{\sigma,p}(u_\ell'(t),\cdot)\|_{V^*} + \|k^{\gamma,q}(u_\ell(t),\cdot)\|_{V^*}) \frac{\|P_\ell v\|_V}{\|v\|_{r,2}} \\
&\le c\left(\|f(t)\|_{0,2} + \|g(t)\|_{(\sigma,p)^*} + \frac{1}{2}|u_\ell'(t)|_{\sigma,p}^{p-1} + \frac{1}{2}|u_\ell(t)|_{\gamma,q}^{q-1}\right).
\end{aligned}$$

Thus, we obtain boundedness of $\{u_\ell''\}$ in $L^1(0,T;(W^{r,2}(\Omega)^d)^*)$. Again, by the Lions–Aubin [29, Lemma 7.7] there holds with $0 < \eta < \min\left\{\frac{1-\sigma}{p-1}, \sigma\right\}$ fixed

$$L^p(0,T;W^{\sigma,p}(\Omega)^d) \cap W^{1,1}(0,T;(W^{r,2}(\Omega)^d)^*) \overset{c}{\hookrightarrow} L^p(0,T;W^{\sigma-\eta,p}(\Omega)^d).$$

Hence, there exists a subsequence (not relabeled) such that

$$u_\ell' \to u' \quad \text{in } L^p(0,T;W^{\sigma-\eta,p}(\Omega)^d),$$

and therefore, we obtain pointwise almost everywhere convergence and the existence of $h \in L^1(0,T)$ such that

$$u_\ell'(t) \to u'(t) \quad \text{in } W^{\sigma-\eta,p}(\Omega)^d, \quad |u_\ell'(t)|_{\sigma-\eta,p}^p \le h(t), \quad \text{a.e. on } (0,T). \tag{4.9}$$

Passage to the limit. We like to proceed to the limit in the following equation:

$$\begin{aligned}
-\int_0^T (u_\ell'(t), v_m)\phi'(t)\,\mathrm{d}t + \int_0^T k^{\sigma,p}(u_\ell'(t), v_m)\phi(t)\,\mathrm{d}t + \int_0^T k^{\gamma,q}(u_\ell(t), v_m)\phi(t)\,\mathrm{d}t \\
= \int_0^T (f(t), v_m)\phi(t)\,\mathrm{d}t + \int_0^T \langle g(t), v_m\rangle\phi(t)\,\mathrm{d}t
\end{aligned}$$

for all $\phi \in C_c^\infty(0,T)$ and $v_m \in V_m$ with $m \in \mathbb{N}$, $m \le \ell$. The first term converges due to (4.6) to $-\int_0^T (u'(t), v_m)\phi'(t)\,\mathrm{d}t$. To get convergence of the second term, we use (4.9), $V_m \subset$

$W^{\sigma+\eta(p-1),p}(\Omega)^d$, and Lemma 2.2, which yield $k^{\sigma,p}(u'_\ell(t), v_m) \to k^{\sigma,p}(u'(t), v_m)$ for $\ell \to \infty$. With Lebesgue's theorem on dominated convergence in combination with (2.4) from Lemma 2.2 and $h \in L^1(0,T)$ from (4.9), we obtain

$$\int_0^T k^{\sigma,p}(u'_\ell(t), v_m)\phi(t)\,\mathrm{d}t \to \int_0^T k^{\sigma,p}(u'(t), v_m)\phi(t)\,\mathrm{d}t.$$

For the third term on the left-hand side, we copy this strategy. With (4.8), $V_m \subset W^{\gamma+\omega(q-1),q}(\Omega)^d$, Lemma 2.2, and Lebesgue's theorem on dominated convergence in combination with the boundedness of $\{u_\ell\}$ in $L^\infty(0,T; W^{\gamma,q}(\Omega)^d)$ we have for $\ell \to \infty$ that

$$\int_0^T k^{\gamma,q}(u_\ell(t), v_m)\phi(t)\,\mathrm{d}t \to \int_0^T k^{\gamma,q}(u(t), v_m)\phi(t)\,\mathrm{d}t.$$

In view of the limited completeness of the Galerkin scheme u solves

$$\begin{aligned}
&-\int_0^T (u'(t), v)\phi'(t)\,\mathrm{d}t + \int_0^T k^{\sigma,p}(u'(t), v)\phi(t)\,\mathrm{d}t + \int_0^T k^{\gamma,q}(u(t), v)\phi(t)\,\mathrm{d}t \\
&\quad = \int_0^T (f(t), v)\phi(t)\,\mathrm{d}t + \int_0^T \langle g(t), v\rangle \phi(t)\,\mathrm{d}t
\end{aligned}$$

for all $\phi \in \mathcal{C}_c^\infty(0,T)$ and $v \in V$. This shows that u' possesses a weak time derivative. Due to the weak* density of $\mathcal{C}_c^\infty(0,T) \otimes V$ in $L^\infty(0,T;V)$, we arrive at

$$u'' + K^{\sigma,p}u' + K^{\gamma,q}u = f + g \quad \text{in } L^1(0,T;V^*).$$

It remains to show that the initial values are taken. First, we observe that

$$u_\ell \rightharpoonup u \quad \text{in } W^{1,2}(0,T; L^2(\Omega)^d) \hookrightarrow \mathcal{C}([0,T]; L^2(\Omega)^d).$$

Because the trace operator $\Gamma : W^{1,2}(0,T; L^2(\Omega)^d) \to L^2(\Omega)^d$, $\Gamma v = v(0)$, is linear and bounded, it is weakly–weakly continuous. Therefore, it follows $u_\ell^0 = u_\ell(0) \rightharpoonup u(0)$. Since $u_\ell^0 \to u_0$ in $W^{\gamma,q}(\Omega)^d$, we obtain $u(0) = u_0$. Furthermore, since $u'_\ell \in W^{1,1}(0,T;V_\ell) \subset \mathcal{AC}([0,T];V_\ell)$ there holds on the one hand for all $v_m \in V_m$

$$\begin{aligned}
-(v_\ell^0, v_m) &= \int_0^T \left[(u'_\ell(t), v_m)\frac{T-t}{T} \right]' \mathrm{d}t \\
&= \int_0^T \left((f(t), v_m) + \langle g(t), v_m\rangle - k^{\sigma,p}(u'_\ell(t), v_m) - k^{\gamma,q}(u_\ell(t), v_m) \right) \frac{T-t}{T}\,\mathrm{d}t \\
&\quad - \frac{1}{T}\int_0^T (u'_\ell(t), v_m)\,\mathrm{d}t
\end{aligned}$$

and on the other hand with $u' \in W^{1,1}(0,T;V^*) \subset \mathcal{AC}([0,T];V^*)$

$$
\begin{aligned}
-(u'(0), v_m) &= \int_0^T \left[(u'(t), v_m) \frac{T-t}{T} \right]' \mathrm{d}t \\
&= \int_0^T ((f(t), v_m) + \langle g(t), v_m \rangle - k^{\sigma,p}(u'(t), v_m) - k^{\gamma,q}(u(t), v_m)) \frac{T-t}{T} \, \mathrm{d}t \\
&\quad - \frac{1}{T} \int_0^T (u'(t), v_m) \, \mathrm{d}t.
\end{aligned}
$$

Hence, with $\ell \to \infty$ we obtain $(v_0, v_m) = (u'(0), v_m)$ for all $v_m \in V_m$, $m \in \mathbb{N}$ and by the limited completeness of the Galerkin scheme $u'(0) = v_0$.

Regularity. By the structure of the equation, there holds in particular

$$u'' \in L^1(0,T;L^2(\Omega)^d) + L^{p^*}(0,T;(W^{\sigma,p}(\Omega)^d)^*) + L^\infty(0,T;(W^{\gamma,q}(\Omega)^d)^*).$$

Furthermore, since $u' \in L^\infty(0,T;L^2(\Omega)^d)$ there holds

$$u \in L^\infty(0,T;W^{\gamma,q}(\Omega)^d) \cap \mathcal{AC}([0,T];L^2(\Omega)^d)$$

and therefore, we obtain with [26, Chapitre 3, Lemme 8.1] that $u \in \mathcal{C}_w([0,T];W^{\gamma,q}(\Omega)^d)$. Moreover, since $u'' \in L^1(0,T;V^*)$ there holds

$$u' \in L^\infty(0,T;L^2(\Omega)^d) \cap \mathcal{AC}([0,T];V^*)$$

and thus, again by [26, Chapitre 3, Lemme 8.1], it follows $u' \in \mathcal{C}_w([0,T];L^2(\Omega)^d)$.

Energy estimate. Note that we are not allowed to test the equation with the derivative of the solution, because $u'(t)$ is not known to be (almost everywhere) an element of $W^{\gamma,q}(\Omega)^d$ and hence of V. Therefore, we start with the a priori estimate (4.2) and obtain due to the weak and weak* sequential lower semicontinuity of the norms and seminorms and the convergence of u and the initial values

$$
\begin{aligned}
&\|u'(t)\|_{0,2}^2 + \int_0^t |u'(s)|_{\sigma,p}^p \, \mathrm{d}s + \frac{1}{q}|u(t)|_{\gamma,q}^q \\
&\leq 2\int_0^t (f(s), u'(s)) \, \mathrm{d}s + 2\int_0^t \langle g(s), u'(s) \rangle \, \mathrm{d}s + \|v_0\|_{0,2}^2 + \frac{1}{q}|u_0|_{\gamma,q}^q. \qquad \blacksquare
\end{aligned}
$$

All the results in this work are also applicable to nonlocal operators that behave like the fractional p-Laplacian but are not monotone, that is, they share the same growth estimates and the same bounds from below. Such kind of operators are studied for instance in [14].

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