

Stability and Convergence of the Two-step BDF for the Incompressible Navier-Stokes Problem

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Abstract

The incompressible Navier-Stokes problem is discretised in time by means of the two-step backward differentiation formula with constant step sizes. Existence and stability of a time discrete solution are proved as well as the convergence of a piecewise polynomial prolongation towards a weak solution. The results presented cover both the two- and three-dimensional case. Furthermore, a linearisation that is based upon a modification of the convective term using a second-order extrapolation is considered.

Keywords: Incompressible Navier-Stokes equation, time discretisation, backward differentiation formula, stability, convergence

1 Introduction

Whereas the spatial approximation of the incompressible Navier-Stokes problem seems to be rather well-understood, only a comparably small number of articles focus on a strict mathematical analysis of time discretisation methods. Methods with constant time steps have been considered e.g. in Temam [14], Girault & Raviart [4], Baker et al. [1], Heywood & Rannacher [7], Müller-Urbaniak [11], Prohl [12], and Hill & Süli [8]. For an overview and the state-of-the-art, we refer to Rannacher [13] and Marion & Temam [10].

Especially, the question of “realistic” higher-order error estimates has become topical since Heywood & Rannacher [7] have proven optimal second-order estimates for the Crank-Nicolson scheme. Higher regularity of the exact solution is equivalent to compatibility conditions on the problem’s data, which lead –due to the divergence-free constraint– to a virtually uncheckable and often violated over-determined Neumann problem for the initial pressure (cf. Heywood [6], Temam [15]). So higher-order estimates should rely upon parabolic smoothing properties. The method

under consideration, therefore, needs to be A- or G-stable.

Although efficient time integration requires adaptive methods, there is, to the best knowledge of the author, no analysis of time discretisations of the Navier-Stokes equations on non-uniform grids available. Only in Prohl [12], discretisations on structured time grids have been considered in order to overcome the incompatibility of fluid flows.

The backward differentiation formulae (BDF), even with variable time steps, have been used by many authors for the integration of (nonlinear) ordinary and partial differential equations. The two-step BDF with constant time steps is known to be formally of second order and zero- as well as A- and G-stable (cf. Hairer & Wanner [5]).

In the context of the incompressible Navier-Stokes problem, the two-step BDF has been firstly studied by Girault & Raviart [4]. They have considered a linearised variant and replaced the convective term $(u^n \cdot \nabla) u^n$, where u^n denotes the approximate velocity at time t_n , by $((2u^{n-1} - u^{n-2}) \cdot \nabla) u^n$. Unfortunately, the optimal error estimate given there relies upon inappropriate higher regularity assumptions. In Baker et al. [1], the three-step BDF has been

analysed, and a second-order error estimate has been postulated for the linearised variant of the two-step BDF under higher regularity assumptions as well as restrictions on the time step size in dependence of the mesh size of an underlying spatial discretisation. Recently, Hill & Süli [8] have proven sub-optimal error estimates of order 1/4 under feasible regularity assumptions. Their result applies to the two-dimensional case with autonomous right-hand side.

Yet, the original nonlinear approximation has not been considered in the literature so far. Moreover, stability of the discrete problem and convergence of a time continuous approximate solution obtained by piecewise polynomial prolongation has not been dealt with. Convergence of time continuous approximate solutions is, beside error estimates, of interest in its own since it does not follow directly from and might be proven under weaker assumptions than error estimates. Furthermore, it answers the question of how to compute approximate values between the discrete time points.

Smoothing error estimates under suitable regularity assumptions that avoid global compatibility conditions are provided in Emmrich [3] for the two- and three-dimensional case: The time-weighted velocity error, measured in the $L^\infty(L^2)$ - and $L^2(H_0^1)$ -norm, is shown to be of second order for the nonlinear approximation and of order 3/2 for the linearised variant. After reintroducing the pressure, its time-weighted error is shown to be of order 1 and 1/2 in the $L^\infty(L^2/\mathbb{R})$ -norm, respectively, for the nonlinear and linearised approximation, respectively.

In this paper, we consider the two-step BDF with constant time steps for the two- and three-dimensional Navier-Stokes problem in its pressure-free variational formulation. Beside the original nonlinear approximation, we also consider the above-mentioned linearised variant. We firstly study solvability. Afterwards, we prove $L^\infty(L^2)$ - and $L^2(H_0^1)$ -stability for the discrete solution as well as estimates for the discrete time derivative. We then construct piecewise polynomial time continuous solutions from the discrete values. Finally, we show under suitable assumptions on the problem's data that

these approximate solutions converge towards a weak solution whenever the time steps tend to zero. The convergence is strong in $L^q(L^2)$ for $q \in [1, \infty)$, weak in $L^2(H_0^1)$, and weak* in $L^\infty(L^2)$.

2 Continuous and time discrete problem

We consider the Navier-Stokes equations describing the non-stationary flow of an incompressible, homogeneous, viscous fluid at constant temperature,

$$\begin{aligned} u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f, \\ \nabla \cdot u &= 0 \text{ in } \Omega \times (0, T), \\ u &= 0 \text{ on } \partial\Omega \times (0, T), \quad u(\cdot, 0) = u_0 \text{ in } \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$, $d = \dim \Omega \in \{2, 3\}$, is a bounded domain with locally Lipschitz continuous boundary $\partial\Omega$, $(0, T)$ is the time interval under consideration, $\nu > 0$ denotes the inverse of the Reynolds number, $u = u(x, t)$ is the d -dimensional velocity vector with prescribed initial velocity $u_0 = u_0(x)$, $p = p(x, t)$ is the pressure, and $f = f(x, t)$ is an outer force per unit mass.

We introduce the solenoidal function spaces

$$\begin{aligned} V &:= \{v \in H_0^1(\Omega)^d : \nabla \cdot v = 0\}, \\ H &:= \{v \in L^2(\Omega)^d : \nabla \cdot v = 0, \gamma_n v = 0\}, \end{aligned}$$

where γ_n denotes the trace operator in normal direction (cf. Temam [14] for more details). As usual, the subspace of $H^1(\Omega)$ -functions vanishing at the boundary is denoted by $H_0^1(\Omega)$. Here, by L^p and $W^{m,p}$ ($W^{m,2} \equiv H^m$), we denote the usual Lebesgue and Sobolev spaces with the natural norms $\|\cdot\|_{0,p}$ and $\|\cdot\|_{m,p}$, respectively. We will not distinguish between the scalar and vector case. With

$$\begin{aligned}
 ((u, v)) &:= \sum_{i,j=1}^d \int_{\Omega} \frac{\partial u_i(x)}{\partial x_j} \frac{\partial v_j(x)}{\partial x_i} dx, \\
 \|u\| &:= ((u, u))^{1/2}, \quad u, v \in V, \\
 (u, v) &:= \sum_{i=1}^d \int_{\Omega} u_i(x) v_i(x) dx, \\
 |u| &:= (u, u)^{1/2}, \quad u, v \in H,
 \end{aligned}$$

the spaces V and H are Hilbert spaces. The space V is dense and compactly embedded in H . Note that V , H , and the dual V^* form a Gelfand triple. The dual pairing between V and V^* is denoted by $\langle \cdot, \cdot \rangle$, the dual norm by $\|\cdot\|_*$, which is different from the $H^{-1}(\Omega)^d$ -norm.

We then consider the weak formulation of the Navier-Stokes problem:

Problem (P) For given $u_0 \in H$ and $f \in L^2(0, T; V^*)$, find $u \in L^2(0, T; V)$ such that for all $v \in V$

$$\begin{aligned}
 \frac{d}{dt}(u(t), v) + \nu((u(t), v)) \\
 + b(u(t), u(t), v) = \langle f(t), v \rangle
 \end{aligned} \tag{1}$$

holds in $(0, T)$ in the distributional sense with $u(0) = u_0$.

Here,

$$b(u, v, w) := ((u \cdot \nabla)v, w)$$

incorporates the nonlinearity. By $L^p(0, T; X)$ with some Banach space X , we denote the usual space of Bochner integrable abstract functions $u: [0, T] \rightarrow X$. The discrete counterpart for functions defined on a time grid is denoted by $l^p(0, T; X)$. The space of m -times continuously differentiable abstract functions will be denoted by $C^m([0, T]; X)$, where X is omitted if $X = \mathbb{R}$. The natural norm of a function space Y will be sometimes denoted by $\|\cdot\|_Y$.

It is known that Problem (P) has at least one solution $u \in L^2(0, T; V) \cap L^{\infty}(0, T; H)$ with $u' \in L^{4/3}(0, T; V^*)$, where u' denotes the time

derivative of the abstract function u in the distributional sense. As then u is almost everywhere equal to a continuous function with values in V^* , $u \in C([0, T]; V^*)$, the initial condition makes sense. In the two-dimensional case, the solution is unique and in $C([0, T]; H)$ with $u' \in L^2(0, T; V^*)$ (cf. Temam [14]). For more regular data ($u_0 \in V$, $f \in L^{\infty}(0, T; H)$, $\partial\Omega \in C^2$), a so-called strong solution, i.e. a unique solution $u \in C([0, T]; V)$, exists in the two-dimensional case for arbitrary T , but in the three-dimensional case only locally up to a (possibly rather small) time T (cf. Temam [16]).

The trilinear form $b(\cdot, \cdot, \cdot)$ satisfies the following well-known properties that will be needed in the sequel (cf. Temam [16]).

Lemma 1. Let $u, v, w \in V$ be arbitrary. Then $b(u, v, w) = -b(u, w, v)$ and for some $\beta > 0$

$$|b(u, v, w)| \leq \beta \begin{cases} \|u\|_{0,4} \|v\|_{0,4} \|w\|, \\ \|u\|^{1/2} \|u\|^{1/2} \|v\| \|w\|, \\ \|u\| \|v\|^{1/2} \|v\|^{1/2} \|w\|, \\ \|u\| \|v\| \|w\|. \end{cases}$$

Moreover, in the two-dimensional case, there holds for some $\tilde{\beta} > 0$

$$|b(u, v, w)| \leq \tilde{\beta} |u|^{1/2} \|u\|^{1/2} \|v\| \|w\|^{1/2} \|w\|^{1/2}.$$

We now come to the time discrete problem. Let the time interval $[0, T]$ for given $N \in \mathbb{N}$ be equidistantly partitioned with the time step Δt and $t_n := n\Delta t$ ($n = 0, \dots, N$). For a grid function $\{u^n\}_{n \in \mathbb{N}}$, we denote by D_1 and D_2 the backward divided differences:

$$\begin{aligned}
 D_1 u^n &:= \frac{u^n - u^{n-1}}{\Delta t}, \\
 D_2 u^n &:= \frac{1}{\Delta t} \left(\frac{3}{2} u^n - 2u^{n-1} + \frac{1}{2} u^{n-2} \right) \\
 &= \frac{3}{2} D_1 u^n - \frac{1}{2} D_1 u^{n-1}.
 \end{aligned}$$

For a Bochner integrable function f , we also consider the natural restrictions

$$R_1^n f := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} f(t) dt,$$

$$R_2^n f := \frac{3}{2} R_1^n f - \frac{1}{2} R_1^{n-1} f.$$

Furthermore, we use the formally second-order extrapolation

$$Eu^n := 2u^{n-1} - u^{n-2}.$$

Note that $R_q^n u' = D_q u(t_n) = u'(t_n) + O((\Delta t)^q)$ ($q \in \{1, 2\}$) and $Eu(t_n) = u(t_n) + O((\Delta t)^2)$ for a smooth function $u = u(t)$.

The time discretisation of Problem (P) by the two-step BDF for computing u^n approximating $u(t_n)$ reads as

Problem (P_{Δt}) For given $u^0, u^1 \in H$ and $f \in L^2(0, T; V^*)$, find $u^n \in V$ ($n = 2, 3, \dots, N$) such that for all $v \in V$

$$(D_2 u^n, v) + \nu((u^n, v)) + b(u^n, u^n, v) = \langle R_2^n f, v \rangle. \tag{2}$$

We also consider the linearised variant:

Problem (LP_{Δt}) For given $u^0, u^1 \in V$ and $f \in L^2(0, T; V^*)$, find $u^n \in V$ ($n = 2, 3, \dots, N$) such that for all $v \in V$

$$(D_2 u^n, v) + \nu((u^n, v)) + b(Eu^n, u^n, v) = \langle R_2^n f, v \rangle. \tag{3}$$

In opposite to the original method, the convective term $b(u^n, u^n, v)$ has been replaced by the formally second-order modification $b(Eu^n, u^n, v)$.

In both problems, the starting values can be obtained by taking $u^0 := u_0$ and computing u^1 from u^0 using the implicit Euler method. The use

of $R_2^n f$ instead of an arbitrary approximation f^n is only for simplicity and avoids to consider an extra error $f^n - R_2^n f$. By standard arguments, it can be shown that

$$\Delta t \sum_{j=2}^n \|R_2^j f\|_*^2 \leq 4 \int_0^{t_n} \|f(t)\|_*^2 dt. \tag{4}$$

Theorem 1. *There is at least one solution to Problem (P_{Δt}) and there is a unique solution to Problem (LP_{Δt}).*

Proof. We start with Problem (P_{Δt}) and wish to apply Brézis' main theorem on pseudomonotone operators (cf. Zeidler [17]). If u^{n-2}, u^{n-1} are known, the problem of determining $u^n \in V$ can be written in operator form,

$$A_{\Delta t}(u^n) = g^n := R_2^n f + \frac{1}{\Delta t} \left(2u^{n-1} - \frac{1}{2}u^{n-2} \right) \in V^*,$$

where

$$A_{\Delta t}(v) := A_{\Delta t}^{(1)}(v) + B(v), \quad A_{\Delta t}^{(1)}(v) := \frac{3}{2\Delta t} v + \nu Av.$$

Here, $A: V \rightarrow V^*$, defined by $\langle Au, v \rangle := ((u, v))$, is the energetic extension of the classical Stokes operator. Furthermore, $B: V \rightarrow V^*$, defined by $\langle B(v), w \rangle := b(v, v, w)$, is the nonlinearity. Obviously, $A_{\Delta t}^{(1)}: V \rightarrow V^*$ is linear, bounded, and strongly monotone since V is continuously embedded in H .

It follows from Hölder's inequality and embedding arguments that $B: V \rightarrow V^*$ is bounded. Furthermore, B is strongly continuous: Let $\{v_k\} \subset V$ be weakly convergent with limit v . Since V is compactly embedded in $L^4(\Omega)^d$, $\{v_k\}$ is strongly convergent and bounded in $L^4(\Omega)^d$. We have for arbitrary $w \in V$ with Hölder's inequality (see Lemma 1)

$$\begin{aligned} & \langle B(v_k) - B(v), w \rangle \\ &= b(v_k - v, v, w) + b(v_k, v_k - v, w) \\ &\leq c \|v_k - v\|_{0,4} \left(\|v\|_{0,4} + \|v_k\|_{0,4} \right) \|w\|, \end{aligned}$$

and thus $\{B(v_k)\}$ converges strongly to $B(v)$ in V^* .

Hence, $A_{\Delta t}$ is pseudomonotone since the sum of a linear monotone and a strongly continuous operator is pseudomonotone. Because of $\langle B(v), v \rangle = b(v, v, v) = 0$, it also follows that $A_{\Delta t}$ is coercive. Finally, the boundedness of $A_{\Delta t}$ is clear. Since V is a separable, reflexive Banach space, Brézis' theorem ensures the existence of a solution to Problem $(P_{\Delta t})$.

For Problem $(LP_{\Delta t})$, the assertion follows from the Lax-Milgram lemma. #

Remark. A solution to Problem $(P_{\Delta t})$ is unique for small data:

$$|u^0|^2 + |u^1|^2 + \frac{1}{\nu} \|f\|_{L^2(0,T;V^*)}^2 \leq C(\nu^5 \Delta t)^{1/2}. \quad (5)$$

Uniqueness can be also obtained for more regular solutions. Furthermore, in the two-dimensional case, uniqueness can be ensured by a small data condition more refined than (5) that is independent on Δt and relies upon higher regularity of the discrete solution. #

Here and in the following, C denotes a generic constant that may depend on the domain Ω and its dimension, on embedding constants, the constant β or $\tilde{\beta}$ from Lemma 1, etc., but not on the Reynolds number, the exact solution, or the initial data or right-hand side. Moreover, let c be a generic constant that does not depend on problem parameters at all.

3 Stability

In the following, let

$$M := \left(|u^0|^2 + |u^1|^2 + \frac{1}{\nu} \int_0^T \|f(t)\|_*^2 dt \right)^{1/2}.$$

We may also use the abbreviation D^2 for the

second divided difference:

$$D^2 u^n := \frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2}.$$

Note that $D^2 u(t_n) = u''(t_n) + O((\Delta t)^2)$ for smooth functions u .

Theorem 2. Any solution to Problem $(P_{\Delta t})$ or $(LP_{\Delta t})$ is stable in $L^\infty(0, T; H)$ and $L^2(0, T; V)$ with

$$|u^n|^2 + (\Delta t)^4 \sum_{j=1}^{n-1} |D^2 u^j|^2 + \nu \Delta t \sum_{j=2}^n \|u^j\|^2 \leq c M^2, \quad n = 2, 3, \dots, N.$$

Proof. Set $v = u^n$ in (2) or (3). Since $b(u^n, u^n, u^n) = b(Eu^n, u^n, u^n) = 0$ and

$$4 \left(\frac{3a}{2} - 2b + \frac{c}{2} \right) a = a^2 + (2a - b)^2 - b^2 - (2b - c)^2 + (a - 2b + c)^2, \quad a, b, c \in \mathbb{R},$$

so that

$$4(D_2 u^n, u^n) = D_1 \left(|u^n|^2 + |Eu^{n+1}|^2 \right) + (\Delta t)^3 |D^2 u^{n-1}|^2, \quad (6)$$

the assertion follows with

$$4 \langle R_2^n f, u^n \rangle \leq 4 \|R_2^n f\|_* \|u^n\| \leq \frac{8}{\nu} \|R_2^n f\|_*^2 + 2\nu \|u^n\|^2$$

and (4) after summation. #

We shall remark that higher regularity (i.e. a priori estimates in $L^\infty(0, T; V)$ and $L^2(0, T; H^2(\Omega)^2 \cap V)$) can be proved in the two-dimensional case for small data. This relies upon a discrete Gronwall-type lemma for resolving a difference inequality with a quadratic term. However, in the three-dimensional case, a cubic term would arise that cannot be handled

similarly. This situation is in accordance with the time continuous case regarding the existence of strong solutions only up to a possibly small time T (cf. Temam [16]).

We now provide some estimates for the discrete time derivative $D_2 u^n$. These results will again reflect the same situation as it appears in the continuous case: Let u be a weak solution to Problem (P). Then $u' \in L^2(0, T; V^*)$ in the two-dimensional case, but only $u' \in L^{4/3}(0, T; V^*)$ in the three-dimensional case.

Theorem 3. *Let $\{u^n\}$ be a solution to Problem $(P_{\Delta t})$. Then*

$$\Delta t \sum_{j=2}^N \|D_2 u^j\|_*^2 \leq \frac{CM^2}{\nu} (\nu^2 + M^2) \quad \text{if } \dim \Omega = 2,$$

$$\Delta t \sum_{j=2}^N \|D_2 u^j\|_*^{4/3} \leq \frac{CM^{2/3}}{\nu} ((T\nu^5 M^2)^{1/3} + M^2) \quad \text{if } \dim \Omega = 3.$$

If $\{u^n\}$ is the solution to Problem $(LP_{\Delta t})$ then

$$\Delta t \sum_{j=2}^N \|D_2 u^j\|_*^2 \leq \frac{CM^2}{\nu} \left(\nu^2 + M^2 + \nu \Delta t (\|u^0\|^2 + \|u^1\|^2) \right) \quad \text{if } \dim \Omega = 2,$$

$$\Delta t \sum_{j=2}^N \|D_2 u^j\|_*^{4/3} \leq \frac{CM^{2/3}}{\nu} \left((T\nu^5 M^2)^{1/3} + M^2 + \nu \Delta t (\|u^0\|^2 + \|u^1\|^2) \right) \quad \text{if } \dim \Omega = 3.$$

Proof. We commence with Problem $(P_{\Delta t})$. From (2), it immediately follows that

$$\|D_2 u^n\|_* \leq \|R_2^n f\|_* + \nu \|u^n\| + \|B(u^n)\|_* \quad (7)$$

Due to Lemma 1, we have

$$\|B(u^n)\|_* \leq \begin{cases} \tilde{\beta} \|u^n\| \|u^n\| & \text{for } \dim \Omega = 2, \\ \beta \|u^n\|^{1/2} \|u^n\|^{3/2} & \text{for } \dim \Omega = 3. \end{cases} \quad (8)$$

If $\dim \Omega = 2$, we thus find from (7) with Theorem 2 and (4)

$$\begin{aligned} \Delta t \sum_{j=2}^N \|D_2 u^j\|_*^2 &\leq c \left(\int_0^T |f(t)|_*^2 dt + \nu^2 \Delta t \sum_{j=2}^N \|u^j\|^2 \right. \\ &\quad \left. + \tilde{\beta}^2 \max_{j=2, \dots, N} \|u^j\|^2 \Delta t \sum_{j=2}^N \|u^j\|^2 \right) \\ &\leq c \left(\nu M^2 + \frac{\tilde{\beta}^2 M^4}{\nu} \right), \end{aligned}$$

which is the assertion.

Because of

$$\Delta t \sum_{j=2}^N \|a_j\|^{4/3} \leq T^{1/3} \left(\Delta t \sum_{j=2}^N \|a_j\|^2 \right)^{2/3},$$

we find in the three-dimensional case that

$$\begin{aligned} \Delta t \sum_{j=2}^N \|D_2 u^j\|_*^{4/3} &\leq c \left(T^{1/3} \left(\Delta t \sum_{j=2}^N \|R_2^j f\|_*^{4/3} \right)^{2/3} \right. \\ &\quad \left. + T^{1/3} \left(\nu^2 \Delta t \sum_{j=2}^N \|u^j\|^2 \right)^{2/3} \right) \\ &\quad + \beta^{4/3} \left(\max_{j=2, \dots, N} \|u^j\|^2 \right)^{1/3} \Delta t \sum_{j=2}^N \|u^j\|^2, \end{aligned}$$

and the assertion follows again with Theorem 2 and (4).

For Problem $(LP_{\Delta t})$, we may use, instead of (8), the estimates

$$|b(Eu^n, u^n, v)| \leq \begin{cases} \tilde{\beta} \|Eu^n\|^{1/2} \|Eu^n\|^{1/2} \|u^n\|^{1/2} \|u^n\|^{1/2} \|v\| & \text{for } \dim \Omega = 2, \\ \beta \|Eu^n\| \|u^n\|^{1/2} \|u^n\|^{1/2} \|v\| & \text{for } \dim \Omega = 3 \end{cases}$$

that follow from Lemma 1. Remembering $Eu^n = 2u^{n-1} - u^{n-2}$, the rest of the proof is nearly the same as for Problem $(P_{\Delta t})$. However, we have

$$\begin{aligned} & \Delta t \sum_{j=2}^N \|Eu^j\| \|Eu^j\| |u^j| |u^j| \\ & \leq c \Delta t \sum_{j=2}^N \left(\|Eu^j\|^2 |u^j|^2 + |u^j|^2 |u^j|^2 \right) \\ & \leq c \max_{j=0, \dots, N} |u^j|^2 \Delta t \sum_{j=0}^N |u^j|^2 \\ & \leq c M^2 \left(\frac{M^2}{\nu} + \Delta t \left(|u^0|^2 + |u^1|^2 \right) \right) \end{aligned}$$

in the two-dimensional case and

$$\begin{aligned} & \Delta t \sum_{j=2}^N |Eu^j|^{4/3} |u^j|^{2/3} |u^j|^{2/3} \\ & \leq c \Delta t \sum_{j=2}^N \left(\|Eu^j\|^2 + |u^j|^2 \right) |u^j|^{2/3} \\ & \leq c \max_{j=2, \dots, N} |u^j|^{2/3} \Delta t \sum_{j=0}^N |u^j|^2 \\ & \leq c M^{2/3} \left(\frac{M^2}{\nu} + \Delta t \left(|u^0|^2 + |u^1|^2 \right) \right) \end{aligned}$$

in the three-dimensional case.

The following result will be needed in order to justify assumptions that are necessary for our convergence result below. #

Proposition 1. Let $\{u^n\}$ be a solution to Problem $(P_{\Delta t})$ or $(LP_{\Delta t})$ with the additional assumption that $u^0, u^1 \in V$ and $f \in L^1(0, T; H)$. Then for $n = 2, 3, \dots, N$

$$\begin{aligned} U_n := & (\Delta t)^2 \sum_{j=2}^n |D_2 u^j|^2 \\ & + \nu \Delta t \left(|u^n|^2 + (\Delta t)^4 \sum_{j=1}^{n-1} \|D^2 u^j\|^2 \right) \end{aligned}$$

can be estimated by

$$\begin{aligned} U_n \leq & C \left((\nu + M) \Delta t \left(|u^0|^2 + |u^1|^2 \right) \right. \\ & \left. + \|f\|_{L^1(0, T; H)}^2 + \frac{M^3}{\nu} \right) \\ & \text{if } \dim \Omega = 2, \end{aligned}$$

$$\begin{aligned} (\Delta t)^{1/4} U_n \leq & C \left(\nu (\Delta t)^{5/4} \left(|u^0|^2 + |u^1|^2 \right) \right. \\ & \left. + M^{1/2} (\Delta t)^{5/4} \left(|u^0|^2 + |u^1|^2 \right)^{5/4} \right. \\ & \left. + (\Delta t)^{1/4} \|f\|_{L^1(0, T; H)}^2 + \frac{M^3}{\nu^{5/4}} \right) \\ & \text{if } \dim \Omega = 3. \end{aligned}$$

Proof. With $\nu = D_2 u^n$ in (2) and the Cauchy-Schwarz as well as Young's inequality, it follows

$$\begin{aligned} & \frac{1}{2} |D_2 u^n|^2 + \nu ((D_2 u^n, u^n)) \\ & \leq \frac{1}{2} |R_2^n f|^2 - \frac{1}{\Delta t} b \left(u^n, 2u^{n-1} - \frac{1}{2} u^{n-2}, u^n \right). \end{aligned}$$

If $\dim \Omega = 2$, we thus find with an identity analogous to (6) for the term $((D_2 u^n, u^n))$ after summing up

$$\begin{aligned} U_n \leq & \nu \Delta t \left(|u^1|^2 + |Eu^2|^2 \right) + 2(\Delta t)^2 \sum_{j=2}^n |R_2^j f|^2 \\ & - 4\Delta t \sum_{j=2}^n b \left(u^j, 2u^{j-1} - \frac{1}{2} u^{j-2}, u^j \right). \end{aligned}$$

With standard arguments, it can be shown that

$$(\Delta t)^2 \sum_{j=2}^n |R_2^j f|^2 \leq \left(\Delta t \sum_{j=2}^n |R_2^j f| \right)^2 \leq 4 \|f\|_{L^1(0, T; H)}^2.$$

Furthermore, we have with Lemma 1 and Theorem 2

$$\begin{aligned} & 4\Delta t \sum_{j=2}^n \left| b \left(u^j, 2u^{j-1} - \frac{1}{2} u^{j-2}, u^j \right) \right| \\ & \leq 4 \tilde{\beta} \Delta t \sum_{j=2}^n |u^j| |u^j| \left| 2u^{j-1} - \frac{1}{2} u^{j-2} \right| \\ & \leq 2 \tilde{\beta} \max_{j=2, \dots, N} |u^j| \Delta t \sum_{j=2}^n \left(|u^j|^2 + \left| 2u^{j-1} - \frac{1}{2} u^{j-2} \right|^2 \right) \\ & \leq \frac{c \tilde{\beta} M^3}{\nu} + c \tilde{\beta} M \Delta t \left(|u^0|^2 + |u^1|^2 \right), \end{aligned}$$

and the assertion follows.

For $\dim \Omega = 3$, we firstly observe analogously to the foregoing case that

$$\begin{aligned}
 (\Delta t)^{1/4} U_n &\leq \nu(\Delta t)^{5/4} \left(|u^1|^2 + |Eu^2|^2 \right) \\
 &\quad + 8(\Delta t)^{1/4} \|f\|_{L^2(0,T;H)}^2 \\
 &\quad - 4(\Delta t)^{5/4} \sum_{j=2}^n b \left(u^j, 2u^{j-1} - \frac{1}{2}u^{j-2}, u^j \right).
 \end{aligned}$$

With Lemma 1, Young's inequality, and Theorem 2, we find

$$\begin{aligned}
 &4(\Delta t)^{5/4} \sum_{j=2}^n \left| b \left(u^j, 2u^{j-1} - \frac{1}{2}u^{j-2}, u^j \right) \right| \\
 &\leq 4\beta(\Delta t)^{5/4} \sum_{j=2}^n |u^j|^{1/2} |u^j|^{3/2} \left| 2u^{j-1} - \frac{1}{2}u^{j-2} \right| \\
 &\leq 2\beta \max_{j=2,\dots,N} |u^j|^{1/2} \times \\
 &\quad \times (\Delta t)^{5/4} \sum_{j=2}^n \left(|u^j|^{5/2} + \left| 2u^{j-1} - \frac{1}{2}u^{j-2} \right|^{5/2} \right) \\
 &\leq c\beta \frac{M^3}{\nu^{5/4}} + c\beta M^{1/2} (\Delta t)^{5/4} \left(|u^0|^{5/2} + |u^1|^{5/2} \right),
 \end{aligned}$$

and the assertion follows. In the last step, we made use of

$$(\Delta t)^{5/4} \sum_{j=2}^n |u^j|^{5/2} \leq \left(\Delta t \sum_{j=2}^n |u^j|^2 \right)^{5/4}.$$

With obvious modifications, the proof for Problem (LP_{Δt}) is the same.

4 Convergence

From the discrete values u^n ($n = 0, 1, \dots, N$), computed by solving Problem (P_{Δt}) or (LP_{Δt}), we now construct a piecewise constant function $U_{\Delta t}$ and a piecewise linear function $V_{\Delta t}$ defined on $[0, T]$:

$$U_{\Delta t}(t) = \begin{cases} u^1 & \text{if } t \in [0, t_1], \\ u^n & \text{if } t \in (t_{n-1}, t_n] \quad (n = 2, 3, \dots, N); \end{cases}$$

$$V_{\Delta t}(t) = \begin{cases} \frac{1}{2}(u^1 + Eu^2) + D_1 u^1 (t - t_1) & \text{if } t \in [0, t_1], \\ \frac{1}{2}(u^n + Eu^{n+1}) + D_2 u^n (t - t_n) & \text{if } t \in (t_{n-1}, t_n] \quad (n = 2, 3, \dots, N). \end{cases}$$

There are other possible prolongations we will not consider here (but see the remarks at the end of this section). The construction of $V_{\Delta t}$ reflects the choice of the method: The value u^1 is thought to be computed by the implicit Euler method. The slope of $V_{\Delta t}$ in $(t_{n-1}, t_n]$ is $D_2 u^n$ for $n = 2, 3, \dots, N$, and the function is continuous. However, $V_{\Delta t}$ does not interpolate.

In the following, we shall assume $u^0, u^1 \in V$ and

$$|u^0|^2 + |u^1|^2 + \nu \Delta t |u^1|^2 + \Delta t |D_1 u^1|_*^{4/3} \leq const, \quad (9)$$

where *const* is independent of Δt and N . This is true if, for instance, u^1 is computed by the implicit Euler method. (The proof follows similar arguments as employed in the foregoing section.)

Proposition 2. *Let $\{(\Delta t)_k\}$ be some sequence of time steps. If (9) holds true then $\{U_{(\Delta t)_k}\}$ and $\{V_{(\Delta t)_k}\}$ are bounded in $L^\infty(0, T; H)$ and $L^2(0, T; V)$. Moreover, the sequence of derivatives $\{V'_{(\Delta t)_k}\}$ is bounded in $L^{4/3}(0, T; V^*)$.*

Proof. The first assertion follows directly from the definition of $U_{\Delta t}$ and $V_{\Delta t}$, and from Theorem 2. The second assertion follows again from the construction of $V_{\Delta t}$, its continuity, and from Theorem 3 because of

$$\int_0^T \|V'_{\Delta t}\|_*^{4/3} dt = \Delta t |D_1 u^1|_*^{4/3} + \Delta t \sum_{j=2}^N |D_2 u^j|_*^{4/3}.$$

#

At this point, it is worth to mention that $U_{(\Delta t)_k}$ possesses derivatives of fractional order less than 1/4 that are uniformly bounded in $L^2(0, T; H)$ if

$$(\Delta t)^2 \sum_{j=2}^{N-1} |D^2 u^j|_* \leq const. \tag{10}$$

To be more precise, $\{U_{(\Delta t)_k}\}$ is bounded in $H^1(0, T; V, H)$ for $\gamma \in (0, 1/4)$, where

$$H^1(0, T; V, H) := \{v \in L^2(0, T; V) : \tau \mapsto |\tau|^\gamma \hat{v}(\tau) \in L^2(\mathbb{R}; H)\}$$

and \hat{v} is the Fourier transform of v . However, due to the nonlinearity, we do not have any proof of the hypothesis (10) at hand.

Corollary 1. *Let $\{(\Delta t)_k\}$ be some sequence of time steps and assume (9). Then there is a subsequence $\{(\Delta t)_k\}$ such that $\{U_{(\Delta t)_k}\}$ is weakly* convergent in $L^\infty(0, T; H)$ and weakly convergent in $L^2(0, T; V)$. Furthermore, $\{V_{(\Delta t)_k}\}$ is weakly* convergent in $L^\infty(0, T; H)$, weakly convergent in $L^2(0, T; V)$, and strongly convergent in $L^q(0, T; H)$ for $q \in [1, \infty)$.*

Proof. The existence of a weakly* in $L^\infty(0, T; H)$ and weakly in $L^2(0, T; V)$ convergent subsequence of $\{U_{(\Delta t)_k}\}$ and $\{V_{(\Delta t)_k}\}$ follows from standard compactness arguments because of Proposition 2 (cf. Brézis [2]). The strong convergence of a subsequence of $\{V_{(\Delta t)_k}\}$ in $L^2(0, T; H)$ follows from a compactness theorem by Aubin and Lions (cf. Lions [9]) since $\{V_{(\Delta t)_k}\}$ is bounded in $L^2(0, T; V)$ and $\{V'_{(\Delta t)_k}\}$ is bounded in $L^{4/3}(0, T; V^*)$ (see Proposition 2). The boundedness in $L^\infty(0, T; H)$ then implies convergence in any $L^q(0, T; H)$ with $q < \infty$. #

Let us remark that, under the hypothesis (10), there is also a strongly in $L^2(0, T; H)$ convergent subsequence of $\{U_{(\Delta t)_k}\}$ since the compact embedding of V in H implies that $H^1(0, T; V, H)$ is compactly embedded in $L^2(0, T; H)$. However, we do not need to assume

(10) since $\{U_{(\Delta t)_k}\}$ and $\{V_{(\Delta t)_k}\}$ must have the same limit in $L^2(0, T; H)$ under an assumption that follows directly from Proposition 1:

Proposition 3. *Let $\{(\Delta t)_k\}$ be a null sequence and assume (9) as well as*

$$(\Delta t)_k^3 \sum_{j=2}^{N_k} |D_2 u^j_{(k)}|^2 \rightarrow 0 \text{ as } k \rightarrow \infty \tag{11}$$

for the solutions $\{u^n_{(k)}\}$ to Problem $(P_{(\Delta t)_k})$ or $(LP_{(\Delta t)_k})$ with $N_k = T/(\Delta t)_k$. If one of the sequences $\{U_{(\Delta t)_k}\}$ or $\{V_{(\Delta t)_k}\}$ converges strongly in $L^2(0, T; H)$ then the other one does so with the same limit U .

Proof. For brevity, we omit the subscript k . With $D_1 u^n = D_2 u^n - \frac{\Delta t}{2} D^2 u^{n-1}$, we find

$$\int_0^T |U_{\Delta t}(t) - V_{\Delta t}(t)|^2 dt \leq \frac{\Delta t}{12} |u^1 - u^0|^2 + \frac{(\Delta t)^3}{6} \sum_{j=2}^N |D_2 u^j|^2 + \frac{(\Delta t)^5}{8} \sum_{j=1}^{N-1} |D^2 u^j|^2.$$

The assertion follows from (9), (11), and Theorem 2. #

Remark. Assumption (11) indeed follows from Proposition 1 if $f \in L^1(0, T; H) \cap L^2(0, T; V^*)$ and $u^0, u^1 \in V$. Since $L^2(0, T; H)$ is a dense subset of $L^2(0, T; V^*)$ and $L^1(0, T; H)$, also all the weak limits in Corollary 1 then coincide with U . #

Theorem 4. *Let $u^0 = u_0 \in V$ and u^1 be computed by the nonlinear or linearised implicit Euler method with right-hand side $R_1^1 f$, i.e. for all $v \in V$*

$$(D_1 u^1, v) + v((u^1, v)) + b(u^1, u^1, v) = \langle R_1^1 f, v \rangle, \tag{12}$$

or $b(u^1, u^1, v)$ being replaced by $b(u^0, u^1, v)$.

such that (9) is fulfilled. Assume further (11) and

$$\frac{1}{2}(u_{(k)}^0 + u_{(k)}^1) \rightarrow u_0 \text{ weakly in } H \text{ as } k \rightarrow \infty \quad (13)$$

for a null sequence $\{(\Delta t)_k\}$. The common limit U of the convergent subsequences $\{U_{(\Delta t)_k}\}$ and $\{V_{(\Delta t)_k}\}$, which exists in view of Corollary 1 and Proposition 3, then is a weak solution to Problem (P). The whole sequences $\{U_{(\Delta t)_k}\}$ and $\{V_{(\Delta t)_k}\}$ converge if Problem (P) admits a unique solution.

Proof. For brevity, we omit the subscript k' indicating the subsequence. Because of (2) and (13) (for the starting phase), we have for $t \in (0, T]$ and all $v \in V$

$$\begin{aligned} & \frac{d}{dt}(V_{\Delta t}(t), v) + v((U_{\Delta t}(t), v)) \\ & + b(U_{\Delta t}(t), U_{\Delta t}(t), v) = \langle f_{\Delta t}(t), v \rangle, \end{aligned}$$

where $f_{\Delta t}(t) = R_2^n f$ for $t \in (t_{n-1}, t_n]$ ($n = 2, 3, \dots, N$) and $f_{\Delta t}(t) = R_1^1 f$ for $t \in [0, \Delta t]$. Testing with a function $\phi = \phi(t) \in C^1([0, T])$, $\phi(T) = 0$, and integration by parts give, because of $V_{\Delta t}(0) = (u^0 + u^1)/2$,

$$\begin{aligned} & - \int_0^T (V_{\Delta t}(t), v) \phi'(t) dt + v \int_0^T ((U_{\Delta t}(t), v)) \phi(t) dt \\ & + \int_0^T b(U_{\Delta t}(t), U_{\Delta t}(t), v) \phi(t) dt \\ & = \int_0^T \langle f_{\Delta t}(t), v \rangle \phi(t) dt + \left(\frac{u^0 + u^1}{2}, v \right) \phi(0). \end{aligned}$$

With (13), we have for all $v \in H \supset V$

$$\left(\frac{u^0 + u^1}{2}, v \right) \rightarrow (u_0, v).$$

Since $U_{\Delta t}, V_{\Delta t} \rightarrow U$ weakly in $L^2(0, T; V)$ and $t \mapsto v \phi'(t) \in C([0, T]; V)$ as well as $t \mapsto Av \phi(t) \in C([0, T]; V^*)$, it follows

$$\int_0^T (V_{\Delta t}(t), v) \phi'(t) dt \rightarrow \int_0^T (U(t), v) \phi'(t) dt$$

and, because of $\langle \cdot, A \cdot \rangle = ((\cdot, \cdot))$,

$$\int_0^T ((U_{\Delta t}(t), v)) \phi(t) dt \rightarrow \int_0^T ((U(t), v)) \phi(t) dt.$$

Furthermore, we find with Lemma 1 and since $\phi \in C([0, T])$

$$\begin{aligned} & \left| \int_0^T (b(U_{\Delta t}(t), U_{\Delta t}(t), v) - b(U(t), U(t), v)) \phi(t) dt \right| \\ & \leq c \int_0^T |b(U_{\Delta t}(t) - U(t), U_{\Delta t}(t), v) \\ & \quad + b(U(t), U_{\Delta t}(t) - U(t), v)| dt \\ & \leq 2\beta c \int_0^T \|U_{\Delta t}(t) - U(t)\|^{1/2} \|U_{\Delta t}(t) - U(t)\|^{1/2} \times \\ & \quad \times (\|U_{\Delta t}(t)\| + \|U(t)\|) \|v\| dt \\ & \leq 2\beta c \|U_{\Delta t} - U\|_{L^2(0, T; H)}^{1/2} \|U_{\Delta t} - U\|_{L^2(0, T; V)}^{1/2} \times \\ & \quad \times (\|U_{\Delta t}\|_{L^2(0, T; V)} + \|U\|_{L^2(0, T; V)}) \|v\|, \end{aligned}$$

and in view of $U \in L^2(0, T; V)$, the boundedness of $\{U_{\Delta t}\}$ in $L^2(0, T; V)$, and the strong convergence $U_{\Delta t} \rightarrow U$ in $L^2(0, T; H)$, we end up with

$$\begin{aligned} & \int_0^T b(U_{\Delta t}(t), U_{\Delta t}(t), v) \phi(t) dt \\ & \rightarrow \int_0^T b(U(t), U(t), v) \phi(t) dt. \end{aligned}$$

It remains to show

$$\int_0^T \langle f_{\Delta t}(t), v \rangle \phi(t) dt \rightarrow \int_0^T \langle f(t), v \rangle \phi(t) dt,$$

for which

$$f_{\Delta t} \rightarrow f \text{ in } L^2(0, T; V^*) \quad (14)$$

is sufficient. The mapping $f \mapsto f_{\Delta t}$ is obviously linear and bounded in $L^2(0, T; V^*)$ since

$$\begin{aligned} \int_0^T \|f_{\Delta t}(t)\|_{V^*}^2 dt &= \Delta t \|R_1^1 f\|_{V^*}^2 + \Delta t \sum_{j=2}^N \|R_2^j f\|_{V^*}^2 \\ &\leq c \|f\|_{L^2(0, T; V^*)}^2. \end{aligned}$$

Furthermore, (14) holds true for all functions of the dense subset $C^2([0, T]; V^*) \subset L^2(0, T; V^*)$, and so it is true for all $f \in L^2(0, T; V^*)$.

The limit U , finally, satisfies for all $\phi \in C^1([0, T])$ with $\phi(T) = 0$

$$\begin{aligned} & - \int_0^T (U(t), v) \phi'(t) dt + v \int_0^T ((U(t), v)) \phi(t) dt \\ & + \int_0^T b(U(t), U(t), v) \phi(t) dt \\ & = \int_0^T \langle f(t), v \rangle \phi(t) dt + (u_0, v) \phi(0), \end{aligned}$$

which shows that U is a weak solution to Problem (P).

The proof for the linearised variant follows the same arguments. The second assertion, regarding the unique solvability, is shown by standard arguments.

Remark. If $u^0 = u_0$ and u^1 is computed by the implicit Euler method then, under suitable regularity assumptions, (13) can be proved. The proof relies upon the pointwise convergence $|u((\Delta t)_k) - u^1_k| \rightarrow 0$ that follows from a (possibly sub-optimal) error estimate and upon the demicontinuity in H of the exact solution u (i.e. $t \mapsto (u(t), v) \in C([0, T])$ for all $v \in H$ and thus $u((\Delta t)_k) \rightarrow u_0$ weakly in H).

Finally, we remark that there are other possible prolongations leading to sequences of approximate solutions that converge towards a weak solution: We may, for instance, consider the interpolating linear spline or the discontinuous interpolating piecewise linear function whose slope in $(t_{n-1}, t_n]$ ($n = 2, 3, \dots, N$) is $D_2 u^n$, or a continuous piecewise quadratic interpolation.

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