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Convergence of a Time Discretisation for Doubly Nonlinear Evolution Equations of Second Order

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Abstract The convergence of a time discretisation with variable time steps is shown for a class of doubly nonlinear evolution equations of second order. This also proves existence of a weak solution. The operator acting on the zero-order term is assumed to be the sum of a linear, bounded, symmetric, strongly positive operator and a nonlinear operator that fulfils a certain growth and a Hölder-type continuity condition. The operator acting on the first-order time derivative is a nonlinear hemicontinuous operator that fulfils a certain growth condition and is (up to some shift) monotone and coercive.

Keywords Evolution equation of second order · Monotone operator · Weak solution · Time discretisation · Variable time grid · Convergence

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1 Introduction

The mathematical description of many applications from mechanics, elasticity theory, molecular dynamics, and quantum mechanics leads to nonlinear evolution equations of second order in time. Examples are

- the equation that appears in the description of vibrating membranes (see [1], [8, p. 165], [13, pp. 38ff., 62ff., 222ff.], [14]) for some $p \geq 2$

$$u_{tt} + |u_t|^{p-2}u_t - \Delta u = f; \tag{1}$$

- the equation (see [16, p. 298], [18, pp. 928ff.])

$$u_{tt} - \nabla \cdot \big(\psi\big(x, t, |\nabla u_t|\big)|\nabla u_t|\big) - \Delta u = f, \tag{2}$$

  where ψ only is a sufficiently smooth, strictly monotonically increasing function with $m \leq \psi \leq M$ for some $m, M > 0$;
- the equation (see [9, 14] as well as [3, pp. 298ff.], [8, p. 238], [13, p. 4] for the non-viscous case) for some $p \geq 2$

$$u_{tt} - \nabla \cdot \big(|\nabla u_t|^{p-2}\nabla u_t\big) - \Delta u + c(u, \nabla u) = f \tag{3}$$

  with an appropriate function c, including viscous regularisations of the Klein–Gordon equation with $c(u, \nabla u) = |u|^\gamma u$ ($\gamma \geq 0$) or of the sine-Gordon equation with $c(u, \nabla u) = \sin u$, both appearing in relativistic quantum mechanics and quantum field theory.

Similar equations arise in elasticity theory and material sciences (see [16, pp. 98ff.]). Further examples can be found, e.g., in [4, 11, 12].

The functional analytic formulation of all these problems leads to the initial value problem

$$u'' + Au' + Bu = f \quad \text{in } (0, T), \qquad u(0) = u_0, \qquad u'(0) = v_0. \tag{4}$$

The operator A is supposed to be the Nemytskii operator corresponding to a family of nonlinear hemicontinuous operators $A(t) : W \to W^*$ ($t \in [0, T]$) that fulfil a certain growth condition. Moreover, $A(t) + \varkappa I : W \to W^*$ (I denotes the identity) is assumed to be coercive and monotone for some $\varkappa \geq 0$, uniformly in $t \in [0, T]$. Here, W is a real, reflexive, separable Banach space that is dense and continuously embedded in a Hilbert space H. If $\varkappa \neq 0$ then W is assumed to be compactly embedded in H.

It would also be possible to incorporate a strongly continuous perturbation of A similarly as is done in [5–7] for first-order equations. In order to keep the presentation brief, we do not consider this more general case here.

The operator B is the Nemytskii operator corresponding to a family of operators $B(t) = B_0 + C(t)$, where $B_0 : V \to V^*$, acting on a Gelfand triple $V \subseteq H \subseteq V^*$ with the same Hilbert space H as above, is assumed to be independent of time as well as linear, bounded, symmetric, and strongly positive. The operators $C(t) : V \to W^*$ are supposed to fulfil a certain growth condition and to be Hölder-type continuous on bounded subsets. If $C(t) \not\equiv 0$ then W is assumed to be compactly embedded in H. Note that the assumptions on B_0 force V to be a Hilbert space.

The foregoing structural assumptions are general enough to cover many interesting applications.

For linear evolution equations of second order, a full theory of existence and uniqueness is given in [8]. Results on the existence, uniqueness, and regularity of solutions to (4) as well as on the convergence of the Galerkin method can be found in [10, Kap. 7] and [18, Chap. 33] for the case $V = W$. Results allowing more involved nonlinearities of the first- and zero-order terms relying upon $V \neq W$ (with $V \cap W$ being dense in V as well as in W) can be found in [3, Chap. V], [14], and [16, pp. 296ff., 342ff.]. See also [9] for a special class of problems of the form (4) and [1, 13] for particular examples.

The evolution problem (4) shall be approximated in time by means of the scheme

$$\frac{2}{\tau_{n+1}+\tau_n}\left(\frac{u^{n+1}-u^n}{\tau_{n+1}} - \frac{u^n-u^{n-1}}{\tau_n}\right) + A(t_n)\frac{u^{n+1}-u^n}{\tau_{n+1}} + B(t_n)u^n = f^n,$$
$$n = 1, 2, \ldots, N-1, \tag{5}$$

on a sequence of variable time grids

$$\mathbb{I}: 0 = t_0 < t_1 < \cdots < t_N = T, \quad \tau_n = t_n - t_{n-1}\ (n = 1, 2, \ldots, N \in \mathbb{N}). \tag{6}$$

For given approximations $u^0 \approx u_0$, $v^0 \approx v_0$, $\{f^n\} \approx f$, this yields approximations $u^n \approx u(t_n)$. Note that, in the case $A \equiv 0$, the scheme (5) is known as the leap-frog scheme, which falls into the class of Newmark schemes and can be interpreted as a partitioned Runge–Kutta method (here as the Störmer–Verlet method).

Although also of interest, an analysis of other numerical methods (in particular, methods of higher order) applied to the above class of doubly nonlinear evolution equations of second order is not yet available. A main difficulty will be, as always, to derive appropriate a priori estimates based upon the stability of the numerical method. Nevertheless, without additional regularity of the exact solution, there is no need for higher-order methods.

Error estimates for a full discretisation combining a finite element method with the Newmark scheme can be found for the linear case in [15, Chap. 8]. To our best knowledge, the only reference for studying the convergence of time discretisations of (4) in the nonlinear case is [4]. The authors also deal with the scheme (5), but on an equidistant time grid and under more restrictive assumptions on the data of the problem. The convergence result in [4] applies to the special case $V = W$ with A being a time-independent maximal monotone operator and B being time-independent, linear, bounded, symmetric, and (up to some shift) strongly positive. The weak convergence results in [4] are somewhat better than the results obtained here, which is due to the stronger assumptions on the initial data and right-hand side (leading to a more regular exact solution). Besides, there is an existence result proven in [2] via a time discretisation for a special class of semilinear equations of the type (4).

In this paper, we show weak convergence of piecewise polynomial prolongations of the time-discrete solutions to (5) towards the weak solution to (4) whenever the maximum time steps of the sequence of variable time grids tend to zero. Moreover, the deviation of the time grids from an equidistant time grid cannot be too large in the sense that

$$\max_{n=3,4,\dots,N}\left(\frac{1}{\tau_n}\max\left(0,\frac{\tau_{n-1}}{\tau_n}-\frac{\tau_{n-2}}{\tau_{n-1}}\right)\right)$$

is bounded, and

$$\sum_{n=2}^{N}\frac{(\tau_n-\tau_{n-1})^2}{\tau_n+\tau_{n-1}}$$

tends to zero when considering a sequence of time grids (6). This is, e.g., fulfilled if $\tau_{n+1}=\tau_n(1+c\tau_n^{1+\varepsilon})$ for some $c,\varepsilon>0$. This, however, seems to be in accordance with the observations in [17] made for the test equation $y''+\omega^2 y=0$ $(\omega>0)$.

Our analysis requires the continuous embedding of W in V, which implies that the operator A dominates the operator B_0 in the sense that there are constants $c_1>0$, $c_2\geq 0$ such that for all $v\in W$, $t\in[0,T]$,

$$\big\langle A(t)v,v\big\rangle\geq c_1\langle B_0v,v\rangle-c_2,$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing.

The demonstration of the convergence of the time discretisation is, indeed, an alternative proof for the existence of a weak solution. Our aim is, however, to rigorously justify a widely used numerical approximation of the problem under consideration. We are not going to prove any error estimates or convergence rates, as those require higher-order regularity of the exact solution (which is unknown in the generic case and cannot be derived from, e.g., the regularity results given in [10]). We should emphasise that, under the rather weak assumptions on the problem data and the governing operators, one cannot expect "more" than a weak solution and, therefore, our weak convergence results are optimal.

In the general situation we consider here, existence cannot be implied from the results known from the respective literature [9, 10, 14, 16, 18] and seems to be new, too.

If W is not continuously embedded in V, a lack of stability of the time discretisation method appears in the sense that the necessary a priori estimates are not at hand. This might be circumvented, however, by employing an inverse inequality based on a suitable spatial discretisation and coupling then the time step size and the spatial discretisation parameter. A corresponding analysis will be the topic of further research.

The paper is organised as follows: in Sect. 2, we introduce some necessary notation and precisely state the main assumptions on the operators appearing in the evolution equation. The numerical scheme for the time discretisation is analysed in Sect. 3. In particular, we provide a priori estimates of the time discrete solution. The main convergence result is formulated and proven in Sect. 4.

2 Time Continuous Problem and Notation

Let $(W,|||\cdot|||)$ be a real, reflexive, separable Banach space that is dense and continuously embedded in the Hilbert space $(V,\|\cdot\|)$, and let $(V,\|\cdot\|)$ be dense and continuously embedded in the Hilbert space $(H,(\cdot,\cdot),|\cdot|)$. By identifying the dual H^*

with H, we come to the scale of spaces with dense and continuous embeddings

$$W \subseteq V \subseteq H \subseteq V^* \subseteq W^*.$$

We always denote the standard dual norm by the subscript $*$, and the dual pairing is denoted by $\langle \cdot, \cdot \rangle$. Note that $V \subseteq H \subseteq V^*$ as well as $W \subseteq H \subseteq W^*$ form a Gelfand triple.

The space of Bochner integrable (for $r = \infty$ Bochner measurable and essentially bounded) abstract functions mapping $[0, T]$ into a Banach space X is denoted by $L^r(0, T; X)$ ($r \in [1, \infty]$) and equipped with the standard norm $\| \cdot \|_{L^r(0,T;X)}$. Moreover, we denote by $\mathcal{C}^r([0, T]; X)$ ($r \in \mathbb{N}$, $\mathcal{C}^0 \equiv \mathcal{C}$) the space of uniformly continuous functions mapping $[0, T]$ into X with uniformly continuous time derivatives up to order r.

In what follows, we always assume $p \in [2, \infty)$ and set $p^* = p/(p-1)$. The dual pairing between $L^p(0, T; V)$ and $(L^p(0, T; V))^* \cong L^{p^*}(0, T; V^*)$ is given by

$$\langle f, v \rangle = \int_0^T \big\langle f(t), v(t) \big\rangle_{V^* \times V} \, \mathrm{d}t.$$

The same applies to the case when V is replaced by W. Moreover, we have $(L^1(0, T; H))^* \cong L^\infty(0, T; H)$ with the dual pairing

$$\langle f, v \rangle = \int_0^T \big(f(t), v(t) \big) \, \mathrm{d}t.$$

We also use the Banach space

$$\mathcal{W} = \big\{ v \in L^p(0, T; W) : v' \in \big(L^p(0, T; W) \big)^* \big\},$$
$$\|v\|_{\mathcal{W}} = \|v\|_{L^p(0,T;W)} + \big\| v' \big\|_{(L^p(0,T;W))^*},$$

with v' denoting the distributional time derivative. The space $\mathcal{W}$ is continuously embedded in $\mathcal{C}([0, T]; H)$. If $W \overset{c}{\hookrightarrow} H$ then, by virtue of the Lions–Aubin compactness theorem, $\mathcal{W}$ is compactly embedded in $L^r(0, T; H)$ for any $r \in [1, \infty)$. The scales $L^p(0, T; W) \subseteq L^2(0, T; H) \subseteq (L^p(0, T; W))^*$ and $L^2(0, T; V) \subseteq L^2(0, T; H) \subseteq (L^2(0, T; V))^*$ also form Gelfand triples.

The structural properties we always assume for A and B read as follows.

Assumption A $\{A(t)\}_{t \in [0,T]}$ is a family of hemicontinuous operators $A(t) : W \to W^*$ such that for all $v \in W$ the mapping $t \mapsto A(t)v : [0, T] \to W^*$ is continuous for almost all $t \in [0, T]$. There is a constant $\varkappa \geq 0$ such that $A(t) + \varkappa I : W \to W^*$ is monotone for all $t \in [0, T]$. For a suitable $p \in [2, \infty)$, there are constants $\mu_A, \beta_A > 0$, $\lambda \geq 0$ such that for all $t \in [0, T]$ and $v \in W$

$$\big\langle (A(t) + \varkappa I)v, v \big\rangle \geq \mu_A |||v|||^p - \lambda, \quad \big|\big|\big| A(t)v \big|\big|\big|_* \leq \beta_A \big(1 + |||v|||^{p-1} \big).$$

With $\{A(t)\}_{t \in [0,T]}$, we associate the Nemytskii operator A that is defined by $(Av)(t) := A(t)v(t)$ ($t \in [0, T]$) for a function $v : [0, T] \to W$. Under Assumption A, the Nemytskii operator A maps $L^p(0,T;W)$ into its dual and is hemicontinuous and bounded. Moreover, $A+\varkappa I : L^p(0,T;W) \to (L^p(0,T;W))^*$ is monotone and coercive.

Assumption B $\{B(t)\}_{t\in[0,T]}$ is a family of operators $B(t) = B_0 + C(t)$, where $B_0 : V \to V^*$ is linear, bounded, symmetric, and strongly positive: There are constants $\mu_B, \beta_B > 0$ such that for all $v \in V$

$$\langle B_0 v, v\rangle \geq \mu_B \|v\|^2, \quad \|B_0 v\|_* \leq \beta_B \|v\|.$$

Moreover, $C(t)$ ($t \in [0,T]$) maps V into W^*, and for all $v \in V$, the mapping $t \mapsto C(t)v : [0,T] \to W^*$ is continuous for almost all $t \in [0,T]$. There is a constant $\beta_C > 0$ and a monotonically increasing function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that for all $t \in [0,T]$ and $v, w \in V$

$$\begin{aligned} &\big\|\big|C(t)v\big\|\big|_* \leq \beta_C\big(1 + \|v\|^{2(p-1)/p}\big), \\ &\big\|\big|C(t)v - C(t)w\big\|\big|_* \leq \alpha\big(\max\big(\|v\|, \|w\|\big)\big)|v-w|^{(p-1)/p}. \end{aligned}$$

As for $\{A(t)\}_{t\in[0,T]}$, we associate with $\{B(t)\}_{t\in[0,T]}$ the Nemytskii operator B. Under Assumption B, the Nemytskii operator corresponding to B_0 maps $L^2(0,T;V)$ into its dual and is linear, bounded, symmetric, and strongly positive, whereas the Nemytskii operator C corresponding to $\{C(t)\}_{t\in[0,T]}$ maps $L^2(0,T;V)$ into $(L^p(0,T;W))^*$ and is bounded and continuous.

Remember that we require $W \overset{c}{\hookrightarrow} H$ if $\varkappa \neq 0$ or if $C(t) \not\equiv 0$. It is, however, *not* necessary to have $V \overset{c}{\hookrightarrow} H$.

Under Assumptions A and B and with $C(t) \equiv 0$, problem (4) possesses for any $u_0 \in V$, $v_0 \in H$, $f \in (L^p(0,T;W))^*$ a unique solution $u \in \mathcal{C}^1([0,T];H) \cap \mathcal{C}([0,T];V)$ with $u' \in \mathcal{W}$ such that the evolution equation holds in $(L^p(0,T;W))^*$. This is a consequence of [14, Theorem 2.1]. In the case $W = V$ with $C(t) \not\equiv 0$, existence of a solution follows from [16, Theorem 11.20(ii) on p. 346], see also [9] for C being a potential operator. Uniqueness can be achieved under additional assumptions on the operators $A(t)$ and $C(t)$ ($t \in [0,T]$).

3 Time Discrete Problem and a Priori Estimates

In this section, we consider an arbitrary but fixed time grid (6). We set $\tau_{n+1/2} := (\tau_n + \tau_{n+1})/2$, $t_{n+1/2} = t_n + \tau_{n+1}/2$, and denote by $r_{n+1} := \tau_{n+1}/\tau_n$ ($n = 1, 2, \ldots, N-1$) the ratio of adjacent step sizes. Moreover, we set

$$\begin{aligned} &\tau_{\max} := \max_{n=1,2,\ldots,N} \tau_n, \qquad r_{\max} := \max_{n=2,3,\ldots,N} r_n, \qquad r_{\min} := \min_{n=2,3,\ldots,N} r_n, \\ &\gamma_n = \max\left(0, \frac{1}{r_n} - \frac{1}{r_{n-1}}\right), \qquad c_\gamma := \max_{n=3,4,\ldots,N} \frac{\gamma_n}{\tau_n}. \end{aligned}$$

Writing (4) as a first-order system

$$\begin{cases} -u'(t) + v(t) = 0, \\ v'(t) + A(t)v(t) + B(t)u(t) = f(t), \end{cases}$$

and applying the explicit and implicit Euler scheme to the first (backward in time) and second (forward in time) equation, respectively, gives

$$\begin{cases} -\frac{1}{\tau_{n+1}}(u^{n+1} - u^n) + v^n = 0, & n = 0, 1, \dots, N-1, \\ \frac{1}{\tau_{n+1/2}}(v^n - v^{n-1}) + A(t_n)v^n + B(t_n)u^n = f^n, & n = 1, 2, \dots, N-1, \end{cases}$$

with given initial approximations $u^0 \approx u_0$ and $v^0 = (u^1 - u^0)/\tau_1 \approx v_0$. Inserting the first into the second equation leads to the scheme (5), which is formally of first order.

Representing now u^n by $\{v^n\}$,

$$u^n = u^0 + \sum_{j=0}^{n-1}\bigl(u^{j+1} - u^j\bigr) = u^0 + \sum_{j=0}^{n-1}\tau_{j+1}v^j =: Lv^n, \quad n = 0, 1, \dots, N, \tag{7}$$

we find

$$\frac{1}{\tau_{n+1/2}}\bigl(v^n - v^{n-1}\bigr) + A(t_n)v^n + B_0Lv^n + C(t_n)u^n = f^n, \\ n = 1, 2, \dots, N-1, \tag{8}$$

which will be the starting point for our analysis. Here, L is a nonlocal operator acting on grid functions. Sometimes, our analysis is also based on another representation using $B(t_n)u^n = B_0Lv^n + C(t_n)u^n = B(t_n)Lv^n$.

Theorem 1 *Let Assumptions A and B be fulfilled and let $u^0, v^0 \in V$ and $\{f^n\}_{n=1}^{N-1} \subseteq V^*$. If $\tau_{\max} < 1/\varkappa$ then there is a unique solution $\{u^n\}_{n=1}^{N} \subseteq V$ to (5) with $\{v^n\}_{n=1}^{N-1} \subseteq W$ $(v^n = (u^{n+1} - u^n)/\tau_{n+1})$.*

Moreover, let $\tau_{\max} < 1/\max(2\varkappa, \beta_B c_{W\hookrightarrow V}^2/\mu_A)$ ($c_{W\hookrightarrow V}$ denotes the constant from the continuous embedding $W \hookrightarrow V$). Then the following a priori estimates hold true for $n = 1, 2, \dots, N-1$:

$$\bigl\|u^{n+1}\bigr\|^2 + \bigl|v^n\bigr|^2 + \sum_{j=1}^{n}\bigl|v^j - v^{j-1}\bigr|^2 + \sum_{j=1}^{n}\tau_{j+1/2}\bigl\|\bigl|v^j\bigr\|\bigr|^p \\ \le c\left(\bigl\|u^0\bigr\|^2 + \bigl|v^0\bigr|^2 + \tau_1^2\bigl\|v^0\bigr\|^2 + \sum_{j=1}^{n}\tau_{j+1/2}\bigl\|\bigl|f^j\bigr\|\bigr|_*^{p^*} + T\right) =: M,$$

where $c > 0$ is a function in $1/r_{\min}, r_{\max}, c_\gamma$, and $1/(1 - \max(2\varkappa, \beta_B c_{W\hookrightarrow V}^2/\mu_A)\tau_{\max})$ that is bounded on bounded subsets, and

$$\sum_{j=1}^{n}\tau_{j+1/2}\left\|\left|\frac{1}{\tau_{j+1/2}}\bigl(v^j - v^{j-1}\bigr)\right\|\right|_*^{p^*} \le M',$$

where $M' > 0$ is a function in M that is bounded on bounded subsets.

Proof In view of (8), the scheme (5) can be written as

$$\frac{1}{\tau_{n+1/2}} v^n + A(t_n)v^n = \frac{1}{\tau_{n+1/2}} v^{n-1} + f^n - B(t_n)Lv^n, \quad n = 1, 2, \dots, N-1.$$

This equation can be solved step by step: The right-hand side is in W^*. In particular, for given $u^0 \in V$ and $\{v^j\}_{j=0}^{n-1} \subseteq V$, we have with (7) that $Lv^n \in V$ and thus $B(t_n)Lv^n \in W^*$. The theorem of Browder–Minty now provides the existence of $v^n \in W \subseteq V$. The uniqueness follows since the operator appearing in each step is strictly monotone, which follows from $\tau_{n+1/2} < 1/\varkappa$. Once $\{v^n\}_{n=0}^{N-1}$ is known, the solution $\{u^n\}_{n=1}^{N}$ can be calculated from (7).

We now come to the first a priori estimate. We test (8) with v^n. Since

$$(a-b)a = \frac{1}{2}\big(a^2 - b^2 + (a-b)^2\big), \quad a, b \in \mathbb{R}, \tag{9}$$

we have

$$\left\langle \frac{1}{\tau_{n+1/2}}\big(v^n - v^{n-1}\big), v^n \right\rangle = \frac{1}{2\tau_{n+1/2}}\big(\big|v^n\big|^2 - \big|v^{n-1}\big|^2 + \big|v^n - v^{n-1}\big|^2\big).$$

Because of the coercivity condition on $A(t_n) + \varkappa I$, we find

$$\big\langle A(t_n)v^n, v^n\big\rangle \geq \mu_A \big\|\big|v^n\big\|\big|^p - \lambda - \varkappa\big|v^n\big|^2.$$

The third term on the left-hand side in (8) is more involved. By Assumption B, the mapping $(v, w) \mapsto \langle B_0 v, w\rangle$, $V \times V \to \mathbb{R}$, defines an inner product in V, and the norm $\|\cdot\|_B = \langle B_0 \cdot, \cdot\rangle^{1/2}$ induced by this inner product is equivalent to the original norm $\|\cdot\|$ such that

$$\mu_B^{1/2}\|v\| \leq \|v\|_B \leq \beta_B^{1/2}\|v\|, \quad v \in V.$$

We, therefore, find with (7) and

$$(a-b)b = \frac{1}{2}\big(a^2 - b^2 - (a-b)^2\big), \quad a, b \in \mathbb{R},$$

that

$$\begin{aligned}
\big\langle B_0 L v^n, v^n\big\rangle &= \frac{1}{\tau_{n+1}}\big\langle B_0 L v^n, L v^{n+1} - L v^n\big\rangle \\
&= \frac{1}{2\tau_{n+1}}\big(\big\|Lv^{n+1}\big\|_B^2 - \big\|Lv^n\big\|_B^2 - \big\|Lv^{n+1} - Lv^n\big\|_B^2\big) \\
&= \frac{1}{2\tau_{n+1}}\big(\big\|Lv^{n+1}\big\|_B^2 - \big\|Lv^n\big\|_B^2 - \tau_{n+1}^2\big\|v^n\big\|_B^2\big).
\end{aligned}$$

For the remaining term with C, we obtain from Young's inequality, together with the corresponding growth condition,

$$\begin{aligned}\left|\left\langle C(t_n)u^n, v^n\right\rangle\right| &\le \left|\!\left|\!\left|C(t_n)u^n\right|\!\right|\!\right|_* \left|\!\left|\!\left|v^n\right|\!\right|\!\right| \le c\left|\!\left|\!\left|C(t_n)u^n\right|\!\right|\!\right|_*^{p^*} + \frac{\mu_A}{4}\left|\!\left|\!\left|v^n\right|\!\right|\!\right|^p \\ &\le c\left(1 + \|u^n\|^2\right) + \frac{\mu_A}{4}\left|\!\left|\!\left|v^n\right|\!\right|\!\right|^p\end{aligned}$$

with some generic positive constant c (that is independent of the time grid).

For the right-hand side in (8), we employ Young's inequality and find

$$\left\langle f^n, v^n\right\rangle \le \left|\!\left|\!\left|f^n\right|\!\right|\!\right|_* \left|\!\left|\!\left|v^n\right|\!\right|\!\right| \le c\left|\!\left|\!\left|f^n\right|\!\right|\!\right|_*^{p^*} + \frac{\mu_A}{4}\left|\!\left|\!\left|v^n\right|\!\right|\!\right|^p.$$

Putting together the foregoing estimates, multiplying by $2\tau_{n+1/2}$, summing up, and taking into account (7) gives

$$\begin{aligned}&\left|v^n\right|^2 + \sum_{j=1}^n \left|v^j - v^{j-1}\right|^2 + \mu_A \sum_{j=1}^n \tau_{j+1/2}\left|\!\left|\!\left|v^j\right|\!\right|\!\right|^p \\ &\quad + \frac{1}{2}\left(1 + \frac{1}{r_{n+1}}\right)\left\|u^{n+1}\right\|_B^2 + \frac{1}{2}\sum_{j=2}^n \left(\frac{1}{r_j} - \frac{1}{r_{j+1}}\right)\left\|u^j\right\|_B^2 \\ &\le \left|v^0\right|^2 + \frac{1}{2}\left(1 + \frac{1}{r_2}\right)\left\|u^1\right\|_B^2 + c\sum_{j=1}^n \tau_{j+1/2}\left|\!\left|\!\left|f^j\right|\!\right|\!\right|_*^{p^*} + cT \\ &\quad + 2\varkappa \sum_{j=1}^n \tau_{j+1/2}\left|v^j\right|^2 + \sum_{j=1}^n \tau_{j+1/2}\tau_{j+1}\left\|v^j\right\|_B^2 + c\sum_{j=1}^n \tau_{j+1/2}\left\|u^j\right\|^2.\end{aligned}$$

Some elementary calculations together with the continuous embedding $W \hookrightarrow V$, the inequality

$$\left\|v^j\right\|_B^2 \le \beta_B\left\|v^j\right\|^2 \le \beta_B c_{W\hookrightarrow V}^2 \left|\!\left|\!\left|v^j\right|\!\right|\!\right|^2 \le \beta_B c_{W\hookrightarrow V}^2\left(1 + \left|\!\left|\!\left|v^j\right|\!\right|\!\right|^p\right),$$

the condition $\beta_B c_{W\hookrightarrow V}^2 \tau_{\max} < \mu_A$, and a discrete Gronwall argument, which requires $2\varkappa\tau_{\max} < 1$, yields

$$\begin{aligned}&\left|v^n\right|^2 + \sum_{j=1}^n \left|v^j - v^{j-1}\right|^2 + \sum_{j=1}^n \tau_{j+1/2}\left|\!\left|\!\left|v^j\right|\!\right|\!\right|^p + \left\|u^{n+1}\right\|^2 \\ &\quad \le c\left(\left|v^0\right|^2 + \tau_1^2\left\|v^0\right\|^2 + \left\|u^0\right\|^2 + \sum_{j=1}^n \tau_{j+1/2}\left|\!\left|\!\left|f^j\right|\!\right|\!\right|_*^{p^*} + T\right),\end{aligned}$$

where $c > 0$ is a function in $1/r_{\min}$, $r_{\max}$, c_γ, and $1/(1 - \max(2\varkappa, \beta_B c_{W\hookrightarrow V}^2/\mu_A)\tau_{\max})$ that is bounded on bounded subsets. This proves the first estimate asserted.

From (8), the growth condition for A and C, the boundedness of B_0, and the inequality (remember $p^* \le 2$ since $p \ge 2$ as well as $V^* \hookrightarrow W^*$)

$$\left|\!\left|\!\left| B(t_j)u^j \right|\!\right|\!\right|_*^{p^*} \le c\left(\left|\!\left|\!\left| B_0 u^j \right|\!\right|\!\right|_*^{p^*} + \left|\!\left|\!\left| C(t_j)u^j \right|\!\right|\!\right|_*^{p^*}\right) \quad \text{with}$$
$$\left|\!\left|\!\left| B_0 u^j \right|\!\right|\!\right|_*^{p^*} \le c\left\| B_0 u^j \right\|_*^{p^*} \le c\beta_B^{p^*}\left\| u^j \right\|^{p^*} \le c\left(1 + \left\| u^j \right\|^2\right),$$
$$\left|\!\left|\!\left| C(t_j)u^j \right|\!\right|\!\right|_*^{p^*} \le \beta_C^{p^*}\left(1 + \left\| u^j \right\|^{2(p-1)/p}\right)^{p^*} \le c\left(1 + \left\| u^j \right\|^2\right),$$

we immediately arrive at

$$\sum_{j=1}^n \tau_{j+1/2}\left|\!\left|\!\left| \frac{1}{\tau_{j+1/2}}\left(v^j - v^{j-1}\right) \right|\!\right|\!\right|_*^{p^*}$$
$$\le c\sum_{j=1}^n \tau_{j+1/2}\left|\!\left|\!\left| A(t_j)v^j \right|\!\right|\!\right|_*^{p^*} + c\sum_{j=1}^n \tau_{j+1/2}\left|\!\left|\!\left| B(t_j)u^j \right|\!\right|\!\right|_*^{p^*} + c\sum_{j=1}^n \tau_{j+1/2}\left|\!\left|\!\left| f^j \right|\!\right|\!\right|_*^{p^*}$$
$$\le c\sum_{j=1}^n \tau_{j+1/2}\left(1 + \left|\!\left|\!\left| v^j \right|\!\right|\!\right|^p\right) + c\sum_{j=1}^n \tau_{j+1/2}\left(1 + \left\| u^j \right\|^2\right) + c\sum_{j=1}^n \tau_{j+1/2}\left|\!\left|\!\left| f^j \right|\!\right|\!\right|_*^{p^*}.$$

This proves, together with the first estimate, the second estimate. □

4 Convergence of the Sequence of Time Discrete Solutions

Here and in the sequel, we often emphasise the dependence of a quantity g on the time grid $\mathbb{I}$ by writing $g(\mathbb{I})$.

We start by introducing piecewise constant and linear prolongations of the numerical solution: For the solution $\{u^n\}_{n=0}^N$, $\{v^n\}_{n=0}^{N-1}$ to (5) corresponding to a time grid $\mathbb{I}$, let

$$u_{\mathbb{I}}(t) := \begin{cases} 0 & \text{for } t \in [0, t_{1/2}], \\ u^n & \text{for } t \in (t_{n-1/2}, t_{n+1/2}] \ (n = 1, 2, \dots, N-1), \\ 0 & \text{for } t \in (t_{N-1/2}, t_N]; \end{cases}$$

$$v_{\mathbb{I}}(t) := \begin{cases} 0 & \text{for } t \in [0, t_{1/2}], \\ v^n & \text{for } t \in (t_{n-1/2}, t_{n+1/2}] \ (n = 1, 2, \dots, N-1), \\ 0 & \text{for } t \in (t_{N-1/2}, t_N]; \end{cases}$$

$$\hat{v}_{\mathbb{I}}(t) := \begin{cases} v^0 & \text{for } t \in [0, t_{1/2}], \\ v^n + \dfrac{t - t_{n+1/2}}{\tau_{n+1/2}}(v^n - v^{n-1}) & \\ \quad \text{for } t \in (t_{n-1/2}, t_{n+1/2}] \ (n = 1, 2, \dots, N-1), & \\ v^{N-1} & \text{for } t \in (t_{N-1/2}, t_N]. \end{cases}$$

Note that $\hat{v}_{\mathbb{I}}$ is piecewise linear and continuous, and thus differentiable in the weak sense.

For simplicity only, we henceforth restrict ourselves to the case $A(t)0 \equiv 0$, $C(t)0 \equiv 0$ ($t \in [0,T]$). This is possible without loss of generality since, otherwise, we can replace $f(t)$ by $f(t) - A(t)0 - C(t)0$ ($t \in [0,T]$). Note that then $\lambda = 0$ in Assumption A.

For the right-hand side, we employ the natural restriction

$$f^n := \frac{1}{\tau_{n+1/2}} \int_{t_{n-1/2}}^{t_{n+1/2}} f(t)\,\mathrm{d}t,$$

which is well defined for $f \in L^{p^*}(0,T;W^*) \cong (L^p(0,T;W))^*$. We also define

$$f_{\mathbb{I}}(t) := \begin{cases} 0 & \text{for } t \in [0, t_{1/2}], \\ f^n & \text{for } t \in (t_{n-1/2}, t_{n+1/2}]\ (n = 1, 2, \dots, N-1), \\ 0 & \text{for } t \in (t_{N-1/2}, t_N]; \end{cases}$$

$$A_{\mathbb{I}}(t) := \begin{cases} A(t_1) & \text{for } t \in [0, t_{1/2}], \\ A(t_n) & \text{for } t \in (t_{n-1/2}, t_{n+1/2}]\ (n = 1, 2, \dots, N-1), \\ A(t_{N-1}) & \text{for } t \in (t_{N-1/2}, t_N]; \end{cases}$$

$$C_{\mathbb{I}}(t) := \begin{cases} C(t_1) & \text{for } t \in [0, t_{1/2}], \\ C(t_n) & \text{for } t \in (t_{n-1/2}, t_{n+1/2}]\ (n = 1, 2, \dots, N-1), \\ C(t_{N-1}) & \text{for } t \in (t_{N-1/2}, t_N]. \end{cases}$$

For $w \in L^2(0,T;V)$, we introduce the operator K via

$$(Kw)(t) := \int_0^t w(s)\,\mathrm{d}s.$$

Obviously, K maps $L^2(0,T;V)$ into itself and is linear and bounded.

Moreover, we set

$$\sigma(\mathbb{I}) := \sum_{n=2}^{N} \tau_{n-1/2} \left(\frac{r_n - 1}{r_n + 1} \right)^2 = \frac{1}{2} \sum_{n=2}^{N} \frac{(\tau_n - \tau_{n-1})^2}{\tau_n + \tau_{n-1}}.$$

In the sequel, we consider a sequence $\{\mathbb{I}_\ell\}_{\ell \in \mathbb{N}}$ of time grids (6). The crucial assumptions are as follows.

Assumption $\mathbb{I}$ $\{\mathbb{I}_\ell\}_{\ell \in \mathbb{N}}$ is a sequence of time grids (6) with

$$\tau_{\max}(\mathbb{I}_\ell) \to 0 \quad \text{as } \ell \to \infty, \qquad \sup_{\ell \in \mathbb{N}} \tau_{\max}(\mathbb{I}_\ell) < \big(\max\big(2\varkappa, \beta_B c_{W \hookrightarrow V}^2 \mu_A^{-1}\big)\big)^{-1},$$

$$\inf_{\ell \in \mathbb{N}} r_{\min}(\mathbb{I}_\ell) > 0, \qquad \sup_{\ell \in \mathbb{N}} r_{\max}(\mathbb{I}_\ell) < \infty, \qquad \sup_{\ell \in \mathbb{N}} c_\gamma(\mathbb{I}_\ell) < \infty,$$

$$\sigma(\mathbb{I}_\ell) \to 0 \quad \text{as } \ell \to \infty.$$

Assumption IC The initial values for (5) satisfy

$$\{u^0(\mathbb{I}_\ell)\} \subseteq V, \quad u^0(\mathbb{I}_\ell) \to u_0 \quad \text{in } V \text{ as } \ell \to \infty,$$
$$\{v^0(\mathbb{I}_\ell)\} \subseteq W, \quad v^0(\mathbb{I}_\ell) \to v_0 \quad \text{in } H \text{ as } \ell \to \infty, \quad \sup_{\ell\in\mathbb{N}} \tau_{\max}(\mathbb{I}_\ell)\big|\!\big|\!\big| v^0(\mathbb{I}_\ell)\big|\!\big|\!\big|^p < \infty.$$

Theorem 2 *Let Assumptions A, B, $\mathbb{I}$, and IC be fulfilled, and let $u_0 \in V$, $v_0 \in H$, and $f \in (L^p(0,T;W))^*$. If $\varkappa \neq 0$ or $C(t) \not\equiv 0$ assume that W is compactly embedded in H. Then there is a subsequence, denoted by ℓ', such that the piecewise constant prolongations $u_{\mathbb{I}_{\ell'}}$ converge weakly* in $L^\infty(0,T;V)$ towards an exact solution $u \in \mathcal{C}([0,T];V)$ to (4). Moreover, the piecewise constant prolongations $v_{\mathbb{I}_{\ell'}}$ as well as the piecewise linear prolongations $\hat{v}_{\mathbb{I}_{\ell'}}$ converge weakly in $L^p(0,T;W)$ and weakly* in $L^\infty(0,T;H)$ towards $u' \in \mathcal{W}$, and $\hat{v}'_{\mathbb{I}_{\ell'}}$ converges weakly in $(L^p(0,T;W))^*$ towards $u'' \in (L^p(0,T;W))^*$.*

If W is compactly embedded in H, then $u_{\mathbb{I}_{\ell'}}$, $v_{\mathbb{I}_{\ell'}}$ and $\hat{v}_{\mathbb{I}_{\ell'}}$ also converge strongly in $L^r(0,T;H)$ for any $r \in [1,\infty)$ towards u and u', respectively.

If a solution to (4) is unique, then convergence takes place for the whole sequence.

The proof of the main theorem uses the following lemma.

Lemma 1 *Under the assumptions of Theorem 2, there is a subsequence, denoted by ℓ', and there are elements*

$$u \in \mathcal{C}\big([0,T];V\big), \qquad v \in \mathcal{W} \quad \text{with } u = u_0 + Kv,$$

such that

$$u_{\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;V), \qquad v_{\mathbb{I}_{\ell'}} \rightharpoonup v \quad \text{in } L^p(0,T;W),$$
$$v_{\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;H), \qquad \hat{v}_{\mathbb{I}_{\ell'}} \rightharpoonup v \quad \text{in } L^p(0,T;W),$$
$$\hat{v}_{\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;H), \qquad \hat{v}'_{\mathbb{I}_{\ell'}} \rightharpoonup v' \quad \text{in } \big(L^p(0,T;W)\big)^*,$$
$$Kv_{\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} Kv \quad \text{in } L^\infty(0,T;W) \text{ as } \ell' \to \infty.$$

Moreover, for any $r \in [1,\infty)$,

$$u_0 + Kv_{\mathbb{I}_\ell} - u_{\mathbb{I}_\ell} \to 0 \quad \text{in } L^r(0,T;V) \text{ as } \ell \to \infty. \tag{10}$$

If W is compactly embedded in H, then for any $r \in [1,\infty)$,

$$u_0 + Kv_{\mathbb{I}_{\ell'}} \to u \quad \text{in } \mathcal{C}\big([0,T];H\big), \qquad u_{\mathbb{I}_{\ell'}} \to u \quad \text{in } L^r(0,T;H),$$
$$v_{\mathbb{I}_{\ell'}} \to v \quad \text{in } L^r(0,T;H), \qquad \hat{v}_{\mathbb{I}_{\ell'}} \to v \quad \text{in } L^r(0,T;H) \text{ as } \ell' \to \infty.$$

Proof A direct consequence of the a priori estimates in Theorem 1 is the boundedness of $\{u_{\mathbb{I}_\ell}\}$ in $L^\infty(0,T;V)$, and of $\{v_{\mathbb{I}_\ell}\}$ and $\{\hat{v}_{\mathbb{I}_\ell}\}$ in $L^p(0,T;W)$ as well as in $L^\infty(0,T;H)$, as a straightforward calculation of the corresponding norms shows. Moreover, $\{\hat{v}'_{\mathbb{I}_\ell}\}$ is bounded in $(L^p(0,T;W))^*$.

By standard arguments, we thus have a subsequence, denoted by ℓ', and elements $u \in L^\infty(0,T;V)$, $v \in L^p(0,T;W) \cap L^\infty(0,T;H)$, $\hat{v} \in \mathcal{W}$ such that

$$u_{\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;V), \qquad v_{\mathbb{I}_{\ell'}} \rightharpoonup v \quad \text{in } L^p(0,T;W),$$
$$v_{\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;H), \qquad \hat{v}_{\mathbb{I}_{\ell'}} \rightharpoonup \hat{v} \quad \text{in } L^p(0,T;W),$$
$$\hat{v}_{\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} \hat{v} \quad \text{in } L^\infty(0,T;H), \qquad \hat{v}'_{\mathbb{I}_{\ell'}} \rightharpoonup \hat{v}' \quad \text{in } \big(L^p(0,T;W)\big)^*.$$

We now prove $v = \hat{v}$. From the definition of the piecewise prolongations, we immediately find

$$\begin{aligned} &\big\| \hat{v}_{\mathbb{I}_\ell} - v_{\mathbb{I}_\ell} \big\|^2_{L^2(0,T;H)} \\ &\quad \le \tau_{\max}(\mathbb{I}_\ell) \left(\big| v^0(\mathbb{I}_\ell) \big|^2 + \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \big| v^n(\mathbb{I}_\ell) - v^{n-1}(\mathbb{I}_\ell) \big|^2 + \big| v^{N-1}(\mathbb{I}_\ell) \big|^2 \right), \end{aligned} \tag{11}$$

but in view of the a priori estimates from Theorem 1, the right-hand side tends to zero as $\ell \to \infty$. This shows, by density, the coincidence of the weak limits v and $\hat{v}$.

For proving

$$K v_{\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} K v \quad \text{in } L^\infty(0,T;W) \cong \big(L^1\big(0,T;W^*\big)\big)^*,$$

let $g \in L^1(0,T;W^*)$ be arbitrary. Then

$$\begin{aligned} \langle g, K v_{\mathbb{I}_{\ell'}} - K v \rangle &= \int_0^T \left\langle g(t), \int_0^t \big(v_{\mathbb{I}_{\ell'}}(s) - v(s) \big) \, \mathrm{d}s \right\rangle \mathrm{d}t \\ &= \int_0^T \int_0^t \big\langle g(t), v_{\mathbb{I}_{\ell'}}(s) - v(s) \big\rangle \, \mathrm{d}s \, \mathrm{d}t. \end{aligned}$$

A change of the integration variables yields

$$\begin{aligned} \langle g, K v_{\mathbb{I}_{\ell'}} - K v \rangle &= \int_0^T \int_s^T \big\langle g(t), v_{\mathbb{I}_{\ell'}}(s) - v(s) \big\rangle \, \mathrm{d}t \, \mathrm{d}s \\ &= \int_0^T \left\langle \int_s^T g(t) \, \mathrm{d}t, v_{\mathbb{I}_{\ell'}}(s) - v(s) \right\rangle \mathrm{d}s \\ &= \langle G, v_{\mathbb{I}_{\ell'}} - v \rangle. \end{aligned}$$

Since

$$G(s) := \int_s^T g(t) \, \mathrm{d}t \in L^\infty\big(0,T;W^*\big)$$

and since

$$v_{\mathbb{I}_{\ell'}} \rightharpoonup v \quad \text{in } L^p(0,T;W),$$

the right-hand side in the foregoing identity tends to zero.

With (7) and Hölder's inequality, we find (without writing out the dependence on $\mathbb{I}_\ell$ for a moment)

$$\begin{aligned}
&\|u_0 + Kv_{\mathbb{I}} - u_{\mathbb{I}}\|^2_{L^2(0,T;V)} \\
&\quad = \int_0^{t_{1/2}} \|u_0\|^2 \,\mathrm{d}t + \sum_{n=1}^{N-1} \int_{t_{n-1/2}}^{t_{n+1/2}} \left\| u_0 + \sum_{j=1}^{n-1} \tau_{j+1/2} v^j + (t - t_{n-1/2}) v^n - u^n \right\|^2 \mathrm{d}t \\
&\qquad + \int_{t_{N-1/2}}^{T} \left\| u_0 + \sum_{j=1}^{N-1} \tau_{j+1/2} v^j \right\|^2 \mathrm{d}t \\
&\quad \le c\tau_{\max} \|u_0\|^2 + c\|u_0 - u^0\|^2 + c\left(\sum_{j=1}^{N-1} |\tau_{j+1/2} - \tau_{j+1}| \|v^j\| \right)^2 \\
&\qquad + c\tau_{\max}^2 \sum_{n=1}^{N-1} \tau_{n+1/2} \|v^n\|^2 + c\tau_{\max}^2 \|v^0\|^2 + c\tau_{\max} \sum_{j=1}^{N-1} \tau_{j+1/2} \|v^j\|^2,
\end{aligned}$$

where

$$\left(\sum_{j=1}^{N-1} |\tau_{j+1/2} - \tau_{j+1}| \, \|v^j\| \right)^2 \le \sigma(\mathbb{I}) \sum_{j=1}^{N-1} \tau_{j+1/2} \|v^j\|^2.$$

In view of the embedding of W in V and since $p \ge 2$, we also have $\|v^j\|^2 \le c(1 + |||v^j|||^p)$. The a priori estimates of Theorem 1 together with the assumptions on the sequence of time grids (in particular, on the deviation from an equidistant grid) and on the initial values now yield the strong convergence of $\{u_0 + Kv_{\mathbb{I}_\ell} - u_{\mathbb{I}_\ell}\}$ towards zero in $L^2(0,T;V)$. Since $\{v_{\mathbb{I}_\ell}\}$ is bounded in $L^p(0,T;W) \hookrightarrow L^p(0,T;V)$, $\{Kv_{\mathbb{I}_\ell}\}$ is bounded in $L^\infty(0,T;V)$. Since also $\{u_{\mathbb{I}_\ell}\}$ is bounded in $L^\infty(0,T;V)$, we thus obtain the strong convergence in $L^r(0,T;V)$ for any $r \in [1,\infty)$, as is claimed in (10).

We now show $u = u_0 + Kv$. However, this is a direct consequence of what was shown before:

$$u_0 + Kv - u = u_0 + Kv - Kv_{\mathbb{I}_{\ell'}} + Kv_{\mathbb{I}_{\ell'}} - u_{\mathbb{I}_{\ell'}} + u_{\mathbb{I}_{\ell'}} - u \rightharpoonup 0 \quad \text{in } L^2(0,T;V).$$

Since $u_0 \in V$ and $Kv \in \mathcal{C}([0,T];V)$, we also find $u \in \mathcal{C}([0,T];V)$.

If W is compactly embedded in H, the Lions–Aubin theorem immediately allows us to conclude from the boundedness of $\{\hat{v}_{\mathbb{I}_\ell}\}$ in $\mathcal{W}$ with the strong convergence of a subsequence of $\{\hat{v}_{\mathbb{I}_\ell}\} \subseteq L^\infty(0,T;H)$ in $L^p(0,T;H)$ and thus in $L^r(0,T;H)$ for any $r \in [1,\infty)$. By density, the limit can only be v. Because of (11) and the a priori estimates from Theorem 1, we already know that $\{\hat{v}_{\mathbb{I}_\ell} - v_{\mathbb{I}_\ell}\}$ converges strongly towards zero in $L^2(0,T;H)$. Hence, $\{v_{\mathbb{I}_{\ell'}}\} \subseteq L^\infty(0,T;H)$ converges strongly towards v in $L^2(0,T;H)$ and thus in $L^r(0,T;H)$ for any $r \in [1,\infty)$.

It is then straightforward to show $Kv_{\mathbb{I}_{\ell'}} \to Kv = u - u_0$ in $\mathcal{C}([0,T];H)$. This together with (10) implies the strong convergence $u_{\mathbb{I}_{\ell'}} \to u$ in $L^r(0,T;H)$ for any $r \in [1,\infty)$. □

Proof of Theorem 2 For readability, we omit the subscripts ℓ and ℓ'.

The numerical scheme (8) can be written as

$$\hat{v}'_{\mathbb{I}} + A_{\mathbb{I}} v_{\mathbb{I}} + B_0 K v_{\mathbb{I}} + C_{\mathbb{I}} u_{\mathbb{I}} = f_{\mathbb{I}} + B_0 (K v_{\mathbb{I}} - u_{\mathbb{I}}). \tag{12}$$

This equation holds in $(L^p(0,T;W))^*$.

The growth condition for A shows that A maps subsets bounded in $L^p(0,T;W)$ into subsets bounded in $(L^p(0,T;W))^*$. Therefore, $\{A_{\mathbb{I}} v_{\mathbb{I}}\}$ is bounded in $(L^p(0,T;W))^*$, and by standard arguments we have a subsequence and an element $a \in (L^p(0,T;W))^*$ such that

$$A_{\mathbb{I}} v_{\mathbb{I}} \rightharpoonup a \quad \text{in } \big(L^p(0,T;W)\big)^*. \tag{13}$$

With respect to B_0, we observe that B_0 is a linear and bounded mapping of $L^2(0,T;V)$ into $L^2(0,T;V^*)$. Since $\{v_{\mathbb{I}}\}$ is bounded in $L^p(0,T;W)$ and thus in $L^2(0,T;V)$ and since $K : L^2(0,T;V) \to L^2(0,T;V)$ is bounded, also $\{B_0 K v_{\mathbb{I}}\}$ is bounded in $(L^2(0,T;V))^*$. Hence, there is a subsequence and an element $b \in (L^2(0,T;V))^*$ such that

$$B_0 K v_{\mathbb{I}} \rightharpoonup b \quad \text{in } \big(L^2(0,T;V)\big)^*. \tag{14}$$

For the term $C_{\mathbb{I}} u_{\mathbb{I}}$, we observe the following. In view of the continuity of $C = C(t)$ with respect to t (see Assumption B), we have

$$\big\|\big|C_{\mathbb{I}}(t)u(t) - C(t)u(t)\big\|\big|_* \to 0$$

for almost all $t \in (0,T)$. Moreover, the growth condition for $C(t)$ ($t \in [0,T]$) leads to

$$\big\|\big|C_{\mathbb{I}}(t)u(t) - C(t)u(t)\big\|\big|_*^{p^*} \le c\big(1 + \|u(t)\|^2\big),$$

and the right-hand side is integrable. Lebesgue's theorem on the dominated convergence now yields the strong convergence

$$C_{\mathbb{I}} u \to Cu \quad \text{in } \big(L^p(0,T;W)\big)^*.$$

From the Hölder-type continuity of $C(t)$ ($t \in [0,T]$) (see Assumption B), we find with Hölder's inequality

$$\begin{aligned} &\|C_{\mathbb{I}} u_{\mathbb{I}} - C_{\mathbb{I}} u\|_{(L^p(0,T;W))^*} \\ &\quad \le \alpha\big(\max\big(\|u_{\mathbb{I}}\|_{L^\infty(0,T;V)}, \|u\|_{L^\infty(0,T;V)}\big)\big)\|u_{\mathbb{I}} - u\|_{L^1(0,T;H)}^{1/p^*}. \end{aligned}$$

This, together with the first a priori estimate in Theorem 1 and the strong convergence result in Lemma 1, shows that $C_{\mathbb{I}} u_{\mathbb{I}} - C_{\mathbb{I}} u \to 0$ in $(L^p(0,T;W))^*$. We thus have

$$C_{\mathbb{I}} u_{\mathbb{I}} \to Cu \quad \text{in } \big(L^p(0,T;W)\big)^*. \tag{15}$$

For the right-hand side in (12), we observe the following. By standard arguments, we obtain

$$f_{\mathbb{I}} \to f \quad \text{in } \big(L^p(0,T;W)\big)^*. \tag{16}$$

Again, since $B_0 : L^2(0,T;V) \to L^2(0,T;V^*)$ is linear and bounded, we conclude from (10) that

$$B_0(Kv_{\mathbb{I}} - u_{\mathbb{I}}) \to -B_0 u_0 \quad \text{in } L^2\big(0,T;V^*\big). \tag{17}$$

Finally, we derive from (12) in the limit

$$v' + a + b + Cu = f - B_0 u_0 \quad \text{in } \big(L^p(0,T;W)\big)^*. \tag{18}$$

It remains to show that $v \in \mathcal{W}$ fulfils the initial condition and that $a + b = Av + B_0 K v$.

With $\hat{v}_{\mathbb{I}}(0) = v^0$ and Assumption IC, we have

$$\hat{v}_{\mathbb{I}}(0) \to v_0 \quad \text{in } H. \tag{19}$$

With $\hat{v}_{\mathbb{I}}(T) = v^{N-1}$ and the first a priori estimate in Theorem 1, we can choose the subsequence such that

$$\hat{v}_{\mathbb{I}}(T) \rightharpoonup \xi \quad \text{in } H \tag{20}$$

for some $\xi \in H$. Since $\hat{v}_{\mathbb{I}} \in \mathcal{W}$, we can employ integration by parts, which yields for all $w \in W$ and $\varphi \in \mathcal{C}^1([0,T])$ by inserting (18) and (12)

$$\begin{aligned}
&\big(v(T),w\big)\varphi(T) - \big(v(0),w\big)\varphi(0)\\
&\quad= \int_0^T \Big(\big\langle v'(t),w\big\rangle\varphi(t) + \langle v(t),w\rangle\varphi'(t)\Big)\,\mathrm{d}t\\
&\quad= \int_0^T \Big(\big\langle f(t) - B_0u_0 - a(t) - b(t) - C(t)u(t),w\big\rangle\varphi(t) + \big\langle v(t),w\big\rangle\varphi'(t)\Big)\,\mathrm{d}t\\
&\quad= \int_0^T \Big(\big\langle f(t) - f_{\mathbb{I}}(t) + \hat{v}_{\mathbb{I}}'(t) + A_{\mathbb{I}}(t)v_{\mathbb{I}}(t) + B_0Kv_{\mathbb{I}}(t) + C_{\mathbb{I}}(t)u_{\mathbb{I}}(t)\\
&\qquad - B_0(Kv_{\mathbb{I}} - u_{\mathbb{I}})(t) - B_0u_0 - a(t) - b(t) - C(t)u(t),w\big\rangle\varphi(t)\\
&\qquad + \big\langle v(t),w\big\rangle\varphi'(t)\Big)\,\mathrm{d}t\\
&\quad= \int_0^T \Big(\big\langle f(t) - f_{\mathbb{I}}(t) + A_{\mathbb{I}}(t)v_{\mathbb{I}}(t) - a(t) + B_0Kv_{\mathbb{I}}(t) - b(t) + C_{\mathbb{I}}(t)u_{\mathbb{I}}(t)\\
&\qquad - C(t)u(t) - B_0(u_0 + Kv_{\mathbb{I}} - u_{\mathbb{I}})(t),w\big\rangle\varphi(t)\\
&\qquad + \big\langle v(t) - \hat{v}_{\mathbb{I}}(t),w\big\rangle\varphi'(t)\Big)\,\mathrm{d}t + \big(\hat{v}_{\mathbb{I}}(T),w\big)\varphi(T) - \big(\hat{v}_{\mathbb{I}}(0),w\big)\varphi(0).
\end{aligned}$$

Taking the limit on the right-hand side, we obtain

$$\big(v(T),w\big)\varphi(T) - \big(v(0),w\big)\varphi(0) = (\xi,w)\varphi(T) - (v_0,w)\varphi(0).$$

Choosing $\varphi(T)=0$ and $\varphi(0)=0$, respectively, we find

$$v(0)=v_0 \quad \text{and} \quad v(T)=\xi \tag{21}$$

in H since $W \ni w$ is dense in H.

In what follows, we wish to employ the monotonicity of $A(t)+\varkappa I : W \to W^*$ ($t \in [0,T]$) and thus of $A_{\mathbb{I}}+\varkappa I$ as a mapping of $L^p(0,T;W)$ into $(L^p(0,T;W))^*$ as well as the positivity of the linear operator B_0K. Indeed, the linear operator B_0K is positive as a mapping of $L^2(0,T;V)$ into $L^2(0,T;V^*)$ and thus also as a mapping of $L^p(0,T;W) \subseteq L^2(0,T;V)$ into $(L^p(0,T;W))^*$: Since the linear operator $B_0 : V \to V^*$ defines an inner product on V and since $(Kw)'=w$ for all $w \in L^2(0,T;V)$, we find with integration by parts

$$\begin{aligned}\langle B_0Kw, w\rangle &= \int_0^T \big\langle B_0(Kw)(t), w(t)\big\rangle\,\mathrm{d}t = \int_0^T \big\langle B_0(Kw)(t), (Kw)'(t)\big\rangle\,\mathrm{d}t\\ &= \frac{1}{2}\Big(\big\langle B_0(Kw)(T), (Kw)(T)\big\rangle - \big\langle B_0(Kw)(0), (Kw)(0)\big\rangle\Big) \ge 0.\end{aligned}$$

In the last step, we have employed the facts that $B_0 : V \to V^*$ is positive and that $(Kw)(0)=0$.

For $\varkappa=0$, we can now proceed similarly as in the proof of [6, Theorem 4.2]. For arbitrary $w \in L^p(0,T;W)$, we find

$$\begin{aligned}\big\langle (A_{\mathbb{I}}+B_0K)v_{\mathbb{I}}, v_{\mathbb{I}}\big\rangle &\ge \big\langle (A_{\mathbb{I}}+B_0K)v_{\mathbb{I}}, v_{\mathbb{I}}\big\rangle\\ &\quad - \big\langle (A_{\mathbb{I}}+B_0K)v_{\mathbb{I}} - (A_{\mathbb{I}}+B_0K)w, v_{\mathbb{I}}-w\big\rangle\\ &= \big\langle (A_{\mathbb{I}}+B_0K)v_{\mathbb{I}}, w\big\rangle + \big\langle (A_{\mathbb{I}}+B_0K)w, v_{\mathbb{I}}-w\big\rangle.\end{aligned}$$

We thus obtain from (12)

$$\begin{aligned}0 &= \big\langle \hat{v}_{\mathbb{I}}' + (A_{\mathbb{I}}+B_0K)v_{\mathbb{I}} + C_{\mathbb{I}}u_{\mathbb{I}} - f_{\mathbb{I}} - B_0(Kv_{\mathbb{I}}-u_{\mathbb{I}}), v_{\mathbb{I}}\big\rangle\\ &\ge \big\langle \hat{v}_{\mathbb{I}}', v_{\mathbb{I}}\big\rangle + \big\langle (A_{\mathbb{I}}+B_0K)v_{\mathbb{I}}, w\big\rangle + \big\langle (A_{\mathbb{I}}+B_0K)w, v_{\mathbb{I}}-w\big\rangle\\ &\quad + \langle C_{\mathbb{I}}u_{\mathbb{I}}, v_{\mathbb{I}}\rangle - \langle f_{\mathbb{I}}, v_{\mathbb{I}}\rangle - \big\langle B_0(Kv_{\mathbb{I}}-u_{\mathbb{I}}), v_{\mathbb{I}}\big\rangle.\end{aligned} \tag{22}$$

For the first term on the right-hand side of (22), we observe with (9)

$$\begin{aligned}\big\langle \hat{v}_{\mathbb{I}}', v_{\mathbb{I}}\big\rangle &= \sum_{n=1}^{N-1} \int_{t_{n-1/2}}^{t_{n+1/2}} \left(\frac{v^n - v^{n-1}}{\tau_{n+1/2}}, v^n\right)\mathrm{d}t = \sum_{n=1}^{N-1} \big(v^n - v^{n-1}, v^n\big)\\ &\ge \frac{1}{2}\big|v^{N-1}\big|^2 - \frac{1}{2}\big|v^0\big|^2 = \frac{1}{2}\big|\hat{v}_{\mathbb{I}}(T)\big|^2 - \frac{1}{2}\big|\hat{v}_{\mathbb{I}}(0)\big|^2.\end{aligned}$$

With (19), (20), (21), we thus find by integration by parts

$$\liminf \big\langle \hat{v}_{\mathbb{I}}', v_{\mathbb{I}}\big\rangle \ge \frac{1}{2}\big|v(T)\big|^2 - \frac{1}{2}\big|v(0)\big|^2 = \big\langle v', v\big\rangle. \tag{23}$$

Because of Assumption A, we have for almost all $t \in (0, T)$

$$A_{\mathbb{I}}(t)w(t) - A(t)w(t) \to 0 \quad \text{in } W^*.$$

On the other hand, we find from the growth condition that for almost all $t \in (0, T)$

$$\big\|\big|A_{\mathbb{I}}(t)w(t) - A(t)w(t)\big\|\big|_*^{p^*} \le c\big(1 + \big\|\big|w(t)\big\|\big|^p\big),$$

and the right-hand side is integrable. Hence, Lebesgue's theorem shows that

$$A_{\mathbb{I}}w - Aw \to 0 \quad \text{in } \big(L^p(0, T; W)\big)^*.$$

With (13), (14), (15), (17), (16), (23), and the weak convergence of $\{v_{\mathbb{I}}\}$ towards v, we now obtain from (22) and (18)

$$\begin{aligned} 0 &\ge \big\langle v', v\big\rangle + \langle a + b, w\rangle + \big\langle (A + B_0K)w, v - w\big\rangle + \langle Cu, v\rangle - \langle f, v\rangle + \langle B_0u_0, v\rangle \\ &= -\langle a + b, v - w\rangle + \big\langle (A + B_0K)w, v - w\big\rangle, \end{aligned}$$

which yields

$$\langle a + b, v - w\rangle \ge \big\langle (A + B_0K)w, v - w\big\rangle.$$

With $w = v \pm sz$ ($z \in L^p(0, T; W)$) and $s \to 0+$, the hemicontinuity of A and the continuity of B_0K prove $a + b = (A + B_0K)v$ in $(L^p(0, T; W))^*$.

For $\varkappa \ne 0$, we only have that $A_{\mathbb{I}} + \varkappa I + B_0K : L^p(0, T; W) \to (L^p(0, T; W))^*$ is monotone and coercive. We thus may replace $A_{\mathbb{I}}$ by $A_{\mathbb{I}} + \varkappa I$ on the left-hand side of (12) and have to "correct" this by considering the term $\varkappa v_{\mathbb{I}}$ on the right-hand side of (12). In view of Lemma 1, this term, however, converges strongly in $L^{p^*}(0, T; H)$ and thus in $L^{p^*}(0, T; W^*) \cong (L^p(0, T; W))^*$ towards $\varkappa v$.

Finally, we have that $v \in \mathcal{W}$ fulfils the initial condition $v(0) = v_0$ and the equation

$$v' + Av + B_0(u_0 + Kv) + Cu = f \quad \text{in } \big(L^p(0, T; W)\big)^*.$$

As was already shown in Lemma 1, we have $u = u_0 + Kv$ and thus $u' = (Kv)' = v \in \mathcal{W}$. This shows $u(0) = u_0$, $u'(0) = v_0$ as well as

$$u'' + Au' + B_0u + Cu = f \quad \text{in } \big(L^p(0, T; W)\big)^*.$$

So, u is a solution to the original problem (4).

By contradiction, we can show that not only a subsequence but the whole sequence converges towards u and v, respectively, if a solution to (4) is unique. □

Remark 1 Assumption IC on the sequence $\{v^0(\mathbb{I}_\ell)\}_{\ell\in\mathbb{N}}$ can always be fulfilled for $v_0 \in H$ since W is dense in H. Assumption $\mathbb{I}$ on σ and c_γ, i.e., on the ratios of adjacent step sizes, is obviously fulfilled for an equidistant partition but also for variable time grids that are a perturbation of an equidistant partition.

Remark 2 Theorem 2 applies to the weak formulation of (2) and (3) in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, supplemented by initial and, e.g., homogeneous Dirichlet boundary conditions, with the function spaces $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, the operator $B_0 : V \to V^*$ defined via

$$\langle B_0 v, w\rangle = \int_\Omega \nabla v(x) \cdot \nabla w(x)\, \mathrm{d}x,$$

and

– for (2) with $p = 2$, $W = H_0^1(\Omega)$, $A(t) : W \to W^*$ ($t \in [0, T]$) defined via

$$\big\langle A(t)v, w\big\rangle = \int_\Omega \psi\big(x, t, \big|\nabla v(x)\big|\big)\nabla v(x) \cdot \nabla w(x)\, \mathrm{d}x,$$

  see also [18, pp. 928ff.] for more details on the weak formulation of (2);
– for (3) with $p \ge 2$, $W = W_0^{1,p}(\Omega)$, $A(t) \equiv A : W \to W^*$ defined via

$$\langle Av, w\rangle = \int_\Omega \big|\nabla v(x)\big|^{p-2}\nabla v(x) \cdot \nabla w(x)\, \mathrm{d}x,$$

  $C(t) \equiv C : V \to W^*$ defined via

$$\langle Cv, w\rangle = \int_\Omega c\big(v(x), \nabla v(x)\big)w(x)\, \mathrm{d}x,$$

  see also [14, pp. 88ff.] for more details on the weak formulation of (3) and a discussion of appropriate boundary conditions (in the case $c \equiv 0$). The requirements on the operator C in Assumption B are fulfilled if, e.g., $c(u, \nabla u) = \sin u$ or $c(u, \nabla u) = |u|^\gamma u$ with $\gamma \le 1 - \frac{2}{p}$ (this rather restrictive assumption ensures that the second time derivative of u is indeed in the dual of the space in which u' is), but more complicated semilinearities would also be allowed.

Equation (1) does not fit into our framework, since in this example $W = L^p(\Omega)$ is not embedded in $V = H_0^1(\Omega)$.

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