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Existence via time discretization for a class of doubly nonlinear operator-differential equations of Barenblatt-type [☆]

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ABSTRACT

The initial value problem for a first order operator-differential equation of type $M(u') + A(u, u') = f$ is studied, where both M and A are nonlinear operators. The equation can be interpreted as the quasistatic limit of a second order evolution equation with a severe coupling of the damping and nondamping term. Existence of a global-in-time weak solution is shown by proving convergence of a suitable time discretization method.

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1. Introduction

We are concerned with initial value problems of the type

$$M(u') + A(u, u') = f \quad \text{in } (0, T), \tag{1.1a}$$

$$u(0) = u_0, \tag{1.1b}$$

where $T > 0$ is the time under consideration and where u_0 and f are given data of the problem. The operator $M : V \rightarrow V'$ is a hemicontinuous, monotone, and coercive operator defined on a real, reflexive, separable Banach space V , whereas $A : V \times V \rightarrow V'$ is strongly continuous. We look for solutions $u : [0, T] \rightarrow V$, for which the time derivative also takes values in V . The coupling of u and u' by A is a severe difficulty in the analysis of the above problem.

An example we have in mind is the initial–boundary value problem for the partial differential equation of Barenblatt-type with $p(x)$ -Laplacian

$$h(\partial_t u) - \operatorname{div}(|\nabla \partial_t u|^{p(\cdot)-2} \nabla \partial_t u) - \operatorname{div} a_1(\cdot, u, \partial_t u) + a_0(\cdot, u, \partial_t u) = f, \tag{1.2}$$

where h is a strictly monotonically increasing, Lipschitz continuous function and where a_0, a_1 are Carathéodory functions satisfying, in particular, suitable growth and continuity conditions.

A much simpler though academic example illustrating our framework is the initial–boundary value problem for the partial differential equation

$$-\partial_x(\partial_{xt}^2 u)^3 + |u|^{7/4} \partial_t u = f \tag{1.3}$$

in one spatial dimension. The principal part of this equation is the 4-Laplacian applied to the first time derivative, which is then described by a uniformly monotone operator in a standard Sobolev space. The lower order term, which couples the solution u and its time derivative, corresponds to the operator A in (1.1a). Employing results on the continuous and compact embedding of Sobolev spaces, which make use of the fact that in this example the underlying domain is one-dimensional, allows to prove growth condition as well as strong continuity required below in the assumptions on A .

Let us consider the following time discretization of (1.1) on an equidistant time grid with abscissae $t_n = n\tau$ ($n = 0, 1, \dots, N \in \mathbb{N}$) and constant step size $\tau = T/N$: We look for approximations $u^n \approx u(t_n)$ such that

$$M\left(\frac{u^n - u^{n-1}}{\tau}\right) + A\left(u^{n-1}, \frac{u^n - u^{n-1}}{\tau}\right) = f^n, \quad n = 1, 2, \dots, N, \tag{1.4}$$

where $u^0 \approx u_0$ and $\{f^n\}_{n=1}^N \approx f$ are given approximations of the initial datum and right-hand side. Note that the numerical scheme above is implicit but explicit in the first argument of A .

Considering a sequence of time discrete problems (1.4) with $\tau \rightarrow 0$, we show convergence of the approximate solutions in a suitable sense. The limit is then shown to be a solution to the original problem (1.1), which proves existence of solutions to (1.1). A crucial assumption in the proof of convergence is the unique solvability of (1.4) from step to step. This uniqueness assumption is fulfilled for the example (1.3). We will also justify this assumption for the more intrigued example (1.2), which then requires to employ an L^1 -type technique.

The uniqueness assumption, of course, means a restriction of the class of problems we can deal with. Such a restriction, however, is expected since (1.1), interpreted as an operator equation posed on a suitable Banach space of time-dependent abstract functions with values in V , misses the standard assumptions for solvability such as pseudomonotonicity of the governing operator. In particular, the strong continuity of $A : V \times V \rightarrow V'$ does not imply strong continuity of the corresponding Nemytskii operator acting on Bochner integrable functions with values in V .

Equations of Barenblatt-type appear in the description of nonlinear and anomalous diffusion, see the classical work of Barenblatt [2]. Abstract problems of Barenblatt-type have been studied, e.g., in Colli [5] and in Bauzet and Vallet [3] (together with a stochastic version), see also the references cited therein. Doubly nonlinear equations including a first-order time derivative are also discussed in detail in Roubíček [20, Chapter 11], see also Gajewski, Gröger and Zacharias [11, Kap. V, §2], Ptashnyk [19], and Seam and Vallet [21] for pseudoparabolic equations. The doubly nonlinear problems considered in, e.g., Hokkanen and Moroşanu [13, Chapter 10] are, however, of a different type since $M(u')$ is to be replaced by $(M(u))'$. In contrast to (1.1), the problems considered so far do not include the coupling of u and u' .

We remark that (1.1) can also be written as the abstract Volterra integral equation

$$M(v) + A(Kv, v) = f, \quad \text{where } (Kv)(t) = u_0 + \int_0^t v(s) ds.$$

However, to the best knowledge of the authors, there are no results available from the theory on evolutionary Volterra equations (see the standard monographs Gripenberg, Londen and Staffans [12] and Průš [18]), which could be applied in our situation.

Finally, we emphasize that (1.1a) can be seen as the quasistatic limit of the second order equation

$$u'' + M(u') + A(u, u') = f \quad \text{in } (0, T).$$

For the analysis of nonlinear evolution equations of second order with damping, we refer in particular to the seminal work of Lions and Strauss [16]. For doubly nonlinear evolution equations of second order, see also Friedman and Nečas [10] as well as Emmrich and Thalhammer [8] and the references cited therein. Again, a coupling of u and u' as in (1.1) is not covered by the aforementioned results. A second order evolution equation with a severe coupling of u and u' , arising in the description of a viscoelastic beam, has recently been considered in Emmrich and Thalhammer [9], though the equation studied there does not fall into the class of problems considered here.

The paper is organized as follows: In Section 2, we introduce the functional analytic setting together with the general assumptions on the operators M and A , and we state the main result. The time discrete problem is studied in Section 3. Existence via convergence is then proved in Section 4. In Sections 5 and 6, we study the example (1.3) and (1.2), respectively.

1.1. Notation

Let $(V, \|\cdot\|)$ be a real, reflexive, separable Banach space and denote by $(V', \|\cdot\|_{V'})$ its dual with the duality pairing denoted by $\langle \cdot, \cdot \rangle$. Note that V' is reflexive and separable since V is reflexive and separable (see, e.g., Brézis [4, Coroll. III.24 on p. 48]).

Bochner–Lebesgue spaces $L^r(0, T; V)$ ($r \in [1, \infty]$) are defined in the usual way and equipped with the standard norm. Denoting by $r' = r/(r-1)$ the conjugate of $r \in (1, \infty)$ with $r' = \infty$ if $r = 1$, we have $(L^r(0, T; V))' = L^{r'}(0, T; V')$ if $r \in [1, \infty)$; the duality pairing is given by

$$\langle g, v \rangle = \int_0^T \langle g(t), v(t) \rangle dt,$$

see, e.g., Diestel and Uhl [7, Thm. 1 on p. 98, Coroll. 13 on p. 76, Thm. 1 on p. 79]. Moreover, $L^r(0, T; V)$ is reflexive if $r \in (1, \infty)$ (see [7, Coroll. 2 on p. 100]) and $L^1(0, T; V')$ is separable.

By $W^{1,r}(0, T; V)$ ($r \in [1, \infty]$), we denote the Banach space of functions $u \in L^r(0, T; V)$ whose distributional time derivative u' is again in $L^r(0, T; V)$; the space is equipped with the standard norm.

Note that if $u \in W^{1,r}(0, T; V)$ ($r \in [1, \infty]$) then u equals almost everywhere a function that is absolutely continuous on $[0, T]$ as a function taking values in V . Moreover, $W^{1,r}(0, T; V)$ ($r \in [1, \infty]$) is continuously embedded in $\mathcal{C}([0, T]; V)$, the Banach space of functions that are continuous on $[0, T]$ as functions with values in V (see, e.g., Roubíček [20, Chapter 7] for more details).

The norm in a Banach space X will always be denoted by $\|\cdot\|_X$, except for the norm in V , where we omit the subscript V . By c , we denote a generic positive constant.

2. Main result: existence via time discretization

The structural properties we assume for the operators M and A read as follows:

Assumption (M). The operator $M : V \rightarrow V'$ is hemicontinuous (i.e., the mapping $t \mapsto \langle M(u + tv), w \rangle$ is continuous on $[0, 1]$ for arbitrary $u, v, w \in V$), monotone, and coercive in the sense that there exist $p \in (1, \infty)$, $\mu > 0$, $\lambda \geq 0$ such that for all $v \in V$

$$\langle M(v), v \rangle \geq \mu \|v\|^p - \lambda. \quad (2.1)$$

Assumption (A). The operator $A : V \times V \rightarrow V'$ is strongly continuous (i.e., in $V \times V$ weakly convergent sequences are mapped into in V' strongly convergent sequences). Moreover, there exist $c > 0$, $q \in [0, p - 1]$ such that for all $u, v \in V$

$$\|A(u, v)\|_{V'} \leq c(1 + \|u\|^{p-1} + \|v\|^q). \quad (2.2)$$

Assumption (M, A). For each $u \in V$ and $b \in V'$ there exists a unique $v \in V$ such that

$$M(v) + A(u, v) = b \quad \text{in } V'. \quad (2.3)$$

Remark 2.1. The strong continuity of $A : V \times V \rightarrow V'$ in Assumption (A) follows, e.g., from the following Hölder-type continuity on bounded subsets: Let V be compactly embedded in the Banach space $(X, \|\cdot\|_X)$ and assume there exist $\delta_1, \delta_2 \in (0, 1]$ such that for any $R > 0$ there exists $C(R) > 0$ and for all $u, \bar{u}, v, \bar{v} \in V$ with $\max(\|u\|, \|\bar{u}\|, \|v\|, \|\bar{v}\|) \leq R$

$$\|A(u, v) - A(\bar{u}, \bar{v})\|_{V'} \leq C(R)(\|u - \bar{u}\|_X^{\delta_1} + \|v - \bar{v}\|_X^{\delta_2}).$$

Remark 2.2. Existence of a solution to (2.3) in Assumption (M, A) follows under Assumptions (M) and (A) immediately from Brézis' theorem on pseudomonotone operators (see, e.g., Roubíček [20, Thm. 2.6 on p. 31] or Zeidler [22, Thm. 27.A on p. 589]) since, in particular, for any $u \in V$ the operator $M(\cdot) + A(u, \cdot) : V \rightarrow V'$ is coercive. This is seen from (2.1) and the growth condition (2.2) together with Young's inequality. Uniqueness will be exemplified, in particular, in Section 6.

We shall remark that the operators M and A can be extended, as usual, to operators acting on functions defined on $[0, T]$ and taking values in V . Indeed, since the operator $M : V \rightarrow V'$ is hemicontinuous and monotone, it is also demicontinuous (see, e.g., Zeidler [22, Propos. 26.4 on p. 555]) and thus maps, in view of the theorem of Pettis (see, e.g., Diestel and Uhl [7, Thm. 2 on p. 42]), a Bochner measurable function $v : [0, T] \rightarrow V$ into a Bochner measurable function $M(v) : [0, T] \rightarrow V'$. Here we have invoked the separability of V' . Since we do not assume a growth condition on M and since M is, as a monotone operator, in general only locally bounded, we can, however, not assure that M maps $L^p(0, T; V)$ into its dual.

Because of the growth condition (2.2) and the continuity of the operator $A : V \times V \rightarrow V'$, it maps $L^p(0, T; V) \times L^p(0, T; V)$ into $(L^p(0, T; V))' = L^{p'}(0, T; V')$. A main difficulty with the operator A is, however, that the strong continuity as a mapping of $V \times V$ into V' does not pass to the time-dependent case with A being an operator mapping $L^p(0, T; V) \times L^p(0, T; V)$ into $L^{p'}(0, T; V')$.

The concept of solution to (1.1) we consider is as follows. We recall that $W^{1,p}(0, T; V)$ is continuously embedded in $\mathcal{C}([0, T]; V)$.

Definition 2.3 (Solution). Let $u_0 \in V$ and $f \in L^{p'}(0, T; V')$ be given. A function $u \in W^{1,p}(0, T; V)$ is said to be a solution to (1.1) if (1.1a) holds true in $L^{p'}(0, T; V')$ and if the initial condition $u(0) = u_0$ is fulfilled in V .

Let $\{N_\ell\}$ be a sequence of integers such that $N_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$. Consider the corresponding sequence of time discrete problems (1.4) with step size $\tau_\ell = T/N_\ell$, starting value $u_\ell^0 \in V$, and right-hand side $\{f^n\}_{n=1}^{N_\ell} \subset V$ given by

$$f^n := \frac{1}{\tau_\ell} \int_{t_{n-1}}^{t_n} f(t) dt, \quad \text{where } t_n = n\tau_\ell; \tag{2.4}$$

as a slight abuse of notation, we do not call the dependence of f^n and the time instances t_n on ℓ .

If $\{u^n\}_{n=1}^{N_\ell} \subset V$ denotes the solution to (1.4) with step size τ_ℓ then let u_ℓ be the piecewise constant function with $u_\ell(t) = u^n$ for $t \in (t_{n-1}, t_n]$ ($n = 1, 2, \dots, N_\ell$). We extend u_ℓ to the left by $u_\ell(t) = u_\ell^0$ for $t \in (-\tau_\ell, 0]$. Moreover, let \hat{u}_ℓ be the piecewise affine-linear interpolation of the points (t_n, u^n) ($n = 0, 1, \dots, N_\ell$) and v_ℓ be the piecewise constant function with $v_\ell(t) = (u^n - u^{n-1})/\tau_\ell$ for $t \in (t_{n-1}, t_n]$ ($n = 1, 2, \dots, N_\ell$), where $u^0 = u_\ell^0$.

We are now able to state the main result:

Theorem 2.4 (Existence via convergence of a time discretization). Let Assumptions (M), (A) and (M, A) hold true and let $u_0 \in V$ and $f \in L^{p'}(0, T; V')$ be given. Moreover, let $\{u_\ell^0\} \subset V$ be such that

$$u_\ell^0 \rightarrow u_0 \quad \text{in } V \text{ as } \ell \rightarrow \infty.$$

Then there is a subsequence (still denoted by ℓ) such that, as $\ell \rightarrow \infty$, the sequences $\{u_\ell\}$ of piecewise constant and $\{\hat{u}_\ell\}$ of piecewise affine-linear interpolants of the time discrete solutions to (1.4) with (2.4) converge weakly* in $L^\infty(0, T; V)$; the limit u is a solution to (1.1) in the sense of Definition 2.3. Moreover, the sequence $\{v_\ell\}$ of piecewise constant interpolants of the discrete time derivatives converges weakly in $L^p(0, T; V)$ towards u' .

If $f \in L^\infty(0, T; V')$ then $u \in W^{1,\infty}(0, T; V)$, $\{v_\ell\}$ converges weakly* in $L^\infty(0, T; V)$ towards u' , and (1.1a) holds in $L^\infty(0, T; V')$.

Let us note that if V is compactly embedded in a Banach space X then, in view of the Arzelà-Ascoli theorem, it easily follows that $\{\hat{u}_\ell\}$ converges strongly in $\mathcal{C}([0, T]; X)$ towards u as $\ell \rightarrow \infty$.

3. A priori estimates for the time discrete problem

In this section, we study the time discretization (1.4) for a given time step $\tau > 0$. Let us introduce, for $n = 1, 2, \dots, N$,

$$v^n := \frac{u^n - u^{n-1}}{\tau}. \tag{3.1a}$$

We then have

$$u^n = u^0 + \tau \sum_{j=1}^n v^j, \tag{3.1b}$$

and the scheme (1.4) can be written as

$$M(v^n) + A(u^{n-1}, v^n) = f^n. \tag{3.2}$$

Proposition 3.1 (Time discrete problem). *Let Assumptions (M), (A) and (M, A) hold true and let $u^0 \in V$ and $\{f^n\}_{n=1}^N \subset V'$ be given. Then there is a unique solution $\{u^n\}_{n=1}^N \subset V$ to (1.4). Moreover, for $n = 1, 2, \dots, N$, the following a priori estimate holds true:*

$$\|u^n\|^p + \tau \sum_{j=1}^n \|v^j\|^p \leq c \left(1 + \|u^0\|^p + \tau \sum_{j=1}^n \|f^j\|_{V'}^{p'} \right). \tag{3.3}$$

Proof. Existence and uniqueness of a solution follow immediately step-by-step from Assumption (M, A). For the a priori estimate, we test (3.2) by v^n and invoke the coercivity of M as well as the growth condition for A . This gives, with Young’s inequality and recalling that $q < p - 1$,

$$\begin{aligned} \mu \|v^n\|^p - \lambda &\leq \|f^n\|_{V'} \|v^n\| + \|A(u^{n-1}, v^n)\|_{V'} \|v^n\| \\ &\leq \|f^n\|_{V'} \|v^n\| + c(1 + \|u^{n-1}\|^{p-1} + \|v^n\|^q) \|v^n\| \\ &\leq \frac{\mu}{2} \|v^n\|^p + c \|f^n\|_{V'}^{p'} + c + c \|u^{n-1}\|^p. \end{aligned}$$

With (3.1) and Hölder’s inequality, we find

$$\|u^{n-1}\|^p \leq c \left(\|u^0\|^p + \tau \sum_{j=1}^{n-1} \|v^j\|^p \right).$$

We therefore come up with

$$\|v^n\|^p \leq c(1 + \|u^0\|^p + \|f^n\|_{V'}^{p'}) + c\tau \sum_{j=1}^{n-1} \|v^j\|^p.$$

Applying a discrete Gronwall-type argument shows that

$$\|v^n\|^p \leq c \left(1 + \|u^0\|^p + \|f^n\|_{V'}^{p'} + \tau \sum_{j=1}^{n-1} \|f^j\|_{V'}^{p'} \right) \tag{3.4}$$

and thus

$$\tau \sum_{j=1}^n \|v^j\|^p \leq c \left(1 + \|u^0\|^p + \tau \sum_{j=1}^n \|f^j\|_{V'}^{p'} \right),$$

which implies the assertion because of $\|u^n\| \leq \|u^0\| + \tau \sum_{j=1}^n \|v^j\|$. \square

We shall remark that, by standard techniques, we may also derive an estimate for $\tau \sum_j \|M(v^j)\|_{V'}^{p'}$ from the growth condition for A together with the above estimate. We will, however, not make use of such an estimate. Note that obtaining an estimate in V' is always problematic when considering a full discretization.

4. Proof of the main result: convergence of the time discretization

We now consider a sequence of time discrete problems (1.4) with step sizes $\tau_\ell = T/N_\ell$, where $N_\ell \in \mathbb{N}$ with $N_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$. For the approximation (2.4) of the right-hand side f , let f_ℓ be the piecewise constant function with $f_\ell = f^n$ if $t \in (t_{n-1}, t_n]$ ($n = 1, 2, \dots, N_\ell$).

Proof of Theorem 2.4. Since the sequence $\{u_\ell^0\}$ of starting values for (1.4) converges strongly in V towards $u_0 \in V$, it is also bounded. For $f \in L^{p'}(0, T; V')$ and $\{f^n\}_{n=1}^{N_\ell}$ given by (2.4), one can show that

$$\|f_\ell\|_{L^{p'}(0, T; V')}^{p'} = \tau_\ell \sum_{j=1}^{N_\ell} \|f^n\|_{V'}^{p'} \leq \|f\|_{L^{p'}(0, T; V')}^{p'}.$$

This shows that the right-hand side of the a priori estimate (3.3) is uniformly bounded with respect to ℓ . Moreover, using a density argument, one can also show that, as $\ell \rightarrow \infty$,

$$f_\ell \rightarrow f \quad \text{in } L^{p'}(0, T; V').$$

It immediately follows that $\{u_\ell\}$ as well as $\{\hat{u}_\ell\}$ are bounded in $L^\infty(0, T; V)$ and that $\{v_\ell\}$ is bounded in $L^p(0, T; V)$. By standard compactness arguments (see, e.g., Brézis [4, Coroll. III.26, Thm. III.27 on p. 50]) together with the separability of $L^1(0, T; V')$ and the reflexivity of $L^p(0, T; V)$, there exists a subsequence (denoted by ℓ') and elements $u, \hat{u} \in L^\infty(0, T; V)$, $v \in L^p(0, T; V)$ such that, as $\ell' \rightarrow \infty$,

$$u_{\ell'} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, T; V), \quad \hat{u}_{\ell'} \overset{*}{\rightharpoonup} \hat{u} \quad \text{in } L^\infty(0, T; V), \quad v_{\ell'} \rightharpoonup v \quad \text{in } L^p(0, T; V).$$

Because of

$$u_\ell(t) - \hat{u}_\ell(t) = (t_n - t)\hat{u}'_\ell(t) = (t_n - t)v_\ell(t), \quad t \in (t_{n-1}, t_n) \quad (n = 1, 2, \dots, N_\ell), \tag{4.1}$$

we find

$$\|u_\ell - \hat{u}_\ell\|_{L^p(0, T; V)} \leq \tau_\ell \|v_\ell\|_{L^p(0, T; V)},$$

which shows (recall that $\{v_\ell\}$ is bounded in $L^p(0, T; V)$) that $u = \hat{u}$. With $u_\ell - u_\ell(\cdot - \tau_\ell) = \tau_\ell v_\ell$, we similarly find

$$u_{\ell'}(\cdot - \tau_{\ell'}) \rightharpoonup u \quad \text{in } L^\infty(0, T; V).$$

Since $\hat{u}'_\ell = v_\ell$ in the weak sense and since $\hat{u}_{\ell'} \rightharpoonup u$ as well as $v_{\ell'} \rightharpoonup v$, both in $L^p(0, T; V)$, we obtain $v = u' \in L^p(0, T; V)$.

Let us recall that $W^{1,p}(0, T; V)$ is continuously embedded in $\mathcal{C}([0, T]; V)$. For any $t \in [0, T]$, we can therefore consider the trace operator $\Gamma_t : W^{1,p}(0, T; V) \rightarrow V$ with $\Gamma_t u = u(t)$. As a linear bounded operator, $\Gamma_t : W^{1,p}(0, T; V) \rightarrow V$ is weak-weak continuous (see, e.g., Brézis [4, Thm. III.9]). Since $\hat{u}_{\ell'} \rightharpoonup u$ in $W^{1,p}(0, T; V)$, we thus obtain

$$\hat{u}_{\ell'}(t) \rightharpoonup u(t) \quad \text{in } V \text{ for all } t \in [0, T].$$

Similarly to (4.1), we find

$$u_\ell(t - \tau_\ell) - \hat{u}_\ell(t) = -(t - t_{n-1})v^n, \quad t \in (t_{n-1}, t_n) \quad (n = 1, 2, \dots, N_\ell),$$

which implies, for all $t \in (0, T]$,

$$\|u_\ell(t - \tau_\ell) - \hat{u}_\ell(t)\| \leq \tau_\ell \max_{j=1, \dots, N_\ell} \|v^j\| \leq \tau_\ell \left(\sum_{j=1}^{N_\ell} \|v^j\|^p \right)^{1/p} \leq c\tau_\ell^{1/p'}.$$

In the last step, we have invoked the a priori estimate (3.3) so that $c > 0$ depends on u_0 and f . This, finally, shows that

$$u_{\ell'}(t - \tau_{\ell'}) \rightharpoonup u(t) \quad \text{in } V \text{ for all } t \in (0, T]. \tag{4.2}$$

Unfortunately, we cannot apply the above argumentation to $\{v_\ell\}$. Nevertheless, we can show the following: Estimate (3.4), together with the boundedness of $\{u_\ell^0\}$ in V , immediately implies, for all $t \in (0, T]$,

$$\|v_\ell(t)\| \leq c(1 + \|f_\ell(t)\|_{V'}).$$

Because of the strong convergence of $\{f_\ell\}$ in $L^{p'}(0, T; V')$, there is a subsequence of the subsequence ℓ' , which we still denote by ℓ' , and a pointwise majorizing function $g \in L^{p'}(0, T)$ such that (see, e.g., Brézis [4, Thm. IV.9] as well as the proof of Gajewski et al. [11, Satz 1.11 on p. 127])

$$f_{\ell'}(t) \rightarrow f(t) \quad \text{in } V', \quad \|f_{\ell'}(t)\|_{V'} \leq g(t) \quad \text{for almost all } t \in (0, T). \tag{4.3}$$

We thus obtain

$$\|v_{\ell'}(t)\| \leq c(1 + g(t)).$$

For almost all $t \in (0, T)$, we therefore have a subsequence of the subsequence ℓ' , which we denote by ℓ'_t and which depends on t , and an element $v_t \in V$ such that

$$v_{\ell'_t}(t) \rightarrow v_t \quad \text{in } V \text{ as } \ell'_t \rightarrow \infty. \tag{4.4}$$

We will later be able to show $v_t = v(t)$ for almost all $t \in (0, T)$ and that the subsequence is indeed independent of t .

The numerical scheme (1.4) (see also (3.2)) can be written as

$$M(v_\ell(t)) + A(u_\ell(t - \tau_\ell), v_\ell(t)) = f_\ell(t) \quad \text{in } V', \quad t \in (0, T). \tag{4.5}$$

The strong continuity of $A : V \times V \rightarrow V'$ (see Assumption (A)) together with (4.2), (4.3) and (4.4) now provides for almost all $t \in (0, T)$ the strong convergence

$$M(v_{\ell'_t}(t)) = f_{\ell'_t}(t) - A(u_{\ell'_t}(t - \tau_{\ell'_t}), v_{\ell'_t}(t)) \rightarrow f(t) - A(u(t), v_t) := m_t \quad \text{in } V' \text{ as } \ell'_t \rightarrow \infty.$$

The monotonicity and hemicontinuity of $M : V \rightarrow V'$ (see Assumption (M)) allows to identify the limit $m_t \in V'$ by employing Minty's trick: Let $z \in V$ be arbitrary. We then find

$$\begin{aligned} \langle M(v_{\ell'_t}(t)), v_{\ell'_t}(t) \rangle &= \langle M(v_{\ell'_t}(t)) - M(z), v_{\ell'_t}(t) - z \rangle + \langle M(v_{\ell'_t}(t)), z \rangle + \langle M(z), v_{\ell'_t}(t) - z \rangle \\ &\geq \langle M(v_{\ell'_t}(t)), z \rangle + \langle M(z), v_{\ell'_t}(t) - z \rangle. \end{aligned}$$

Taking the limit, we obtain

$$\langle m_t, v_t \rangle \geq \langle m_t, z \rangle + \langle M(z), v_t - z \rangle.$$

With $z = v_t \pm \theta w$ for arbitrary $w \in V$ and $\theta \in (0, 1]$, we thus have

$$\pm \langle M(v_t \pm \theta w), w \rangle \geq \pm \langle m_t, w \rangle,$$

and the hemicontinuity shows for $\theta \rightarrow 0$ that $m_t = M(v_t)$ in V' .

Because of the unique solvability of (2.3), the element $v_t \in V$ is the unique solution to

$$M(v_t) + A(u(t), v_t) = f(t) \quad \text{in } V'$$

for given $u(t) \in V$ and $f(t) \in V'$. This now shows, by contradiction, that the convergence (4.4) not only takes place for a subsequence ℓ'_t depending on t but for the whole sequence ℓ' . This, however, means that $\{v_{\ell'_t}(t)\}$ converges weakly in V towards v_t for almost all $t \in (0, T)$, whereas $\{v_{\ell'_t}\}$ converges weakly in $L^p(0, T; V)$ towards u' , which implies that for all $y \in V'$ the sequence of functions $t \mapsto \langle y, v_{\ell'_t}(t) \rangle$ converges almost everywhere to $t \mapsto \langle y, v_t \rangle$ as well as weakly in $L^p(0, T)$ towards $t \mapsto \langle y, u'(t) \rangle$. This proves $\langle y, v_t \rangle = \langle y, u'(t) \rangle$ for all $y \in V'$ and almost all $t \in (0, T)$ (see, e.g., Gajewski et al. [11, Lemma 1.19 on p. 27]) and thus $v_t = u'(t)$ in V' for almost all $t \in (0, T)$ with $u' \in L^p(0, T; V)$.

Therefore,

$$M(u'(t)) + A(u(t), u'(t)) = f(t) \quad \text{in } V'$$

holds for almost all $t \in (0, T)$. Since $f \in L^p(0, T; V')$ and since, in view of the growth condition (2.2), also $A(u, u') \in L^p(0, T; V')$, the limit $u \in W^{1,p}(0, T; V)$ is a solution to (1.1) in the sense of Definition 2.3.

Regarding the regularity statement, we observe the following: If $f \in L^\infty(0, T; V')$ then (3.4) already implies the boundedness and thus (passing to a subsequence if necessary) the weak*-convergence of $\{v_{\ell'_t}\}$ in $L^\infty(0, T; V)$ such that $u' \in L^\infty(0, T; V)$. Hence, in view of the growth condition (2.2), it follows $A(u, u') \in L^\infty(0, T; V')$. This, finally, implies $M(u') \in L^\infty(0, T; V')$. \square

5. A first example

In this section, we consider the scalar equation (1.3) in the interval $(0, 1)$ supplemented by homogeneous Dirichlet boundary conditions and the initial condition $u(\cdot, 0) = u_0$ in $(0, 1)$. We wish to apply Theorem 2.4.

In what follows, we rely on the usual notation of function spaces. Let $V = W_0^{1,4}(0, 1)$. The operators M and A are defined via

$$\langle M(v), w \rangle = \int_0^1 (\partial_x v)^3 \partial_x w \, dx, \quad \langle A(u, v), w \rangle = \int_0^1 |u|^{7/4} v w \, dx, \quad u, v, w \in V.$$

Since $W_0^{1,4}(0, 1)$ is continuously embedded in $\mathcal{C}([0, 1])$, the operators M and A indeed are mappings of V into V' .

Moreover, $M : V \rightarrow V'$ is the 4-Laplacian and is known to be hemicontinuous, uniformly monotone with

$$\langle M(v) - M(\bar{v}), v - \bar{v} \rangle \geq \frac{1}{4} \|v - \bar{v}\|^4, \quad v, \bar{v} \in V,$$

and also coercive with $p = 4$ (see, e.g., Lindqvist [15]). Assumption (M) is thus fulfilled.

The operator $A : V \rightarrow V'$ satisfies the growth condition

$$\|A(u, v)\|_{V'} \leq c(\|u\|^3 + \|v\|^{12/5}), \quad u, v \in V,$$

as is seen from the continuous embedding of V into $\mathcal{C}([0, 1])$ and Young’s inequality. Again employing the continuous embedding of V into $\mathcal{C}([0, 1])$, we find for all $u, v \in V$

$$\|A(u, v) - A(\bar{u}, \bar{v})\|_{V'} \leq \int_0^1 \left| |u|^{7/4} - |\bar{u}|^{7/4} \right| |v| \, dx + \int_0^1 |\bar{u}|^{7/4} \left| |v| - |\bar{v}| \right| \, dx.$$

Since $W_0^{1,4}(0, 1)$ is indeed compactly embedded in $\mathcal{C}([0, 1])$, the foregoing estimate shows the strong continuity of $A : V \times V \rightarrow V'$. Hence, Assumption (A) is fulfilled.

Since existence of a solution to (2.3) in Assumption (M, A) already follows from Assumption (M) and (A), it remains to show uniqueness. However, for $u \in V$ fixed and arbitrary $v, \bar{v} \in V$, we have

$$\langle M(v) + A(u, v) - M(\bar{v}) - A(u, \bar{v}), v - \bar{v} \rangle \geq \frac{1}{4} \|v - \bar{v}\|^4 + \int_0^1 |u|^{7/4} |v - \bar{v}|^2 \, dx \geq \frac{1}{4} \|v - \bar{v}\|^4,$$

which immediately implies uniqueness. Assumption (M, A) is also fulfilled.

Finally, we can apply Theorem 2.4 and obtain the following result.

Corollary 5.1. *Let $u_0 \in V := W_0^{1,4}(0, 1)$ and $f \in L^{4/3}(0, T; V')$. Then there exists a solution $u \in W^{1,4}(0, T, V)$ to (1.3) subject to homogeneous Dirichlet boundary conditions and $u(\cdot, 0) = u_0$. If $f \in L^\infty(0, T, V')$ then $u \in W^{1,\infty}(0, T, V)$.*

6. An example involving the $p(x)$ -Laplacian

In this section, we wish to apply Theorem 2.4 to the scalar partial differential equation (1.2). Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded Lipschitz domain. We consider (1.2) subject to homogeneous Dirichlet boundary conditions and the initial condition $u(\cdot, 0) = u_0$ in Ω .

We assume that the variable exponent $p = p(x) : \Omega \rightarrow (1, \infty)$ satisfies the following log-Hölder continuity condition: there exists $c > 0$ such that for all $x, y \in \Omega$

$$|p(x) - p(y)| \leq \frac{c}{\ln(e + \frac{1}{|x-y|})}. \tag{6.1}$$

It easily follows that also the conjugate exponent $x \mapsto p(x)' = p(x)/(p(x) - 1)$ is log-Hölder continuous. Moreover, we assume that

$$p_- := \inf_{x \in \Omega} p(x) \geq \frac{2d}{d+1}, \quad p_+ := \sup_{x \in \Omega} p(x) < \infty, \quad p_- + \frac{1}{p_+} > 2.$$

The last condition is not needed if the functions a_0 and a_1 are bounded by a constant.

Furthermore, we assume that $h : \mathbb{R} \rightarrow \mathbb{R}$ is strictly monotonically increasing and Lipschitz continuous with $h(0) = 0$. For the Carathéodory functions $a_0 : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $a_1 : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^d$, we assume the following growth condition: there exist $c > 0$, $q \in [1 - 1/p_+, p_- - 1)$ such that for almost all $x \in \Omega$ and all $u, v \in \mathbb{R}$

$$|a_0(x, u, v)| + |a_1(x, u, v)| \leq c(1 + |u|^{p_- - 1} + |v|^q).$$

Moreover, we assume that there is an Osgood function θ ($\theta \in \mathcal{C}([0, \infty))$, $\theta(0) = 0$, $\theta(\omega) > 0$ if $\omega > 0$) such that for almost all $x \in \Omega$ and all $u, v, \bar{v} \in \mathbb{R}$

$$|a_1(x, u, v) - a_1(x, u, \bar{v})| \leq \theta(|v - \bar{v}|), \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \theta^{-\min(2, (p_+)^')}(\omega) d\omega = \infty. \tag{6.2}$$

Finally, a_0 is supposed to be monotonically increasing in its third argument.

In what follows, we shortly recall some properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, respectively. We refer, in particular, to Antontsev and Shmarev [1], Diening, Harjulehto, Hästö and Růžička [6], Kováčik and Rákosník [14], Musielak [17] and the references cited therein.

Under the above assumptions on $p = p(x)$, we can define $L^{p(\cdot)}$ as the set of (equivalence classes of almost everywhere equal) Lebesgue measurable functions $w : \Omega \rightarrow \mathbb{R}$ such that $x \mapsto |w(x)|^{p(x)}$ is Lebesgue integrable on Ω . Equipped with the Luxemburg norm

$$\|w\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} |\lambda^{-1} w(x)|^{p(x)} dx \leq 1 \right\},$$

$L^{p(\cdot)}$ is a reflexive, separable Banach space with $\mathcal{C}_c^\infty(\Omega)$, the space of infinitely times differentiable functions with compact support, being dense in $L^{p(\cdot)}$ and with $(L^{p(\cdot)}(\Omega))' = L^{p(\cdot)' }(\Omega)$; there holds the generalized Hölder inequality

$$\int_{\Omega} |v(x)w(x)| dx \leq c \|v\|_{L^{p(\cdot)}(\Omega)} \|w\|_{L^{p(\cdot)' }(\Omega)}, \quad v \in L^{p(\cdot)}(\Omega), w \in L^{p(\cdot)' }(\Omega), \tag{6.3}$$

where $c = 1 + 1/p_- - 1/p_+ \leq 2$ (see, e.g., [14, Coroll. 2.7, Thm. 2.11, Coroll. 2.12]). Note that

$$\|v\|_{L^{p(\cdot)}(\Omega)} \leq \|w\|_{L^{p(\cdot)}(\Omega)} \quad \text{if } 0 \leq v(x) \leq w(x) \text{ for almost all } x \in \Omega. \tag{6.4}$$

It can easily be shown that

$$\int_{\Omega} \left| \frac{w(x)}{\|w\|_{L^{p(\cdot)}(\Omega)}} \right|^{p(x)} dx = 1 \quad \text{if } w \neq 0, \tag{6.5}$$

which implies for all $w \in L^{p(\cdot)}(\Omega)$

$$\int_{\Omega} |w(x)|^{p(x)} dx \geq \|w\|_{L^{p(\cdot)}(\Omega)}^{p_-} - 1. \tag{6.6}$$

The space $W^{1,p(\cdot)}(\Omega)$ is defined, as usual, as the set of functions in $L^{p(\cdot)}$ whose distributional derivatives of first order are again in $L^{p(\cdot)}$. With the norm $w \mapsto \sum_{|\alpha| \leq 1} \|D^\alpha w\|_{L^{p(\cdot)}(\Omega)}$, $W^{1,p(\cdot)}(\Omega)$ is a reflexive, separable Banach space (see, e.g., [14, Thm. 3.1] or [6, Thm. 8.1.6]).

The closure of $\mathcal{C}_c^\infty(\Omega)$ with respect to the norm of $W^{1,p(\cdot)}(\Omega)$ is denoted by $W_0^{1,p(\cdot)}(\Omega)$. The space $W_0^{1,p(\cdot)}(\Omega)$ is a Banach space for the norm $w \mapsto \sum_{|\alpha|=1} \|D^\alpha w\|_{L^{p(\cdot)}(\Omega)}$, which follows from the Poincaré–Friedrichs inequality: there exists $c > 0$ such that $\|w\|_{L^{p(\cdot)}(\Omega)} \leq c \sum_{|\alpha|=1} \|D^\alpha w\|_{L^{p(\cdot)}(\Omega)}$ for all $w \in W_0^{1,p(\cdot)}(\Omega)$ (see, e.g., [14, Thms. 3.10, 3.11] or [6, Thm. 8.2.4]). Because of $|\nabla w| := (\sum_{|\alpha|=1} |D^\alpha w|^2)^{1/2} \leq \sum_{|\alpha|=1} |D^\alpha w|$ and (6.4), we see that

$$\begin{aligned} \|\nabla w\|_{L^{p(\cdot)}(\Omega)} &\leq \left\| \sum_{|\alpha|=1} |D^\alpha w| \right\|_{L^{p(\cdot)}(\Omega)} \leq \sum_{|\alpha|=1} \|D^\alpha w\|_{L^{p(\cdot)}(\Omega)} \\ &\leq \sum_{|\alpha|=1} \|\nabla w\|_{L^{p(\cdot)}(\Omega)} = d \|\nabla w\|_{L^{p(\cdot)}(\Omega)}. \end{aligned}$$

This shows that $w \mapsto \|\nabla w\|_{L^{p(\cdot)}(\Omega)}$ is an equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$. In what follows, we only write $\|\nabla w\|_{L^{p(\cdot)}(\Omega)}$ instead of $\|\nabla w\|_{L^{p(\cdot)}(\Omega)}$.

Finally, we note that $W_0^{1,p(\cdot)}(\Omega)$ is continuously embedded in $L^{r(\cdot)}(\Omega)$ for any $r = r(x)$ that is bounded and satisfies $r(x) \leq dp(x)/(d - p(x))$ for all $x \in \Omega$ with $p(x) < d$. Moreover, $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{p(\cdot)}(\Omega)$ (see [6, Coroll. 8.3.2, Thm. 8.4.2]).

Starting from the weak formulation, our example (1.2) can now be written as (1.1), where $V = W_0^{1,p(\cdot)}(\Omega)$ and where M and A are given by

$$\begin{aligned} \langle M(v), w \rangle &= \int_{\Omega} (h(v(x))w(x) + |\nabla v(x)|^{p(x)-2} \nabla v(x) \cdot \nabla w(x)) \, dx, \\ \langle A(u, v), w \rangle &= \int_{\Omega} (a_1(u(x), v(x)) \cdot \nabla w(x) + a_0(u(x), v(x))w(x)) \, dx, \quad u, v, w \in V. \end{aligned}$$

Indeed, with the growth of h (as a globally Lipschitz continuous function, h is linearly bounded), the Hölder inequality (6.3), and the continuous embedding of V into $L^{p(\cdot)}(\Omega)$ and $L^{p(\cdot)' }(\Omega)$, we find for $v, w \in V$

$$\begin{aligned} |\langle M(v), w \rangle| &\leq c \|v\|_{L^{p(\cdot)' }(\Omega)} \|w\|_{L^{p(\cdot)}(\Omega)} + c \|\nabla v\|_{L^{p(\cdot)}(\Omega)}^{\bar{p}-1} \|\nabla w\|_{L^{p(\cdot)}(\Omega)} \\ &\leq c (\|\nabla v\|_{L^{p(\cdot)}(\Omega)} + \|\nabla v\|_{L^{p(\cdot)}(\Omega)}^{\bar{p}-1}) \|\nabla w\|_{L^{p(\cdot)}(\Omega)}, \end{aligned} \tag{6.7}$$

where $\bar{p} = p^+$ if $\|\nabla v\|_{L^{p(\cdot)}(\Omega)} \geq 1$ and $\bar{p} = p^-$ if $\|\nabla v\|_{L^{p(\cdot)}(\Omega)} < 1$. Here we have employed Young’s inequality and (6.5), which shows for $v, w \neq 0$ that

$$\begin{aligned} \int_{\Omega} \frac{|\nabla v(x)|^{p(x)-1}}{\|\nabla v\|_{L^{p(\cdot)}(\Omega)}^{\bar{p}-1}} \frac{|\nabla w(x)|}{\|\nabla w\|_{L^{p(\cdot)}(\Omega)}} \, dx &\leq \int_{\Omega} \left(\frac{|\nabla v(x)|}{\|\nabla v\|_{L^{p(\cdot)}(\Omega)}} \right)^{p(x)-1} \frac{|\nabla w(x)|}{\|\nabla w\|_{L^{p(\cdot)}(\Omega)}} \, dx \\ &\leq \int_{\Omega} \left(\frac{1}{p(x)'} \left(\frac{|\nabla v(x)|}{\|\nabla v\|_{L^{p(\cdot)}(\Omega)}} \right)^{p(x)} + \frac{1}{p(x)} \left(\frac{|\nabla w(x)|}{\|\nabla w\|_{L^{p(\cdot)}(\Omega)}} \right)^{p(x)} \right) \, dx \\ &\leq 1 - \frac{1}{p_+} + \frac{1}{p_-} \leq 2. \end{aligned} \tag{6.8}$$

The estimate (6.7) shows that M maps V into V' . Hemicontinuity as well as monotonicity of $M : V \rightarrow V'$ can be proved straightforward. The estimate (6.6) together with the monotonicity of h and $h(0) = 0$ implies for $v \in V$

$$\langle M(v), v \rangle = \int_{\Omega} (h(v(x))v(x) + |\nabla v(x)|^{p(x)}) dx \geq \|\nabla v\|_{L^{p(\cdot)}(\Omega)}^{p_-} - 1,$$

which finally shows that Assumption (M) is fulfilled.

From the growth condition for a_0 and a_1 , the Hölder inequality (6.3), an argumentation analogous to (6.8), and the continuous embedding of V into $L^{\alpha(\cdot)}(\Omega)$ for $\alpha = p(\cdot), (p_- - 1)p(\cdot)', qp(\cdot)'$, where $p(x)' = p(x)/(p(x) - 1)$ and

$$1 \leq qp(x)' \leq (p_- - 1)p(x)' \leq p(x) < \frac{dp(x)}{d-1} < \frac{dp(x)}{d-p_-} \leq \frac{dp(x)}{d-p(x)} \quad \text{if } p(x) < d, \tag{6.9}$$

we obtain for $u, v, w \in V$

$$\begin{aligned} |\langle A(u, v), w \rangle| &\leq c \int_{\Omega} ((1 + |u(x)|^{p_- - 1} + |v(x)|^q) |\nabla w(x)| \\ &\quad + (1 + |u(x)|^{p_- - 1} + |v(x)|^q) |w(x)|) dx \\ &\leq c(1 + \|u\|_{L^{(p_- - 1)p(\cdot)'(\Omega)}^{p_- - 1} + \|v\|_{L^{qp(\cdot)'(\Omega)}^q) \|\nabla w\|_{L^{p(\cdot)}(\Omega)} \\ &\quad + c(1 + \|u\|_{L^{(p_- - 1)p(\cdot)'(\Omega)}^{p_- - 1} + \|v\|_{L^{qp(\cdot)'(\Omega)}^q) \|w\|_{L^{p(\cdot)}(\Omega)} \\ &\leq c(1 + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}^{p_- - 1} + \|\nabla v\|_{L^{p(\cdot)}(\Omega)}^q) \|\nabla w\|_{L^{p(\cdot)}(\Omega)}. \end{aligned}$$

This shows that A maps $V \times V$ into V' and fulfills the growth condition of Assumption (A).

The continuity of the Nemytskii operator associated with a_0 and a_1 , respectively, as a mapping of $L^{(p_- - 1)p(\cdot)'(\Omega)} \times L^{qp(\cdot)'(\Omega)}$ into $L^{p(\cdot)'(\Omega)}$ (see [14, Thms. 4.1, 4.2]) implies that A is continuous as a mapping of $L^{(p_- - 1)p(\cdot)'(\Omega)} \times L^{qp(\cdot)'(\Omega)}$ into V' . Since V is compactly embedded in $L^{p(\cdot)}(\Omega)$ and thus in $L^{(p_- - 1)p(\cdot)'(\Omega)}$ and $L^{qp(\cdot)'(\Omega)}$ (see also (6.9)), this shows the strong continuity of A as a mapping of $V \times V$ into V' and thus Assumption (A).

It remains to show uniqueness of a solution to (2.3) in order to show that Assumption (M, A) is fulfilled. Remember that, for given $u \in V$ and $b \in V'$, existence of a solution $v \in V$ to (2.3) follows from Brézis' theorem on pseudomonotone operators.

We first recall the following algebraic relations: for all $r \in (1, \infty)$ and all $a, b \in \mathbb{R}^d$ there holds

$$(|a|^{r-2}a - |b|^{r-2}b) \cdot (a - b) \geq \begin{cases} 2^{2-r}|a - b|^r & \text{if } r \geq 2, \\ (r - 1)(1 + |a| + |b|)^{-2+r}|a - b|^2 & \text{if } 1 < r < 2, \end{cases}$$

see, e.g., Lindqvist [15, pp. 71 and 74]. Note that $C_p := \min(2^{2-\max(2, p_+)}, \min(2, p_-) - 1)$ is a lower bound for the coefficients in the above estimate if $r = p(x)$. For all $x \in \Omega$ and all $a, b \in \mathbb{R}^d$, we therefore have

$$(|a|^{p(x)-2}a - |b|^{p(x)-2}b) \cdot (a - b) \geq C_p \begin{cases} |a - b|^{p(x)} & \text{if } p(x) \geq 2, \\ (1 + |a| + |b|)^{-2+p(x)}|a - b|^2 & \text{if } 1 < p(x) < 2. \end{cases} \tag{6.10}$$

We prove uniqueness by contradiction. Let $v \in V$ and $\bar{v} \in V$ be two different solutions such that

$$M(v) - M(\bar{v}) + A(u, v) - A(u, \bar{v}) = 0 \quad \text{in } V'. \tag{6.11}$$

We wish to test Eq. (6.11) by $\Psi_\eta(v - \bar{v})$, where Ψ_η is a suitable approximation of the Heaviside function. In view of (6.2), there is for any $\eta > 0$ a number $\varepsilon(\eta) \in (0, \eta)$ such that $\int_{\varepsilon(\eta)}^\eta \theta^{-\min(2, (p_+)')}(\omega) d\omega = 1$. Let $\Psi_\eta \in \mathcal{C}^1([0, \infty))$ be given by

$$\Psi_\eta(z) := \begin{cases} 0 & \text{if } z < \varepsilon(\eta), \\ \int_{\varepsilon(\eta)}^z \theta^{-\min(2, (p_+)')}(\omega) d\omega & \text{if } \varepsilon(\eta) \leq z \leq \eta, \\ 1 & \text{if } z > \eta. \end{cases}$$

Note that Ψ_η is monotonically increasing and that $\Psi_\eta(v - \bar{v}) \in V$, which follows from the boundedness of p . We set

$$\Omega_1 := \{x \in \Omega : p(x) < 2\}, \quad \Omega_2 := \{x \in \Omega : p(x) \geq 2\}.$$

With (6.10), the monotonicity of h as well as of a_0 with respect to its third argument, (6.2), and Young’s inequality, we then find (for readability, we omit the argument x in the following)

$$\begin{aligned} & \int_{\Omega} (h(v) - h(\bar{v}))\Psi_\eta(v - \bar{v}) dx + C_p \int_{\Omega_1} (1 + |\nabla v| + |\nabla \bar{v}|)^{-2+p(\cdot)} |\nabla v - \nabla \bar{v}|^2 \Psi'_\eta(v - \bar{v}) dx \\ & + C_p \int_{\Omega_2} |\nabla v - \nabla \bar{v}|^{p(\cdot)} \Psi'_\eta(v - \bar{v}) dx \\ & \leq \int_{\Omega} ((h(v) - h(\bar{v}))\Psi_\eta(v - \bar{v}) + (|\nabla v|^{p(\cdot)-2} \nabla v - |\nabla \bar{v}|^{p(\cdot)-2} \nabla \bar{v}) \cdot (\nabla v - \nabla \bar{v})) \Psi'_\eta(v - \bar{v}) dx \\ & = \langle M(v) - M(\bar{v}), \Psi_\eta(v - \bar{v}) \rangle = -\langle A(u, v) - A(u, \bar{v}), \Psi_\eta(v - \bar{v}) \rangle \\ & = - \int_{\Omega} (a_1(\cdot, u, v) - a_1(\cdot, u, \bar{v})) \cdot (\nabla v - \nabla \bar{v}) \Psi'_\eta(v - \bar{v}) dx \\ & \quad - \int_{\Omega} (a_0(\cdot, u, v) - a_0(\cdot, u, \bar{v})) \Psi_\eta(v - \bar{v}) dx \\ & \leq \int_{\Omega} \theta(|v - \bar{v}|) |\nabla v - \nabla \bar{v}| \Psi'_\eta(v - \bar{v}) dx \\ & \leq c \int_{\Omega_1} \theta(|v - \bar{v}|)^2 \Psi'_\eta(v - \bar{v}) (1 + |\nabla v| + |\nabla \bar{v}|)^{2-p(\cdot)} dx \\ & \quad + \frac{C_p}{2} \int_{\Omega_1} (1 + |\nabla v| + |\nabla \bar{v}|)^{-2+p(\cdot)} |\nabla v - \nabla \bar{v}|^2 \Psi'_\eta(v - \bar{v}) dx \\ & \quad + c \int_{\Omega_2} \theta(|v - \bar{v}|)^{p(\cdot)'} \Psi'_\eta(v - \bar{v}) dx + \frac{C_p}{2} \int_{\Omega_2} |\nabla v - \nabla \bar{v}|^{p(\cdot)} \Psi'_\eta(v - \bar{v}) dx. \end{aligned}$$

Because of $\Psi'_\eta(z) = \theta^{-\min(2, (p_+)')}(z)$ for $\varepsilon(\eta) \leq z \leq \eta$ and $\Psi'_\eta(z) = 0$ otherwise, we thus come up with

$$\begin{aligned} 0 &\leq \int_{\Omega} (h(v) - h(\bar{v})) \Psi_\eta(v - \bar{v}) \, dx \\ &\leq c \int_{\Omega_{1,\eta}} \theta(|v - \bar{v}|)^{2-\min(2, (p_+)')} (1 + |\nabla v| + |\nabla \bar{v}|)^{2-p(\cdot)} \, dx + c \int_{\Omega_{2,\eta}} \theta(|v - \bar{v}|)^{p(\cdot)' - \min(2, (p_+)')} \, dx, \end{aligned}$$

where

$$\Omega_{i,\eta} := \Omega_i \cap \{x \in \Omega : \varepsilon(\eta) \leq v(x) - \bar{v}(x) \leq \eta\}, \quad i = 1, 2.$$

Since $2 - \min(2, (p_+)') \geq 0$, $p(x)' - \min(2, (p_+)') \geq (p_+)' - \min(2, (p_+)') \geq 0$ for all $x \in \Omega$ and since $\theta \in \mathcal{C}([0, \infty))$ is bounded on $[0, \eta]$, it follows that $\theta(|v - \bar{v}|)^{2-\min(2, (p_+)')}$ and $\theta(|v - \bar{v}|)^{p(\cdot)' - \min(2, (p_+)')}$ is bounded on Ω_1 and Ω_2 , respectively. Since $v, \bar{v} \in V$, we also know that $(1 + |\nabla v| + |\nabla \bar{v}|)^{2-p(\cdot)}$ is integrable on Ω_1 as long as $2 - p(\cdot) \leq p(\cdot)$, which is fulfilled since $p_- > 1$. Finally, we note that the measure of $\Omega_{i,\eta}$ ($i = 1, 2$) tends to zero as $\eta \rightarrow 0$.

This shows that $h(v(x)) = h(\bar{v}(x))$ for almost all $x \in \Omega$ for which $v(x) \geq \bar{v}(x)$. The function h , however, is strictly monotonically increasing and thus $v(x) \geq \bar{v}(x)$ implies $h(v(x)) > h(\bar{v}(x))$, which is a contradiction. Modifying the definition of Ψ_η appropriately, we can derive a contradiction when $v(x) \leq \bar{v}(x)$. This, finally, proves uniqueness; Assumption (M, A) is thus fulfilled.

An immediate consequence of Theorem 2.4 is now the following result.

Corollary 6.1. *Let $u_0 \in V := W_0^{1,p(\cdot)}(\Omega)$ and $f \in L^{(p_+)' }(\Omega, T; V')$. Under the above assumptions on h, a_0, a_1 and the variable exponent p , there exists a solution $u \in W^{1,p_-(\cdot)}(\Omega, T; V)$ to (1.2) subject to homogeneous Dirichlet boundary conditions and $u(\cdot, 0) = u_0$. If $f \in L^\infty(\Omega, T; V')$ then $u \in W^{1,\infty}(\Omega, T; V)$.*

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