Seminar Paper

Minimising Sequences, Young Measures and Oscillations

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Abstract. In the calculus of variations, where one wants to minimise a functional over a function space, the direct method gives a broad criterion for the existence of a minimiser. For this method to apply, the functional has to be weak-lower-semicontinuous. When this is the case, a weakly convergent minimising sequence will converge to a minimiser. This property is difficult to establish, the key observation being that limits of weakly convergent sequences do not commute with continuous, nonlinear functions. Failure of commutativity is common when the minimising sequence is highly oscillating. The Young measure is a tool for understanding the weak limits of such compositions. In this seminar paper, we will establish three results relating to the Young measure. First, we will show existence. Second, we will establish a broad criterion, expressed in terms of the Young measure, for functionals to be weak-lower-semi-continuous. Third, we will compute the Young measure for a sample problem that doesn't admit a minimiser.

Contents

0. Introduction

In this report we will consider the general problem of the calculus of variations. In its most general description, X is a separable Banach space consisting of functions $u : \mathbb{R}^k \to \mathbb{R}^m$ and J a non-linear, continuous functional $J : X \to \mathbb{R}$. Assume this functional J is bounded below. One is then interested in finding a function u such that

$$J(u) = \inf\{J(u)|u \in X\}.$$

u is called a *minimiser* of the functional J and, in general, no such u may exist.

In many physical applications J will be the energy associated with a particular function u. For example, in the theory of nonlinear-elasticity $U \subset \mathbb{R}^3$ is a bounded domain representing the shape of a material under no stress and $u : U \to \mathbb{R}^3$ is a deformation of the material. Here $J(u)$ is the energy of the deformation. Another common example is letting M be a surface in $\mathbb{R}^3$, $u : [0, 1] \to M$ a curve and $J(u)$ its length. In this case X is the set of curves with fixed end points and minimisers of J will be geodesics of the surface.

In some problems, for example in classical mechanics, one derives the Euler-Lagrange equations, the solutions of which might be extrema of J. However there must always exist a sequence u_n in X such that $J(u_n) \to \inf\{J(u)|u \in X\}$. Such a sequence is called a *minimising sequence*. Analysing the minimising sequence itself is called the *direct method* of the calculus of variations and this is the technique we will be treating in this seminar paper.

In the first section we will discuss the direct method in a general setting. The method is based on the simple observation that if u_n is a minimising that is bounded in X then it contains a weakly convergent subsequence. If u is the weak limit then u is a minimiser if

$$u_n \rightharpoonup u \text{ in X implies that } I(u) \leq \liminf_{n\to\infty} I(u_n). \tag{0.1}$$

This property is called *weak-lower-semi-continuity* or *wlsc* for short. Note that this idea mimics the case when X is finite dimensional and J is continuous.

A typical setting for the problems we consider is when X is a Sobolev space and

$$J(u) = \int_\Omega F(\nabla u)\, dx, \text{ where } u \in X$$

What can we say about wlsc for such a problem? Establishing wlsc is in general very difficult. In the first section we will discuss in detail the obstacles to wlsc. Of particular importance are oscillating sequences. If u is the weak limit of a highly oscillating sequence u_n, then typically $F(u_n) \not\rightharpoonup F(u)$. However, there is a tool for understanding weak limits under such compositions; the Young measure.

In section 2 we prove the fundamental existence theorem for Young measures. This theorem requires a great deal of auxiliary technical results that are important in their own right. Therfore we will treat these results in some detail and provide many references for further reading. Following the existence theorem are some alternative formulations as well as some heuristic interpretations of the Young measure.

In section 3 we will return to the topic of wlsc in a broader setting. In the context of section 1, J is wlsc if it is convex, continuous and defined on a closed, convex set. In section 3 we will generalise the notion of convexity to establish a converse to this statement. It is well known that any convex function satisfies Jensen's inequality with respect to a probablility measure. In section 3 we will show that the functional J is wlsc if and only if the function F satisfies Jensen's inequality with respect to a Young measure. This is the broader notion of convexity which we will discuss.

Up to this point we would have only treated the problem in the setting of J being convex in some sense. However there are physical applications where J cannot be convex in the sense of section 3. Consider again the example from the theory of non-linear elasticity. In this context, a minimiser is called a well. Let $J(u)$ be the energy functional and suppose that J is invariant under rotations i.e. for any $R \in SO(3)$, $J(Ru) = J(u)$. Let u and Ru be two wells related by a rotation. If J is convex then J is also minimising along a segment joining u and Ru. In the theory of nonlinear elasticity, this is a physically unreasonable conclusion. So either J is not convex or J is not invariant under rotations. From a physical standpoint, the later should certainly be true, hence J cannot be convex. See [Ped00] proposition 4.2 for more details.

This example demonstrates the need to consider a theory where convexity fails in some sense. In this field there are many competing notions of convexity, but we will concern ourselves only with the notions discussed in section 1 and section 3. Section 4 considers a sample problem for a function F that is not convex in the sense of section 3 and is therefore not wlsc. The functional J will have no minimiser but remarkably, the Young measure still has applications in this context. We will calculate the Young measure for this problem and show how it can be used to explicitly construct a minimising sequence. As one would expect for a functional that is not wlsc, the minimising sequence will exhibit oscillatory behaviour.

1. The Direct Method, Weak Limits and Oscillations

As motivated in the introduction, one is interested in finding a minimiser of a given functional. If the classical methods, such as determining the zeroes of the first derivative or the Euler-Lagrange equation do not apply, the *direct method* might be a helpful alternative. This is in some sense the most natural way to treat the problem, because it directly uses an object that exists for any functional; a sequence of points in the function space whose values converge to the functional's infimum. We will consider minimising sequences whose norm is bounded, to deduce the existence of a weakly convergent subsequence. This is an analogue of the finite dimensional case. The main ingredients in the direct method is wlsc. In the second subsection we will see why this property is the most difficult one to establish. In the third subsection we will observe behaviour that is typical of functionals that are not wlsc; oscillations.
The following theorems and examples can be found in [Ped97, p.1-8] and [Chi00, p.131-134]. The rest is the observation of the authors.

1.1. **The Direct Method.**

The following theorem is the most common statement of the direct method. The subsequent remarks contain some informative variations. The direct method is sometimes called the *generalised Weierstrass Theorem*, because, like the Weierstrass Extreme Value Theorem, it is an existence result about the extrema of a function, but in an infinite dimensional setting.

Theorem 1.1 (The direct method)**.** *Let X be a reflexive Banach space and $V \subseteq X$ a weakly closed subset of X*[1]*. Furthermore, let $J\colon V \to \mathbb{R}$ be a function with the following properties:*

- *J has a minimising sequence that is bounded in V,*
- *J is wlsc.*

Then there exists a $u \in V$ such that

$$J(u) = \inf_{v \in V} J(v).$$

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded minimising sequence, i.e. $J(u_n) \to \inf_{v\in V} J(v)$. Because X is reflexive and $(u_n)_{n\in\mathbb{N}}$ is bounded, the Banach-Alaoglu theorem ensures the existence of a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a $u \in X$ with

$$u_{n_k} \rightharpoonup u, \quad \text{for } k \to \infty.$$

Since V is weakly closed, and therefore sequentially weak closed, we have that $u \in V$. Furthermore, because J is wlsc

$$\inf_{v\in V} J(v) = \liminf_{k\to\infty} J(u_{n_k}) \geq J(u). \tag{1.1}$$

We have found a $u \in V$ with $J(u) \leq J(v)$ for all $v \in V$. This completes the proof. □

Remark 1.2. i) Technically one doesn't require that J is bounded below. To see this, suppose that J is not bounded below, i.e. $\inf_{v\in V} J(v) = -\infty$, but that we still have a bounded minimising sequence in V. Following the steps of the proof above, equation (1.1) implies the existence of a $u \in V$ with

$$-\infty = \inf_{v\in V} J(v) = \liminf_{k\to\infty} J(u_{n_k}) \geq J(u),$$

This is clearly a contradiction. But if J is considered as taking values in $\overline{\mathbb{R}}$ (which is often the case) this contradictions vanishes. However, the proof remains valid in this case, but one might get a trivial minimiser, i.e. one with value $-\infty$.

[1] A set $V \subseteq X$ is said to be *weakly closed*, if it is closed with respect to the weak topology. V is said to *sequentially weak closed* if for every sequence $(v_n)_{n\in\mathbb{N}} \subseteq V$ with $v_n \rightharpoonup v$ for $n \to \infty$ the inclusion $v \in V$ holds. Every weak closed set is sequentially weak closed.

ii) We are imitating the finite dimensional case when we apply the Banach-Alaoglu theorem to get a weakly convergent subsequence. In a finite dimensional Banach space, any bounded sequence contains a convergent subsequence. If the function f is lower semi-continuous and the set V closed, an almost identically proof would ensure the existence of a minimiser.
iii) Note that a closed and convex set is always weak closed, see [Alt06, p.239].
iv) The existence of a bounded minimising sequence is often a consequence of a *coercitivity* condition. For example

$$J(u) \geq C\|u\|_X, \quad \text{for all } u \in V,$$

or

$$\lim_{\|u\|\to\infty} J(u) = \infty.$$

v) Though it hard to show that J is wlsc it holds if J is convex, continuous and defined on a closed and convex set, see [TT90, p.269].

1.2. **Weak Limits.** Consider the following typical problem

$$J\colon W^{1,\infty}(\Omega) \to \mathbb{R},\ J(u) := \int_\Omega F(\nabla u(x))\, dx,$$

where $\Omega \subseteq \mathbb{R}^k$ is some bounded domain, and $F\colon \mathbb{R}^k \to \mathbb{R}$ a continuous function. We want to find out under which circumstances J could be *wlsc*. This discussion isn't intended to be formal. Rather, it should illustrate why *wlsc* is so difficult to establish and how this leads to the *Young measure*. Let $(u_n)_{n\in\mathbb{N}} \subseteq W^{1,\infty}(\Omega)$ and $u \in W^{1,\infty}(\Omega)$ with

$$u_n \overset{*}{\rightharpoonup} u, \quad \text{as } n \to \infty.$$

Here we use weak-$*$ convergence because $L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$ are not reflexive. Even though we consider *w∗lsc* convergence for technical reasons, the distinction is not central to the discussion. Now, using the dual space of $W^{1,\infty}(\Omega)$, one can show that

$$\nabla u_n \overset{*}{\rightharpoonup} \nabla u, \quad \text{as } n \to \infty \text{ in } L^\infty(\Omega)^k.$$

By convergence in $L^\infty(\Omega)^k$ we mean convergence in each component of ∇u_n. Let us further assume that F is bounded below by 0 (or some other constant). Then we can apply Fatou's lemma to get

$$\liminf_{n\to\infty} \int_\Omega F(\nabla u_n(x))\, dx \geq \int_\Omega \liminf_{n\to\infty} F(\nabla u_n(x))\, dx.$$

One now wants

$$\int_\Omega \liminf_{n\to\infty} F(\nabla u_n(x))\, dx = \int_\Omega F(\nabla u(x))\, dx, \tag{1.2}$$

which implies $\liminf_{n\to\infty} J(u_n) \geq J(u)$. The problem is that $(\nabla u_n)_{n\in\mathbb{N}}$ is only weak-$*$ convergent. But since weak-$*$ convergence in the dual of a Banach space implies that the sequence is bounded (see [Alt06, p.227]), we get that $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)^k$. The continuity of F implies that the sequence $(F \circ \nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. And so the Banach-Alaoglu theorem A.3 ensures the existence of some $g \in L^\infty(\Omega)$ and of a (not relabelled) subsequence such that

$$(F \circ \nabla u_n) \overset{*}{\rightharpoonup} g, \quad \text{as } n \to \infty.$$

To justify equation (1.2) it would suffice to have

$$F \circ \left(\lim_{n\to\infty} \text{-}* \nabla u_n\right) = F \circ \nabla u \overset{(*)}{=} g = \lim_{n\to\infty} \text{-}* (F \circ \nabla u_n).$$

This is an informal discussion so we have simply written lim instead liminf for clarity. Unfortunately, in general, $(*)$ is not valid. Failure is usually due to the presence of oscillations, however $(*)$ holds if F is 'convex', where the notion of convex is defined using the Young measure (see [Ped97] and section 3).

1.3. **Oscillations.** Let $\Omega \subseteq \mathbb{R}^k$ be some domain, $f \in C(\mathbb{R})$ and $(u_n)_{n\in\mathbb{N}} \subseteq L^\infty(\Omega)$ with

$$u_n \overset{*}{\rightharpoonup} u, \text{ as } n \to \infty,$$

for some $u \in L^\infty(\Omega)$. Then $(f \circ u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\Omega)$ and the Banach-Alaoglu theorem A.3 implies that there is a $g \in L^\infty(\Omega)$ and a (not relabelled) subsequence with

$$f \circ u_n \overset{*}{\rightharpoonup} g, \text{ for } n \to \infty.$$

We will show that in general $f \circ u \neq g$. We need the following lemma (see [JMR96, p.146]):

Lemma 1.3. *Let $\overline{u} \in L^\infty(\mathbb{R})$ be 2π-periodic. Define the sequence $(u_n)_{n\in\mathbb{N}} \subseteq L^\infty(0, 2\pi)$ by*

$$u_n(x) := \overline{u}(nx), \quad \text{on } (0, 2\pi).$$

Then

$$u_n \overset{*}{\rightharpoonup} \frac{a_0}{2}, \quad \text{in } L^\infty(0, 2\pi) \text{ as } n \to \infty,$$

where $a_0 := \frac{1}{\pi}\int_0^{2\pi} \overline{u}(x)\, dx$.

Proof. See Appendix B. □

Applying the Lemma to $\overline{u} = \sin \in L^\infty(\mathbb{R})$ we get

$$u_n \overset{*}{\rightharpoonup} \frac{1}{2\pi}\int_0^{2\pi} \sin x\, dx = 0, \text{ in } L^\infty(0, 2\pi) \text{ as } n \to \infty.$$

For the continuous function $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ the Lemma shows that

$$f \circ u_n \overset{*}{\rightharpoonup} \frac{1}{2\pi}\int_0^{2\pi} \sin^2(x)\, dx = \frac{1}{2}, \text{ in } L^\infty(0, 2\pi) \text{ for } n \to \infty.$$

The two results together imply that

$$f \circ \left(\lim_{n\to\infty} \text{-} * u_n\right) = 0 < \frac{1}{2} = \lim_{n\to\infty} \text{-} * (f \circ u_n), \text{ in } L^\infty(\Omega),$$

which is what we wanted to show. This inequality, which would help us to prove wlsc, does not always hold. For example, for the non-convex function $h: \mathbb{R} \to \mathbb{R}, x \mapsto \sqrt{x}$ we get

$$h \circ f \circ u_n \overset{*}{\rightharpoonup} \frac{1}{2\pi}\int_0^{2\pi} |\sin(x)|\, dx = \frac{2}{\pi}, \text{ in } L^\infty(0, 2\pi) \text{ for } n \to \infty,$$

and therefore

$$h \circ \left(\lim_{n\to\infty} \text{-} * (f \circ u_n)\right) = \frac{1}{\sqrt{2}} > \frac{2}{\pi} = \lim_{n\to\infty} \text{-} * (h \circ f \circ u_n), \text{ in } L^\infty(\Omega).$$

These examples demonstrate that if one composes a highly oscillating sequence with a continuous, nonlinear functional, the limit processes do not commute. In the next chapter we will see how the *Young measure* might help us in certain situations to deduce a relationship between f, u_n, u and g.

2. The Existence of Young Measures

In this section we prove the basic existence theorem for Young measures. We mainly follow the presentation in [Ped97, p.95-101]. Other sources used here are: [JMR96, p.148-154], [Bal89, p.207-215] and [Web13]. Several technical notions are needed for this result; In particular, the *Bochner integral*, the duals of the *Bochner-Lebesgue spaces* and the dual of the space of continuous functions which vanish at infinity.

2.1. Auxiliary Results.

Because the Bochner integral is such a fundamental tool in mathematical analysis, let us make some general remarks. Essentially, the Bochner integral extends the Lebesgue integral to functions which take values in a Banach space.

> It is a straightforward abstraction of the Lebesgue integral. Indeed, some have said that the Bochner integral is only the Lebesgue integral with absolute value signs replaced by norm signs. We shall see that this is often the case, and sometimes it is a totally ignorant appraisal of the Bochner integral. [DU77, p.44]

In the scalar case a function is measurable, iff it is the pointwise limit of a sequence of step functions[2]. This concept is used to define the Lebesgue integral. A modification is used to define the vector valued Bochner integral. A complete introduction to this topic can be found in [AE08, chapter X], shorter expositions in [DU77] or [Ruz04]. In the following let Ω be a measurable subset of $\mathbb{R}^n$ and X a Banach Space.

Definition 2.1 (Simple functions)**.** A function $f\colon \Omega \to X$ is said to be *simple*, if there exists $k \in \mathbb{N}$, $x_1, \dots, x_k \in X$ and $E_1, \dots, E_k \in \mathcal{B}(\Omega)$, where $\lambda(E_i) < \infty$ for each $i \in \{1, \dots, k\}$, with the property that

$$f = \sum_{i=1}^{k} x_i \mathbb{1}_{E_i},$$

where $\mathbb{1}_E$ is the indicator function of E and $\mathcal{B}(\Omega)$ is the Borel-σ-Algebra on Ω, with Borel-Lebesgue measure λ.[3]

Definition 2.2 (Strongly, weakly and weakly-$*$ measurable functions)**.** A function $f : \Omega \to X$ is said to be *strongly measurable* (or μ-measurable or Bochner-measurable) if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that

$$\|f_n(\omega) - f(\omega)\|_X \overset{n\to\infty}{\longrightarrow} 0 \quad \text{for a.e. } \omega \in \Omega.$$

A function $f\colon \Omega \to X$ is said to be *weakly measurable* (or weakly μ−measurable or weakly Bochner-measurable) if for every $T \in X'$ the composition

$$\omega \mapsto \langle f(\omega), T\rangle = T(f(\omega))$$

is measurable (with respect to the Borel-σ-Algebra on $\mathbb{R}$). Analogously, a function $f\colon \Omega \to X'$ is said to be *weakly-$*$-measurable* if the function

$$\omega \mapsto \langle x, f(\omega)\rangle = (f(\omega))(x)$$

is measurable for all $x \in X$ (with respect to the Borel-σ-algebra on $\mathbb{R}$).

Remark 2.3 (Bochner integral). For the sake of completeness, we will define the Bochner Integral, even though it is not needed here. See [AE08, chapter X] for the missing details. For a simple function f as in Definition 2.1, we define the *Bochner integral* as

$$\int_\Omega f(\omega)\, d\omega := \sum_{i=1}^{k} \lambda(E_i) x_i \in X.$$

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[2]This fact is not necessarily true in a non-separable Banach space. Therefore a slightly different measurability concept is used for the construction of the Bochner integral.

[3]If $x_i \neq 0$ for all $i \in \{1, \dots, k\}$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, this is the so called *standard representation* of the simple function f. Every simple function has a unique standard representation.

Here one has to check that the definition is independent of the the representation of f.
A strongly measurable function $f\colon \Omega \to X$ is said to be *Bochner integrable* if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that

$$\|f_n(\omega) - f(\omega)\|_X \overset{n\to\infty}{\longrightarrow} 0 \quad \text{for a.e. } \omega \in \Omega,$$

and

$$\int_\Omega \|f_n(\omega) - f(\omega)\|_X \, d\omega \overset{n\to\infty}{\longrightarrow} 0.$$

In this case the *Bochner integral* is defined by

$$\int_\Omega f(\omega)\, d\omega := \lim_{n\to\infty} \int_\Omega f_n(\omega)\, d\omega. \tag{2.1}$$

To justify this definition, one should check that $\|f_n(\cdot) - f(\cdot)\|_X$ is measurable, that the limit in (2.1) exists and that this limit is independent of the choice of the sequence $(f_n)_{n\in\mathbb{N}}$.

Now we are able to define the *Bochner-Lebesgue spaces*, which are the generalizations of the scalar Lebesgue spaces.

Definition 2.4. For $1 \le p < \infty$ we define the *Bochner-Lebesgue space* by

$$L^p(\Omega, X) := \left\{ [f] \,|\, f\colon \Omega \to X \text{ is strongly measurable and } \|f\|_p := \left(\int_\Omega \|f(\omega)\|_X^p \, d\omega \right)^{\frac{1}{p}} < \infty \right\},$$

where $[\cdot]$ denotes equivalence classes of functions that are equal almost everywhere. In what follows, we will drop this notation and write f instead of $[f]$. For $p = \infty$ we have, as per the scalar case

$$L^\infty(\Omega, X) := \left\{ [f] \,|\, f\colon \Omega \to X \text{ is strongly measurable and } \|f\|_\infty := \operatorname{ess\,sup}_{\omega\in\Omega} \|f(\omega)\|_X < \infty \right\}.$$

Remark 2.5. $L^p(\Omega, X)$ with norm $\|\cdot\|_p$ is a Banach space for all $p \in [1, \infty]$ (see [AE08, p.123]). For $1 \le p < \infty$ the set of simple functions is dense in $L^p(\Omega, X)$ (see [Ruz04, p.39]) and therefore $L^p(\Omega, X)$ is separable if X is separable.

It is well known that the *dual space* of the scalar Lebesgue space $L^p(\Omega)$ is $L^q(\Omega)$ where $1 = \frac{1}{q} + \frac{1}{p}$ and $1 \le p < \infty$ (q is then called the conjugate exponent of p). If X' has the so called *Radon-Nikodým property* (this is the case, for example , if X is reflexive) then we have the following duality result

$$L^p(\Omega, X)' \cong L^q(\Omega, X').$$

To define the Young measure we need a more general result, because X' does not necessarily have the Radon-Nikodým property. All these results are credited to *N.Dunford and B.J. Pettis* ([DP39]) and *A. and C. Ionescu Tulcea* ([AT69]). The notation is quite dated and the results dispersed, so a more concise source is [CM97, p.3, p.23-40].

Before proceeding, we have to define another function space:

Definition 2.6. Let X be separable. For $1 \le p \le \infty$ we define the space

$$L^p_{w^*}(\Omega, X') := \left\{ [f] \,|\, f\colon \Omega \to X' \text{ weakly-}*\text{-measurable and } \|f(\cdot)\|_{X'} \in L^p(\Omega) \right\},$$

with the norm

$$\|f\|_{L^p_{w^*}(\Omega,X')} = \| \, \|f(\cdot)\|_{X'} \|_p.$$

Remark 2.7. It is also possible to define this space if X is not necessarily separable. But this is done in different ways in the literature, see [Ped97, p.113] and [CM97, p.38]. However these definitions coincide if X is separable and we only require this case. The reason we write w^* is because the target space is the dual of X. If we map into X we get the space

$$L^p_w(\Omega, X) := \left\{ [f] \,|\, f\colon \Omega \to X \text{ weakly measurable and } \|f(\cdot)\|_X \in L^p(\Omega) \right\}.$$

The following theorem is central in showing the existence of Young measures. A proof can be found in [CM97, p.3, p.23-40]. This proof does not need the separability of X, however $L^p_{w^*}(\Omega, X')$ needs to be defined in a slightly different way. A self-contained proof can be found in [JMR96, p.149-152], but the authors use a slightly different definition of weak-$*$-measurability.

Theorem 2.8. *Let X be a separable Banach space, $1 \leq p < \infty$ and q the conjugate exponent to p. Then*

$$(L^p(\Omega, X))' \cong L^q_{w^*}(\Omega, X'),$$

under the duality

$$\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\omega,$$

for $f \in L^p(\Omega, X)$ and $g \in L^q_{w^*}(\Omega, X')$.

We only need two more spaces and their duals.

Definition 2.9. By $C_0(\mathbb{R}^m)$ we denote the separable Banach space

$$C_0(\mathbb{R}^m) := \left\{ f \in C(\mathbb{R}^m, \mathbb{R}) : \lim_{|x|\to\infty} f(x) = 0 \right\},$$

under the norm $\|f\|_{C_0} := \|f\|_\infty = \sup_{x\in\mathbb{R}^m} |f(x)|$.

For the next definition, one requires some knowledge of measure theory. The following summary can be found in the the book [Els09, chapter 7 and 8].
Let $(\Omega, \mathcal{A})$ be a measurable space. A function $\nu \colon \mathcal{A} \to \overline{\mathbb{R}}$ is called a *signed measure* if $\nu(\emptyset) = 0$ and

$$\nu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \nu(A_n),$$

for mutually disjoint $A_n \in \mathbb{A}$, $n \in \mathbb{N}$. For such a signed measure we define its *variation*, as a function $\mathcal{A} \to \mathbb{R}$, by setting for $A \in \mathcal{A}$

$$|\nu|(A) := \sup\left\{ \sum_{i=1}^n |\nu(A_i)| \colon A_1, \dots, A_n \in \mathcal{A}, \text{disjoint}, A = \bigcup_{i=1}^n A_i \right\},$$

the *positive variation* by

$$\nu^+(A) := \sup\{\nu(B) \colon B \in \mathcal{A}, B \subseteq A\}$$

and the *negative variation* by

$$\nu^-(A) := -\inf\{\nu(B) \colon B \in \mathcal{A}, B \subseteq A\}.$$

The maps $|\nu|, \nu^+, \nu^- \colon \mathcal{A} \to \overline{\mathbb{R}}$ are measures on $\mathcal{A}$, where $|\nu| = \nu^+ + \nu^-$ and $\nu = \nu^+ - \nu^-$, see [Els09, p.269-273]. A signed measure $\nu \colon \mathcal{B}(\mathbb{R}^m) \to \mathbb{R}$ is said to be *regular* if for every $A \in \mathcal{B}(\mathbb{R}^m)$ and $\varepsilon > 0$ there exists a compact set K and an open set U such that $K \subseteq A \subseteq U$ and $|\nu|(U \backslash K) < \varepsilon$. The set of all regular signed measures on $\mathcal{B}(\mathbb{R}^m)$ is denoted by $\mathcal{M}(\mathbb{R}^m)$. With the total variation $\|\nu\| = |\nu|(\Omega)$ for $\nu \in \mathcal{M}(\mathbb{R}^m)$ as a norm the set $\mathcal{M}(\mathbb{R}^m)$ forms a Banach space (see [Els09, p.346]).
We are now able to integrate with respect to a signed measure ν by setting $\mathcal{L}^1(\nu) := \mathcal{L}^1(\nu^+) \cap \mathcal{L}^1(\nu^-)$ and

$$\int_{\mathbb{R}^m} f(x)\, d\nu(x) := \int_{\mathbb{R}^m} f(x)\, d\nu^+(x) - \int_{\mathbb{R}^m} f(x)\, d\nu^-(x),$$

for $f \in \mathcal{L}^1(\nu)$.

We can now state the following theorem, the proof of which can be found in [Els09, p.349].

Theorem 2.10 (Riesz representation theorem)**.** *The map*

$$\Phi\colon \mathcal{M}(\mathbb{R}^m)) \to (C_0(\mathbb{R}^m))',$$
$$(\Phi(\nu))(f) := \int_{\mathbb{R}^m} f(x)\, d\nu(x)$$

for $\nu \in \mathcal{M}(\mathbb{R}^m))$ *and* $f \in C_0(\mathbb{R}^m)$ *is a norm preserving isomorphism.*

Remark 2.11. i) Let us define the set

$$C_0^+(\mathbb{R}^m) := \{f \in C_0(\mathbb{R}^m)\colon f \geq 0\}.$$

Every element Γ of $(C_0(\mathbb{R}^m))'$ possesses a unique partition into positive $\Gamma^+, \Gamma^- \in (C_0(\mathbb{R}^m))'$, i.e. for all $f \in C_0^+(\mathbb{R}^m)$ we have $\Gamma^\pm(f) \geq 0$, and $\Gamma = \Gamma^+ - \Gamma^-$, whereby

$$\Gamma^+(f) = \sup\left\{\Gamma(h)\colon h \in C_0^+(\mathbb{R}^m), h \leq f\right\},$$

for $f \in C_0^+(\mathbb{R}^m)$. The Riesz representation theorem also says that for $\nu \in \mathcal{M}(\mathbb{R}^m)$ the components ν^+ and $(\Phi(\nu))^+$, respectively ν^- and $(\Phi(\nu))^-$ correspond to each other. This shows that if $\Phi(\nu)$ is a positive, continuous linear form for some $\nu \in \mathcal{M}(\mathbb{R}^m)$, then we have $\nu \geq 0$. The content of this remark can also be found in [Els09, p.348-349].

ii) Regular signed measures are sometimes called Radon measures in the literature.

With this background we are finally able to state and prove the existence theorem for Young measures.

2.2. **The Existence Theorem.**

Theorem 2.12 (Existence of Young measures)**.** *Let* $\Omega \subseteq \mathbb{R}^n$ *be a measurable set and let* $z_i\colon \Omega \to \mathbb{R}^m$ *be measurable functions for* $i \in \mathbb{N}$ *such that*

$$\sup_{i\in\mathbb{N}} \int_\Omega g(|z_i(x)|)\, dx < \infty, \tag{2.2}$$

where $g\colon [0,\infty) \to [0,\infty]$ *is a continuous, non-decreasing function with* $\lim_{t\to\infty} g(t) = \infty$. *Furthermore, let* $\psi\colon \Omega \times \mathbb{R}^m \to \overline{\mathbb{R}}$ *be a Carathéodory function*[4]*. Then there exists a subsequence* $(z_{i_k})_{k\in\mathbb{N}}$ *and a family of probability measures* $\nu = \{\nu_x\}_{x\in\Omega}$ *on* $\mathbb{R}^m$ *such that, whenever* $(\psi(\cdot, z_{i_k}(\cdot)))_{k\in\mathbb{N}} \rightharpoonup \overline{\psi}$ *as* $k \to \infty$ *in* $L^1(\Omega)$*, we have*

$$\overline{\psi}(x) = \int_{\mathbb{R}^m} \psi(x,\lambda)\, d\nu_x(\lambda), \quad \text{for a.e. } x \in \Omega. \tag{2.3}$$

Proof. i) First let $\psi \in L^1(\Omega, C_0(\mathbb{R}^m))$. It is clear that such a ψ can be understood as a Carathéodory function, by defining $\overline{\psi}(x,\lambda) := (\psi(x))(\lambda)$. For the sake of simplicity we identify $\overline{\psi}$ with ψ. Because of Theorem 2.8 and 2.10 we have

$$(L^1(\Omega, C_0(\mathbb{R}^m)))' \cong L^\infty_{w^*}(\Omega, \mathcal{M}(\mathbb{R}^m)),$$

under the duality

$$\langle f, \nu\rangle = \int_\Omega \langle f(x), \nu_x\rangle\, dx = \int_\Omega \int_{\mathbb{R}^m} f(x,\lambda)\, d\nu_x(\lambda)\, dx,$$

where $f \in L^1(\Omega, C_0(\mathbb{R}^m))$ and $\nu \in L^\infty_{w^*}(\Omega, \mathcal{M}(\mathbb{R}^m))$. For each $i \in \mathbb{N}$ we define ν_i for $x \in \Omega$ by

$$\nu_i(x) := \delta_{z_i(x)},$$

[4] A function which is measurable in the first argument and continuous in the second is called *Carathéodory function*.

where δ_λ is the usual Dirac measure at $\lambda \in \mathbb{R}^m$, i.e. for $A \in \mathcal{B}(\mathbb{R}^m)$ we have $\delta_\lambda(A) = \mathbb{1}_A(\lambda)$. For any $f \in C_0(\mathbb{R}^m)$ we get the following composition of measurable functions

$$\langle f, \nu_i(\cdot)\rangle = \int_\Omega f(\lambda)\, d\delta_{z_i(\cdot)}(\lambda) = f(z_i(\cdot)),$$

and therefore ν_i is weakly-∗-measurable. Furthermore we have

$$\|\, \|\nu_i(\cdot)\|_{\mathcal{M}(\mathbb{R}^m)}\|_\infty = \operatorname{ess\,sup}_{x\in\Omega} \|\delta_{z_i(x)}\|_{\mathcal{M}(\mathbb{R}^m)} = \operatorname{ess\,sup}_{x\in\Omega} \left(\sup_{\substack{f\in C_0(\mathbb{R}^m),\\ \|f\|_\infty=1}} |f(z_i(x))| \right) = 1,$$

and therefore $\nu_i \in L^\infty_{w^*}(\Omega, \mathcal{M}(\mathbb{R}^m))$ for each $i \in \mathbb{N}$. Since $C_0(\mathbb{R}^m)$ is separable, Remark 2.5 implies that $L^1(\Omega, C_0(\mathbb{R}^m))$ is separable and by the Banach-Alaoglu theorem there exists a subsequence $(\nu_{i_k})_{k\in\mathbb{N}}$ and a $\nu \in L^\infty_{w^*}(\Omega, \mathcal{M}(\mathbb{R}^m))$ such that

$$\nu_{i_k} \overset{*}{\rightharpoonup} \nu, \text{ as } k \to \infty \text{ and } \|\nu\|_{L^\infty_{w^*}(\Omega,\mathcal{M}(\mathbb{R}^m))} \le 1.$$

For the chosen $\psi \in L^1(\Omega, C_0(\mathbb{R}^m))$ this gives us

$$\lim_{k\to\infty} \int_\Omega \psi(x, z_{i_k}(x))\, dx = \lim_{k\to\infty} \langle \psi, \nu_{i_k}\rangle = \langle \psi, \nu\rangle = \int_\Omega \int_{\mathbb{R}^m} \psi(x,\lambda)\, d\nu_x(\lambda)\, dx \tag{2.4}$$

Now, for any $h \in C_0(\mathbb{R}^m)$ and $\phi \in L^1(\Omega)$ such that $h, \phi \ge 0$, we define $f \in L^1(\Omega, C_0(\mathbb{R}^m))$ by $f(x) := \phi(x)h$. Applying (2.4) to this particular f we obtain

$$0 \le \lim_{k\to\infty} \int_\Omega \phi(x) h(z_{i_k}(x))\, dx = \int_\Omega \phi(x) \int_{\mathbb{R}^m} h(\lambda)\, d\nu_x(\lambda)\, dx. \tag{2.5}$$

This shows that

$$\int_{\mathbb{R}^m} h(\lambda)\, d\nu_x(\lambda) \ge 0$$

for a.e. $x \in \Omega$ and therefore the map $\Gamma\colon C_0(\mathbb{R}^m) \to \mathbb{R}, h \mapsto \int_{\mathbb{R}^m} h(\lambda)\, d\nu_x(\lambda)$ is a positive linear form. Remark 2.11 implies that the signed measure ν_x is for a.e. $x \in \Omega$ a measure.

ii) The next step is very technical. Let ψ be a nonnegative, Carathéodory function such that the sequence $(\psi(\cdot, z_{i_k}(\cdot)))_{k\in\mathbb{N}}$ converges weakly in $L^1(\Omega)$. By Theorem A.5 we obtain

$$\lim_{l\to\infty} \sup_{k\in\mathbb{N}} \int_{\{\psi(x, z_{i_k}(x))\ge l\}} \psi(x, z_{i_k}(x))\, dx = 0. \tag{2.6}$$

Since g is non-decreasing we have

$$g(l) \sup_{k\in\mathbb{N}} \lambda(\{|z_{i_k}| \ge l\}) \le \sup_{k\in\mathbb{N}} \int_\Omega g(|z_{i_k}(x)|)\, dx < \infty,$$

and because of $\lim_{l\to\infty} g(l) = \infty$ this implies that

$$\lim_{l\to\infty} \sup_{k\in\mathbb{N}} \lambda(\{|z_{i_k}| \ge l\}) = 0. \tag{2.7}$$

This condition, which is sometimes called a *tightness condition*, tells us that we can find a sequence $(m_l)_{l\in\mathbb{N}}$ such that $m_l \ge l$ for all $l \in \mathbb{N}$ and

$$\sup_{k\in\mathbb{N}} \lambda(\{|z_{i_k}| \ge m_l\}) \le \frac{1}{l^2},$$

and so

$$l \sup_{k\in\mathbb{N}} \lambda(\{|z_{i_k}| \ge m_l\}) \to 0, \text{ for } l \to \infty.$$

This last condition will be needed later. Finally let us define the following cut-off functions $\phi^l\colon [0,\infty) \to [0,1]$ for $l > 0$ by

$$\phi^l(t) := \mathbb{1}_{[0,l)}(t) + (1 - t + l)\mathbb{1}_{[l,l+1)}(t).$$

Further, let ψ^l be defined by

$$\psi^l(x,\lambda) := \phi^l(|\lambda|)\phi^l(\psi(x,\lambda))\psi(x,\lambda).$$

So ψ^l is basically ψ, but gets continuously cut off if $|\lambda|$ or ψ is large enough. We immediately get the following five properties which we will need later:

a) $\psi^l = \psi$ for $\psi \leq l$ and $|\lambda| \leq l$,
b) $\psi^l \in L^1(\Omega, C_0(\mathbb{R}^m))$ for all $l \in \mathbb{N}$, if $\lambda(\Omega) < \infty$,
c) $0 \leq \psi^l \leq \psi$ for all $l \in \mathbb{N}$,
d) $(\psi_l)_{l\in\mathbb{N}}$ is a non-decreasing sequence,
e) $\lim_{l\to\infty} \psi^l = \psi$ pointwise.

iii) In this step, equation (2.4) will be generalized to ψ as chosen in step ii). Let us assume from now on that Ω has finite measure. First we estimate, using (2.6) and (2.7) in the last step,

$$\begin{aligned}
&\left|\int_\Omega (\psi^{m_l}(x, z_{i_k}(x)) - \psi(x, z_{i_k}(x)))\,dx\right| \\
&\leq \int_{\{|z_{i_k}|\geq m_l\}\cup\{\psi(x,z_{i_k}(x))\geq m_l\}} \psi(x, z_{i_k}(x))\,dx \\
&\leq \int_{\{|z_{i_k}|\geq m_l\}\cup\{\psi(x,z_{i_k}(x))\geq l\}} \psi(x, z_{i_k}(x))\,dx \\
&\leq \int_{\{\psi(x,z_{i_k}(x))\geq l\}} \psi(x, z_{i_k}(x))\,dx + \int_{\{|z_{i_k}|\geq m_l\}\cap\{\psi(x,z_{i_k}(x))\leq l\}} \psi(x, z_{i_k}(x))\,dx \\
&\leq \sup_{k\in\mathbb{N}} \int_{\{\psi(x,z_{i_k}(x))\geq l\}} \psi(x, z_{i_k}(x))\,dx + l \sup_{k\in\mathbb{N}} \lambda(\{|z_{i_k} \geq m_l\}) \overset{l\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Hence we have

$$\lim_{l\to\infty} \int_\Omega \psi^{m_l}(x, z_{i_k}(x))\,dx = \int_\Omega \psi(x, z_{i_k}(x))\,dx, \tag{2.8}$$

and because we took the supremum over k the above limit converges uniformly with respect to k. This tells us that

$$\lim_{k\to\infty}\lim_{l\to\infty} \int_\Omega \psi^{m_l}(x, z_{i_k}(x))\,dx = \lim_{l\to\infty}\lim_{k\to\infty} \int_\Omega \psi^{m_l}(x, z_{i_k}(x))\,dx.$$

We are allowed to change the order of the limits because $\lim_{l\to\infty} \int_\Omega \psi^{m_l}(x, z_{i_k}(x))\,dx$ and $\lim_{k\to\infty} \int_\Omega \psi^{m_l}(x, z_{i_k}(x))\,dx$ both exist[5] and because of the above mentioned uniform convergence (see [Ger02, Theorem 1.6.5]). We obtain

$$\begin{aligned}
\lim_{k\to\infty} \int_\Omega \psi(x, z_{i_k}(x))\,dx &\overset{(2.8)}{=} \lim_{k\to\infty}\lim_{l\to\infty} \int_\Omega \psi^{m_l}(x, z_{i_k}(x))\,dx \\
&= \lim_{l\to\infty}\lim_{k\to\infty} \int_\Omega \psi^{m_l}(x, z_{i_k}(x))\,dx \\
&\overset{(*)}{=} \lim_{l\to\infty} \int_\Omega \int_{\mathbb{R}^m} \psi^{m_l}(x, \lambda)\,d\nu_x(\lambda)\,dx \\
&\overset{(**)}{=} \int_\Omega \int_{\mathbb{R}^m} \psi(x, \lambda)\,d\nu_x(\lambda)\,dx.
\end{aligned}$$

Note that $(*)$ is valid because $\psi^{m_l} \in L^1(\Omega, C_0(\mathbb{R}^m))$ for all $l \in \mathbb{N}$, hence we are able to apply equation (2.4). Equation $(**)$ is valid because of the properties c), d) and e) and the monotone convergence theorem, which is applicable because ν_x is a measure for almost every $x \in \Omega$.

iv) In this step we will show that we can drop the non-negativity condition for ψ and we will see how (2.4) gives us the claimed weak limit. So let ψ be any Carathéodory function fulfilling the weak convergence condition.

---

[5]Indeed, the first limit exists because of (2.8) and $(\psi(,\cdot, z_{i_k}(\cdot)))_{k\in\mathbb{N}}$ is weakly convergent in $L^1(\Omega)$. The second because $\psi^{m_l} \in L^1(\Omega, C_0(\mathbb{R}^m))$ and (2.4).

We can split it into its positive part $\psi^+ := \sup\{\psi, 0\}$ and its negative part $\psi^- := \sup\{-\psi, 0\}$. Step ii) and iii) are now applicable to these two parts, because if $(\psi(\cdot, z_{i_k}(\cdot)))_{k\in\mathbb{N}}$ converges weakly in $L^1(\Omega)$, the same is true for $(|\psi(\cdot, z_{i_k}(\cdot))|)_{k\in\mathbb{N}}$, which follows immediately from the Dunford-Pettis Theorem A.5. Together with the identities $\psi^+ = \frac{1}{2}(|\psi| + \psi)$ and $\psi^- = \frac{1}{2}(|\psi| - \psi)$ this implies the same for these two sequences. So equation (2.4) also holds for ψ.

For any $\xi \in L^\infty(\Omega)$ we define $\phi(x, \lambda) := \xi(x)\psi(x, \lambda)$. Note that $\phi\colon \Omega \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is also a Carathéodory function. The Dunford-Pettis theorem again implies that the sequence $(\xi(\cdot)\psi(\cdot, z_{i_k}(\cdot)))_{k\in\mathbb{N}}$ is weakly convergent in $L^1(\Omega)$. Hence equation (2.4) gives us

$$\lim_{k\to\infty} \int_\Omega \xi(x)\psi(x, z_{i_k}(x))\, dx = \int_\Omega \xi(x) \int_{\mathbb{R}^m} \psi(x, \lambda)\, d\nu_x(\lambda)\, dx.$$

Because ξ was arbitrary this shows the claimed form of the weak limit $\overline{\psi}$. Note that this is still valid when $\lambda(\Omega) = \infty$, since it holds on every subset with finite measure. Hence we can drop the assumption $\lambda(\Omega) < \infty$.

v) It remains to check that ν_x is a probability measures for a.e. $x \in \Omega$. We already know that $\nu_x \geq 0$ and $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} \leq 1$ for a.e. $x \in \Omega$. For any measurable $B \subseteq \Omega$, consider equation (2.4) for $\psi(x, \lambda) := \mathbb{1}_B(x)$

$$\begin{aligned} \lambda(B) &= \lim_{k\to\infty} \int_\Omega \mathbb{1}_B(x)\, dx = \int_\Omega \int_{\mathbb{R}^m} \mathbb{1}_B(x)\, d\nu_x(\lambda)\, dx \\ &= \int_B \nu_x(\mathbb{R}^m)\, dx = \int_B |\nu_x|(\mathbb{R}^m)\, dx = \int_B \|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)}\, dx, \end{aligned}$$

and since B was arbitrary this shows that $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = |\nu_x|(\mathbb{R}^m) = 1$ for a.e. $x \in \Omega$. Together with $\nu_x \geq 0$ this implies that (after changes on a null set) $\nu = \{\nu_x\}_{x\in\Omega}$ is a family of probability measures. This completes the proof. □

Remark 2.13. i) The family of probabilty measures $\nu = \{\nu_x\}_{x\in\Omega}$, which depends weakly-$*$ measurable on x is called the *Young measure* associated to (or generated by) the sequence $(z_{i_k})_{k\in\mathbb{N}}$. Note that neither the Young measure nor the subsequence is dependent on ψ.

ii) The Young measure can be interpreted in the following way: ν_x can be thought of as the limiting probability distribution of the values of z_{i_k} near the point x. A further explanation can be found in [Bal89, remark 4]. This suggests that we should have strong convergence of $(z_{i_k})_{k\in\mathbb{N}}$ if and only if the Young measure is the Dirac measure $\delta_{z(x)}$ for a.e. $x \in \Omega$. Indeed the following holds (see [Ped97, p.111]): Let $\Omega \subseteq \mathbb{R}^n$ be measurable, $1 \leq p < \infty$ and $(z_i)_{i\in\mathbb{N}} \subseteq L^p(\Omega)$ such that $(|z_i|^p)_{i\in\mathbb{N}}$ is weakly convergent in $L^1(\Omega)$. Let $\nu = \{\nu_x\}_{x\in\Omega}$ be the Young measure associated to $(z_i)_{i\in\mathbb{N}}$. Then the following two statements are equivalent for some $z \in L^p(\Omega)$:
   - $z_i \longrightarrow z$ in $L^p(\Omega)$,
   - $\nu_x = \delta_{z(x)}$ for a.e. $x \in \Omega$.

iii) Condition (2.2) is only one possible formulation. In the proof it was necessary to get the tightness condition (2.7), which some authors use directly in the formulation of the theorem, for example [Bal89, p.209, remark 1.]. An important example for the choice of g is $g(t) = t^p$ for $1 \leq p < \infty$. Therefore, every bounded sequence in $L^p(\Omega)$ contains a subsequence, which generates a Young measure.

iv) Let us present a slightly different version of the fundamental theorem that will be used in a later chapter.
Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set and let $z_i\colon \Omega \to \mathbb{R}^m$ be a sequence of functions in $L^\infty(\Omega, \mathbb{R}^m)$ that is uniformly bounded, i.e.

$$\|\,|z_i|\,\|_\infty \leq C, \quad \text{for } i \in \mathbb{N},$$

where $C > 0$ is some constant, $|\cdot|$ the norm in $\mathbb{R}^m$ and $\|\cdot\|$ the usual $L^\infty(\Omega)$-norm.

Then there exists a subsequence of $(z_i)_{i\in\mathbb{N}}$, not relabelled, and a family of probability measures $\{\nu_x\}_{x\in\Omega}$ on $\mathbb{R}^m$ such that for every $F \in C(\mathbb{R}^m, \mathbb{R})$

$$(F \circ z_i) \overset{*}{\rightharpoonup} \overline{F}, \text{ in } L^\infty(\Omega),$$

where

$$\overline{F}(x) = \int_{\mathbb{R}^m} F(\lambda)\, d\nu_x(\lambda), \text{ for a.e. } x \in \Omega.$$

The proof of this version is more or less contained in the previous proof. Part i) stays exactly the same and the equality in (2.5) shows the claimed form of $\overline{F}$ for a given $F \in C_0(\mathbb{R}^m)$. Because of the boundedness of the sequence $(u_i)_{i\in\mathbb{N}}$, we get the tightness condition (2.7) and therefore the rest of the proof can be repeated with only minor alterations to show that the measures are probability measures. We also get that (2.4) is valid for any Carathéodory function and together with (2.5) this shows the claim for $F \in C(\mathbb{R}^m)$. Note that in this case we do not have to assume that $(F \circ u_i)_{i\in\mathbb{N}}$ is weak-$*$ convergent in $L^\infty(\Omega)$, because this is automatically true.

3. Young Measures and weak-lower-semi-continuity

In this section we consider a typical problem in the calculus of variations. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $u : \Omega \to \mathbb{R}$. Let $M^{n\times N}$ be the set of $n \times N$ matricies with real entries. Let $F : M^{m\times N} \to \mathbb{R}$ be a continuous function that is bounded below. We are interested in the functional

$$J(u) = \int_\Omega F(\nabla u(x))\, dx \text{ where } u \in W^{1,p} \text{ for some } 1 < p < \infty \tag{3.1}$$

The following theorem establishes that J is wlsc if and only if F satisfies Jensen's inequality with respect to the Young measure.

Theorem 3.1. *Consider* (3.1). *Let u_n be a sequence in $W^{1,p}(\Omega)$ such that $|J(u_n)|$ is bounded and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$. Let ν_x be the Young measures associated to the sequence ∇u_n i.e. if $F(x,\lambda)$ is a Cartheodory function, $E \subset \Omega$ is a measurable subset and if $F(\cdot, \nabla u_n(\cdot))$ converges weakly in $L_1(E)$, then the limit is given by*

$$\overline{F}(x) = \int_{\mathbb{R}^m} F(x,\lambda)\, d\nu_x(\lambda)\, dx \text{ a.e. } x \in \Omega. \tag{3.2}$$

Under these assumptions we have the following. For all measurable $E \subset \Omega$,

$$\int_E F(\nabla u)\, dx \le \liminf_{n\to\infty} \int_E F(\nabla u_n)\, dx \tag{3.3}$$

if and only if F satisfies,

$$F(\nabla u) \le \int_{M^{m\times N}} F(A)\, d\nu_x. \tag{3.4}$$

We noted in section 1 that J is wlsc if it is convex, continuous and defined on a closed, convex set. The condition in Theorem 3.1 can be thought of as a different notion of convexity. Indeed, any convex function F satisfies Jensen's inequality with respect to any probability measure. The Young measure is distinguished in the sense that Jensen's inequality implies a sufficient condition for the functional J to be wlsc.

3.1. Auxiliary Results.

The proof of 3.1 requires several lemmas. We will not prove those which do not directly relate to the Young measure.

Lemma 3.2 (Chacon's Biting Lemma)**.** *Let f_n be a bounded sequence in $L^1(\Omega)$. Then there exists a subsequence, not relabelled, and a sequence of measurable sets Ω_j, $j \in \mathbb{N}$ such that $\Omega_{j+1} \subset \Omega_j \subset \Omega$ with $|\Omega_j| \searrow 0$ and a function $f \in L^1(\Omega)$ s.t. for each fixed j*

$$f_n \rightharpoonup f \text{ in } L^1(\Omega \backslash \Omega_j). \tag{3.5}$$

We say that f_n converges to f in the biting sense and write $f_n \overset{bit.}{\rightharpoonup} f$.

Corollary 3.3 (Corollary to theorem on existence of Young measures)**.** *Let $u_{n n\in\mathbb{N}}$ be a sequence of measurable functions $u_n : \Omega \to \mathbb{R}^m$ with associated Young measures $\{\nu_x\}_{x\in\Omega}$. If $F : \Omega\times\mathbb{R}^m \to \mathbb{R}$ is a Cartheodory function such that the sequence $F(x,u_n(x))$ is bounded in $L^1(\Omega)$, then there exists a subsequence, not relabelled, such that*

$$F(x,u_n(x)) \overset{bit.}{\rightharpoonup} \overline{F}(x) = \int_{\mathbb{R}^m} F(x,\lambda)\, d\nu_x(\lambda)\, dx \tag{3.6}$$

Proof. Because $F(x,u_n(x))$ is bounded in $L^1(\Omega)$, Lemma 3.2 implies that there exists a sequence of sets $\{\Omega_j\}_{j\in\Omega}$ with the properties described in the lemma. In particular, there exists a subsequence, not relabelled, that converges weakly to some F_j on $\Omega \backslash \Omega_j$. By theorem 2.12, F_j is given by 3.6 on each $\Omega \backslash \Omega_j$, thus 3.6 is the biting limit of the sequence. □

Lemma 3.4. *Let $f_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of measurable functions. Suppose there are two constants $C_1 \geq 0$ and $C_2 \in \mathbb{R}$ such that*

(1) $\|f_n\|_{L^1(\Omega)} \geq C_1$ for all $n \in \mathbb{N}$.
(2) $f_n(x) \geq C_2$ for all $n \in \mathbb{N}$ and all $x \in \Omega$.

Further suppose that $f_n \overset{bit.}{\rightharpoonup} f$. Then there exists a subsequence that converges weakly to f if and only if

$$\liminf_{n\to\infty} \int_\Omega f_n(x)\, dx \leq \int_\Omega f(x)\, dx \tag{3.7}$$

Lemma 3.5. *Let u_n, $n \in \mathbb{N}$, be a sequence of measurable functions $u_n : \Omega \to \mathbb{R}^m$ with associated Young measures $\{\nu_x\}_{x\in\Omega}$. If $F : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carthéodory function that is bounded below, then for every measurable subset $E \subset \Omega$*

$$\int_E \int_{\mathbb{R}^m} F(x,\lambda)\, d\nu_x(\lambda) dx \leq \liminf_{n\to\infty} \int_E F(x, u_n(x))\, dx \tag{3.8}$$

Proof. If the RHS of (3.8) is infinity then there is nothing to prove. If not then there exists a subsequence, also labelled $F(x, u_n(x))$, that is bounded in $L^1(E)$. By Corollary 3.3, $F(x, u_n(x)) \overset{bit.}{\rightharpoonup} \overline{F}$ where $\overline{F}$ is given by

$$\overline{F}(x) = \int_{\mathbb{R}^m} F(x,\lambda)\, d\nu_x(\lambda) \tag{3.9}$$

Assume (3.8) is false i.e.

$$\int_E \overline{F}(x) dx > \liminf_{n\to\infty} \int_E F(x, u_n(x))\, dx \tag{3.10}$$

Because F is bounded below and the sequence $\{F(x, u_n(x))\}_{n\in\mathbb{N}}$, is bounded in $L_1(E)$, we can apply Lemma 3.4. In particular, inequality (3.10) implies that there exists a subsequence, this time relabelled u_{n_j}, such that $F(x, u_{n_j}(x)) \rightharpoonup \overline{F}$. Because Ω is bounded, the constant function 1 is in $L^1(\Omega)$ and therefore

$$\liminf_{n\to\infty} \int_E F(x, u_n(x))\, dx \leq \lim_{j\to\infty} \int_E F(x, u_{n_j}(x)) \cdot 1\, dx = \int_E \overline{F}(x)\, dx. \tag{3.11}$$

This contradicts (3.10). □

3.2. **Proof of Theorem 3.1.** Before proving theorem 3.1, note that the continuous function $F : M^{m\times N} \to \mathbb{R}$ can be written as a function $F : \Omega \times \mathbb{R}^{mN} \to \mathbb{R}$ by letting $F(x,\lambda) = F(\lambda)$ and interpreting $\lambda \in \mathbb{R}^{nM}$ as an $n \times M$ matrix. This function is indeed measurable in x and continuous in λ. In particular (3.2) becomes

$$\overline{F}(x) = \int_{\mathbb{R}^m} F(x,\lambda)\, d\nu_x(\lambda)\, dx \;= \int_{M^{m\times N}} F(A) d\nu_x(A)$$

Proof of Theorem. First assume (3.4). Then by lemma 3.5

$$\liminf_{n\to\infty} \int_E F(\nabla u_n(x))\, dx \geq \int_E \int_{M^{m\times N}} F(A) d\nu_x(A) dx \geq \int_E F(\nabla u(x))\, dx \tag{3.12}$$

For the converse note that the sequence $F(\nabla u_n)$ is bounded in $L^1(\Omega)$ and so by Lemma 3.2 and Corollary 3.3, there exists a subsequence, not relabelled, such that $F(\nabla u_n) \overset{bit.}{\rightharpoonup} \overline{F}(x) = \int_{M^{m\times N}} F(A) d\nu_x(A)$. In detail, this means there exists a sequence of sets $\{\Omega_j\}_{j\in\mathbb{N}}$ such that $\Omega_{j+1} \subset \Omega_j \subset \Omega$, $|\Omega_j| \searrow 0$ and $F(\nabla u_n) \rightharpoonup \overline{F}$ on $\Omega\backslash\Omega_j$. Note that this is also true on any $E\backslash\Omega_j$ where $E \subset \Omega$ is any measurable set. Using (3.4) and noting that lim inf is clearly less than or equal to lim,

$$\int_{E\backslash\Omega_j} F(\nabla u_n)\,dx \quad \leq \quad \lim_{n\to\infty}\int_{E\backslash\Omega_j} F(\nabla u_n)\cdot 1\,dx \tag{3.13}$$

$$= \quad \int_{E\backslash\Omega_j} \overline{F}(x)\,dx \tag{3.14}$$

Taking $E = \Omega$ in (3.8) proves that $\overline{F} \in L^1(\Omega)$. Similarly, taking $E = \Omega$ in (3.3) and recalling that the LHS of (3.3) is bounded by assumption, we also have $F(\nabla u) \in L^1(\Omega)$. Clearly both are also in $L_1(E)$. Let $\chi_{E\backslash\Omega_j}$ be the indicator function on $E\backslash\Omega_j$, then $\overline{F}(x) \geq \chi_{E\backslash\Omega_j}\overline{F}(x)$ and $F(\nabla u(x)) \geq \chi_{E\backslash\Omega_j}F(\nabla u(x))$. Using the indicator function to write (3.13) and (3.14) as integrals over E, we can apply the Lebesgue dominated convergence theorem to both sides of the inequality to obtain

$$\int_E F(\nabla u)\,dx \leq \int_E \overline{F}(x)\,dx = \int_E\int_{M^{m\times N}} F(A)d\nu_x dx \tag{3.15}$$

Since E was arbitrary we have (3.3). □

4. A Non-Convex Example

To this point we have only considered problems in the setting of some sort of convexity assumption. We have already noted in the introduction the importance in applications of functionals that are not convex. Here we treat a functional that satisfies none of the convexity assumptions we have used so far, which means, by Theorem 3.1, that the functional under consideration is not wlsc. This functional has no minimiser, but the results of section 2 still apply and we will use these results to calculate the Young measure. We will conclude this paper by revisiting our heuristic interpretation of the Young measure by using it to explicitly construct a minmising sequence that is, as expected, oscillatory in nature.

4.1. **Non-Existence of Minimisers.**

Example. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(\lambda_1, \lambda_2) = \lambda_1^2 + (1-\lambda_2)^2$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $u \in W_0^{1,\infty}(\Omega)$. Consider the functional*

$$J(u) = \int_\Omega F(\nabla u(x))\,dx = \int_\Omega u_{x_1}^2 + (1-u_{x_2})^2\,dx \tag{4.1}$$

Proposition 4.1. $\inf_{u\in W_0^{1,\infty}} J(u) = 0$ *and there does not exist a $u \in W_0^{1,\infty}$ s.t. $J(u) = 0$*

Proof. We use the following result from [Dac08]: For any continuous function $\varphi:\mathbb{R}^2 \to \mathbb{R}$

$$\inf_{u\in W_0^{1,\infty}(\Omega)}\int_\Omega \varphi(\nabla u(x))\,dx \leq |\Omega|\varphi^{**}(0,0) \tag{4.2}$$

where φ^{**} is the largest convex function that is everywhere less than or equal to φ. For our particular example 0 is convex and $0 \leq F$ so

$$0 \leq F^{**}(0,\pm 1) \leq F(0,\pm 1) = 0 \tag{4.3}$$

Using (4.3)

$$0 \leq F^{**}(0,0) = F^{**}(\frac{1}{2}(0,1) + \frac{1}{2}(0,-1)) \leq \frac{1}{2}F^{**}(0,1) + \frac{1}{2}F^{**}(0,-1) = 0 \tag{4.4}$$

Thus $F^{**}(0,0) = 0$. That the infima is 0 follows from (4.2) and the fact that $J(u) \geq 0$.

Next we claim that the infima is never reached. Indeed, suppose there exists some $u \in W_0^{1,\infty}(\Omega)$ such that $J(u) = 0$. Then $u_{x_1} = 0$ and thus u is constant along the x_1-coordinate-direction. Note that $W_0^{1,\infty}(\Omega)$ is continuously embedded into the space of Lipschitz continuous functions on Ω, hence the trace of u is equal to its restriction to the boundary of Ω. Because u has zero trace, it is identically zero on the boundary of u and therefore identically 0 everywhere. But $J(0) > 0$; a contradiction.

□

4.2. **The Young Measure of a Minimising Sequence.** Even though no minimiser exists, a bounded minimising sequence does (since it can easily be checked that J is coercive). Therefore the results of section 2 are applicable. We use one of the alternative forumlations found in Remark 2.13. First, we need a lemma.

Lemma 4.2. *Let u_n be a minimising sequence that is bounded in $W_0^{1,\infty}(\Omega)$. Then*

$$\nabla u_n \overset{*}{\rightharpoonup} 0 \tag{4.5}$$

and

$$u_n \to 0. \tag{4.6}$$

In (4.5) *the convergence is in $(L^\infty(\Omega))^2 = L^\infty(\Omega) \times L^\infty(\Omega)$ i.e. for $j = 1, 2$, $u_{n\,x_j} : \Omega \to \mathbb{R}$ converges in the weak-$*$ sense to zero. In* (4.6) *the convergence is in $L^\infty(\Omega)$.*

Proof. We know that $W_0^{1,\infty} \hookrightarrow C(\overline{\Omega})$ is a compact embedding. This fact, plus the Alaoglu theorem, implies that there is a subsequence, not relabelled, with the following two properties:

$$\nabla u_n \overset{*}{\rightharpoonup} w \text{ in } (L^\infty(\Omega))^2 \tag{4.7}$$

and

$$u_n \to u \text{ in } C(\overline{\Omega}). \tag{4.8}$$

We claim that $u \in W_0^{1,\infty}(\Omega)$ and that $w = \nabla u$. To see this let $\psi \in C^\infty(\Omega)$ have compact support and, for convenience, use the notation $u_{x_1}\psi + u_{x_2}\psi = \nabla u \cdot \psi$. Integrating by parts one obtains,

$$\lim_{n\to\infty} \int_\Omega \nabla u_n(x)\psi(x)\,dx = -\lim_{n\to\infty} \int_\Omega u_n(x)\nabla\psi(x)\,dx \tag{4.9}$$

(4.7) implies that the RHS of (4.9) equals

$$\int_\Omega w(x) \cdot \psi(x)\,dx \tag{4.10}$$

(4.8) implies that the LHS of 4.9 equals

$$-\int_\Omega u(x) \cdot \nabla\psi(x)\,dx. \tag{4.11}$$

By (4.10) and (4.11), $w = \nabla u$. Because the u_n are continuous with zero trace, (4.8) implies that u also has zero trace.

Now let us show (4.5) and (4.6). Because u_n is minimising, $u_{nx_1} \to 0$ in $L^2(\Omega)$. Moreo-ever, $u_{nx_1} \overset{*}{\rightharpoonup} u_{x_1}$ in $L^\infty(\Omega)$. Because $u \in L^\infty(\Omega)$ it is also in $L^2(\Omega)$ and $L^1(\Omega)$. Using these facts and the Cauchy-Schwartz inequality,

$$0 \le \int_\Omega u_{x_1}^2(x)\,dx = \lim_{n\to\infty} \int_\Omega u_{nx_1}(x)u_{x_1}(x)\,dx \le \lim_{n\to\infty} ||u_{nx_1}||_{L^2(\Omega)}||\,||u_{x_1}||_{L^2(\Omega)} = 0 \tag{4.12}$$

Hence u is a constant in the x_1 coordinate direction. That $u = 0$ follows because u has zero trace. □

Theorem 4.3. *For a.e. $x \in \mathbb{R}^2$, the Young measure of any bounded minmising sequence u_n is*

$$\nu_x = \frac{1}{2}\delta_{(0,-1)} + \frac{1}{2}\delta_{(0,+1)} \tag{4.13}$$

where $\delta_{(0,\pm 1)}$ are the Dirac delta measures at the points $(0, \pm 1) \in \mathbb{R}^2$ respectively.

Proof. Consider the continuous function $F(\lambda_1, \lambda_2) = \lambda_1^2 + (1 - \lambda_2)^2$. Since u_n is a bounded sequence Remark 2.13 implies the existence of a subsequence, not relabelled, such that

$$F(\nabla u_n) \stackrel{*}{\rightharpoonup} \int_{\mathbb{R}^2} F(\lambda_1, \lambda_2)\, d\nu_x(\lambda) \tag{4.14}$$

Because u_n is minimising

$$\lim_{n\to\infty} \int_\Omega F(\nabla u_n) dx = 0 \tag{4.15}$$

Rewriting (4.15) but using (4.14) to represent the weak-star limit,

$$0 = \lim_{n\to\infty} \int_\Omega F(\nabla u_n) \cdot 1 dx = \int_\Omega \int_{\mathbb{R}^2} F(\lambda_1, \lambda_2) d\nu_x(\lambda)\, dx \tag{4.16}$$

The outer integral is taken w.r.t the standard Lebesgue measure, so the integrand, in this case the inner integral, is a.e. equal to zero. So for a.e. $x \in \mathbb{R}^2$

$$\int_{\mathbb{R}^2} F(\lambda_1, \lambda_2) d\nu_x(\lambda) = 0 \tag{4.17}$$

In particular, (4.17) implies that $\nu_x(E) = 0$ for any set E on which F is non-zero. Because ν_x is a probability measure, ν_x is a convex combination of $\delta_{(0,-1)}$ and $\delta_{(0,+1)}$. So there exists $\alpha(x) \in [0, 1]$ s.t.

$$\nu_x = \alpha(x)\delta_{(0,-1)} + (1 - \alpha(x))\delta_{(0,+1)} \tag{4.18}$$

We claim that $\alpha(x) = \frac{1}{2}$ for a.e. $x \in \mathbb{R}^2$. To see this, consider the auxiliary function $G(\lambda_1, \lambda_2) = \lambda_2$. As in (4.14), we can represent the weak limit via the Young measure

$$G(\nabla u_n) \stackrel{*}{\rightharpoonup} \int_{\mathbb{R}^2} G(\lambda_1, \lambda_2)\, d\nu_x(\lambda) \tag{4.19}$$

But by Lemma 4.2 we also know that

$$G(\nabla u_n) = u_{nx_2} \stackrel{*}{\rightharpoonup} 0 \tag{4.20}$$

(4.19) and (4.20) together imply that

$$0 = \int_{\mathbb{R}^2} \lambda_2 d\nu_x(\lambda) = -\alpha(x) + 1 - \alpha(x) \tag{4.21}$$

Thus $\alpha(x) = \frac{1}{2}$. □

4.3. **A Minimising Sequence.** Theorem 4.3 says that for any minimising sequence, the gradient ∇u_n should take on the values $(0, \pm 1)$ with equal probability as $n \to \infty$. This gives us a starting point for explicitly constructing a minimising sequence.

Consider the sequence $v_n : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$v_n(x_1, x_2) = \begin{cases} x_2 & x_2 \in (0, \frac{1}{n}) \\ \frac{2}{n} - x_2 & x_2 \in (\frac{1}{n}, \frac{2}{n}) \end{cases} \tag{4.22}$$

where the v_n is extended periodically to the whole of $\mathbb{R}^2$. Clearly $F(\nabla v_n) = 0$, however v_n restricted to Ω may not have zero trace. Define u_n by

$$u_n(x) = \min\{v_n(x), \operatorname{dist}(x, \partial\Omega)\} \tag{4.23}$$

Let $\Omega(n) := \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) \leq \frac{1}{n}\}$. Also, note that $\|\nabla \operatorname{dist}(x, \partial\Omega)\| \leq 1$. Hence there is a constant C such that $F(\nabla u_n) \leq C$. It's then clear that

$$\int_\Omega F(\nabla u_n)\, dx = \int_{\Omega(n)} F(\nabla u_n)\, dx \leq C|\Omega(n)|. \tag{4.24}$$

This approaches zero as $n \to \infty$. We conclude with the brief but significant observation that the sequence produces finer and finer oscillatory behaviour as $n \to \infty$, as we have come to expect.

Appendix A. Weak Convergence and the Banach-Alaoglu Theorems

This appendix contains the requisite knowledge of the Banach-Alaoglu theorems, weak convergence and the Dunford-Pettis criterion. The following material can be found in [Die84], unless specified otherwise.

Definition A.1 (Weak convergence)**.** Let X be a Banach space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges *weakly* to $x \in X$ (Notation: $x_n \rightharpoonup x$, as $n \to \infty$) iff for all $\phi \in X'$

$$\langle x_n, \phi\rangle = \phi(x) \to \langle x, \phi\rangle = \phi(x) \text{ as } n \to \infty.$$

A sequence $(\phi_n)_{n\in\mathbb{N}} \subseteq X'$ converges in the *weak-∗ sense* to $\phi \in X'$ (Notation: $\phi_n \overset{*}{\rightharpoonup} \phi, \quad \text{as } n \to \infty$) iff for all $x \in X$

$$\langle x, \phi_n\rangle = \phi_n(x) \to \langle x, \phi\rangle = \phi(x) \text{ as } n \to \infty.$$

A set $M \subseteq X$ ($M \subseteq X'$) is called *weak sequentially compact* (*weak-∗ sequentially compact*), iff every sequence in M has a weak (weak-∗) convergent subsequence with a weak (weak-∗) limit in M.

If X is reflexive then weak convergence in X' is the same as weak-∗ convergence. Every weak (or weak-∗) convergent sequence is bounded, which is a simple corollary of the Banach-Steinhaus theorem.
The following theorems are commonly referred to as the Banach-Alaoglu theorems. We formulate several versions.

Theorem A.2 (Banach-Alaoglu, version 1)**.** *Let X be a reflexive Banach space. Then the closed unit ball $\overline{B(0,1)} \subseteq X$ is weak sequentially compact.*

For a proof of this, see [Alt06, p.234]. Obviously this is also true for every closed ball in X. More importantly, the ball itself does not play any role, but the convexity. The statement is also true for a bounded, convex and closed subset of X.

Theorem A.3 (Banach-Alaoglu, version 2)**.** *Let X be a separable Banach space. Then the closed unit ball $\overline{B(0,1)} \subseteq X'$ is weak-∗ sequentially compact.*

The proof of this version uses the Cantors's diagonal argument (see [Alt06, p.229] for the details). This implies that every bounded sequence in the dual of a separable Banach space has a weak-∗ convergent subsequence.

Theorem A.4 (Banach-Alaoglu, version 3)**.** *Let X be a Banach space. Then the closed unit ball $\overline{B(0,1)} \subseteq X'$ is weakly-∗ compact, i.e. compact in the weak-∗ topology.*

This is the original and most general version. A proof can be found in [Die84, p.13]. Note that this is weak-∗ compactness and not weak-∗ sequential compactness.

For $1 < p < \infty$, it is well known that every bounded sequence in $L^p(\Omega)$ is weakly relative compact because these spaces are reflexive. $L^1(\Omega)$ is not reflexive, however the following theorem holds:

Theorem A.5 (Dunford-Pettis)**.** *Let $M \subseteq L^1(\Omega, \mathbb{R}^m)$ be bounded. Then the following statements are equivalent:*

i) M is relatively weakly compact in $L^1(\Omega, \mathbb{R}^m)$,
ii) M is uniformly integrable*, i.e.*

$$\forall \varepsilon > 0 \ \exists C > 0\colon \sup_{u\in M} \int_{\{x\in\Omega\colon |u(x)|\geq C\}} |u(x)|\, dx \leq \varepsilon,$$

iii) M is equiintegrable *or* equi-absolutely-continuous*, i.e.*

$$\forall \varepsilon > 0 \ \exists \delta > 0\colon \sup_{u\in M} \sup_{\lambda(A)\leq\delta} \int_A |u(x)|\, dx \leq \varepsilon.$$

Proof. See [DU77, p.76] or [Rou13, p.14]. □

Appendix B. Auxiliary Results

Here we give a proof of Lemma 1.3, which can be found in [JMR96, p.146]:

Proof. Define $u \in L^\infty(0, 2\pi)$ as the restriction of $\bar{u}$ to $(0, 2\pi)$. Since $L^\infty(0, 2\pi) \subseteq L^2(0, 2\pi)$ we have $u \in L^2(0, 2\pi)$ and therefore we are able to use the Fourier expansion to write

$$u(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx)), \quad \text{for a.e. } x \in (0, 2\pi),$$

where $a_k = \frac{1}{\pi}\int_0^{2\pi} u(x)\cos(kx)\,dx$ and $b_k = \frac{1}{\pi}\int_0^{2\pi} u(x)\sin(kx)\,dx$. For $n \in \mathbb{N}$ and a.e. $x \in (0, 2\pi)$ this implies

$$u_n(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(knx) + b_k \sin(knx)).$$

If $0 \le a < b \le 2\pi$ we estimate

$$\left|\int_a^b \sum_{k=1}^{\infty} a_k \cos(knx)\,dx\right| \overset{a)}{=} \left|\sum_{k=1}^{\infty} a_k \int_a^b \cos(knx)\,dx\right| \overset{b)}{\le} \sum_{k=1}^{\infty} |a_k| \frac{C}{kn}$$

$$\overset{c)}{\le} \frac{C}{n}\left(\sum_{k=1}^{\infty}|a_k|^2\right)^{1/2}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{1/2} \overset{n\to\infty}{\longrightarrow} 0.$$

a) follows from the fact that

$$\frac{\cos k\cdot}{\sqrt{\pi}}, \quad \text{for } k \in \mathbb{N},$$

forms a orthonormal system of $L^2(0, 2\pi)$. This implies

$$\sum_{k=1}^{\infty} a_k \cos(kn\cdot) \in L^2(0, 2\pi) \subseteq L^1(0, 2\pi),$$

and

$$\left\|\sum_{k=1}^{\infty} a_k \cos(kn\cdot) - \sum_{k=1}^{l} a_k \cos(kn\cdot)\right\|_{L^1(0,2\pi)} \le C_1 \left\|\sum_{k=1}^{\infty} a_k \cos(kn\cdot) - \sum_{k=1}^{l} a_k \cos(kn\cdot)\right\|_{L^2(0,2\pi)} \overset{l\to\infty}{\longrightarrow} 0,$$

for some $C_1 > 0$. b) follows by a change of variables for some constant $C > 0$ and c) is an application of Hölder's inequality. Note that $\left(\sum_{k=1}^{\infty}|a_k|^2\right) < \infty$ is ensured by Parseval's identity for Fourier series. The same holds for the sin-part and therefore we get

$$\int_0^{2\pi} u_n(x)\mathbb{1}_{[a,b]}(x)\,dx \to \int_0^{2\pi} \frac{a_0}{2}\mathbb{1}_{[a,b]}(x)\,dx, \quad \text{for } n \to \infty. \tag{B.1}$$

Since step functions are dense in $L^1(0, 2\pi)$ and (B.1) is valid for such functions, this shows the claim. ◻

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