Numerical methods for a fractional diffusion/anti-diffusion equation

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Outline

1. Introduction
2. Finite difference schemes - Collaboration : Pascal Azerad (Univ. Montp2)
3. Splitting methods - Collaboration : Rémi Carles (CNRS & Univ. Montp2)

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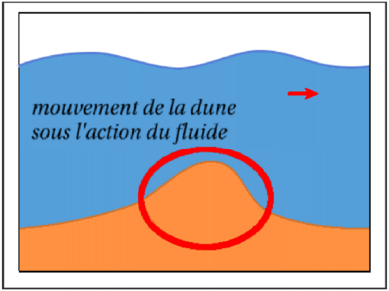
Model for dune morphodynamics

A conservative nonlinear model

For all $t \in (0, T)$ et $x \in \mathbb{R}$,

$$\begin{cases} u_t(t,x) + \left(\frac{u^2}{2}\right)_x (t,x) - u_{xx}(t,x) + \mathcal{I}[u(t,\cdot)](x) = 0, \\ u(0,x) = u_0(x). \end{cases}$$

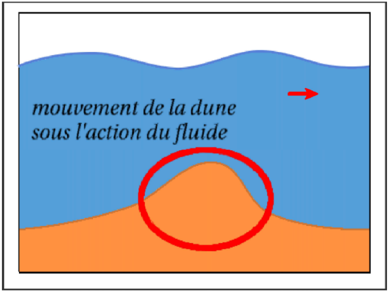


Model for dune morphodynamics

A conservative nonlinear and nonlocal model (A.C Fowler, Oxford)

For all $t \in (0, T)$ et $x \in \mathbb{R}$,

$$\begin{cases} u_t(t,x) + \left(\frac{u^2}{2}\right)_x (t,x) - u_{xx}(t,x) + \int_0^{+\infty} |\xi|^{-1/3} u_{xx}(t, x-\xi)\, d\xi = 0, \\ u(0,x) = u_0(x). \end{cases}$$



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Remark : This nonlocal term also appears in the work of P.-Y. Lagrée (Paris VI).

References :
P-Y Lagrée, *Asymptotic Methods in Fluid Mechanics : Survey and Recent Advances*, lecture notes 523, CISM International Centre for Mechanical Sciences Udine, H. STEINRÜCK Ed., Springer, (2010).
A.C. Fowler, *Mathematics and environment*, lecture note, 2006.

Some expressions for $\mathcal{I}$

- **Integral formula :**
  For all $\varphi \in \mathcal{S}(\mathbb{R})$ and all $x \in \mathbb{R}$

$$\mathcal{I}[\varphi](x) = \frac{4}{9} \int_{-\infty}^{0} \frac{\varphi(x+z) - \varphi(x) - \varphi'(x)z}{|z|^{7/3}} \, dz$$

- **Fractional derivative :**
  For causal functions (i.e. $\varphi(x) = 0$ for $x < 0$)

$$\frac{1}{\Gamma(2/3)} \int_{0}^{+\infty} \frac{\varphi''(x-\xi)}{|\xi|^{1/3}} d\xi = \frac{d^{-2/3}}{dx^{-2/3}} \varphi''(x) = \frac{d^{4/3}}{dx^{4/3}} \varphi(x)$$

Anti-diffusive effect

- **Pseudo-differential formula**
  For all $\varphi \in \mathcal{S}(\mathbb{R})$ and all $\xi \in \mathbb{R}$

$$\mathcal{F}\left(\mathcal{I}[\varphi]\right)(\xi) = -(a_{\mathcal{I}} \pm b_{\mathcal{I}} i)|\xi|^{4/3}\mathcal{F}\varphi\,(\xi)\,,$$

Anti-diffusive effect

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$$\Longrightarrow \mathcal{I} \propto -(-\Delta)^{\frac{\lambda}{2}} \text{ with } \lambda = \frac{4}{3}.$$

Some results

- Existence, uniqueness and continuous dependence of the solution u w.r.t. initial datum in $L^2(\mathbb{R})$,
- Failure of maximum principle,
- Existence of travelling waves $\phi \in C_b^1(\mathbb{R})$,
- Global existence of L^2 perturbations of travelling waves,
- Instability of constant solutions.

References :

- N. Alibaud, P. Azerad, D. Isèbe, *A non-monotone conservation law for dune morphodynamics*, Differential Integral Equations, 2010.
- B. Alvarez-Samaniego, P. Azerad, *Travelling wave solutions of the Fowler equation*, Discrete and Continuous Dynamical Systems, B, 2009.
- A.B. *On the instability of a nonlocal scalar conservation law*, Disc. Cont. Dyn. Syst., Ser. S Vol. 5, no 3 (2012).
- A.B. *Global existence of solutions to the Fowler equation in a neighbourhood of travelling-waves*, Int. J. Diff. Eq. (2011).

Linearized problem

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + \mathcal{I}[v(t,\cdot)](x) = 0, \\ v(0,x) = v_0(x). \end{cases} \implies v(t,x) = K(t,\cdot) * v_0,$$

Kernel of $\mathcal{I} - \partial_{xx}^2$

$K(t,\cdot) = \mathcal{F}^{-1}\left(e^{-t\psi_{\mathcal{I}}}\right)$ with
$\psi_{\mathcal{I}}(\xi) = 4\pi^2\xi^2 - a_{\mathcal{I}}|\xi|^{4/3} + b_{\mathcal{I}} i\xi|\xi|^{1/3}$

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Kernel of $\mathcal{I} - \partial^2_{xx}$

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Properties of $K(t,\cdot), t > 0$

- **C^0-semi-group :**

$$K(t) * K(s) = K(t+s)$$
$$\forall u_0 \in L^2(\mathbb{R}), \lim_{t\to 0} K(t) * u_0 = u_0$$

- **Regularity**

$$K(t,x) \in C^\infty((0,+\infty) \times \mathbb{R})$$

- **Estimates for the gradient :**

$$||\partial_x K(t)||_{L^2} \leq Ct^{-3/4}$$
$$||\partial_x K(t)||_{L^1} \leq Ct^{-1/2}$$

Linearized problem

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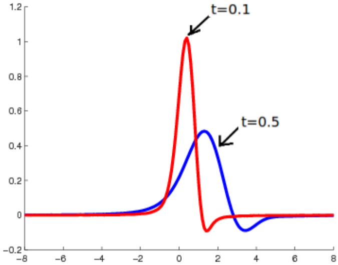

Evolution of K for $t = 0.1$ and $t = 0.5$.

$\implies K(t,\cdot)$ is not positive!

Outline

Explicit finite differences

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + F(u_{j-1}^n, u_j^n, u_{j+1}^n) - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \mathcal{I}_{\Delta x}[u^n]_j = 0$$

Two discretizations for the nonlocal term :

$$\mathcal{I}^1_{\Delta x}[\varphi]_j = \Delta x^{-4/3} \sum_{l=1}^{+\infty} l^{-1/3} \left(\varphi_{j-l+1} - 2\varphi_{j-l} + \varphi_{j-l-1}\right)$$

$$\mathcal{I}^2_{\Delta x}[\varphi]_j = \frac{4}{9} \Delta x^{-4/3} \sum_{l=1}^{+\infty} l^{-7/3} \left(\varphi_{j-l} - \varphi_j + \frac{\varphi_{j+1} - \varphi_{j-1}}{2} l\right)$$

Linearization

For $v > 0$, we have

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + v\frac{u_j^n - u_{j-1}^n}{\Delta x} - \epsilon\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \eta\mathcal{I}_{\Delta x}[u^n]_j = 0$$

Numerical scheme reads for $\mathcal{I}^1_{\Delta x}$

$$\begin{aligned} u_j^{n+1} &= \frac{\epsilon\,\Delta t}{\Delta x^2}\,u_{j+1}^n + \left(1 - \frac{v\,\Delta t}{\Delta x} - 2\frac{\epsilon\,\Delta t}{\Delta x^2} - \frac{\eta\,\Delta t}{\Delta x^{4/3}}\right)\,u_j^n \\ &+ \left(\frac{v\,\Delta t}{\Delta x} + \frac{\epsilon\,\Delta t}{\Delta x^2} + (2 - 2^{-1/3})\frac{\eta\,\Delta t}{\Delta x^{4/3}}\right)\,u_{j-1}^n \\ &- \frac{\eta\,\Delta t}{\Delta x^{4/3}}\sum_{l=2}^{+\infty}\left[(l+1)^{-1/3} - 2l^{-1/3} + (l-1)^{-1/3}\right]\,u_{j-l}^n. \end{aligned}$$

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For $v > 0$, we have

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Numerical scheme reads for $\mathcal{I}_{\Delta x}^1$

$$\begin{aligned} u_j^{n+1} &= \frac{\epsilon\,\Delta t}{\Delta x^2}\,u_{j+1}^n + \left(1 - \frac{v\,\Delta t}{\Delta x} - 2\frac{\epsilon\,\Delta t}{\Delta x^2} - \frac{\eta\,\Delta t}{\Delta x^{4/3}}\right)\,u_j^n \\ &+ \left(\frac{v\,\Delta t}{\Delta x} + \frac{\epsilon\,\Delta t}{\Delta x^2} + (2 - 2^{-1/3})\frac{\eta\,\Delta t}{\Delta x^{4/3}}\right)\,u_{j-1}^n \\ &- \frac{\eta\,\Delta t}{\Delta x^{4/3}}\sum_{l=2}^{+\infty}\left[(l+1)^{-1/3} - 2l^{-1/3} + (l-1)^{-1/3}\right]\,u_{j-l}^n. \end{aligned}$$

Note : u_j^{n+1} is *not* a convex combination of $(u_j^n)_{j\in\mathbb{N}}$

Amplification factor

- $u(t,x) = e^{ikx+\sigma t}$ is solution of

$$u_t + vu_x - \epsilon u_{xx} + \eta \mathcal{I}[u] = 0,$$

iff
dispersion relationship

$$\sigma + i\mu k + \epsilon k^2 - \eta k^{4/3} \Gamma\left(\frac{2}{3}\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 0$$

- Exact amplification factor

$$G_{cont} = e^{\Delta t\left(-\epsilon k^2 + \eta k^{4/3} \frac{1}{2}\Gamma\left(\frac{2}{3}\right)\right)} e^{-i\Delta t\left(\mu k + \eta k^{4/3} \frac{\sqrt{3}}{2}\Gamma\left(\frac{2}{3}\right)\right)}$$

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Strategy

$\hookrightarrow$ High frequencies are responsible of numerical instabilities

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Definition (High Frequency stability)

We say that a numerical scheme is HF-stable if the high frequencies are strongly stable that is to say :

$$\exists 0 < \theta_0 < \pi \text{ such that } \forall (\theta_0, \pi], |g(\Delta x, \Delta t, \theta)| < 1,$$

where g is the discrete amplification factor.

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$\hookrightarrow$ Von-Neumann method :

Input : one single Fourier mode $u_j^n = \hat{u}_k^n e^{ikx_j}$

Output : $\hat{u}_k^{n+1} = g(\Delta x, \Delta t, k)\hat{u}_k^n$

Stability criteria

Proposition

A scheme is HF-stable if Δx and Δt satisfy the following conditions :

- *For $\mathcal{I}^1_{\Delta x}$.*

$$\frac{v\Delta t}{\Delta x} + \frac{2\epsilon\Delta t}{\Delta x^2} + (2 - 2^{-1/3})\eta\,\frac{\Delta t}{\Delta x^{4/3}} < 1,$$

$$(1 - 2^{-1/3})\frac{\eta\Delta t}{\Delta x^{4/3}} \lesssim \frac{2\epsilon\Delta t}{\Delta x^2}.$$

- *For $\mathcal{I}^2_{\Delta x}$.*

$$\frac{v\Delta t}{\Delta x} + 2\epsilon\frac{\Delta t}{\Delta x^2} + \frac{4}{9}\left(\zeta(\frac{4}{3}) - 1\right)\frac{\eta\Delta t}{\Delta x^{4/3}} < 1,$$

$$\frac{4}{9}\left(\zeta(\frac{7}{3}) - 1 + \zeta(\frac{4}{3})\right)\frac{\eta\Delta t}{\Delta x^{4/3}} \lesssim \frac{2\epsilon\Delta t}{\Delta x^2}.$$

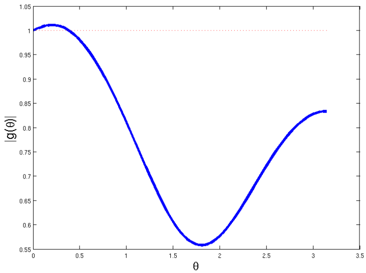
where ζ is the Riemann zeta function.

Azerad P. , B. A., *Finite difference approximations for a diffusion/anti-diffusion equation*, submitted

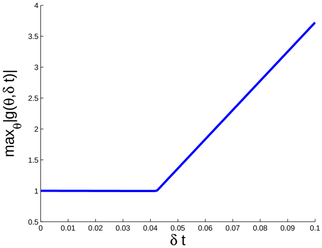
Numerical experiments

Modified CFL

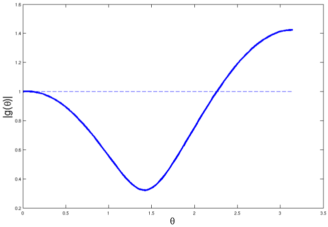
$$\frac{\mu\Delta t}{\Delta x}+2\frac{\epsilon\Delta t}{\Delta x^2}+(2-2^{-1/3})\eta\frac{\Delta t}{\Delta x^{4/3}} < 1$$



$\theta = k\delta x$, $CFL_{mod} \approx 0.94$.



$\delta t_{\max} = 0.042$, $CFL_{mod} \approx 0.99$



$CFL_{mod} \approx 1.22$

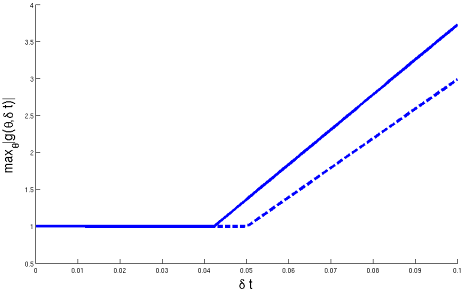
Numerical experiments

Modified CFL

- For $\mathcal{I}^1_{\Delta x}$: $\quad \frac{v\Delta t}{\Delta x} + 2\frac{\epsilon\Delta t}{\Delta x^2} + (2 - 2^{-1/3})\eta\frac{\Delta t}{\Delta x^{4/3}} < 1$

VS.

- For $\mathcal{I}^2_{\Delta x}$: $\quad \frac{v\Delta t}{\Delta x} + 2\frac{\epsilon\Delta t}{\Delta x^2} + \frac{4}{9}\left(\zeta(\frac{4}{9}) - 1\right)\frac{\eta\Delta t}{\Delta x^{4/3}} < 1$



Amplification factors for $\mathcal{I}^1_{\Delta x}$ (blue line) and $\mathcal{I}^2_{\Delta x}$ (dashed line)

Outline

Splitting method

$$\begin{cases} u_t(t,x) + \left(\frac{u^2}{2}\right)_x(t,x) - u_{xx}(t,x) + \mathcal{I}[u(t,\cdot)](x) = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Notation : $u(t,\cdot) = S^t u_0$

↙ ↘

$$\begin{cases} v_t + \left(\frac{v^2}{2}\right)_x - \epsilon\, v_{xx} = 0 \\ v(0,\cdot) = v_0, \end{cases}$$

Notation : $v(t,\cdot) = Y^t v_0$

$$\begin{cases} w_t + \mathcal{I}[w] - \eta\, w_{xx} = 0 \\ w(0,\cdot) = w_0, \end{cases}$$

Notation : $w(t,\cdot) = X^t w_0$

Lie method : $Z_L^t = X^t Y^t$

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Note : H. Holden, C. Lubich, N.-H. Risebro ; *Operator splitting for partial differential equations with Burgers nonlinearity*, to appear, Math. Comp (2012).

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$\Downarrow$ Finite difference method

$\Downarrow$ FFT

Expressions for the flows

- Linear flow

$$X^t v_0 = D(t,\cdot) * v_0,$$

where $D(t,\cdot) = \mathcal{F}^{-1}\left(e^{-t\,\phi_{\mathcal{I}}}\right)$ with $\phi_{\mathcal{I}}(\xi) = 4\pi^2\eta\xi^2 - a_{\mathcal{I}}|\xi|^{4/3} + b_{\mathcal{I}}\xi|\xi|^{1/3}$

- Nonlinear flow (viscous Burgers' equation)

$$Y^t w_0 = G(t,\cdot) * w_0 - \frac{1}{2}\int_0^t \partial_x G(t-s,\cdot) * \left(Y^s w_0\right)^2\, ds,$$

where G is the heat kernel.

- Exact flow

$$S^t u_0 = K(t,\cdot) * u_0 - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * \left(S^s u_0\right)^2\, ds$$

- Splitting operator

$$Z_L^t u_0 = K(t) * u_0 - \frac{1}{2}\int_0^t D(t) * G(t-s) * \partial_x (Y^s u_0)^2\, ds$$

L^2 local error estimate

Proposition

Let $u_0 \in H^3(\mathbb{R})$*. There exists* $C\left(\|u_0\|_{L^2(\mathbb{R})}\right)$ *such that for all* $t \in [0,1]$,

$$\|Z_L^t u_0 - S^t u_0\|_{L^2(\mathbb{R})} \leqslant C\left(\|u_0\|_{L^2(\mathbb{R})}\right) t^2 \|u_0\|^2_{H^3(\mathbb{R})}.$$

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Ingredients of the proof

- $Z_L^t u_0 - S^t u_0 = \frac{1}{2}\int_0^t \partial_x K(t-s) * \left((S^s u_0)^2 - (Z_L^s u_0)^2\right) ds + R(t)$,
- The remainder $R(t)$ is written as $R(t) = \frac{1}{2}\int_0^t R_1(s) ds$, with

$$R_1(s) = \partial_x K(t-s) * (Z_L^s u_0)^2 - D(t) * \partial_x G(t-s,\cdot) * (Y^s u_0)^2,$$

and satisfies :

$$\|R(t)\|_{L^2(\mathbb{R})} \leqslant C\left(\|u_0\|_{L^2(\mathbb{R})}\right) \; t^2 \|u_0\|^2_{H^3(\mathbb{R})}.$$

Modified fractional Gronwall Lemma

Lemma

Let $\phi : [0, T] \to \mathbb{R}_+$ be a bounded measurable function and P be a polynomial with positive coefficients and no constant term. We assume there exists two positive constants C and $\theta \in]0, 1[$ such that for all $t \in [0, T]$,

$$0 \leqslant \phi(t) \leqslant \phi(0) + P(t) + C \frac{d^{-\theta}}{dt^{-\theta}} \phi(t).$$

Then there exists $C_T(\theta)$ such that for all $t \in [0, T]$,

$$\phi(t) \leqslant C_T(\theta)\, \phi(0) + C_T(\theta)\, P(t).$$

Riemann-Liouville operator :

$$\frac{d^{-\theta}}{dt^{-\theta}} \phi(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t - s)^{\theta - 1} \phi(s)\, ds.$$

We will also need the fact that the flow map S^t is uniformly Lipschitzean on balls of $H^2(\mathbb{R})$.

Proposition

Let $T, R > 0$. There exists $K = K(R, T) < \infty$ such that if

$$\|u_0\|_{H^2(\mathbb{R})} \leqslant R, \ \|v_0\|_{H^2(\mathbb{R})} \leqslant R,$$

then

$$\|S^t u_0 - S^t v_0\|_{L^2(\mathbb{R})} \leqslant K\|u_0 - v_0\|_{L^2(\mathbb{R})}, \quad \forall t \in [0, T].$$

Convergence

Theorem

For all $u_0 \in H^3(\mathbb{R})$ *and for all* $T > 0$, *there exist positive constants* c_1, c_2 *and* Δt_0 *such that for all* $\Delta t \in]0, \Delta t_0]$ *and for all* $n \in \mathbf{N}$ *such that* $0 \leqslant n\Delta t \leqslant T$,

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2(\mathbb{R})} \leqslant c_1 \, \Delta t \quad \text{and} \quad \|(Z_L^{\Delta t})^n u_0\|_{H^3(\mathbb{R})} \leqslant c_2.$$

Here, c_1, c_2 *and* Δt_0 *depend only on* T, $\rho = \max_{t \in [0,T]} \|S^t u_0\|_{H^2(\mathbb{R})}$, *and* $\|u_0\|_{H^3(\mathbb{R})}$.

B.A., Carles R., *Splitting methods for the nonlocal Fowler equation*, Math. Comp (2012), to appear.

Proof

- The proof follows the same idea as in [1,2] for instance.
- We prove by induction that there exists $\gamma, \Delta t_0$ such that if $0 < \Delta t \leqslant \Delta t_0$, for all $n \in \mathbf{N}$ with $n\Delta t \leqslant T$,

$$\|(Z_L^{\Delta t})^n u_0\|_{L^2(\mathbb{R})} \leqslant 2\rho, \quad \|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2(\mathbb{R})} \leqslant \gamma \Delta t.$$

- The triangle inequality yields

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2} \leqslant \sum_{k=0}^{n-1} \left\| S^{(n-k-1)\Delta t} \left(Z_L^{\Delta t} u_k \right) - S^{(n-k-1)\Delta t} \left(S^{\Delta t} u_k \right) \right\|_{L^2},$$

with $u_k = (Z_L^{\Delta t})^k$

References :
[1] C. Besse, B. Bidegaray, S. Descombes ; *Order estimates in time of splitting methods for the nonlinear Schrödinger equation*, SIAM J. Numer. Anal., 40 (2002).
[2] H. Holden, C. Lubich, N.-H. Risebro ; *Operator splitting for partial differential equations with Burgers nonlinearity*, to appear, Math. Comp (2012).

Proof

- Lipschitz property of S^t yields

$$\left\| S^{(n-k-1)\Delta t}\left(Z_L^{\Delta t} u_k\right) - S^{(n-k-1)\Delta t}\left(S^{\Delta t} u_k\right)\right\|_{L^2} \leq K \left\| Z_L^{\Delta t} u_k - S^{\Delta t} u_k \right\|_{L^2}$$

- From L^2 local error estimate, we infer

$$\left\| S^{(n-k-1)\Delta t}\left(Z_L^{\Delta t} u_k\right) - S^{(n-k-1)\Delta t}\left(S^{\Delta t} u_k\right)\right\|_{L^2} \leqslant CK(\Delta t)^2 \|u_0\|_{H^3}^2,$$

  for some constant C.

- Therefore,

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2} \leqslant nCK(\Delta t)^2 \|u_0\|_{H^3}^2 \leqslant CTK\Delta t,$$

  which yields the two estimates of the induction, provided one takes $\gamma = CTK$, which is uniform in n and Δt.

Numerical experiments : initial data

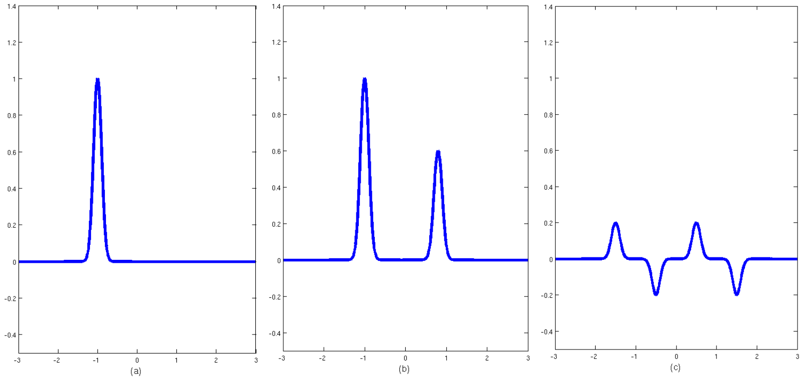

FIGURE: Initial data used for numerical experiments.

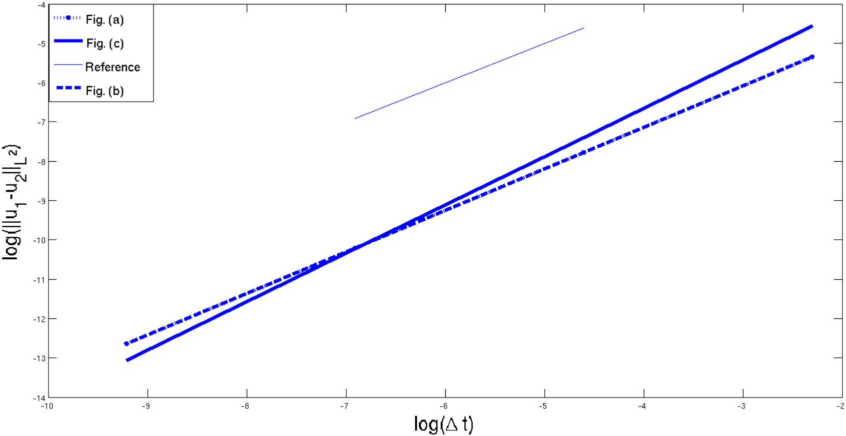

FIGURE: Lie method

Strang method

Strang operator

$$Z_S^t = X^{t/2} Y^t X^{t/2}$$

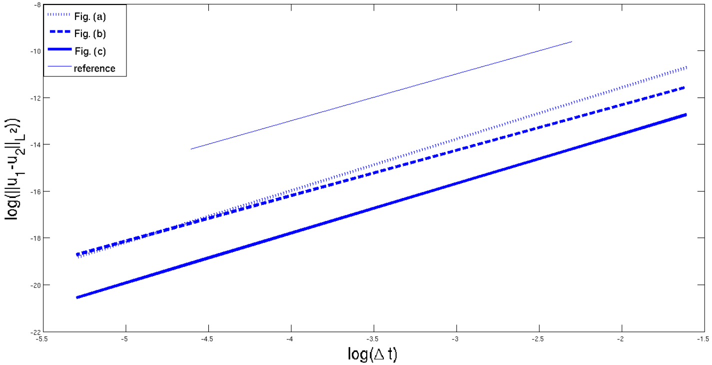

FIGURE: Strang method

Merci, Thanks, Danke !