On some nonlocal nonlinear wave equations

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In collaboration with

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Outline

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  - Peridynamic model
  - Double dispersive model
- The Cauchy problem
  - Local well posedness
  - Global existence vs. finite time blow up
- Travelling wave solutions.
- Further questions

Locally nonlinear, nonlocal elastic model

Equation of Motion

$$\frac{\partial^2}{\partial t^2} v(x,t) = \nabla \cdot \int \beta(x-y) f(\nabla v(y,t)) dy$$

- x : space variable,
- $v\,(x,t)$: displacement at time t,
- ∇v : strain,
- $f\,(p) = D_p W\,(p)$: stress,
  a general smooth nonlinear function with $f(0) = 0$
- $W\,(\nabla v)$: strain energy function

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The convolution with β causes the nonlocal effect.

When f is linear, one obtains Eringen's nonlocal model.

Locally nonlinear, nonlocal elastic model

- 1-d longitudal motion: with $u = v_x$:

$$u_{tt} = [\beta * f(u)]_{xx}$$

Duruk, HA Erbay, Erkip: Nonlinearity 2010

- *1-d transverse motion:*

$$\begin{aligned} u_{1tt} &= [\beta_1 * f_1(u_1, u_2)]_{xx} \\ u_{2tt} &= [\beta_2 * f_2(u_1, u_2)]_{xx} \end{aligned}$$

Duruk, HA Erbay, Erkip: JDE 2011

- *2-d anti-plane shear motion:*

$$w_{tt} = [\beta * f_1(w_x, w_y)]_x + [\beta * f_2(w_x, w_y)]_y$$

HA Erbay, S Erbay, Erkip: Nonlinearity 2011

Examples for the kernel

Triangular kernel

$$\beta(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & |x| \geq 1. \end{cases}$$

$$[\beta * v]_{xx}(x) = v(x+1) - 2v(x) + v(x-1) = \Delta^2 v(x),$$

one gets the **differential-difference equation**

$$u_{tt} = \Delta^2 (u + g(u)).$$

Exponential kernel

$$\beta(x) = \frac{1}{2} e^{-|x|}.$$

$$[\beta * v]_{xx}(x) = (1 - \partial_x^2)^{-1} v,$$

one gets the **Improved Boussinesq equation**

$$u_{tt} - u_{xx} - u_{ttxx} = g(u)_{xx}.$$

Gaussian Kernel $\beta(x) = e^{-x^2}$ gives an **integro-differential equation.**

Preidynamic model

- Formulation of elasticity allowing for discontinuities
  Silling: JMPS 2000.
- Analysis of the linear Cauchy problem
  Emmrich, Weckner: Comm.Math Sci. 2007,
  Du, Zhou, M2AN: 2011
- Analysis of the nonlinear Cauchy problem on $\mathbb{R}$

$$u_{tt} = \int \alpha\,(x-y)\,w(u(y)-u(x))dy.$$

  H.A.Erbay, Erkip, Muslu: JDE 2012

- Well posedness of the nonlinear Cauchy problem

$$u_{tt} = \int_{\Omega} f(u(y)-u(x), x-y)dy.$$

  Emmrich, Puhst: 2012

Double dispersive model

Babaoglu, H. A. Erbay, Erkip: Nonlinear Anal. TMA 2013

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}$$

where L and B are linear pseudodifferential operators (in x) defined via Fourier transform

$$\mathcal{F}\left(Lv\right)(\xi) = l(\xi)\mathcal{F}(v)(\xi), \quad \mathcal{F}\left(Bv\right)(\xi) = b(\xi)\mathcal{F}(v)(\xi),$$

of orders ρ and $-r \leq 0$ respectively (i.e. $|l(\xi)| = O\left|\xi\right|^r$ as $r \to \infty$.)

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The more familiar Boussinesq family,

$$u_{tt} - \widetilde{L}u_{xx} + \widetilde{M}u_{tt} = g(u)_{xx},$$

is of the above form with

$$B = (1 + \widetilde{M})^{-1}, \quad L = \widetilde{L}((1 + \widetilde{M})^{-1}.$$

Local well posedness: Locally nonlinear nonlocal model

$$\begin{aligned} u_{tt} &= [\beta * f(u)]_{xx}, \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= \varphi(x), \quad u_t(x,0) = \psi(x), \qquad x \in \mathbb{R}. \end{aligned}$$

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Assumption: $b(\xi) = \widehat{\beta}(\xi) \leq C(1+\xi^2)^{-\frac{r}{2}}$. Regard the convolution as a pseudodifferential operator with symbol $b(\xi)$ of order $-r$.

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For $r \geq 2$; $D_x^2 B$ is of negative order; hence maps $H^s \to H^s$. For $s > 1/2$, $f(u)$ is locally Lipschitz on H^s. Then we have an H^s valued ODE.

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Theorem: Let X be a Banach space; and let $T : X \to X$ be locally Lipschitz. Then there is some $T > 0$ so that initial value problem for the X valued ODE $U'' = T(U)$, is well posed with solution in $C^2([0,T], X)$ for initial data $U_0, U_1 \in X$.

Local well posedness: Peridynamic model

Theorem: Let $s > \frac{1}{2}$ and $r \geq 2$. There is some $T > 0$ such that the Cauchy problem is well-posed with solution $u \in C^2([0, T], H^s(\mathbb{R}))$ for initial data $\varphi, \psi \in H^s(\mathbb{R})$.

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Extensions:

For bounded data, smoothness can be pulled down to $s \geq 0$ if *beta* is more regular; i.e.:

- when $r > \frac{r}{2}$
- when β_{xx} is a finite measure.
- when β_{xx} is a finite measure, also works for $W^{k,p}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ and $C_b^k(\mathbb{R})$, with interger k.

Local well posedness: Peridynamic model

$$u_{tt} = \int \alpha(x-y)w(u(y)-u(x))dy, \qquad x \in \mathbb{R}, \quad t > 0$$

Under integrability conditions on α the peridynamic equation is a Banach space valued ODE.

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Theorem: Assume that $\alpha \in L^1$. Then there is some $T > 0$ such that the Cauchy problem is well posed with solution in $C^2([0;T];X)$ for initial data in X, where X is any of the spaces

- $W^{k,p}(\mathbb{R})$, $C_b^k(\mathbb{R})$, with interger k, if $w \in C^{k+1}$.
- $H^s(\mathbb{R})$, $s > 0$ if w is a polynomial.

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With suitable smoothness and integrability conditions on f, extends to the general bond based (?) problem on $\mathbb{R}^n$ and on a domain Ω. (Emmrich-Pusht)

The Lipschitz condition seems essential (Emmrich-Pusht)

Local well posedness: Double dispersive model

$$u_{tt} - Lu_{xx} = Bg(u)_{xx},$$

Recall: L is of order ρ, B is of order $-r \leq 0$.

- When $\rho + 2 > 0$; the equation is **not an ODE.**
- When L is coercive, there is hyperbolic behavior due to the semigroup action $S(t) = (-D_x^2 L)^{-\frac{1}{2}} \sin((-D_x^2 L)^{\frac{1}{2}} t)$.
- $S(t)$ has a smoothing effect of order $1 + \frac{\rho}{2}$

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Theorem: Local Existence Let L be coercive, $\rho + 2 > 0$, $\frac{\rho}{2} + 1 + r \geq 2$ and $s > \frac{1}{2}$. There is some $T > 0$ such that the Cauchy problem

$$\begin{aligned} u_{tt} - Lu_{xx} &= Bg(u)_{xx}, \qquad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= \varphi(x), \quad u_t(x,0) = \psi(x) \quad -\infty < x < \infty, \end{aligned}$$

is well posed with solution in $C([0;T]; H^s) \cap C^1([0;T]; H^{s-1-\frac{\rho}{2}})$ for initial data $(\varphi, \psi) \in H^s \times H^{s-1-\frac{\rho}{2}}$.

Local well-posedness: Double dispersive model

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is well posed with solution in $C^1([0;T];H^s)$ for initial data $\varphi, \psi \in H^s$

There is no loss of derivatives.

Global Existence vs. Blow-up

In all three models, global existence / blow-up is controlled by the nonlinearity; in turn by the L^∞ norm of $u(t)$.

Lemma: For sufficiently smooth f, and $v \in H^s \cap L^\infty, s \geq 0$,

$$\|v\|_{H^s} \leq C(\|v\|L^\infty)\|v\|_{H^s}$$

Theorem: (Global existence criterion) The solution of the Cauchy problem exists for all times if and only if for any $T > 0$

$$\lim \sup_{t \to T^-} \|u(t)\|_{L^\infty} < \infty.$$

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The L^∞ control is achieved via energy indentities.

We can define several "energy" terms $E(t)$, with $E'(t) = 0$.

Energy identities (conserved quantities)

For the locally nonlinear nonlocal equation

Recall, $f = DW$,

$$E_1(t) = \frac{1}{2}\|u_t(t)\|_{L^2}^2 + \int\int \beta_{xx}(x-y)W(u(y,t))dydx.$$

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In terms of displacement v, $(u = v_x)$ $\;v_{tt} = [\beta * f(v_x)]_x$,

$$E_2(t) = \frac{1}{2}\|v_t(t)\|^2_{L^2} + \int\int \beta(x-y)W(u(y,t))dydx.$$

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When $\widehat{\beta}(\xi) > 0$; via $\;Ph = \mathcal{F}^{-1}(\widehat{\beta}(\xi)^{-1/2}\widehat{h}(\xi))$,

$$E_3(t) = \frac{1}{2}\|Pv_t(t)\|_{L^2}^2 + \int W(u(x,t))dx.$$

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For peridynamic equation, $u_{tt} = \int \alpha(x-y)w(u(y)-u(x))dy$,

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Global Existence vs. Blow-up

The locally nonlinear nonlocal equation

Theorem: Suppose $r > 3$ and initial data is sufficiently smooth. If there is some $k > 0$ so that $W(r) \geq -kr^2$. Then the Cauchy problem has a global solution.

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Theorem: Let the kernel satisfiy $\beta_{xx} * v = h * v - \lambda v$ for some $\lambda \geq 0$ and for some $h \in L^1 \cap L^\infty$. Suppose initial data is sufficiently smooth. If there is some $C > 0$ and $q > 1$ so that $|f(r)|^q \leq CW(r)$. Then the Cauchy problem has a global solution.

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Peridynamic equation

Theorem: Suppose $w(\eta) = |\eta|^{\upsilon-1}\eta$ with $\upsilon \leq 3$ and $\alpha \in L^1 \cap L^\infty$ with $\alpha \geq 0$ a.e.. Then the Cauchy problem has a global solution.

Travelling wave solutions

HA Erbay, S Erbay, Erkip (in preparation)

We assume L and B are coercive. Consider the double dispersive equation

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \qquad g(u) = \pm|u|^{p-1}u.$$

- *A travelling wave solution $u(x,t) = \varphi(x - ct)$ with velocity c satisfies*
$$(L - c^2 I)B^{-1}\varphi + g(\varphi) = 0.$$
- *Variational problem: Extremal points of:*
$$I(v) = \frac{1}{2}(\|L^{1/2}B^{-1/2}v\|_{L^2}^2 - c^2\|B^{-1/2}v\|_{L^2}^2)$$
subject to $\|v\|_{L^{p+1}} = 1$, are travelling waves
- *We use Lion's "concentration compactness" principle.*
- *In general travelling waves are not unique even up to translation.*

Travelling wave solutions

Theorem: Let L and B be coercive of orders ρ and $-r \leq 0$ respectively. Then there are constants C, D (depending on the symbols $l(\xi), b(\xi)$) so that the double dispersive equation has travelling wave solutions with velocity c,

- for all c with for all $c^2 \leq C^2$, when $\rho \geq 0$,
- for all c with for all $c^2 \geq D^2$, when $\rho \leq 0$.

When $\rho \geq 0$, orbitally stability depends on the concavity of a certain "degree function". In particular, travelling wave solutions are orbitally stable for sufficiently small c.

When $\rho \leq 0$ and $c^2 \geq D^2$, this is not the case; extreme points are not ground states of the energy.

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Stubbe, Portugaliae Mathematica, 1989.

Travelling wave solutions

In the "Boussineq form": $u_{tt} - \widetilde{L}u_{xx} + \widetilde{M}u_{tt} = g(u)_{xx}$,
with $\widetilde{L}, \widetilde{M}$ of orders $s_L, s_M \geq 0$, respectively.
$B = (1 + \widetilde{M})^{-1}, \quad L = \widetilde{L}((1 + \widetilde{M})^{-1}.$, we have $\rho = s_L - s_M$. So:

- When $s_L - s_M > 0$, there are travelling waves for $c^2 < C^2$, possible orbital stability.
- When $s_L - s_M < 0$, there are travelling waves for $c^2 < D^2$, no orbital stability.
- When $s_L - s_M = 0$, both regimes may occur.

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Examples:

The good Boussinesq equation $\widetilde{L} = 1 - D_x^2, \widetilde{M} = 0$, $s_L = 2, s_M = 0$.
Improved Boussinesq equation $\widetilde{L} = 1, \widetilde{M} = 1 - D_x^2$, $s_L = 0, s_M = 2$.

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Improved Boussinesq equation $\widetilde{L} = 1, \widetilde{M} = 1 - D_x^2,\ s_L = 0, s_M = 2$.

Stubbe observed this for the case $\widetilde{L} = a_0 + a|D_x|^\mu, \widetilde{M} = 1 + p|D_x|^\mu$

Further Questions, Nonlocal problems on a domain

An example The Improved Boussineq equation on $\mathbb{R}$ can be written as $u_{tt} = [\int_{\mathbb{R}} \frac{1}{2} e^{-|x-y|} f(u(y)) dy]_{xx}$.

On an interval [a,b] one can consider either

$$u_{tt} = [\int_a^b \frac{1}{2} e^{-|x-y|} f(u(y)) dy]_{xx}, \qquad (I)$$

or

$$u_{tt} = [\int_a^b (\frac{1}{2} e^{-|x-y|} + k(x,y)) f(u(y)) dy]_{xx}, \qquad (II)$$

where $G(x,y) = \frac{1}{2} e^{-|x-y|} + k(x,y)$ is the Green's function for $1 - D_x^2$ with suitable boundary conditions; both relate to the Improved Boussinesq equation on $[a,b]$.

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The interpretation in (I) does not allow consideration of smooth solutions; whereas for (II) one can use eigenfunctions of $1 - D_x^2$ with the boundary conditions for a full analysis.

Further Questions

Scaling properties

Let $\beta_\lambda(x) = \lambda^{-n}\beta(x\lambda^{-1})$.

- As $\lambda \to 0$, the "λ" problem tends to the equation of classical elasticity.
- What happens to the solution?
- Same question (with different scaling) for the nonlinear peridynamic problem.

Further Questions

Scaling properties

Let $\beta_\lambda(x) = \lambda^{-n}\beta(x\lambda^{-1})$.

- As $\lambda \to 0$, the "λ" problem tends to the equation of classical elasticity.
- What happens to the solution?
- Same question (with different scaling) for the nonlinear peridynamic problem.

Limiting properies

More generally suppose the kernels β_j tend to a certain β_0 in some sense. What happens to the solutions?

- Partial result in "good" cases.
- Peridynamic problem?

Further Questions

Numerical results

- Some initial experiments.
- May shed light to the previous questions.

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- The double dispersive problem is promising.
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- Non ODE cases: Bad kernels; The case $r < 2$ for the locally nonlinear nonlocal problem.

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Thank you

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