Convergent numerical methods for fractional conservation laws and fractional degenerate parabolic equations.

Espen R. Jakobsen

Department of Mathematical Sciences
Norwegian University of Science and Technology (NTNU)

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Joint work with Simone Cifani (NTNU).

Fractional Degenerate Parabolic Equations (FDE)

$$\text{(FDE)} \qquad \partial_t u + \nabla \cdot f(u) = \mathcal{L}[A(u)] \quad \text{in} \quad \mathbb{R}^d \times (0,T) =: Q_T.$$

Fractional/fractal conservation law: $\mathcal{L}[A(u)] = -(-\Delta)^{\frac{\alpha}{2}} u$.

Fractional Laplacian:

$$-(-\Delta)^{\frac{\alpha}{2}} \phi(x) = \int_{|z|>0} [\phi(x+z) - \phi(x) - z \cdot D\phi(x) 1_{|z|<1}] \frac{c_\alpha dz}{|z|^{d+\alpha}}.$$

Generator of pure jump Levy process:

$$\mathcal{L}[\phi](x) = \int_{|z|>0} [\phi(x+z) - \phi(x) - z \cdot D\phi(x) 1_{|z|<1}] \, \mu(dz),$$

for a positive Radon measure μ satisfying $\int_{|z|>0} |z|^2 \wedge 1 \, \mu(dz) < \infty$.

Assumptions:

(i) $f = (f_1, \ldots, f_d)$ and A are locally Lipschitz,

(ii) A is non-decreasing, possibly strongly degenerate.

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   Gas detonations, semiconductor growth, radiation hydrodynamics ...

   Biler, Karch, Woyczynski, Imbert, Droniou, Alibaud, Rohde, Chan, Czubak, Silvestre, Karlsen, Kiselev, Nazarov, Shterenberg, Dong, Du, Li ...

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4) Porous medium equations: $f \equiv 0$; $A(u) = u^m$, $m \geq 1$,

$$\partial_t u = -(-\Delta)^{\alpha/2} u^m.$$

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5) New degenerate equations, e.g. nonlocal versions of degenerate local eq'ns

$$\partial_t u + \nabla \cdot f(u) = \Delta A(u).$$

   Local equations: Sedimentation, reservoir simulations, traffic flow...

Some background

General facts:

1) Smooth solutions may develop shocks in finite time.
2) Uniqueness (of weak solutions) is then lost.
3) Entropy conditions can single out physical relevant solution (from the theory of conservation laws ($A \equiv 0$)).
4) $\mathcal{L}$ can be split into a singular and a non-singular part $\mathcal{L} = \mathcal{L}_r + \mathcal{L}^r$.

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Entropy solutions:

u entropy solution of (FDE) if the entropy conditions

$$\eta(u)_t + \nabla \cdot q(u) \leq \mathcal{L}_r[\eta(A(u))] + \eta'(u)\mathcal{L}^r[A(u)] \quad \text{in} \quad \mathcal{D}'$$

hold for all smooth convex η (an entropy) where $q_i'(u) = \eta'(u) f_i'(u)$.

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L^1-contraction: $\int |u(x,t) - v(x,t)| dx \leq \int |u_0(x) - v_0(x)| dx$.

Entropy solutions u, v typically satisfy L^1-contraction and are unique.

On entropy solutions

1) Conservation laws:
   [Kruzkov 70]: L^1-contraction + well-posedness in L^∞

2) Fractal conservation laws ($\alpha < 1$):
   [Alibaud 07]: L^1-contraction + well-posedness in L^∞

3) Fractional degenerate parabolic equations, general case (FDE):
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$\Rightarrow$ there is still a "local" L^1-contraction result.

[Cifani-Jakobsen 11]: Only global L^1-contraction (as far as we know)
$\Rightarrow$ uniqueness only in $L^1 \cap L^\infty$
(consistent with results for $\alpha = 2$ ... Carrillo, Karlsen–Risebro, ...)

On numerical methods

In general few and only recent results.

Results/schemes that can handle non-smooth entropy solutions:

1. Dedner and Rhode 2004. Finite volume scheme for radiation hydrodynamics equation ("$\alpha = 0$").
2. Droniou 2010. Finite difference schemes for fractional conservation laws.
3. Cifani, Jakobsen, and Karlsen 2011. Discontinuous Galerkin method for fractional conservation laws ($\alpha < 1$).
4. Cifani and Jakobsen, to appear in MCOMP. Spectral vanishing viscosity methods for for fractional conservation laws.
5. Cifani and Jakobsen, submitted (THIS TALK). Finite volume schemes for fractional degenerate parabolic equations.

Construction of schemes, convergence results, and except for Droniou, also error estimates.

Plan of the rest of the talk

1. Monotone discretization of the Levy oprator $\mathcal{L}$.
2. A finite volume scheme for (FDE).
3. A priori and error estimates.
4. Ideas of the proofs.

Discretization of $\mathcal{L}$

The Levy operator:

$$\mathcal{L}[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z \cdot D\phi(x) 1_{|z|<1} \; \mu(dz)$$

Discretized operator:

$$\hat{\mathcal{L}}[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x) \; \hat{\mu}(dz) + b_{\Delta x} \cdot \hat{D}^b \phi(x).$$

- Truncated Levy measure:

$$\hat{\mu}(dz) = \mathbf{1}_{|z|>\Delta x} \, \mu(dz) \qquad \text{and} \qquad b_{\Delta x} = \int_{|z|>0} z \, 1_{|z|<1} \, \hat{\mu}(dz).$$

- Upwind difference approximation of "drift":

$$b_{\Delta x,i} D_i \phi \approx b_{\Delta x,i} \hat{D}_i^b \phi = b_{\Delta x,i} \begin{cases} \frac{\phi(x+\Delta x e_i) - \phi(x)}{\Delta x} & \text{when } b_{\Delta x,i} \geq 0, \\ \frac{\phi(x) - \phi(x - \Delta x e_i)}{\Delta x} & \text{when } b_{\Delta x,i} < 0. \end{cases}$$

Properties of $\hat{\mathcal{L}}$

$$\hat{\mathcal{L}}[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x)\ \hat{\mu}(dz) + b_{\Delta x} \cdot \hat{D}^b \phi(x).$$

Monotone/positive:

$$\phi(x_0) \geq \phi(x) \text{ for all } x \in \mathbb{R}^d \quad \Rightarrow \quad \hat{\mathcal{L}}[\phi](x_0) \leq 0.$$

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Consistent with truncation error (in L^1 and L^∞):

$$\|(\hat{\mathcal{L}} - \mathcal{L})[\phi]\| \leq \|D^2\phi\| \Big(\underbrace{\int_{|z|<\Delta x} |z|^2 \mu(dz)}_{\mu-\hat{\mu} \text{ error}} + \underbrace{\Delta x \int_{\Delta x<|z|<1} |z|\, \mu(dz)}_{D-\hat{D}^b \text{ error}} \Big),$$

for fractional Laplace like $\mathcal{L}$ where $0 \leq \frac{d\mu}{dz} 1_{|z|<1} \leq \frac{C}{|z|^{d+\alpha}}$,

$$\leq C\|D^2\phi\| \begin{cases} \Delta x^{2-\alpha}, & \alpha \in (1,2), \\ \Delta x|\ln \Delta x|, & \alpha = 1, \\ \Delta x, & \alpha \in (0,1). \end{cases}$$

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Finite volume approximation:

Find piecewise constant weak solutions

$$U(x,t) = \sum_{\beta\in\mathbb{Z}^d} U_\beta(t)\,\mathbf{1}_{R_\beta}(x),$$

of the approximate equation (AE).

The numerical method for (FDE)

Semidiscrete scheme:

$$\begin{cases} \partial_t U_\alpha = -(D^- \cdot F)(U_\alpha, U_{\alpha+\cdot}) + \sum_{\beta\neq 0} A(U_\beta)\hat{\mathcal{L}}^\alpha_\beta, \\ U_\alpha(0) = \frac{1}{\Delta x}\int_{R_\alpha} u_0(x)\ dx, \end{cases}$$

where

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We consider implicit and explicit time discretizations.

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- Monotone (CFL), consistent, conservative methods (by construction)
- Existence of unique solution in l^1 (fixed point argument in implicit case)
- The limit of any L^1_{loc}-converging sequence $\{U^{\Delta x}\}_{\Delta x > 0}$ of solutions is an entropy solution of (FDE) (cell entropy inequality).

A priori estimates for the implicit scheme

The solution of the implicit scheme $\bar{u} = \sum_\alpha U_\alpha 1_{R_\alpha}$ satisfies

$$\begin{aligned}
&\|\bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)},\\
&\|\bar{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)},\\
&|\bar{u}(\cdot,t)|_{BV(\mathbb{R}^d)} \leq |u_0|_{BV(\mathbb{R}^d)},\\
&\|\bar{u}(\cdot,s)-\bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \sigma_\mu(|s-t|+\Delta t)
\end{aligned}$$

where

$$\sigma_\mu(r) = \begin{cases} r & \text{if } \int_{|z|>0} |z|\wedge 1\ \mu(dz) < \infty,\\ \sqrt{r} & \text{otherwise,}\end{cases}$$

or in fractional Laplace like cases $0 \leq \frac{d\mu}{dz} 1_{|z|<1} \leq \frac{C}{|z|^{d+\alpha}}$,

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Remark: Consistent with (FDE), time reg. involved (weak Lip+approx)

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Error estimates for the implicit method

Assume: $u_0 \in L^1 \cap L^\infty \cap BV \quad (\Rightarrow \quad u(\cdot, t) \in L^1 \cap L^\infty \cap BV)$

Theorem: u entropy solution of (FDE) with initial data u_0 and $\bar{u}$ solution of implicit method with initial data $\bar{u}_0$ (average of u_0 on each element):

$$\|(u - U)(t)\|_{L^1} \leq \hat{\sigma}_T(\Delta x, \Delta t),$$

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where

$$\hat{\sigma}(\Delta x, \Delta t) = C_T \min_{r,\varepsilon,\delta>0} \Big\{ \epsilon + \sigma_\mu(\delta) + \frac{1}{\epsilon}\int_{|z|\leq r} |z|^2\,\mu(dz) + \Big(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta}\Big)\Big(1 + \int_{r<|z|\leq 1} |z|\,\mu(dz) + \int_{|z|>1} \mu(dz)\Big)\Big\}.$$

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$$\hat{\sigma}(\Delta x, \Delta t) = C_T \min_{r,\varepsilon,\delta>0} \Big\{ \epsilon + \sigma_\mu(\delta) + \frac{1}{\epsilon}\int_{|z|\leq r} |z|^2 \, \mu(dz) + \Big(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta}\Big)\Big(1 + \int_{r<|z|\leq 1} |z| \, \mu(dz) + \int_{|z|>1} \mu(dz)\Big)\Big\}.$$

For $\mu \equiv 0$, this is Kuznetsov's original result for conservation laws.

Error estimates – fractional Laplace like cases

Corollary: When $0 \leq \frac{d\mu}{dz} 1_{|z|<1} \leq \frac{C}{|z|^{d+\alpha}}$ and $\Delta t = \Delta x^{\alpha \vee 1}$, then

$$\hat{\sigma}(\Delta x, \Delta x^{\alpha \vee 1}) = C \begin{cases} \Delta x^{\frac{1}{2}}, & \alpha \in (0,1), \\ \Delta x^{\frac{1}{2}} |\ln(\Delta x)|, & \alpha = 1, \\ \Delta x^{\frac{2-\alpha}{2}}, & \alpha \in (1,2). \end{cases}$$

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$$\hat{\sigma}(\Delta x, \Delta x^{\alpha}) \leq C \Delta x^{\frac{2-\alpha}{2}} \quad \text{if} \quad r = \Delta x, \;\; \varepsilon^2 = \Delta x^{2-\alpha} \quad \text{and} \quad \delta = \Delta x^{\frac{\alpha}{2}} \; \Box$$

Remarks on the error estimates

- For smooth solutions, $\hat{\sigma} = C\Delta x^{2-\alpha}$ for $\alpha > 1$
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Remark:

There is no such general theory for degenerate equations of order 2, only special results due to Karlsen et al.

Plan for the proof

1. The definition of entropy solutions of (FDE).
2. A nonlocal Kuznetsov lemma.
3. The cell entropy inequality.
4. The start of the proof.
5. Some estimates.
6. The conclusion.

Definition of entropy solutions

Kruzkov entropies: $\eta_k(u) = |u - k|$ / $q_k(u) = \eta_k'(u)(f(u) - f(k))$.

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Write $\mathcal{L}[\phi] = \mathcal{L}_r[\phi] + \mathcal{L}^r[\phi] + \operatorname{div}(b_r\,\phi)$ where

$$\mathcal{L}_r[\phi](x) = \int_{0<|z|\le r} \phi(x+z) - \phi(x) - z\cdot D\phi(x)\,\mathbf{1}_{|z|\le 1}\,\mu(dz),$$

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Definition: u is entropy solution of (FDE) if

i) $u \in L^\infty(Q_T) \cap C(0,T;L^1(\mathbb{R}^d))$;

ii) for all $k\in\mathbb{R}$, all $r>0$, and all test functions $0\le\varphi\in C_c^\infty(Q_T)$,

$$\iint_{Q_T} \eta_k(u)\partial_t\varphi + (q_k(u)+b_r)\cdot\nabla\varphi + |A(u)-A(k)|\,\mathcal{L}_r[\varphi] + \eta_k'(u)\,\mathcal{L}^r[A(u)]\,\varphi \ge 0;$$

iii) $\lim_{t\to 0}\|u(\cdot,t)-u_0(\cdot)\|_{L^1(\mathbb{R}^d)} = 0$.

A nonlocal Kuznetsov Lemma

If $u, v \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty(0,T; BV(\mathbb{R}^d))$ and u is the entropy solution of (FDE), then for any $\epsilon, r > 0$ and $0 < \delta < T$,

$$
\begin{aligned}
&\|u(\cdot,T) - v(\cdot,T)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + C\left(\epsilon + \mathcal{E}_\delta(u) \vee \mathcal{E}_\delta(v)\right) \\
&\quad - \iint_{Q_T} \iint_{Q_T} \eta(v(x,t), u(y,s))\, \partial_t \varphi^{\epsilon,\delta}(x,y,t,s)\; dx\, dt\, dy\, ds \\
&\quad - \iint_{Q_T} \iint_{Q_T} q(v(x,t), u(y,s)) \cdot \nabla_x \varphi^{\epsilon,\delta}(x,y,t,s)\; dx\, dt\, dy\, ds \\
&\quad + \iint_{Q_T} \iint_{Q_T} \eta(A(v(x,t)), A(u(y,s)))\, \mathcal{L}_r^{\mu^*}[\varphi^{\epsilon,\delta}(x,\cdot,t,s)](y)\; dxdtdyds \\
&\quad - \iint_{Q_T} \iint_{Q_T} \eta'(v(x,t), u(y,s))\, \mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\, \varphi^{\epsilon,\delta}(x,y,t,s)\; dxdtdyds \\
&\quad - \iint_{Q_T} \iint_{Q_T} \eta(A(v(x,t)), A(u(y,s)))\, \gamma^{\mu^*,r} \cdot \nabla_x \varphi^{\epsilon,\delta}(x,y,t,s)\; dxdtdyds \\
&\quad + \text{boundary terms.}
\end{aligned}
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An entropy solution u is compared to an arbitrary function v

A nonlocal Kuznetsov Lemma...

- The modulus in time: $\mathcal{E}_\delta(v) = \sup_{\substack{|t-s|<\delta \\ t,s\in[0,T]}} \|v(\cdot,t) - v(\cdot,s)\|_{L^1(\mathbb{R}^d)}$.
- $\varphi^{\varepsilon,\delta}$ smooth approximate δ-function (mollifier)
- Proved in Alibaud, Cifani, and Jakobsen 2012.
- Nontrivial extension of Kuznetsov´s 1976 result for scalar conservation laws.
- The proof:
  - Kruzkov type doubling of variables argument.
  - Essentially the first part of L^1-uniqueness argument
    – but one of the functions to be compared is not a solution.

A nonlocal Kuznetsov Lemma – the proof

Idea for L^1-estimates:

- $I := \frac{d}{dt} \int |(u-v)(t)| dx = \int \mathrm{sgn}(u-v)(u_t - v_t)(t) dx$
- Use equation to estimate u_t – difficult for nonlinear equations!

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$$I_{\varepsilon,\delta} := \iint \iint \text{sgn}(u(x,t) - v(y,s))(u_t(x,t) - v_s(y,s))\phi_\varepsilon(x-y)\psi_\delta(t-s).$$

- By equation for u (entropy inequality in rigorous proof)

$$I_{\varepsilon,\delta} \leq \iint \iint \text{sgn}(u-v)\Big(\cdots + \mathcal{L}[A(u)](x,t) - v_s(y,s)\Big)\phi\psi \, dxdtdyds$$

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Adapt ideas from viscosity solutions (Alibaud):

- Split $\mathcal{L} = \mathcal{L}_r + \mathcal{L}^r$ and use the Kato inequality:

$$\int \operatorname{sgn}(u-v)\mathcal{L}_r[A(u)]\phi_\varepsilon dx \leq \int \mathcal{L}_r[|A(u)-A(v)|]\phi_\varepsilon dx = \int |A(u)-A(v)|\mathcal{L}_r^*[\phi_\varepsilon]d$$

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Conclusion:

- Integrate by parts, change variables, estimate the boundary terms...

The cell entropy inequality of the implicit scheme

The implicit scheme:

$$U_\alpha^{n+1} = U_\alpha^n - \Delta t (D^- \cdot F)(U_\alpha^{n+1}, U_{\alpha+\cdot}^{n+1}) + \Delta t \sum_{\beta \neq 0} A(U_\beta^{n+1}) \hat{\mathcal{L}}_\beta^\alpha.$$

Split $\hat{\mathcal{L}}_\beta^\alpha = \hat{\mathcal{L}}_{\beta,r}^\alpha + \hat{\mathcal{L}}_\beta^{\alpha,r}$, subtract k, multiply by $\eta'(U_\alpha^{n+1}, k) = \mathrm{sgn}(U_\alpha^{n+1} - k)$, note $\eta(u,k) = |u-k| = (u-k)\eta'(u,k)$, consider the different cases...

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The cell entropy inequality

$$\begin{aligned}\eta(U_\alpha^{n+1}, k) \leq \eta(U_\alpha^n, k) &+ \text{entropy flux terms} \\ &+ \Delta t \sum_{\beta \in \mathbb{Z}^d} \eta(A(U_\beta^{n+1}), A(k)) \hat{\mathcal{L}}_\beta^{\alpha,r} \\ &+ \Delta t \, \eta'(U_\alpha^{n+1}, k) \sum_{\beta \in \mathbb{Z}^d} A(U_\beta^{n+1}) \hat{\mathcal{L}}_{\beta,r}^\alpha.\end{aligned}$$

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Remark: Ideas from Droniou 2010.

The start of the proof of the error estimate:

- Kuznetsov Lemma with $v = \bar{u}$, the solution of the implicit scheme.
- Estimate φ_t-term (v_s): Integrate by parts + cell entropy inequality.

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&+ \text{conservation law terms} \\
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&+ \underbrace{\iint_{Q_T} \iint_{Q_T} \eta(A(\bar{u}(x,t)), A(u(y,s)))\, \hat{\mathcal{L}}_r^{\mu^*}[\bar{\varphi}^{\epsilon,\delta}(\cdot,y,t,s)](x)}_{H_2} \\
&+ \underbrace{\iint_{Q_T} \iint_{Q_T} \eta'(\bar{u}(x,t), u(y,s))\, \mathcal{L}^{\mu,r}[A(\bar{u}(\cdot,t))](x)\, (\bar{\varphi}^{\epsilon,\delta} - \varphi^{\epsilon,\delta})(x,y,t,s)}_{H_3} \\
&+ \underbrace{\iint_{Q_T} \iint_{Q_T} \eta(A(\bar{u}(x,t)), A(u(y,s)))\, \gamma^{\mu^*,r} \cdot (\hat{D}\bar{\varphi}^{\epsilon,\delta} - \nabla_x \varphi^{\epsilon,\delta})(x,y,t,s)}_{H_4}.
\end{aligned}
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Estimation of the terms H_1 and H_3:

H1: By Taylor,

$$\mathcal{L}_r^{\mu^*}[\varphi](x) = \int_{|z|<r} \int_0^t (1-s) z^T D^2 \varphi(x+sz) z \, ds \mu(dz),$$

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and then integration by parts, Fubini, and BV-regularity of η-term, yield

$$H_1 \leq |\eta(\dots)|_{BV_y} \|D\varphi^{\varepsilon,\delta}\|_{L^1} \int_{|z|<r} |z|^2 \mu(dz) \leq L_A |u|_{BV} \frac{C}{\varepsilon} \int_{|z|<r} |z|^2 \mu(dz).$$

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H3: By Taylor and BV,

$$\iint_{Q_T}|\bar{\varphi}^{\varepsilon,\delta}-\varphi^{\varepsilon,\delta}|\, dyds \leq \|D\varphi^{\varepsilon,\delta}\|_{L^1}\Delta x + \|\varphi_t^{\varepsilon,\delta}\|_{L^1}\Delta t \leq C\Big(\frac{\Delta x}{\epsilon}+\frac{\Delta t}{\delta}\Big),$$

$$|\mathcal{L}^r[A(\bar{u}(\cdot,t))](x)| \leq \int_{|z|>r}|\bar{u}(x+z,t)-\bar{u}(x,t)|\mu(dz)$$

$$\leq |\bar{u}|_{BV}\int_{r<|z|\leq 1}|z|\,\mu(dz) + 2\|\bar{u}\|_{L^1}\int_{|z|>1}\mu(dz),$$

Estimation of the terms H_1 and H_3:

H1: By Taylor,

$$\mathcal{L}_r^{\mu^*}[\varphi](x) = \int_{|z|<r} \int_0^t (1-s) z^T D^2 \varphi(x+sz) z \, ds \mu(dz),$$

and then integration by parts, Fubini, and BV-regularity of η-term, yield

$$H_1 \leq |\eta(\dots)|_{BV_y} \|D\varphi^{\varepsilon,\delta}\|_{L^1} \int_{|z|<r} |z|^2 \mu(dz) \leq L_A |u|_{BV} \frac{C}{\varepsilon} \int_{|z|<r} |z|^2 \mu(dz).$$

H3: By Taylor and BV,

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and then by Fubini,

$$|H_3| \leq C\Big(\frac{\Delta x}{\epsilon} + \frac{\Delta x}{\delta}\Big)\Big(\int_{r<|z|\leq 1} |z| \mu(dz) + \int_{|z|>1} \mu(dz)\Big).$$

Conclusion

H2: Same as $H1$, but extra approximation step.

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1. Conservation law terms $\leq C\left(\frac{\Delta x}{\varepsilon} + \frac{\Delta t}{\delta}\right)$ (Kuznetsov)
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4. Insert estimates in the Kuznetsov result, and the result follows:

$$
\begin{aligned}
&\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1} \\
&\leq C_T\left(\Delta x + \epsilon + \sigma_\mu(\delta + \Delta t) + \frac{\Delta x}{\varepsilon}\right) + H_1 + H_2 + H_3 + H_4 \\
&\leq C_T\Bigg(\Delta x + \epsilon + \sigma_\mu(\delta + \Delta t) + \frac{1}{\epsilon}\int_{|z|\leq r} |z|^2\,\mu(dz) \\
&\quad + \left(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta}\right)\left(1 + \int_{r<|z|\leq 1} |z|\,\mu(dz) + \int_{|z|>1}\mu(dz)\right)\Bigg).
\end{aligned}
$$

Publications:

Results of this talk can be found in:

1. S. Cifani and E. R. Jakobsen.
   On numerical methods for degenerate fractional convection-diffusion equations.
   Submitted, preprint on arXiv.

Preprint at www.math.ntnu.no/∼erj or www.arXiv.org

Other selected publications:

1. S. Cifani and E. R. Jakobsen.
   Entropy solution theory for fractional degenerate convection-diffusion equations.
   Annales de l'Institut H. Poincare - Analyse non lineaire 28(3): 413-441, 2011.
2. N. Alibaud, S. Cifani, and E. R. Jakobsen.
   Continuous dependence estimates for nonlinear fractional convection-diffusion equations.
   SIAM Journal of Mathematical Analysis, 44(2): 603-632, 2012.
3. N. Alibaud: Entropy formulation for fractal conservation laws.
   J. Evol. Equ., 7(1):145-175, 2007.
4. S. N. Kružkov: First order quasi-linear equations in several independent variables.
   Math. USSR Sbornik, 10(2):217–243, 1970.
5. D. Applebaum: *Lévy processes and stochastic calculus. Second edition.*
   Cambridge University Press, Cambridge, 2009.

Motivation and key observation

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$$\partial_t u + \nabla \cdot f(u) = \mathcal{L}[A(u)] = \mathcal{L}_r[A(u)] + \mathcal{L}^r[A(u)].$$

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2. Let $\eta_k(u) = |u - k|$. Multiply by $\eta_k'(u) = \mathrm{sgn}(u - k)$:

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$$\partial_t \eta_k(u) + \cdots \leq \mathcal{L}_r[|A(u) - A(k)|] + \eta_k'(u)\mathcal{L}^r[A(u)].$$

4. Multiply by $0 \leq \varphi \in C_c^\infty(Q_T)$, integrate:

$$\iint_{Q_T} \eta_k(u)\varphi_t + \cdots + |A(u) - A(k)|\mathcal{L}_r^*[\varphi] + \eta_k'(u)\mathcal{L}^r[A(u)]\varphi \; dxdt \geq 0.$$

OBS: $\mathcal{L}_r^*$ adjoint operator of $\mathcal{L}_r$.