# Dynamical spike solutions in a nonlocal model of pattern formation

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# Reaction-diffusion equations

**The point of departure:**
a general system of reaction-diffusion (Ordinary-PDE) equations:

$$
\begin{aligned}
u_t &= f(u,v), && \text{for} \quad x \in \overline{\Omega},\ t>0 \\
v_t &= D\Delta v + g(u,v) && \text{for} \quad x \in \Omega,\ t>0
\end{aligned}
$$

in a bounded domain $\Omega \subset \mathbb{R}^n$.
The Neumann boundary condition:

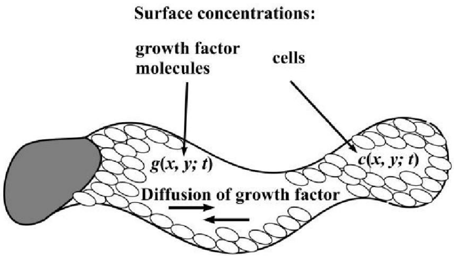
$$\partial_n v = 0 \qquad \text{for} \quad x \in \partial\Omega,\ t>0$$

Initial data:

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x).$$

- $D > 0$ – a constant diffusion coefficient.
- arbitrary $C^1$-nonlinearities $f = f(u,v)$ and $g = g(u,v)$.

# Biological system




- Cell proliferation is influenced by growth factor
- Growth factor is externally supplied or produced by the cells
- Growth factor diffuses along the structure formed by the cells and binds to cell membrane receptors
- Hypothesis: The diffusion of this growth factor may significantly influence the dynamics of the whole cell population

# Derivation of “SHADOW SYSTEM”

**Lemma**

*Let* $T > 0$*. Assume that* $v^D = v^D(x,t)$ *are solutions for each* $D > 0$ *of*

$$\begin{aligned} &v_t = D\Delta v + F \\ &\partial_n v = 0 \\ &v(x,0) = v_0(x). \end{aligned}$$

*in a bounded domain* $\Omega \subset \mathbb{R}^n$ *and for* $t \in [0, T]$*.*
*Some nice assumptions of* $u_0$ *and* $F = F(x,t)$*.*

*Then, for every* $t > 0$*, the limit*

$$\lim_{D\to\infty} v^D(\cdot, t) = \xi(t)$$

*exists where*

$$\frac{d}{dt}\xi(t) = \int_\Omega F(x,t)\,dx, \quad \xi(0) = \int_\Omega v_0(x)\,dx.$$

# Idea of the proof

First, we consider the solution $v^D = v^D(x,t)$ of

$$
\begin{aligned}
v_t &= D\Delta v \\
\partial_n v &= 0 \\
v(x,0) &= v_0(x).
\end{aligned}
$$

Then $v^D(\cdot,t)$ converges towards a constant function

$$\lim_{D\to\infty} v^D(\cdot,t) \to \int_\Omega v_0(x)\,dx \equiv \xi(0) \quad \text{each} \quad t>0.$$

**Hint:** rescale the time and use the identity $v^D(x,t) = v^1(x,Dt)$.

# Idea of the proof

First, we consider the solution $v^D = v^D(x,t)$ of

$$\begin{aligned} v_t &= D\Delta v \\ \partial_n v &= 0 \\ v(x,0) &= v_0(x). \end{aligned}$$

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**Hint:** rescale the time and use the identity $v^D(x,t) = v^1(x, Dt)$.

For the inhomogeneous problem, we use the Duhamel principle

$$v(x,t) = e^{tD\Delta} v_0(x) + \int_0^t e^{(t-s)D\Delta} F(x,s)\, ds.$$

In the limit, we obtain the function

$$\xi(t) = \int_\Omega v_0(x)\, dx + \int_0^t \int_\Omega F(x,t)\, dx.$$

# Shadow problem

The solutions $(u^D, v^D) = (u^D(x,t), v^D(x,t))$ of ordinary-PDE problem

$$\begin{aligned} u_t &= f(u,v), \\ v_t &= D\Delta v + g(u,v) \end{aligned}$$

converge as $D \to \infty$ towards a solution

$$(u,\xi) = (u(x,t), \xi(t))$$

of the following **shadow system**

$$\begin{aligned} u_t &= f(u,\xi), && \text{for} \quad x \in \overline{\Omega},\ t > 0 \\ \xi_t &= \int_\Omega g\big(u(x,t), \xi(t)\big)\, dx && \text{for} \quad t > 0 \end{aligned}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ supplemented with initial data

$$u(x,0) = u_0(x), \qquad \xi(0) = \xi_0.$$

# Local existence of solutions

Shadow problem

$$u_t = f(u,\xi), \qquad \text{for} \quad x \in \overline{\Omega},\ t > 0$$
$$\xi_t = \int_\Omega g\big(u(x,t),\xi(t)\big)\,dx \qquad \text{for} \quad t > 0$$

with initial data: $u(x,0) = u_0(x), \quad \xi(0) = \xi_0.$

Local existence of solutions by standard methods via the integral equations

$$\begin{aligned} u(x,t) &= u_0(x) + \int_0^t f(u(x,s),v(x,s))\,dx \\ \xi(t) &= \xi_0 + \int_0^t \int_\Omega g\big(u(x,s),\xi(s)\big)\,dx\,ds. \end{aligned}$$

# Stationary solutions

Shadow system

$$\begin{aligned} u_t &= f(u, \xi), \\ \xi_t &= \int_\Omega g\big(u(x,t), \xi(t)\big)\, dx \end{aligned}$$

We calculate a stationary solution $(U(x), \xi_0)$ from equations

$$f(U(x), \xi_0) = 0, \quad \int_\Omega g\big(U(x), \xi_0\big)\, dx = 0.$$

If the equation $f(U(x), \xi_0) = 0$ has a unique solution, we obtain only constant stationary solutions.

**Remark**
One can also consider non-constant stationary solutions. They have to be constant on subsets of $\Omega$.

# Instability of steady states

**Theorem**

*Assume that the constant vector $(\bar{u}, \bar{\xi})$ is a solution of the initial-boundary value problem for the system*

$$\begin{aligned} u_t &= f(u, \xi), \\ \xi_t &= \int_\Omega g\big(u(x,t), \xi(t)\big)\, dx, \end{aligned}$$

*which means that $f(\bar{u}, \bar{\xi}) = 0$ and $g(\bar{u}, \bar{\xi}) = 0$. If*

$$f_u(\bar{u}, \bar{\xi}) > 0,$$

*then $(\bar{u}, \bar{\xi})$ is unstable solution of this initial-boundary value problem.*

- The inequality $f_u(\bar{u}, \bar{\xi}) > 0$ can be interpreted as the autocatalysis of $u$ at the steady state $(\bar{u}, \bar{\xi})$.
- It appeares in a natural way in models describing diffusion-driven instability.

# Idea of the proof

*Step 1. Linear instability.*
The linearized system at the steady state $(\bar{u}, \bar{\xi})$:

$$U_t = f_u(\bar{u}, \bar{\xi})U + f_\xi(\bar{u}, \bar{\xi})\Phi$$
$$\Phi_t = g_u(\bar{u}, \bar{\xi}) \int_\Omega U \, dx + g_\xi(\bar{u}, \bar{\xi})\Phi$$

The unbounded solution is given explicitly

$$U(x,t) = e^{tf_u(\bar{u},\bar{\xi})} U_0(x), \qquad \Phi(t) \equiv 0,$$

where $U_0(x)$ is any function such that $\int_\Omega U_0(x) \, dx = 0$.

*Step 2. Nonlinear instability.*
One should adapt classical methods from reaction-diffusion equations.

# Model example

We explain Instability theorem in the case of the problem

$$\begin{aligned} &u_t = -u + u^2\xi, && \text{for} \quad x \in \overline{\Omega},\ t > 0 \\ &\xi_t = -\xi - k\xi \int_\Omega u^2(x,t)\,dx + B && \text{for} \quad t > 0 \\ &u(x,0) = u_0(x), \qquad \xi(0) = \xi_0 \end{aligned}$$

where $k, B \in \mathbb{R}$ are fixed positive parameters. Here,

$$g(u,\xi) = -\xi - k\xi u^2 + B.$$

For simplicity of notation, we assume that $|\Omega| = 1$.

# Space homogeneous solutions

**Theorem**

*All solutions $(u(t), \xi(t))$ of the following initial value problem for ordinary differential equations*

$$\frac{d}{dt}u = -u + u^2\xi, \qquad \frac{d}{dt}\xi = -\xi - ku^2\xi + B$$
$$u(0) = u_0 \geq 0, \qquad \xi(0) = \xi_0 \geq 0$$

*are nonnegative, global-in-time, and uniformly bounded for $t > 0$.*

*Proof.*

We observe that

$$\frac{d}{dt}\big(ku(t) + \xi(t)\big) = -\big(ku(t) + \xi(t)\big) + B.$$

Hence, as long as $u(t)$ and $\xi(t)$ are nonnegative, they have to be uniformly bounded for $t > 0$.

# Space homogeneous solutions
# Constant stationary solutions

We solve the system of equations

$$-u + u^2\xi = 0, \qquad -\xi - ku^2\xi + B = 0$$

So,

$$\text{either} \qquad u = 0, \quad \xi = B \qquad \text{or} \qquad u = \frac{1}{\xi}, \quad -\xi - \frac{k}{\xi} + B = 0$$

**Remark**
The trivial solution $(0, B)$ is an asymptotically stable solution of ODE.

**Remark**
Two nontrivial positive solutions exist if $B > 0$ is large enough.
One of them is stable as a solution of ODE. Another one is unstable.

# Shadow problem

**Remark**
The trivial solution $(0, B)$ is also an asymptotically stable solution of the shadow problem.

**Remark**
Autocatalysis condition:
For $f_u(u, \xi) = -1 + 2u\xi$ at the nontrivial constant solution $u = 1/\xi$ we have

$$f_u(u, \xi) = 1 > 0.$$

**Nontrivial constant solutions are unstable as solutions of the shadow problem.**

This is an example of the Diffusion-Driven Instability or the Turing instability in the case of the shadow problem.

# Question

Nonnegative space homogeneous solutions of the shadow problem

$$u_t = -u + u^2\xi, \qquad \text{for} \quad x \in \overline{\Omega},\ t > 0$$

$$\xi_t = -\xi - k\xi \int_\Omega u^2(x,t)\, dx + B \qquad \text{for} \quad t > 0$$

$$u(x,0) = u_0(x), \qquad \xi(0) = \xi_0$$

are bounded and global-in-time.

What is the large time behavior of space inhomogeneous solutions ?

# Question

Nonnegative space homogeneous solutions of the shadow problem

$$u_t = -u + u^2\xi, \qquad \text{for} \quad x \in \overline{\Omega}, \ t > 0$$

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$$u(x,0) = u_0(x), \qquad \xi(0) = \xi_0$$

are bounded and global-in-time.

What is the large time behavior of space inhomogeneous solutions ?

Some space inhomogeneous solutions of this shadow problem may blow up in finite time.

# Blowup of solutions

**Theorem**

*Let* $0 \in \Omega$ *and assume that* $u_0 \in C(\Omega)$ *satisfies:*

$$u_0(0) = 1 \qquad \text{and} \qquad 0 \le u_0(x) < 1 \quad \text{for} \quad x \neq 0$$

*and*

$$A_0 \equiv \int_\Omega \left( \frac{u_0(x)}{1 - u_0(x)} \right)^2 \, dx < \infty.$$

*Assume also that*

$$\min \left\{ \xi_0, \frac{B}{1 + kA_0} \right\} > 1.$$

*Then, the corresponding solution of shadow problem for the system*

$$u_t = -u + u^2 \xi, \qquad \xi_t = -\xi - k\xi \int_\Omega u^2(x,t) \, dx + B$$

**blows up in a finite time at** $x = 0$**.**

# Idea of the proof.

For fixed $\xi(t)$ and for each $x \in \Omega$, we solve the equation $u_t = -u + u^2\xi$:

$$u(x,t) = \frac{e^{-t}}{\frac{1}{u_0(x)} - \int_0^t \xi(s)e^{-s}\, ds}.$$

Note that

$$T_{max} = \sup\left\{ t > 0 \ : \ \int_0^t \xi(s)e^{-s}\, ds < 1 \right\}$$

because

$$u_0(0) = 1 \qquad \text{and} \qquad 0 \le u_0(x) < 1 \quad \text{for} \quad x \neq 0$$

Hence, we have an estimate up to the blowup point:

$$u(x,t) \le \frac{e^{-t}}{\frac{1}{u_0(x)} - 1} = \frac{u_0(x)e^{-t}}{1 - u_0(x)} \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}).$$

# Idea of the proof.

Next, using the estimate of $u(x,t)$ we deduce from the equation for $\xi$ the following differential inequality

$$\xi_t \geq -(1+kA_0)\xi + B \qquad \text{for all} \quad t \in [0, T_{max})$$

which implies the lower bound

$$\xi(t) \geq \min\left\{\xi_0, \frac{B}{1+kA_0}\right\} \qquad \text{for all} \quad t \in [0, T_{max}).$$

Choosing parameters correctly, we obtain the blowup.

# Activator-inhibitor system

Gierer and Meinhardt system:

$$u_t = \varepsilon \Delta u - u + \frac{u^p}{v^q},$$
$$\tau v_t = D \Delta v - v + \frac{u^r}{v^s},$$

with the Neumann boundary conditions and with positive initial data.

One assumes that $\varepsilon > 0$ is very small and $D > 0$ is large (comparing with the domain size).

In the two extreme cases, we have the shadow problem

$$u_t = -u + \frac{u^p}{\xi^q},$$
$$\tau \xi_t = -\xi + \int_\Omega \frac{u^r}{\xi^s}\, dx.$$

**Caution:** In the literature, the expression "shadow problem" is used in the case of the Gierer–Meinhardt system with $D = +\infty$ and $\varepsilon > 0$.

# Activator-inhibitor system

$$
\begin{aligned}
u_t &= -u + \frac{u^p}{\xi^q}, \\
\tau \xi_t &= -\xi + \int_\Omega \frac{u^r}{\xi^s}\, dx.
\end{aligned}
$$

We assume that $|\Omega| = 1$.

**Results**

- The vector $(1, 1)$ is a constant stationary solution.

# Activator-inhibitor system

$$u_t = -u + \frac{u^p}{\xi^q},$$
$$\tau\xi_t = -\xi + \int_\Omega \frac{u^r}{\xi^s}\,dx.$$

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**Results**

- The vector $(1,1)$ is a constant stationary solution.
- It is asymptotically stable as a solution of ODE, provided $\tau > 0$ is small.

# Activator-inhibitor system

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- The vector $(1, 1)$ is a constant stationary solution.
- It is asymptotically stable as a solution of ODE, provided $\tau > 0$ is small.
- The vector $(1, 1)$ is unstable as a solution of the shadow problem.

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**Results**

- The vector $(1,1)$ is a constant stationary solution.
- It is asymptotically stable as a solution of ODE, provided $\tau > 0$ is small.
- The vector $(1,1)$ is unstable as a solution of the shadow problem.
- For some range of parameters $p, q, r, s$, solutions of the ODE system are bounded and global-in-time.

# Activator-inhibitor system

$$u_t = -u + \frac{u^p}{\xi^q},$$
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We assume that $|\Omega| = 1$.

**Results**

- The vector $(1,1)$ is a constant stationary solution.
- It is asymptotically stable as a solution of ODE, provided $\tau > 0$ is small.
- The vector $(1,1)$ is unstable as a solution of the shadow problem.
- For some range of parameters $p, q, r, s$, solutions of the ODE system are bounded and global-in-time.
- For the same range of parameters, some space inhomogeneous solutions blow up in finite time.

# Model of early carcinogenesis

$$u_t = \left(\frac{au\xi}{1+u\xi} - 1\right)u$$

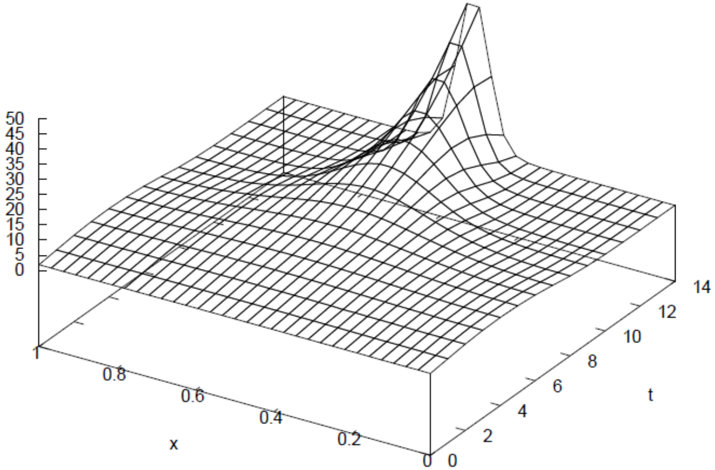
$$\xi_t = -\xi - k\xi \int_\Omega u^2 \, dx + B$$

in a bounded set $\Omega \subset \mathbb{R}^n$. We complete this system with nonnegative initial conditions

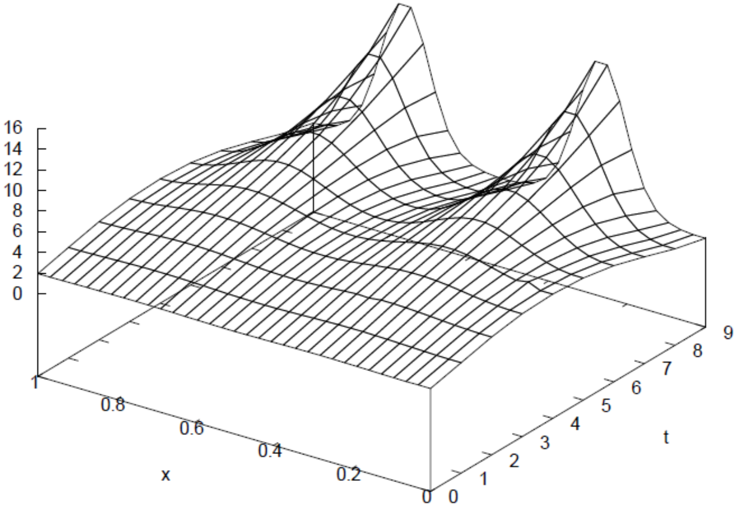
$$u(0,x) = u_0(x), \qquad \xi(0) = \xi_0.$$

- Here, nonnegative solutions are global-in-time.
- Blowup of solutions appears when $t \to \infty$.

# One point blowup



# Two points blowup



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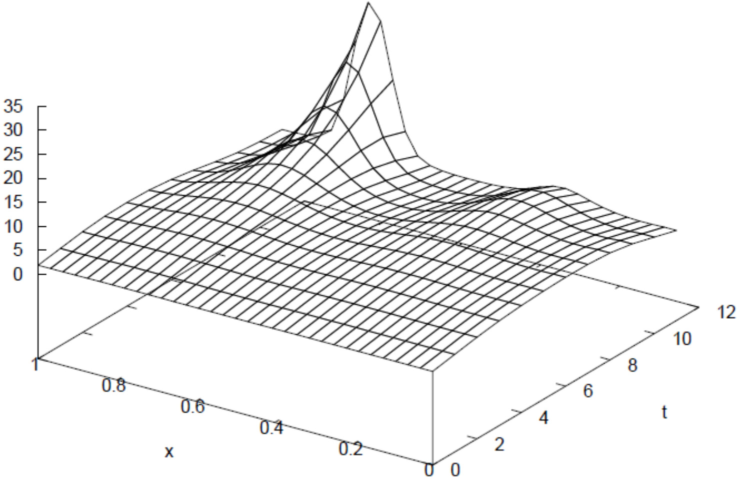


# Initial datum which lead to blowup

Assumption for blowup:

Let $0 \in \Omega$ and assume that $u_0 \in C(\Omega)$ satisfies:

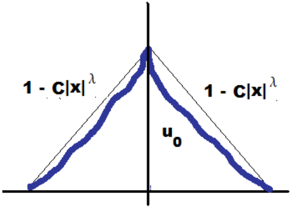
$$u_0(0) = 1 \quad \text{and} \quad 0 \le u_0(x) < 1 \quad \text{for} \quad x \neq 0$$

and

$$A_0 \equiv \int_\Omega \left( \frac{u_0(x)}{1 - u_0(x)} \right)^2 dx < \infty.$$

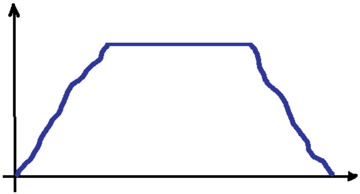
**Remark**

The constant $A_0$ is finite if there exist constants $C > 0$ and $\lambda \in (0, n/2)$ such that $u_0(x) \le 1 - C|x|^\lambda$ for all $x \in \Omega$.



# Initial datum with NO blowup

The following initial datum cannot generate blowup.



For the system

$$
\begin{aligned}
&u_t = -u + u^2\xi, && \text{for} \quad x \in \overline{\Omega},\ t > 0 \\
&\xi_t = -\xi - k\xi \int_\Omega u^2(x,t)\,dx + B && \text{for} \quad t > 0 \\
&u(x,0) = u_0(x), \qquad \xi(0) = \xi_0
\end{aligned}
$$

the "mass" is bounded:

$$\sup_{t\in[0,T_{max})} \int_\Omega u(x,t)\,dx < +\infty.$$

THANK YOU