Doubly Dispersive Nonlocal Nonlinear Wave Equations

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Outline

- The doubly dispersive Boussinesq equation:
  - Double dispersive nature.
- A general class of nonlocal nonlinear wave equations:
  - Spatial nonlocality and dispersive regularization.
- Main results for the Cauchy problem:
  - Local existence.
  - Global existence.
  - Blow-up.

THE DOUBLY DISPERSIVE BOUSSINESQ EQUATION

The Doubly Dispersive Boussinesq Equation

$$u_{tt} - u_{xx} + u_{xxxx} - u_{xxtt} = (g(u))_{xx}, \quad x \in \mathbb{R}, \quad t > 0.$$

- It describes the propagation of long waves in various continuous media.
- The waves travel both to the right and to the left.
- $g(u)$: a sufficiently smooth nonlinear function (nonlinear waves).
- There exists a double effect of dispersion due to u_{xxxx} and u_{xxtt}.
- Special cases:
  - The Boussinesq equation.
  - The improved (regularized) Boussinesq equation.

A historical remark

- The Boussinesq Equation

$$u_{tt} - u_{xx} + u_{xxxx} = (g(u))_{xx}.$$

differs from Boussinesq's Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxxx} = (g(u))_{xx}.$$

- Boussinesq (1872): Long surface waves in shallow water.
- u: free surface elevation.
- It is called the "bad" Boussinesq equation due to amplification of small perturbations. (linearly unstable)
- Zabusky and Kruskal (1965): Longitudinal waves in continuum approximation of an anharmonic chain of particles (The FPU (Fermi-Pasta-Ulam) problem).

The Improved Boussinesq Equation

$$u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx}.$$

- Longitudinal (extensional) waves in an elastic rod of circular cross section with free surface.
- Long waves: The wavelength is large compared with the radius ($\frac{radius}{wavelength} << 1$).
- u: longitudinal (axial) strain.

The Improved Boussinesq Equation: Derivation

$$u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx}.$$

- Love hypothesis: $V = -\nu r U_x$
  where V is the radial displacement, U is the axial displacement, r is the radial coordinate, and ν is the Poisson coefficient. (Poisson contraction effect)
- The dispersion term u_{xxtt} accounts for the transverse motion.
- The finiteness of the rod radius causes dispersion (geometric dispersion).
- Characteristic length: rod radius.
- Ostrowski and Sutin (1977), Sorensen et al (1984), Clarkson et al (1986).

The Doubly Dispersive Boussinesq Equation

$$u_{tt} - u_{xx} + u_{xxxx} - u_{xxtt} = (g(u))_{xx}.$$

- The new dispersion term u_{xxtt} introduces an additional characteristic length.
- Several possible origins of the additional dispersion:
  - Higher-order gradient elasticity (physical dispersion)
  - An elastic rod embedded in another elastic medium (physical dispersion)
  - A narrowing rod (geometrical dispersion)
- Samsonov (1984), Samsonov and Sokurinskaya (1988).

A GENERAL CLASS OF NONLOCAL NONLINEAR WAVE EQUATIONS

A Nonlocal Nonlinear Wave Equation

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \quad x \in \mathbb{R}, \quad t > 0.$$

- g : a sufficiently smooth nonlinear function,
- L and B : linear pseudodifferential operators,
    - L : regularization through the linear dispersion,
    - B : regularization through the smoothing of the nonlinearity.
- A double effect of dispersion (spatial nonlocality).

Notation

- $l(\xi)$ and $b(\xi)$: symbols of L and B, respectively.

$$\mathcal{F}(Lv)(\xi) = l(\xi)\mathcal{F}(v)(\xi), \quad \mathcal{F}(Bv)(\xi) = b(\xi)\mathcal{F}(v)(\xi),$$

- $\mathcal{F}$: the Fourier transform with respect to x.
- The Fourier transform of $u(x)$: $\widehat{u}(\xi)$
- The L^2 Sobolev space on $\mathbb{R}$: $H^s = H^s(\mathbb{R})$
- The H^s norm:

$$\|u\|_s^2 = \int (1+\xi^2)^s |\widehat{u}(\xi)|^2 d\xi$$

- The L^∞ norm: $\|u\|_{L^\infty}$
- The L^2 norm: $\|u\|$

Main Assumptions

There are positive constants c_1, c_2 and c_3 so that for all $\xi \in \mathbb{R}$,

- $$c_1^2(1+\xi^2)^{\rho/2} \leq l(\xi) \leq c_2^2(1+\xi^2)^{\rho/2},$$
- $$0 < b(\xi) \leq c_3^2(1+\xi^2)^{-r/2}.$$

This implies that

- L is an elliptic coercive operator of order ρ with $\rho \geq 0$,
- B is an elliptic positive operator of order $-r$ with $r \geq 0$.

A Different Form of the Wave Equation

$$u_{tt} - \tilde{L}u_{xx} + Mu_{tt} = (g(u))_{xx}, \quad x \in \mathbb{R}, \quad t > 0.$$

where

- $\tilde{L} = B^{-1}L$,
- $M = B^{-1} - I$,
- $\tilde{l}(\xi)$ and $m(\xi)$ are the symbols of $\tilde{L}$ and M, respectively,
- M is an elliptic positive pseudodifferential operator of order $r > 0$.

The Linear Dispersion Relation

- The linearized wave equation:

$$u_{tt} - \tilde{L} u_{xx} + M u_{tt} = 0$$

- The linear dispersion relation:

$$\xi \quad \mapsto \quad \omega^2(\xi) = \frac{\xi^2 \tilde{l}(\xi)}{1 + m(\xi)}$$

- Wavelike solutions if $\frac{\tilde{l}(\xi)}{1+m(\xi)} \geq 0$, that is, $l(\xi) \geq 0$,

- The double nature of dispersion:
    - "Numerator-based" dispersion,
    - "Denominator-based" dispersion.

Special Cases

$\tilde{L} = I - \partial_x^2$ and $M = -\partial_x^2$

The double dispersion equation:

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = (g(u))_{xx}.$$

$\tilde{L} = 1 - \partial_x^2$ and $M = 0$ (the zero operator)

The Boussinesq equation:

$$u_{tt} - u_{xx} + u_{xxxx} = (g(u))_{xx}$$

Further Special Cases

$\tilde{L} = I$ and $M = -\partial_x^2$

The improved (or regularized) Boussinesq equation:

$$u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx}$$

$L = 0$ and B is a convolution

$$(Bv)(x) = (\beta * v)(x) = \int \beta(x-y)v(y)dy, \qquad \beta(x) = \mathcal{F}^{-1}(b(\xi))$$

The nonlocal nonlinear wave equation:

$$u_{tt} = \left(\int \beta(x-y) g(u(y,t)) dy \right)_{xx}.$$

THE CAUCHY PROBLEM: MAIN RESULTS

The Cauchy Problem for the Linearized Equation

Theorem

Let

- $T > 0, \quad s \in \mathbb{R}$, and
- $\varphi \in H^s, \;\; \psi \in H^{s-1-\frac{\rho}{2}}, \;\; h \in L^1([0, T], H^{s+1-\frac{\rho}{2}})$.

Part A: Then the Cauchy problem

$$u_{tt} - Lu_{xx} = (h(x,t))_{xx}, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),$$

has a unique solution $u \in C([0, T], H^s) \cap C^1([0, T], H^{s-1-\frac{\rho}{2}})$.
Part B: For $0 \le t \le T$ and some positive constants A_1 and A_2,

$$\|u(t)\|_s + \|u_t(t)\|_{s-1-\frac{\rho}{2}}$$
$$\le (A_1 + A_2 T)\left(\|\varphi\|_s + \|\psi\|_{s-1-\frac{\rho}{2}} + \int_0^t \|h(\tau)\|_{s+1-\frac{\rho}{2}}\, d\tau\right).$$

Sketch of the Proof

Let $K = L^{1/2}$ and $k(\xi) = \sqrt{l(\xi)}$. Then

$$c_1(1+\xi^2)^{\rho/4} \le k(\xi) \le c_2(1+\xi^2)^{\rho/4}.$$

Applying the Fourier transform

$$\begin{aligned} &\widehat{u}_{tt} + (\xi k(\xi))^2\,\widehat{u} = -\xi^2\widehat{h}(\xi,t), \\ &\widehat{u}(\xi,0) = \widehat{\varphi}(\xi), \quad \widehat{u}_t(\xi,0) = \widehat{\psi}(\xi). \end{aligned}$$

The solution is generated by the semigroup

$$\mathcal{S}(t)v = \mathcal{F}^{-1}\left(\frac{\sin(\xi k(\xi) t)}{\xi k(\xi)}\widehat{v}(\xi)\right).$$

Sketch of the Proof (continued)

The solution is

$$u(t) = \partial_t \mathcal{S}(t)\varphi + \mathcal{S}(t)\psi + \int_0^t \partial_x^2 \mathcal{S}(t-\tau)h(\tau)d\tau.$$

- The estimate for the first term is $\|\partial_t \mathcal{S}(t)v\|_s^2 \leq \|v\|_s^2$.
- For the second term,

$$\|\mathcal{S}(t)v\|_s^2 \leq \left(t^2 2^{1+\frac{\rho}{2}} + \frac{2}{c_1^2}\right) \|v\|_{s-1-\frac{\rho}{2}}^2 .$$

- The third term can be estimated via Minkowski's inequality

$$\left\|\int_0^t \partial_x^2 \mathcal{S}(t-\tau)v(\tau)d\tau\right\|_s \leq \left(t^2 2^{1+\frac{\rho}{2}} + \frac{2}{c_1^2}\right)^{1/2} \int_0^t \|v(\tau)\|_{s+1-\frac{\rho}{2}}\, d\tau.$$

Sketch of the Proof (continued)

- Summing up the estimates, we obtain

$$\|u(t)\|_s \leq \|\varphi\|_s + \left(t^2 2^{1+\frac{\rho}{2}} + \frac{2}{c_1^2}\right)^{1/2} \left(\|\psi\|_{s-1-\frac{\rho}{2}} + \int_0^t \|h(\tau)\|_{s+1-\frac{\rho}{2}}\, d\tau\right)$$

  for $0 \leq t \leq T$.

- Differentiating the solution with respect to t, we get

$$u_t(t) = \partial_t^2 \mathcal{S}(t)\varphi + \partial_t \mathcal{S}(t)\psi + \int_0^t \partial_x^2 \partial_t \mathcal{S}(t-\tau) h(\tau) d\tau.$$

- For the first term we have the estimate

$$\left\|\partial_t^2 \mathcal{S}(t) v\right\|_{s-1-\frac{\rho}{2}}^2 \leq c_2^2 \|v\|_s^2 .$$

Sketch of the Proof (continued)

- For the second term, $\|\partial_t \mathcal{S}(t)v\|_s \leq \|v\|_s$.
- For the third term,

$$\left\| \int_0^t \partial_x^2 \partial_t \mathcal{S}(t-\tau)v(\tau)d\tau \right\|_{s-1-\frac{\rho}{2}} \leq \left(t^2 2^{1+\frac{\rho}{2}} + \frac{2}{c_1^2} \right)^{1/2} \int_0^t \|v(\tau)\|_{s+1-\frac{\rho}{2}} \, d\tau.$$

- The use of these estimates leads to

$$\|u_t(t)\|_{s-1-\frac{\rho}{2}} \leq c_2 \|\varphi\|_s + \|\psi\|_{s-1-\frac{\rho}{2}} + \int_0^t \|h(\tau)\|_{s+1-\frac{\rho}{2}} \, d\tau$$

for $0 \leq t \leq T$.

Local Existence for the Nonlinear Problem

Theorem

Assume that

- $\frac{\rho}{2} + r \geq 1, \quad s > \frac{1}{2}, \quad g \in C^{[s]+1}(\mathbb{R}),$
- $\varphi \in H^s$, $\psi \in H^{s-1-\frac{\rho}{2}}$.

Then there is some $T > 0$ *such that the Cauchy problem has a unique solution* $u \in C\left([0,T], H^s\right) \cap C^1\left([0,T], H^{s-1-\frac{\rho}{2}}\right)$.

Regularization / Smoothing

- The larger the order ρ the smaller the order of Sobolev space.
- Increasing ρ adds more regularization into the problem.
- More regularization allows less smooth initial data.
- The combined effect of two dispersive effects must be greater than a critical value.

Sketch of the Proof

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

- Assume that $\varphi \in H^s$ and $\psi \in H^{s-1-\frac{\rho}{2}}$ for some fixed $s > 1/2$.
- Define the Banach space

$$X(T) = \{u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1-\frac{\rho}{2}})\}$$

endowed with the norm

$$\|u\|_{X(T)} = \max_{t\in[0,T]} \left(\|u(t)\|_s + \|u_t(t)\|_{s-1-\frac{\rho}{2}}\right)$$

for a fixed $T > 0$.

Sketch of the Proof (continued)

- By the Sobolev Embedding Theorem,

$$H^s(\mathbb{R}) \subset L^\infty(\mathbb{R}) \text{ for s ¿1/2.}$$

- Then $u \in C([0, T], L^\infty)$ whenever $u \in X(T)$.
- By the Sobolev Embedding Theorem, there is a constant d such that

$$\|u(t)\|_{L^\infty} \leq d \, \|u(t)\|_{X(T)} \text{ (for } s > 1/2)$$

- Consider a closed subset $Y(T)$ of $X(T)$

$$Y(T) = \{u \in X(T) : \|u\|_{X(T)} \leq A\}$$

for some constant $A > 0$ to be determined later.

Sketch of the Proof (continued)

- Consider the initial-value problem

$$u_{tt} - Lu_{xx} = (g(w))_{xx},$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)$$

  with $w \in Y(T)$.

- By the theorem for the linearized problem, the above problem has a unique solution $u(x,t)$.
- The map $\mathcal{S}$ carries $w \in Y(T)$ into the unique solution $u(x,t)$: $u(x,t) = \mathcal{S}(w)$.

Sketch of the Proof (continued)

- We prove that, for appropriately chosen T and A, the map $\mathcal{S}$ has a unique fixed point in $Y(T)$.
- This will be done in three steps.
  - In the first step we establish that the range of $Y(T)$ under the map $\mathcal{S}$ belongs to the space $X(T)$.
  - Secondly, we derive suitable estimates on $\|\mathcal{S}(w)\|_{X(T)}$ so that $\mathcal{S}(Y(T)) \subset Y(T)$.
  - The third step is to show that the mapping $\mathcal{S}$ is a contraction mapping.

Sketch of the Proof (continued)

Lemma: Local Bound

Assume that $f \in C^k(\mathbb{R})$, $f(0) = 0$, $u \in H^s \cap L^\infty$ and $k = [s] + 1$, where $s \geq 0$. Then, we have

$$\|f(u)\|_s \leq C_1(M) \|u\|_s$$

if $\|u\|_{L^\infty} \leq M$, where $C_1(M)$ is a constant dependent on M.

Lemma: Lipschitz Condition

Assume that $f \in C^k(\mathbb{R})$, $u, v \in H^s \cap L^\infty$ and $k = [s] + 1$, where $s \geq 0$. Then, we have

$$\|f(u) - f(v)\|_s \leq C_2(M) \|u - v\|_s$$

if $\|u\|_{L^\infty} \leq M$, $\|v\|_{L^\infty} \leq M$, $\|u\|_s \leq M$, and $\|v\|_s \leq M$, where $C_2(M)$ is a constant dependent on M and s.

Sketch of the Proof (continued)

Lemma

Assume that $\rho \geq 2$, $s > 1/2$, $\varphi \in H^s$, $\psi \in H^{s-1-\frac{\rho}{2}}$ *and* $g \in C^{[s]+1}(\mathbb{R})$. *Then for suitably chosen* A *and sufficiently small* T, *the map* $\mathcal{S}$ *is a contractive mapping from* $Y(T)$ *into itself.*

- we need to show that $\mathcal{S}(Y(T)) \subset Y(T)$.
- we use a standard contraction argument.
- With the choice of T the mapping $\mathcal{S}$ becomes contractive.
- By the Banach Fixed Point Theorem there is a unique $u \in Y(T)$ such that $S(u) = u$.

Global Existence for the Nonlinear Problem

Theorem

Assume that

- $\rho \geq 2, \quad s > \frac{1}{2}, \quad g \in C^{[s]+1}(\mathbb{R}),$
- $\varphi \in H^s, \ \psi \in H^{s-1-\frac{\rho}{2}}$
- the unique solution of the Cauchy problem is defined on the maximal time interval $[0, T_{\max})$.

If the maximal time is finite, i.e. $T_{\max} < \infty$, then

$$\limsup_{t \to T_{\max}^-} \left[\|u(t)\|_s + \|u_t(t)\|_{s-1-\frac{\rho}{2}} \right] = \infty.$$

Sketch of the Proof

The main approach is to look for the local solutions of the Cauchy problems on finite time intervals and then is to patch those local solutions together in a continuous manner.

Global Existence for the Nonlinear Problem

Theorem (Energy Identity)

- Suppose that $u(x,t)$ is a solution of the Cauchy problem on some interval $[0,T)$.
- Let $K = L^{1/2}$, $G(u) = \int_0^u g(p)dp$.
- Let $\Lambda^{-\alpha} w = \mathcal{F}^{-1}[|\xi|^{-\alpha}\mathcal{F}w]$ and $B^{-1/2}w = \mathcal{F}^{-1}[(b(\xi))^{-1/2}\mathcal{F}w]$.

If $B^{-1/2}\Lambda^{-1}\psi \in L^2$, $B^{-1/2}K\varphi \in L^2$ and $G(\varphi) \in L^1$, then, for any $t \in [0,T)$, the energy identity

$$E(t) = \left\|B^{-1/2}\Lambda^{-1}u_t\right\|^2 + \left\|B^{-1/2}Ku\right\|^2 + 2\int_{\mathbb{R}} G(u)dx = E(0)$$

is satisfied.

Global Existence for the Nonlinear Problem

Theorem A

Assume that

- $r + \frac{\rho}{2} \geq 1, \quad s = \frac{r}{2} + \frac{\rho}{2}, \quad g \in C^{[\frac{r}{2}+\frac{\rho}{2}]+1}(\mathbb{R}),$
- $\varphi \in H^{\frac{r}{2}+\frac{\rho}{2}}, \quad \psi \in H^{\frac{r}{2}-1},$
- $\Lambda^{-1}\psi \in L^2, \quad G(\varphi) \in L^1$ and $G(u) \geq 0$ for all $u \in \mathbb{R}$.

Then the Cauchy problem has a unique global solution

$$u \in C\left([0,\infty), H^{\frac{r}{2}+\frac{\rho}{2}}\right) \cap C^1\left([0,\infty), H^{\frac{r}{2}-1}\right).$$

Global Existence for the Nonlinear Problem

Theorem B

Assume that

- $r + \frac{\rho}{2} \geq 1, \;\; \frac{r}{2} + \frac{\rho}{2} > 1/2, \;\; s > 1/2, \;\; g \in C^{[s]+1}(\mathbb{R})$,
- $\varphi \in H^s$, $\psi \in H^{s-1-\frac{\rho}{2}}$,
- $G(\varphi) \in L^1$ and $G(u) \geq 0$ for all $u \in \mathbb{R}$.

Then the Cauchy problem has a unique global solution

$$u \in C\left([0,\infty), H^s\right) \cap C^1\left([0,\infty), H^{s-1-\frac{\rho}{2}}\right).$$

Blow-up in Finite Time

Theorem

Assume that

- $B^{-1/2}K\varphi \in L^2, \quad B^{-1/2}\Lambda^{-1}\psi \in L^2, \quad G(\varphi) \in L^1.$

If there is some $\nu > 0$ such that

$$pg(p) \le 2(1+2\nu)G(p) \text{ for all } p \in \mathbb{R},$$

and

$$E(0) = \left\| B^{-1/2}\Lambda^{-1}\psi \right\|^2 + \left\| B^{-1/2}K\varphi \right\|^2 + 2\int_{\mathbb{R}} G(\varphi)dx < 0,$$

then the solution $u(x,t)$ of the Cauchy problem blows up in finite time.

Sketch of the Proof

Lemma (Levine)

- Suppose that $H(t)$, $t \geq 0$ is a positive, twice differentiable function satisfying $H''H - (1+\nu)(H')^2 \geq 0$ where $\nu > 0$.
- If $H(0) > 0$ and $H'(0) > 0$, then $H(t) \to \infty$ as $t \to t_1$ for some $t_1 \leq H(0)/(\nu H'(0))$.

ONGOING STUDY

Ongoing Study

Present problem

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \quad x \in \mathbb{R}, \quad t > 0,$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

- Assumptions:

$$c_1^2(1+\xi^2)^{\rho/2} \le l(\xi) \le c_2^2(1+\xi^2)^{\rho/2},$$

$$0 < b(\xi) \le c_3^2(1+\xi^2)^{-r/2}.$$

- Question: What happens for small amplitude solutions ?