Mathematical analysis of the linear peridynamics model

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Nonlocal models and Peridynamics
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(Joint work with Qiang Du)

The classical equations

- The classical system of equations of motion

$$\rho\, \mathbf{u}_{tt}(\mathbf{x}, t) = \operatorname{div} \sigma(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t),$$

  is a system of differential equations.
- The model inherently assumes that fields have some sort of smoothness.
- The model can be used to study problems that involve discontinuities, such as fracture or phase transition. But either problems have to be recast or additional conditions have to be included.
- This creates analytical and numerical inconveniences.

A unifying framework?

What if there is a theory that models continuous media and those with discontinuities with a single mathematical framework?

Peridynamics attempts to do that!

The peridynamic model

- The equations of motion have a similar structure

$$\rho\, \mathbf{u}_{tt}(\mathbf{x}, t) = \mathcal{P}_{\mathbf{u}}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t)$$

  where $\mathcal{P}_{\mathbf{u}}(\mathbf{x}, t)$ is the internal force density and is nonlinear in $\mathbf{u}$ in general.
- It is a continuum model $\mathcal{P}_{\mathbf{u}}$ being an integral operator, as opposed to a differential operator for classical models.
- By default the formulation requires very minimal regularity of fields to start with. No spatial derivative is involved and thus the same equation can be applied even on discontinuities.

Outline

- Describe the bond-based model.
- Formulate the linearization of the bond-based model.
- Show well posedness of the linear model.
- Demonstrate the convergence of the linear PD system to the Classical Navier system in the event of vanishing nonlocality.

A nonlocal model

- Peridynamics belongs to a class of nonlocal models: nonlocality arising from the assumption that material points of finite distance apart are assumed to interact.
- As such forces are long range as opposed to only contact forces for local models.

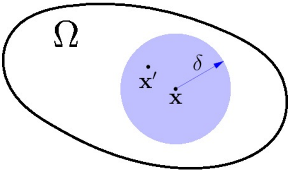


Figure: Material point x interacts with all x' within horizon δ.

Internal forces

How do you compute the internal forces in the body?

A simple and intuitive model is the **Bond-based peridynamic model**. It treats a material as a complex mass-spring system and

- any two nearby material particles are connected through "bond", and interact through a long range force up to a certain fixed extent, called *horizon*,
- any action on two different bonds are independent (this is a limitation!).

Bond-based Peridynamics

The system of equations of motion in the bond-based model is

$$\rho\,\mathbf{u}_{tt}(\mathbf{x},t) = \mathcal{P}_{\mathbf{u}}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t)$$

where

$$\mathcal{P}_{\mathbf{u}}(x,t) = \int_{B_\delta(x)\cap\Omega} \mathbf{f}(\mathbf{u}(\mathbf{x}',t) - \mathbf{u}(\mathbf{x},t), \mathbf{x}' - \mathbf{x})\,dV_{\mathbf{x}'}.$$

x: in a reference configuration

$\mathbf{u}(x,t)$: displacement of x at time t

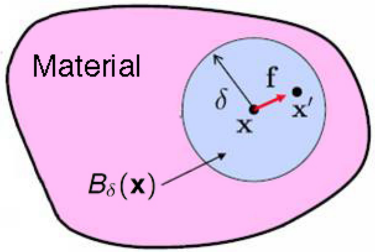
$\mathbf{f}(\eta,\boldsymbol{\xi})$: pairwise force density

δ: material horizon (interact range)

$B_\delta(x)$: effective neighborhood of x

ρ: material density

$\mathbf{b}$: loading force density



- The quantity

$$\mathcal{P}_{\mathbf{u}}(x,t) = \int_{B_\delta(x)\cap\Omega} \mathbf{f}(\mathbf{u}(\mathbf{x}',t) - \mathbf{u}(\mathbf{x},t), \mathbf{x}' - \mathbf{x})\, dV_{\mathbf{x}'}$$

  represents the force per unit reference volume at $\mathbf{x}$ due to interaction with particles in $B_\delta(x)$.

- The force density function $\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\xi})$ is material dependant and contains all material properties.
  - $\boldsymbol{\eta} = \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})$ -relative displacement
  - $\boldsymbol{\xi} = \mathbf{x}' - \mathbf{x}$ -relative position vector in the reference configuration
  - $|\boldsymbol{\eta} + \boldsymbol{\xi}|$ -relative position vector in the deformed configuration

The laws of physics

- Restrictions on the force density function (balance of linear and angular momentum):

$$\left.\begin{array}{c}\mathbf{f}(-\boldsymbol{\eta},-\boldsymbol{\xi})=-\mathbf{f}(\boldsymbol{\eta},\boldsymbol{\xi})\\(\boldsymbol{\eta}+\boldsymbol{\xi})\times\mathbf{f}(\boldsymbol{\eta},\boldsymbol{\xi})=0\end{array}\right\}\Longrightarrow\mathbf{f}(\boldsymbol{\eta},\boldsymbol{\xi})=F(\boldsymbol{\eta},\boldsymbol{\xi})(\boldsymbol{\eta}+\boldsymbol{\xi}).$$

  where F is a scalar function.
- For elastic materials there is a central potential, $w(\boldsymbol{\eta},\boldsymbol{\xi})$, through which material points are interacting, $\mathbf{f}=\nabla_{\boldsymbol{\eta}}w$,
- Observe that no spacial regularity $\mathbf{u}$ or $\mathbf{f}$ is required in the formulation.

Example: bond stretch model

- "Force is proportional to the relative change" (Hooke's Law)

$$\mathbf{f}(\boldsymbol{\eta},\boldsymbol{\xi}) = c(\delta,d)\left(\frac{|\boldsymbol{\eta}+\boldsymbol{\xi}|-|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|}\right)\frac{\boldsymbol{\eta}+\boldsymbol{\xi}}{|\boldsymbol{\eta}+\boldsymbol{\xi}|}\chi_{[0,\delta](|\boldsymbol{\xi}|)}$$

- Note that $s(\boldsymbol{\eta},\boldsymbol{\xi}) = \frac{|\boldsymbol{\eta}+\boldsymbol{\xi}|-|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|}$ is the relative change of a bond (bond-stretch).
- The central potential for this material is

$$w(\boldsymbol{\eta},\boldsymbol{\xi}) = \frac{c(\delta,d)}{2}\left(\frac{|\boldsymbol{\eta}+\boldsymbol{\xi}|}{|\boldsymbol{\xi}|}-1\right)^2|\boldsymbol{\xi}|$$

- More generally, one may consider

$$w(\boldsymbol{\eta},\boldsymbol{\xi}) = \frac{c(\delta,d)}{2}\left(\frac{|\boldsymbol{\eta}+\boldsymbol{\xi}|}{|\boldsymbol{\xi}|}-1\right)^2 g(|\boldsymbol{\xi}|),$$

  where g specifies whether bonds with different reference lengths can have different elastic response.

Our mathematical study of the PD model focuses when the relative displacement

$$\sup_{|\mathbf{x}-\mathbf{x}'|<\delta} |\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})| << 1$$

In this case the model can be well approximated by a linear system.

Linearization

- The linear model is obtained by linearizing the force density w.r.t relative displacement:

$$\mathbf{f}(\boldsymbol{\eta},\boldsymbol{\xi}) = f(0,\boldsymbol{\xi}) + \nabla_{\boldsymbol{\eta}}\mathbf{f}(0,\boldsymbol{\xi})\boldsymbol{\eta} + h.o.t.$$

- In this case, the linearized force per unit volume at a material point x is given by

$$\mathcal{P}(\mathbf{u})(\mathbf{x}) := \int_{B_\delta(\mathbf{x})\cap\Omega} \mathbb{C}(\mathbf{x}-\mathbf{x}')\left(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})\right) d\mathbf{x}'.$$

- $\mathbb{C}(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\eta}}\mathbf{f}(0,\boldsymbol{\xi})$ is called the micromodulus matrix function.

The micromodulus function

- Materials with pairwise force function $\mathbb{C}(\boldsymbol{\xi})\boldsymbol{\eta}$ are called linear materials.
- For linear and elastic materials, using the fact $\mathbf{f}(\boldsymbol{\eta},\boldsymbol{\xi}) = F(\boldsymbol{\eta},\boldsymbol{\xi})(\boldsymbol{\eta}+\boldsymbol{\xi})$, one can show that $\mathbb{C}(\boldsymbol{\xi})$ has the form
$$\mathbb{C}(\boldsymbol{\xi}) = 2\lambda(\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi} + 2F_0(\boldsymbol{\xi})\mathbb{I},$$
- Note that $\mathbf{f}(0,\boldsymbol{\xi}) = F_0(\boldsymbol{\xi})\boldsymbol{\xi}$, and for isotropic materials (when there is no preferred material direction) the functions λ and F_0 are radial function.

Linearized materials

- When

$$\mathbf{f}(\boldsymbol{\eta},\boldsymbol{\xi}) = c(\delta,d)\left(\frac{|\boldsymbol{\eta}+\boldsymbol{\xi}|-|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|}\right)\frac{\boldsymbol{\eta}+\boldsymbol{\xi}}{|\boldsymbol{\eta}+\boldsymbol{\xi}|}\chi_{[0,\delta]}(|\boldsymbol{\xi}|),$$

  then

$$\mathbb{C}(\boldsymbol{\xi}) = c(\delta,d)\frac{1}{|\boldsymbol{\xi}|^3}\boldsymbol{\xi}\otimes\boldsymbol{\xi}\chi_{[0,\delta]}(|\boldsymbol{\xi}|); \qquad \left(\lambda(|\boldsymbol{\xi}|) = \frac{1}{|\boldsymbol{\xi}|}\right)$$

- In general, when $w(\boldsymbol{\eta},\boldsymbol{\xi}) = \frac{c(\delta,d)}{2}\left(\frac{|\boldsymbol{\eta}+\boldsymbol{\xi}|}{|\boldsymbol{\xi}|}-1\right)^2 g(|\boldsymbol{\xi}|)$, then

$$\mathbb{C}(\boldsymbol{\xi}) = c(\delta,d)\frac{g(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^3}\boldsymbol{\xi}\otimes\boldsymbol{\xi} \qquad \left(\lambda(|\boldsymbol{\xi}|) = \frac{g(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}\right).$$

Linear kernels

- In general, $\mathbb{C}(\boldsymbol{\xi})$ may not be integrable. In fact, as we will see its integrability and/or nonintegrability determines the 'regularity' of the displacement field.
- When $F_0 \equiv 0$, $\mathbb{C}(\mathbf{x} - \mathbf{x}')$ - "spring constant" of the bond between $\mathbf{x}'$ and $\mathbf{x}$. [Network of Complex Hookean spring]
- However "there is no reason to assume that this condition would exist in real materials."
- In fact, it is shown that $F_0(|\boldsymbol{\xi}|)$ must change sign in modeling realistic **unstressed** materials.

The linearized PD system as a nonlocal BVP

- Equations of motion

$$\begin{cases} \rho(\mathbf{x})\mathbf{u}_{tt}(\mathbf{x},t) = \mathcal{P}\,\mathbf{u}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t), \quad \forall(\mathbf{x},t) \in \Omega \times (0,T) \\ \mathbf{u}_t(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \\ \mathbf{u}(\cdot,t) \in V, \quad \forall t \in [0,T]. \end{cases} \tag{E-O-M}$$

- Equilibrium equations

$$\begin{cases} -\mathcal{P}\,\mathbf{u} = \mathbf{b} \\ \mathbf{u} \in V \end{cases}$$

V is the space of vector fields satisfying volume constraints.

For linear microelastic and isotropic materials

- The nonlocal operator $\mathcal{P}$ is given by

$$\mathcal{P}\,\mathbf{u}(\mathbf{x},t) = \int_{B_\delta(\mathbf{x})\cap\Omega} \mathbb{C}(\mathbf{x}'-\mathbf{x})(\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t))d\mathbf{x}'$$

  the micromodulus tensor being

$$\mathbb{C}(\boldsymbol{\xi}) = 2\lambda(|\boldsymbol{\xi}|)\boldsymbol{\xi}\otimes\boldsymbol{\xi} + 2F_0(|\boldsymbol{\xi}|)\mathbb{I}$$

- Basic assumptions on $\mathbb{C}(\boldsymbol{\xi})$
  - λ is a nonnegative radial function and

$$\lambda(r) > 0, \quad \text{near } r = 0 \text{ and } \lambda(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^4 \in L^1_{loc}(\mathbb{R}^d).$$

  - F_0 is radial and $F_0(|\boldsymbol{\xi}|) \in L^1_{loc}(\mathbb{R}^d)$

The equilibrium equation

- The linear peridynamic equilibrium equation as a nonlocal boundary problem: Find $\mathbf{u}$

$$\begin{cases} -\mathcal{P}\mathbf{u} = \mathbf{b} \\ \mathbf{u} \ \text{ satisfy a "boundary conditions"} \end{cases}$$

  Since $\mathcal{P}$ is a nonlocal operator, the "boundary condition" is a nonlocal volumetric condition: say $\mathbf{u}(\mathbf{x}) = 0$ for all $x \in \omega \subset \Omega$ and $|\omega| > 0$.

- One may compare this with the local equation:

$$\begin{cases} -\text{div}(\mathfrak{C}\nabla\mathbf{u}) = \mathbf{b}, \\ \mathbf{u} = 0 \quad x \in \partial\Omega \end{cases}$$

  where $\mathfrak{C}$ is the classical elasticity tensor.

The operator and solution space

We have to address two questions:

- How should we understand the operator $\mathcal{P}$ when $\mathbb{C}(\boldsymbol{\xi})$ is not integrable?
- Associated with each $\mathbb{C}$, what is an appropriate solution space to solve the equations?

Well-posedness: when $F_0(|\xi|) = 0$

In this case the operator $\mathcal{P}$ reduces to

$$\mathcal{L}\mathbf{u}(x) = 2\int_{B_\delta(\mathbf{x})\cap\Omega} \lambda(|\mathbf{x}' - \mathbf{x}|)(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))d\mathbf{x}'$$

So the goal is to solve,

$$-\mathcal{L}\mathbf{u} = \mathbf{b}$$

in the variational sense.

When $\mathbb{C}$ is integrable

In this case:

- Observe that

$$(\mathcal{L}\mathbf{u})_j = \sum_{i=1}^{d} L_{ij}(\mathbf{u})$$

where

$$L_{ij}(\mathbf{u}) = \int_\Omega K_{ij}(\mathbf{x}' - \mathbf{x})(u_i(\mathbf{x}') - u_i(\mathbf{x}))d\mathbf{x}'$$

and

$$K_{ij}(\mathbf{x}' - \mathbf{x}) = 2\lambda(|\mathbf{x}' - \mathbf{x}|)(x_i' - x_i)(x_j' - x_j).$$

Hilbert Space setting

- The linear operator

$$\mathcal{L} : L^2(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{R}^d)$$

is bounded i.e. for any $\mathbf{u} \in L^2(\Omega;\mathbb{R}^d)$, $\mathcal{L}\mathbf{u} \in L^2(\Omega;\mathbb{R}^d)$ and

$$\|\mathcal{L}\mathbf{u}\|_{L^2} \leq C\|\mathbf{u}\|_{L^2}$$

This is so because the operator componentwise is (a sum of) convolution type with a locally integrable kernel.

- The operator is also self-adjoint, i.e.

$$(-\mathcal{L}\mathbf{u},\mathbf{w})_{L^2} = (\mathbf{u},-\mathcal{L}\mathbf{w})_{L^2}.$$

In fact, after change of variables, we have

$$(-\mathcal{L}\mathbf{u},\mathbf{w}) = \int_\Omega\int_\Omega \lambda(|\mathbf{x}'-\mathbf{x}|)((\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{w}(\mathbf{x}')-\mathbf{w}(\mathbf{x}))\mathbf{dx}'\mathbf{dx}.$$

which defines a bilinear form on $L^2(\Omega;\mathbb{R}^d)$.

Well-posedness using Lax-Milgram

- The associated energy is

$$(-\mathcal{L}\mathbf{u},\mathbf{u}) = \int_\Omega \int_\Omega \lambda(|\mathbf{x}-\mathbf{x}'|) \left[(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))\right]^2 d\mathbf{x}' d\mathbf{x}$$

- So given a closed subspace V of $L^2(\Omega;\mathbb{R}^d)$, that supports the Poincaré-type inequality

$$(\mathcal{L}\mathbf{u},\mathbf{u})_{L^2} \geq \lambda \|\mathbf{u}\|^2_{L^2}, \quad \text{for all } \mathbf{u} \in V$$

  for some $\lambda > 0$, then the equation $\mathcal{L}\mathbf{u} = \mathbf{b}$ will have a unique variational solution in V for any $b \in L^2(\Omega;\mathbb{R}^d)$.

- Moreover, the solution minimizes

$$\min_{\mathbf{u}\in V\subset L^2(\Omega;\mathbb{R}^d)} E(\mathbf{u}), \qquad E(\mathbf{u}) = \frac{1}{2}(\mathcal{L}\mathbf{u},\mathbf{u})_{L^2} - (\mathbf{b},\mathbf{u})_{\mathbf{L^2}}$$

- Question: What kind of subspaces of L^2 support the nonlocal Poincaré-type inequality?
- We will answer this question shortly, but let's go ahead obtain the correct variational formulation for the case when $\mathbb{C}$ is not integrable.

When the kernel $\mathbb{C}$ is not integrable

- The linear operator $\mathcal{L}$ is an unbounded operator on $L^2(\Omega;\mathbb{R}^d)$.
- In this case one has to work on a smaller set of vector fields: The energy space

$$\mathcal{S} = \quad \{\mathbf{u} \in L^2 : \int_\Omega \int_\Omega \lambda(\mathbf{x}-\mathbf{x}')\,|(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))|^2\, d\mathbf{x}'\, d\mathbf{x} < \infty\}$$

- In general, $\mathcal{S}$ is a proper subset of $L^2(\Omega;\mathbb{R}^d)$.

The solution space $\mathcal{S}$

- Turns out $\mathcal{S}$ is a Hilbert space with the inner product $(\mathbf{u},\mathbf{w}) + (\mathbf{u},\mathbf{w})_s$ where

$$(\mathbf{u},\mathbf{w})_s = \int\int_\Omega \lambda(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{w}(\mathbf{x}')-\mathbf{w}(\mathbf{x}))d\mathbf{x}'d\mathbf{x},$$

semi-norm $|\mathbf{u}|_s^2 = (\mathbf{u},\mathbf{u})_s$ and norm

$$\|\mathbf{u}\|_s^2 = \|\mathbf{u}\|_{L^2}^2 + |\mathbf{u}|_s^2$$

Relations with standard function spaces

- $C(\overline{\Omega}; \mathbb{R}^d)$ is dense in $\mathcal{S}(\Omega)$;
- $\mathcal{S}$ contains smooth functions

$$H^1(\Omega; \mathbb{R}^d) \hookrightarrow \mathcal{S}, \quad \text{and } |\mathbf{u}|_s \leq C(\Omega, d, \lambda)|\mathbf{u}|_1.$$

- Therefore, $\mathcal{S}$ is dense in $L^2(\Omega)$.
- Note that $\mathcal{S}$ coincides with $L^2(\Omega; \mathbb{R}^d)$, when $\lambda(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^2$ is integrable.

The nonlocal operator

- Turns out that when $\mathbb{C}$ is not integrable

$$\mathcal{L}\mathbf{u} = \text{P.V.}2\int_{B_\delta(\mathbf{x})\cap\Omega} \lambda(|\mathbf{x}'-\mathbf{x}|)(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))d\mathbf{x}' \quad \text{in } \mathcal{S}'(\Omega),$$

- Moreover, if one defines the bilinear form

$$B(\mathbf{u},\mathbf{w}) = \int_\Omega\int_\Omega \lambda(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{w}(\mathbf{x}')-\mathbf{w}(\mathbf{x}))d\mathbf{x}'d\mathbf{x}$$

$$\mathcal{L} : \mathcal{S} \to \mathcal{S}'$$

and satisfies

$$\langle -\mathcal{L}\mathbf{u}, \mathbf{w}\rangle = B(\mathbf{u},\mathbf{w})$$

(In exactly the same way as the Laplacian: $-\Delta$)

- If we show that $B(\mathbf{u}, \mathbf{w})$ is a coercive bilinear form on a closed subset of $\mathcal{S}$, we may apply Lax-Milgram for well-posedness of

$$-\mathcal{L}\mathbf{u} = \mathbf{b}$$

on the subspace.
- Proving coercivity is precisely proving a nonlocal Poincaré-type inequality.

Proving nonlocal Poincaré-type inequality

- We first observe that

$$B(\mathbf{u},\mathbf{u}) = \int_\Omega\int_\Omega \lambda(\mathbf{x}'-\mathbf{x})|(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} = 0$$

  if $\mathbf{u}(\mathbf{x}) = Q\mathbf{x}+\mathbf{a}, \quad Q^T = -Q, \quad \mathbf{a}\in\mathbb{R}^d$ (Rigid deformations).
- We denote the set of rigid deformations by Π.
- Is the converse true? i.e Is $B(\mathbf{u},\mathbf{u}) = 0$ iff $\mathbf{u}$ is a rigid deformations?

Lemma (nonlocal char. of rigid deformations)

If $\mathbf{u}\in L^2(\Omega;\mathbb{R}^d)$ *and* $B(\mathbf{u},\mathbf{u}) = 0,$ *then* $\mathbf{u}\in\Pi$.

Poincaré-type inequality for a special subspace

Will give a subspace of $\mathcal{S}$ that avoid nontrivial rigid deformations.
Let us define a boundary layer of thickness $\epsilon > 0$

$$b\Omega_\epsilon = \{x \in \Omega : \quad \text{dist}(x, \partial\Omega) \leq \epsilon\}$$

Define also the subspace

$$V_0^\epsilon = \{\mathbf{u} \in L^2(\Omega; \mathbb{R}^d) : \mathbf{u}|_{b\Omega_\epsilon} = 0\}$$

We will denote $\mathcal{V}_0^\epsilon = \text{Closure}(V_0^\epsilon)$ in $\|\cdot\|_s$ norm.
I will use $V_0^\epsilon \cap \mathcal{S}$ in stead of $\mathcal{V}_0^\epsilon$.

Lemma

There exists $\kappa = \kappa(V_0^\epsilon) > 0$ such that

$$B(\mathbf{u}, \mathbf{u}) \geq \kappa \|\mathbf{u}\|_{L^2}^2 \quad \forall \mathbf{u} \in V_0^\epsilon.$$

Compactness

The proof of the nonlocal Poincaré-type inequality uses the following compactness result:

Lemma (L^2-Compactness)

If $\{\mathbf{u}_n\}$ *is a bounded sequence in* $L^2(\Omega;\mathbb{R}^d)$, $\mathbf{u}_n|_{b\Omega_\epsilon} = 0$, *and* $\lim_{n\to\infty} B(\mathbf{u}_n,\mathbf{u}_n) = 0$, *then* $\{\mathbf{u}_n\}$ *is precompact in* $L^2(\Omega;\mathbb{R}^d)$.

Well-posedness

Theorem

The variational problem: given $b \in L^2$ *find* $\mathbf{u} \in V_0^\epsilon \cap \mathcal{S}$ *such that*

$$\langle -\mathcal{L}\mathbf{u}, \mathbf{v} \rangle = (b, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V_0^\epsilon \cap \mathcal{S}$$

has a unique solution with the estimate

$$|\mathbf{u}|_s \leq C \|b\|_{L^2}.$$

Moreover,

$$\mathbf{u} = arg \min_{\mathbf{v} \in V_0^\epsilon \cap \mathcal{S}} E(\mathbf{v}), \qquad E(\mathbf{v}) = \frac{1}{2}\langle -\mathcal{L}\mathbf{v}, \mathbf{v} \rangle - (\mathbf{b}, \mathbf{v}).$$

Well-posedness (general)

Theorem

Let V be any closed subspace of $L^2(\Omega;\mathbb{R}^d)$ such that $V \cap \Pi = \{\mathbf{O}\}$. The variational problem: given $b \in L^2$ find $\mathbf{u} \in V \cap \mathcal{S}$ such that

$$\langle -\mathcal{L}\mathbf{u}, \mathbf{v}\rangle = (b, \mathbf{v}) \quad \textit{for all } \mathbf{v} \in V \cap \mathcal{S}$$

has a unique solution with the estimate

$$|\mathbf{u}|_s \leq C\|b\|_{L^2}.$$

Examples of spaces of solutions

- Vector fields that vanish on a set with nonzero measure

$$V = \{\mathbf{u} \in L^2(\Omega; \mathbb{R}^d) : \mathbf{u}|_\omega = 0, \quad \text{for } \omega \subset \Omega \text{ with } |\omega| > 0\}.$$

- Some vector fields in R^3

$$V = \{\mathbf{u} = u(1, -1, 0) \in L^2(B_1(0); \mathbb{R}^3) : \int_\Omega u(\mathbf{x}) d\mathbf{x} = 0\}.$$

A remark on regularity of solutions

- When $\mathbb{C}$ is integrable, then
$$\mathbf{b} \in L^2 \text{ (and no more!)} \Longrightarrow \mathbf{u} = \mathcal{L}^{-1}\mathbf{b} \in L^2 \text{ (and no more!)}.$$
In other words, regularity of solutions is about the same as the regularity of the right hand side.
- (Work of Du&Zhou:) Using Fourier expansion (in 1D) one may write the PD equations as
$$\sum_k \eta_\delta(k) u_k \sin(kx) = \sum_k b_k \sin(kx),$$
where
$$\eta_\delta(k) = \int_{-\delta}^{\delta} (1 - \cos(ky))\lambda(|y|)|y|^2 dy.$$
-
$$\lambda(|y|)|y|^2 \in L^2 \Longrightarrow c_1(\delta) \leq \eta_\delta(k) \leq c_2(\delta) \quad \forall k \geq 1.$$
- Then
$$u_k = \frac{b_k}{\eta_\delta(k)} \approx b_k, \quad \forall k$$

- When $\mathbb{C}$ is not integrable, the solution gains some kind of regularity. Indeed,

$$\mathbf{b} \in L^2(\Omega;\mathbb{R}^d)\,\text{(and no more!)} \Longrightarrow \mathbf{u} = \mathcal{L}^{-1}\mathbf{b} \in \mathcal{S}$$

as one may think of $\mathcal{S}$ as a fractional Sobolev space.

The case: when $F_0(|\xi|)$ is sign changing (and integrable)

- Recall that

$$\mathbb{C}(\xi) = 2\lambda(|\xi|)\xi \otimes \xi + 2F_0(|\xi|)\mathbb{I}.$$

- Note that for each ξ, $\mathbb{C}(\xi)$ has two eigenvalues $\lambda(|\xi|)|\xi|^2 + F_0(|\xi|)$ and $F_0(|\xi|)$.Thus the matrix $\mathbb{C}(\xi)$ could be negative definite for some ξ when F is sign changing.
- Therefore the quadratic energy

$$\langle -\mathcal{P}\mathbf{u}, \mathbf{u} \rangle$$

could be negative. So we cannot expect the operator to be "positive definite" always.

The operator as perturbation

- When $F_0 \not\equiv 0$, we interpret $\mathcal{P}$ as a perturbation of $\mathcal{L}$

$$\mathcal{P} = \mathcal{L} + \mathcal{F}$$

where $\mathcal{F}$ is the linear map defined as

$$\mathcal{F}\mathbf{u}(x) = 2\int_{B_\delta(\mathbf{x})\cap\Omega} F_0(|\mathbf{x}' - \mathbf{x}|)(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))d\mathbf{x}'.$$

- Note that $\mathcal{F} : L^2(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{R}^d)$ is a bounded operator, and

$$(-\mathcal{F}\mathbf{u}, \mathbf{u}) = \int_\Omega\int_\Omega F_0(|\mathbf{x}' - \mathbf{x}|)|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^2 d\mathbf{x}' d\mathbf{x}$$

- Let us impose a condition on a condition on F_0 so that $\langle -\mathcal{P}\mathbf{u}, \mathbf{u}\rangle$ is positive definite.
- To that end, we estimate that for any $\mathbf{u} \in V_0^\epsilon$,

$$\frac{\langle -\mathcal{P}\mathbf{u}, \mathbf{u}\rangle}{\|\mathbf{u}\|_s^2} \geq 1/(1+\kappa)\left(1 - 2\kappa(\|F_0\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^\infty(\Omega)})\right),$$

where $f(\mathbf{x}) = \int_\Omega F_0(|\mathbf{x}' - \mathbf{x}|)d\mathbf{x}'$ and κ is the "Poincaré" constant corresponding to V_0.

- Then if $\kappa(\|F_0\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^\infty(\Omega)}) < \frac{1}{2}$, then given $\mathbf{b} \in L^2(\Omega;\mathbb{R}^d)$,

$$-\mathcal{P}\mathbf{u} = \mathbf{b} \quad \text{in}$$

has a unique variational solution in $V_0^\epsilon \cap \mathcal{S}$ Moreover,

$$\mathbf{u} = arg \min_{\mathbf{u}\in V_0^\epsilon \cap \mathcal{S}} E(\mathbf{u}) \qquad E(\mathbf{u}) = \frac{1}{2}\langle -\mathcal{L}\mathbf{u}, \mathbf{u}\rangle + \frac{1}{2}(-\mathcal{F}\mathbf{u}, \mathbf{u}) - (\mathbf{b}, \mathbf{u}).$$

On other subspaces

Well-posedness via minimization works well over any closed subspace that does not intersect rigid deformations.

Necessity for minimization

In the case where $\mathbb{C}$ is integrable, and

$$V = \{\mathbf{u} \in L^2(\Omega; \mathbb{R}^d) : u|_\omega = 0 \quad \omega \subset \Omega, \ |\omega| > 0\}$$

a necessary condition for minimization is the matrix

$$\Lambda(\mathbf{x}) = \int_{B_\delta(\mathbf{x})\cap\Omega} \lambda(|\mathbf{x}' - \mathbf{x}|)(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) + F_0(|\mathbf{x}' - \mathbf{x}|)\mathbb{I} d\mathbf{x}'$$

is uniformly positive definite in $\Omega \setminus \omega$.

Nonlinear perturbation:

- We could study a nonlinear perturbation, but compact, of $\mathcal{L}$ resulting in the energy

$$E_{\mathbf{u}} = \frac{1}{2}\langle -\mathcal{L}\mathbf{u}, \mathbf{u}\rangle + \psi(\mathbf{u}) - (\mathbf{b}, \mathbf{u}),$$

where ψ is a is continuous functional in V_0 w.r.t the L^2 norm satisfying

$$\psi(\mathbf{u}) \geq C - \theta \|\mathbf{u}\|_s^2.$$

- If θ is small, then $E_{\mathbf{u}}$ will have a minimum in V_0; direct method of calculus of variations.
- An example of such functional is

$$\psi(\mathbf{u}) = \int_\Omega \int_\Omega F_0(|\mathbf{x}'-\mathbf{x}|)|\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})|^2 d\mathbf{x}' d\mathbf{x} + \int_\Omega h_\alpha(\mathbf{x}, \mathbf{u}(\mathbf{x}))d\mathbf{x},$$

$h_\alpha(\mathbf{x}, \boldsymbol{\eta})$ is continuous in $\boldsymbol{\eta}$, and has the growth condition

$$|h_\alpha(\mathbf{x}, \boldsymbol{\eta})| \leq c(k(\mathbf{x}) + |\boldsymbol{\eta}|^\alpha),$$

with $0 < \alpha < 2$ and $k(\mathbf{x}) \in L^1(\Omega)$.

Well-posedness via Fredholm Alternative Theorem

Recall that $\mathcal{L} : V_0^\epsilon \cap \mathcal{S} \to \mathcal{S}'$ is positive definite. Therefore invertible and the inverse

$$\mathcal{L}^{-1} : L^2(\Omega; \mathbb{R}^d) \to \mathcal{S}$$

is linear and bounded.
Now given $\mathbf{b} \in L^2(\Omega; \mathbb{R}^d)$ if $\mathbf{u} \in V_0^\epsilon$ is a solution to

$$-(\mathcal{L} + \mathcal{F})\mathbf{u} = \mathbf{b} \iff (\mathcal{I} + \mathcal{L}^{-1}\mathcal{F})u = \mathcal{L}^{-1}(-\mathbf{b}) = \tilde{\mathbf{b}}$$

Denote $\mathcal{K} := \mathcal{L}^{-1}\mathcal{F}$. Then

$$\mathcal{K} : L^2(\Omega; \mathbb{R}^d) \to V_0^\epsilon$$

is a bounded linear operator and

$$|\mathcal{K}u|_s \leq C\|u\|.$$

Applying Fredholm Alternative

So we need to study the solvability of

$$(\mathcal{I} + \mathcal{K})(\mathbf{u}) = \tilde{\mathbf{b}}.$$

Idea: Apply Fredholm Alternative Theorem.
For that we need to show that $\mathcal{K} : L^2 \to L^2$ is a COMPACT operator.
Turns out that the compactness of the operator depends on the integrability of the kernel.

Take the solution space to be $V_0^\epsilon \cap \mathcal{S}$. Then $\mathcal{K} = \mathcal{L}^{-1}\mathcal{F}$:

$$\text{Range}(\mathcal{K}) \subset V_0^\epsilon \cap \mathcal{S}.$$

- If λ satisfies the following density condition

$$\lim_{\tau\to 0} \frac{\tau^2}{\int_{B(0,\tau)} \lambda(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^4 d\boldsymbol{\xi}} = 0, \qquad (*)$$

  then the embedding

$$V_0^\epsilon \cap \mathcal{S} \hookrightarrow L^2(\Omega;\mathbb{R}^d)$$

  is compact, from which the compactness of $\mathcal{K}$ follows.

- If $(*)$ holds, then $\lambda(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^2 \notin L^1_{loc}$. Therefore,in this case, $\mathcal{S} \subsetneq L^2(\Omega;\mathbb{R}^d)$.

- However, there are still λ that do not satisfy $(*)$ and yet $\lambda(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^2 \notin L^1_{loc}$.

Some examples

1. For any $0 < s < 1$,

$$\lambda(|\xi|) = \frac{1}{|\xi|^{d+2+2s}}$$

satisfies $(*)$. And, of course, any other function that can be bounded from below and above by a constant multiple.

2.

$$\lambda(|\xi|) = \chi_{B(0,1)}(|\xi|)\frac{-\ln(|\xi|)}{|\xi|^{d+2}}$$

satisfies $(*)$.

3. The function

$$\lambda(|\xi|) = \frac{1}{|\xi|^{d+2}}$$

does not satisfy $(*)$, and yet $\lambda(|\xi|)|\xi|^2 \notin L^1_{loc}$.

- Do we have compactness for other boundary conditions?

  Ans: We are working on it!

- What happens to the compactness of $\mathcal{K}$ when $\mathcal{S} = L^2(\Omega; \mathbb{R}^d)$?

  Ans: In general, $\mathcal{K}$ is not compact!

non-compactness

In fact we can prove the following:

If $\mathcal{S} = L^2(\Omega; \mathbb{R}^d)$ and V is any closed subspace such that $dim(V) = \infty$, then for $\mathcal{K}$ to be compact $\lambda(r)r^2 - F(r)$ must change sign.

Is all lost in this case?

Compactness

Take V_0^ϵ as your solution space (nonlocal 0 Dirichlet boundary condition).

Theorem

Assume that $F_0(r)$ has support $(0, \epsilon)$. Then the operator $\mathcal{K} = L^2(\Omega; \mathbb{R}^d) \to V_0^\epsilon$ is compact if and only if

$$\int_{B_\epsilon(0)} F_0(|\xi|) d\xi = 0.$$

The main idea is when $\int_{B_\epsilon(0)} F(|\xi|) d\xi = 0$, then the operator $\mathcal{F}$ is essentially a convolution operator (which is a compact operator!) So $\mathcal{K}$ is a composition of continuous and compact operators.

Well-posedness

Theorem (Existence theorem)

In the case where $\mathcal{K}$ is a compact operator, either

$$\mathcal{P}\mathbf{u} = 0$$

has nontrivial solution or

$$\mathcal{P}\mathbf{u} = \mathbf{b}$$

is wellposed for all $\mathbf{b} \in L^2(\Omega; \mathbb{R}^d)$ in V_0^ϵ.

Another form

If $F_0(r)$ is given in parametric form as in $F_0(r) = \theta F(r)$, $\mathcal{P}_\theta = \mathcal{L} + \theta\mathcal{F}$. In this case when $\mathcal{K}$ is a compact operator, we have the following

Theorem

- *There exists a countable set $\Sigma \subset \mathbb{R}$, (spectrum of $\mathcal{K}$,) such that the variational equation*

$$\mathcal{P}_\theta \mathbf{u} = \mathbf{b}, \quad \mathbf{u} \in V$$

  has a unique solution for each $\mathbf{b} \in L^2(\Omega; \mathbb{R}^d)$ if and only if $\theta \notin \Sigma$.

- *If Σ is countably infinite, then $\Sigma = \{\theta_k\}_{k=1}^{\infty}$ such that*

$$|\theta_k| \to \infty \quad \text{as } k \to \infty.$$

- *If $|\theta|$ is small, then the solution $\mathbf{u}$ minimizes a potential energy.*

The time-dependent equation

Would like to solve the initial value problem:

$$\begin{cases} \rho(\mathbf{x})\mathbf{u}_{tt}(\mathbf{x},t) = \mathcal{P}\,\mathbf{u}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t), \ \ \forall(\mathbf{x},t) \in \Omega \times (0,T) \\ \qquad \mathbf{u}_t(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) \\ \qquad \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \\ \qquad \mathbf{u}(\cdot,t) \in V, \quad \forall t \in [0,T]. \end{cases}$$

V is the space of functions satisfying certain volumetric boundary conditions.
We consider two cases: $V = \mathcal{S}$ and $V = V_0^{\epsilon}$.

Variational techniques

Definition

Suppose that $\mathbf{b} \in L^2((0,T); L^2(\Omega))$, $\mathbf{u}_0 \in \mathcal{S}$, and $\mathbf{v}_0 \in L^2(\Omega)$. We say $\mathbf{u}$ is a solution if $\mathbf{u} \in L^2((0,T); \mathcal{S})$, $\mathbf{u}_t \in L^2((0,T); L^2(\Omega))$ and $\mathbf{u}_{tt} \in L^2((0,T); \mathcal{S}')$ such that $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u}_t(0) = \mathbf{v}_0$ and

$$\frac{d}{dt}(m\mathbf{u}_t, \mathbf{v}) + \langle \mathcal{P}\mathbf{u}, \mathbf{v} \rangle = (\mathbf{b}, \mathbf{v})$$

for all $\mathbf{v} \in \mathcal{S}$ in the sense of distribution in $[0, T)$.

Key properties to verify

- The triple
$$\mathcal{S}(\Omega) \subset L^2(\Omega; \mathbb{R}^d) \subset \mathcal{S}'(\Omega).$$
is an evolution triple. That is $\mathcal{S}(\Omega)$ is a separable Hilbert space densely embedded in $L^2(\Omega; \mathbb{R}^d)$.
- $\mathcal{P} : \mathcal{S}(\Omega) \to \mathcal{S}'(\Omega)$ is a linear, bounded, self-adjoint operator satisfying Gårding-type inequality:
$$\langle \mathcal{P}\mathbf{u}, \mathbf{u} \rangle + c\|\mathbf{u}\|^2 \geq C\|\mathbf{u}\|_s^2 \quad \forall \mathbf{u} \in \mathcal{S}.$$

Variational techniques: existence of a unique solution

Given $\mathbf{b} \in L^2((0,T); L^2(\Omega))$, $\mathbf{u}_0 \in \mathcal{S}(\Omega)$, and $\mathbf{v}_0 \in L^2(\Omega)$, the system

$$\begin{cases} \rho(\mathbf{x})\mathbf{u}_{tt}(\mathbf{x},t) = \mathcal{P}\,\mathbf{u}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t), \quad \forall(\mathbf{x},t) \in \Omega \times (0,T) \\ \mathbf{u}_t(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \\ \mathbf{u}(\cdot,t) \in \mathcal{S}(\Omega), \quad \forall t \in [0,T]. \end{cases}$$

has a unique solution.

- PD model introduces a length scale, the horizon δ.
- For linear PD models, $\mathbb{C}(\boldsymbol{\xi}) = \mathbb{C}(\boldsymbol{\xi};\delta)$
- For linear isotropic, microelastic materials by comparison with classical parameters $\lambda(\boldsymbol{\xi};\delta) = \frac{1}{\delta^{d+4}}\tilde{\lambda}(\frac{|\boldsymbol{\xi}|}{\delta})$, and $F_0(\boldsymbol{\xi};\delta) = \frac{1}{\delta^d}\tilde{F}_0(\frac{|\boldsymbol{\xi}|}{\delta})$ where

$$\int_{B(0,1)} \tilde{\lambda}(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^4 d\boldsymbol{\xi} = 1, \quad \tilde{F}_0 \in L^1(B(0,1))$$

- What happens when this length scale $\delta \to 0$? Intuitively, we should expect to get the classical linearized elasticity. Let us verify that.

Vanishing nonlocality

Fix λ a nonnegative radial function such that

$$Supp(\lambda) \subset B(0,1), \text{and } \int_{B(0,1)} \lambda(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^4 d\boldsymbol{\xi} = 1.$$

We denote

$$\lambda_\delta(|\boldsymbol{\xi}|) = \frac{1}{\delta^{d+4}} \lambda\left(\frac{|\boldsymbol{\xi}|}{\delta}\right)$$

so that $Supp(\lambda_\delta) \subset B(0,\delta)$ and $\int_{B(0,\delta)} \lambda_\delta(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^4 d\boldsymbol{\xi} = 1$.
Finally denote:

$$F_0^\delta = \frac{1}{\delta^d} F_0\left(\frac{|\boldsymbol{\xi}|}{\delta}\right)$$

Let $\mathbf{b} \in L^2(\Omega;\mathbb{R}^d)$ be given. Let $\mathbf{u}_\delta$ be the solution to the equilibrium system

$$\begin{cases} -\mathcal{P}_\delta\, \mathbf{u} = \mathbf{b} \\ \mathbf{u} \in V_0(\Omega_\delta) \end{cases}$$

Question: what happens when $\delta \to 0$?

- Answer: $\mathbf{u}_\delta \to \mathbf{u}$ strongly in $L^2(\Omega; \mathbb{R}^d)$ where $\mathbf{u}$ solves the Navier system

$$\begin{cases} -\mu\Delta\mathbf{u}(\mathbf{x}) - 2\mu\nabla\text{div}\,\mathbf{u}(\mathbf{x}) = \mathbf{b}(\mathbf{x}) \quad a.e \quad \mathbf{x} \in \Omega \\ \mathbf{u}(\mathbf{x}) = 0 \quad \text{on } \partial\Omega. \end{cases}$$

- The same happens if $F_0 \equiv 0$, implying that the macroscopic equation, which is the Navier system, does not see small disturbances, while the PD equation indeed detects the effect of the presence of F_0^δ.
- Observe that the PDE is a Navier system with Poisson ratio $\frac{1}{4}$.

Lemma

For all $\mathbf{w} \in C_c^\infty(\Omega; \mathbb{R}^d)$, *for all* $\mathbf{x} \in \Omega$, *and all* $\lambda \in \mathbb{R}$ *we have*

$$-(\mathcal{L}^\delta + \mathcal{F}^\delta)\mathbf{w}(\mathbf{x}) \longrightarrow -\mathcal{L}_0\mathbf{w}(\mathbf{x}), \quad \text{as } \delta \to 0,$$

where $-\mathcal{L}_0$ *is the (local) Navier operator*

$$-\mathcal{L}_0\mathbf{w}(\mathbf{x}) = -\mu\Delta\mathbf{w}(\mathbf{x}) - 2\mu\nabla \mathrm{div}\mathbf{w}(\mathbf{x}), \quad (\mu = \frac{\omega_d}{d+2}, \omega_\delta = |B_1(0)|).$$

Moreover, there exists a constant $C = C(d, \mathbf{w})$ such that

$$\sup_{\delta>0} \sup_{x\in\Omega} |(\mathcal{L}^\delta + \lambda\mathcal{F}^\delta)\mathbf{w}(\mathbf{x})| \leq C.$$

Improved Poincaré-type inequality

Lemma

There exists δ_0 *and* $C(\delta_0)$ *such that for all* $\delta \in (0, \delta_0]$, *such that for all* $\mathbf{u} \in V_0(\Omega_\delta)$

$$\|\mathbf{u}\|_{L^2}^2 \leq C(\delta_0) \int_\Omega \int_\Omega \lambda^\delta(|\mathbf{x}' - \mathbf{x}|)|(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^2 d\mathbf{x}' d\mathbf{x}.$$

If $\mathbf{u}_\delta$ solves

$$\begin{cases} -\mathcal{P}_\delta\,\mathbf{u} = \mathbf{b} \\ \qquad \mathbf{u} \in V_0(\Omega_\delta) \end{cases},$$

turns out that

$$\sup_\delta \int_\Omega \int_\Omega \lambda^\delta(|\mathbf{x}' - \mathbf{x}|)|(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^2 d\mathbf{x}' d\mathbf{x} < \infty$$

and using the improved Poincaré inequality that

$$\sup_\delta \|\mathbf{u}_\delta\|_{L^2} < \infty.$$

Applying a compactness result: $\mathbf{u}_\delta$ is precompact in L^2 with any limit point being in $W_0^{1,2}(\Omega; \mathbb{R}^d)$.

proof

Follows from

$$\mathbf{u}_\delta \to \mathbf{u} \quad \text{strongly in } L^2$$

and

$$-\mathcal{P}^\delta \mathbf{w} \to -\mu\Delta\mathbf{w}(\mathbf{x}) - 2\mu\nabla div\mathbf{w}(\mathbf{x}) \quad forall \mathbf{w} \in C_0^\infty(\Omega; \mathbb{R}^d.)$$

Summary

- Investigated the solvability of the linear bond-based peridynamic model.
- Studied the associated function space.
- Studied the associated operator.
- demonstrated that the classical Navier equation (with Poisson ration $1/4$) can be approximated by a nonlocal system.

Thank you for your attention.