Analysis and approximations of nonlocal diffusion and peridynamic models

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Joint work with
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Outline

- Recap of model equations:
  peridynamics (Silling 2000), nonlocal diffusion, linear models

- Some elements of mathematics analysis:
  nonlocal calculus,
  volume constrained problems,
  local limit

- Recent results on numerical approximations:
  FD/FV/FE, Galerkin projection/collocation,
  convergence-conditioning,
  a priori/posteriori error analysis,
  adaptive FEM
Related works

- **Simulations/experiments with PD**
  Silling/Askari (2005): meshfree methods
  Parks/Lehoucq/Plimpton/Silling (2008): PD implemented via MD
  Seleson/Parks/Gunzburger/Lehoucq (2009): PD as upscaling of MD
  Chen/Gunzburger (2010): finite element for PD
  Parks/Seleson (2011): role of influence function

- **Mathematical Analysis of PD**
  Emmerich/Weckner 2006,2007: $L^p$ theory
  Lehoucq/Silling 2005, 2007: connection with Elasticity
  Du/Zhou 2010: $L^2$ and $H^s$ theory, periodic boundary
  Alali/Lipton 2010: $L^p$ theory/homogenization
  Lipton/Mengesha 2011: homogeneization
  Aksolyu/Mengesha 2011: $L^2$ theory
  Mengesha/Du 2012: sign-changing kernel, bond-based system

- **Numerical analysis of PD/nonlocal diffusion**
  Zhou/Du 2011: FEM convergence/conditioning/error estimates
  Aksoylu/Parks 2011: domain decomposition
  Aksolyu/Mengesha 2010: conditioning, spectral analysis
  Du/Gunzburger/Lehoucq/Zhou 2012: FEM convergence
  Du/Ju/Tian/Zhou 2011: a posteriori error estimation
  Du/Tian/Zhao 2012: convergence of Adaptive FEM
Recap of model equations

We only consider linear models: for displacement field $u$,

$$\rho \ddot{u}(x, t) = \int_{H_x} C(x', x)(u(x', t) - u(x, t)) \, dx' + b(x, t)$$

$\rho$: material density,

$b$: external body force,

$C(x', x)$: micromodulus function.

$\delta$: material horizon,

$H_x = B_\delta(x)$: family of $x$

(Silling 2000, 2007)

Mathematical studies of some nonlinear/nonlocal models can be found in
(Du/Kamm/Lehoucq/Parks 2011 SIAM Appl Math)
Three linear models

\[ \rho \ddot{u}(x, t) = \int_{H_x} C(x', x)(u(x', t) - u(x, t)) \, dx' + b(x, t) \]

- PD bond based model: force only depends the particles that form the bond: \( C(x, x') = K_1(x, x') \) where \( K_1(x, x') \) represents direct interaction between \( x \) and \( x' \)

- PD state based model: force is also determined through intermediate particles, \( C(x, x') = K_1(x, x') + C_0(x, x') \) where \( C_0(x, x') \) represents indirect interaction

- Scalar nonlocal PD/diffusion model: \( u(x) \) and \( C(x, x') \) are scalar fields, with \( C(x, x') \) possibly changing sign
Some analytical issues being addressed

- Well-posedness of linear variational problems:
  Scalar model and bond-based system on bounded domain, linear state-based peridynamic Navier equation

- Well-posedness of linear dynamic equations:
  Initial and constrained volume value problems

- Local limits of the nonlocal models (as $\delta \to 0$):
  convergence in distribution sense for smooth functions, convergence in $L^2$ for weak solution

- Key: a nonlocal calculus, providing a unifying framework; completeness/separability/compactness of energy space
Some algorithmic/numerical issues being addressed

- Finite difference/Finite Volume/Finite Element: Galerkin projection/collocation/quadrature
  - Similarities and differences
  - Discretization with maximum principle
  - Convergence/Error analysis for fixed horizon
  - Convergence/divergence for fixed horizon/mesh size ratio

- Finite element methods:
  - Convergence of conforming methods
  - A priori error analysis in energy space
  - Residual based a posteriori error estimator: reliability and efficiency for integrable kernel
  - For mildly singular non-integrable kernel: reliability and convergence of AFEM
References

Bond-based PD operator

A bond-based PD operator $L_b$ defined as, $\forall \mathbf{x} \in \Omega_s \subset \mathbb{R}^n$ (solution domain)

$$L_b \mathbf{u}(\mathbf{x}) = \int_{\Omega_s \cup \Omega_I} \mathbb{C}(\mathbf{x}, \mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \, d\mathbf{y}$$

$$= \int_{\Omega_s \cup \Omega_I} \omega(|\mathbf{y} - \mathbf{x}|) \alpha(\mathbf{y}, \mathbf{x}) \otimes \alpha(\mathbf{y}, \mathbf{x})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \, d\mathbf{y}.$$ 

$\omega(\cdot)$: influence function; $\Omega_I$: interaction domain; $\alpha(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}$.

Reformulation:

$\boxed{L_b \mathbf{u} = -\mathcal{D}(\omega \mathcal{G}^* \mathbf{u})}$ for symmetric $\omega(\mathbf{x}, \mathbf{y}) = \omega(\mathbf{y}, \mathbf{x})$

with nonlocal divergence and its dual (adjoint) defined by, $\forall \mathbf{x} \in \Omega_s$,

$$(\mathcal{D} \Psi)(\mathbf{x}) = \int_{\mathbb{R}^n} (\Psi(\mathbf{x}, \mathbf{y}) + \Psi(\mathbf{y}, \mathbf{x})) \cdot \alpha(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}, \quad \forall \Psi,$$

$$(\mathcal{D}^* \mathbf{v})(\mathbf{x}, \mathbf{y}) = - (\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x})) \otimes \alpha(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{v},$$

and $\mathcal{G}^* = Tr(\mathcal{D}^*)$, for anti-symmetric vector field $\alpha(\mathbf{x}, \mathbf{y}) = -\alpha(\mathbf{y}, \mathbf{x})$.
Nonlocal vector calculus

- $\mathcal{D}, \mathcal{D}^*, \mathcal{G}^*$ along with other nonlocal gradient/curl are part of a nonlocal vector calculus recently developed in Du/Gunzburger/Lehoucq/Zhou 2011 (to appear in $M^3$AS, 2013)

- Motivation of the new calculus: to represent nonlocal balance laws

- An axiomatic approach to define nonlocal quantities and relations

- Nonlocal operators like $\mathcal{D}/\mathcal{D}^*$ map from 1-point to 2-point functions or vice-versa, in the spirit of maps between 0th and 1st order forms

- Past studies on nonlocal operators (nonlocal gradient, nonlocal mean, nonlocal graph Laplacian, ...) focus mostly on scalar fields, our vector and tensor versions systematically generalize existing works.

Eg. For scalar $u$, consider the nonlocal diffusion operator $\mathcal{L}_d$

$$\mathcal{L}_d u (x) = \int_{\Omega_s \cup \Omega_l} \omega(x,y)(u(y) - u(x)) \, dy.$$ 

Reformulation:

$$\mathcal{L}_d u = -\mathcal{D}(\omega \mathcal{D}^* u) \quad \forall x \in \Omega_s$$

Du/Gunzburger/Lehoucq/Zhou 2012, *SIAM Rev*
Nonlocal diffusion, variational problem

Consider the nonlocal operator $\mathcal{N}_d$,

$$\mathcal{N}_d u (x) = \int_{\Omega_S \cup \Omega_I} \omega(x, y) (u(x) - u(y)) \, dy, \quad \forall x \in \Omega_I.$$

Nonlocal (generalized) Green’s identity

$$\int_{\Omega_S} v \mathcal{D}(\omega \mathcal{D}^*(u)) \, dx - \int_{\Omega_S \cup \Omega_I} \int_{\Omega_S \cup \Omega_I} \omega \mathcal{D}^*(v) \cdot \mathcal{D}^*(u) \, dy \, dx = \int_{\Omega_I} v \mathcal{N}_d(u) \, dx.$$

Volume Constrained Problems

<table>
<thead>
<tr>
<th>NL-Dirichlet</th>
<th>NL-Neumann</th>
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<tbody>
<tr>
<td>$-\mathcal{L}_d u = f$ in $\Omega_S$</td>
<td>$-\mathcal{L}_d u = f$ in $\Omega_S$</td>
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<tr>
<td>$u = h$ in $\Omega_I$</td>
<td>$\mathcal{N}_d u = g$ in $\Omega_I$</td>
</tr>
</tbody>
</table>

Du/Gunzburger/Lehoucq/Zhou 2012, *SIAM Rev*
Assumptions

To study variational problems, we make assumptions that

- $\Omega_S$: solution domain, bounded with interior cone condition
- $\Omega_I$: set of $y \notin \Omega$ with $\omega(x - y) \neq 0$ for some $x \in \Omega$
- $\Omega_C$: domain of constraints
- $\Omega = (\bar{\Omega}_S \cup \bar{\Omega}_I)^o$, a connected set

For influence function $\omega(x, y) = \omega(y, x)$, we typically have

- $\int_{B_\delta(x)} |y - x|^2 \omega(x, y) \, dy < \infty$
  (a necessary/sufficient condition for finite elastic moduli/diffusion coefficient)
- $\omega(x, y) \geq 0$, $\forall y \in B_\delta(x)$, and $\omega(x, y) = 0$ otherwise
- $\omega(x, y) \geq \omega_0 > 0$, $\forall y \in B_{\delta/2}(x)$
  (may be relaxed to include negative parts, see Mengesha-Du)
Nonlocal diffusion: variational formulation

For $-\mathcal{L}_d u = \mathcal{D}(\omega \mathcal{D}^* u) = f$, possibly with a volume constraint

Energy space:
$$V = \{u \in L^2(\Omega), \sqrt{\omega} \mathcal{D}^* u \in L^2(\Omega^2)\}$$

Bilinear form:
$$B(u, v) = \int_\Omega \int_\Omega \omega \mathcal{D}^* u \mathcal{D}^* v \, dy \, dx,$$

Weak form: find $u \in V_c$ (constrained subspace of $V$)
$$B(u, v) = \int_\Omega fv \, dx, \quad \forall v \in V_c$$

Kernel space: $Z = \{u \in V, B(u, u) = 0\}$.

Under suitable conditions on $\omega$ and $V_c$, nonlocal Poincare’s inequalities can be established, thus, $B(\cdot, \cdot)$ gives a well-defined inner product on $V_c$ or $V/Z$, which leads to well-posedness of nonlocal constrained value problems.
Well-posedness

For $V_c$ that is continuously imbedded in $L^2(\Omega)$ and $V_c \cap Z = \{0\}$, the variational problem has a unique solution in $V_c$ if there is a nonlocal Poincare's inequality.

Examples:

1) $V = L^2(\Omega)$ if $\omega \in L^2(\Omega^2)$ or $\omega(x, y) = \omega(x - y) \in L^1(\Omega)$.
   (the case of bounded $L^d$ in $L^2$ with no elliptic smoothing)

2) $V = H^s(\Omega)$ for $s \in (0, 1)$, if there are constants $\gamma^*, \gamma_* > 0$, $\forall x \in \mathbb{R}^n$,
   \[
   \omega(x, y) \leq \frac{\gamma^*}{|y - x|^{n+2s}}, \quad \forall y \in B_\delta(x), \quad \omega(x, y) \geq \frac{\gamma_*}{|y - x|^{n+2s}}, \quad \forall y \in B_{\frac{\delta}{2}}(x).
   \]
   (unbounded $L^d$ in $L^2$, elliptic smoothing with order $2s$) or for $\omega(x, y) = \omega(x - y)$,
   \[
   \delta^{-2} \int_{B_\delta(0)} |x|^2 \omega(|x|) dx \to \infty, \quad \text{as} \quad \delta \to 0
   \]
   ( more details are presented in Mengesha-Du)
State-based PD model

Strain energy of a linear PD solid (Silling 2007): for deformation state \( \mathbf{Y} \),

\[
W(\mathbf{Y}) = \frac{\kappa}{2} \vartheta^2 + \frac{\eta}{2} \int_{\mathbb{R}^3} \left( \mathbf{e} \langle \xi \rangle - \frac{\vartheta |\xi|}{3} \right)^2 d\xi.
\]

\( \mathbf{e} \): extension state; \( \vartheta \): dilatation; \( \varpi \): influence function; \( \kappa \) and \( \eta \): bulk/shear moduli.

Define, for any point function \( \mathbf{U} \) and a 2-point function \( \mathbf{u} \), weighted operators:

\[
D_\omega (\mathbf{U})(\mathbf{x}) = D(\omega \mathbf{U}(\mathbf{x}))(\mathbf{x}) \quad \text{and} \quad D^*_\omega (\mathbf{u})(\mathbf{x}) = \int_{\mathbb{R}^3} D^*(\mathbf{u})(\mathbf{x}, \mathbf{y}) \omega d\mathbf{y}, \quad \forall \mathbf{x} \in \mathbb{R}^3.
\]

The operator \( D_\omega \) resembles nonlocal divergence of point functions, indeed, it converges to the conventional divergence in the local limit. These \textit{weighted} nonlocal operators are presented in \textit{Du/Gunzburger/Lehoucq/Zhou 2013 M^3AS}.

\( D_\omega \) and \( D^*_\omega \) can be used to reformulate a linearized PD state model.
Peridynamic Navier equation

Taking 2nd order approximation of $W(Y)$ w.r.t. the displacement field $u$:

$$
\tilde{W}(u) = \frac{\kappa}{2} \left( \text{Tr}(\mathcal{D}_\omega^* u) \right)^2 + \frac{\eta}{2} \int_{\mathbb{R}^3} \omega(x, y) \left( \text{Tr}(\mathcal{D}_T u) - \frac{\text{Tr}(\mathcal{D}_\omega^* u)}{3} |y - x| \right)^2 \, dy.
$$

Consider:

$$
\min \int_{\Omega} \tilde{W}(u) \, dx - \int_{\Omega_S} u \cdot b \, dx, \quad \text{subject to } u = h_b, \quad \text{in } \Omega_I.
$$

$\Rightarrow$ PD Navier equation

$$
\begin{cases}
\eta \mathcal{D}(\omega(D_T^*(u))^T) + \sigma \mathcal{D}_\omega(\text{Tr}(\mathcal{D}_\omega^* u)) I = b, & \text{in } \Omega_s, \\
u = h_d, & \text{in } \Omega_c.
\end{cases}
$$

• Well-posedness in energy space $V$: nonlocal Korn’s inequality

Example: $V = L^2(\Omega)$ if $\int_{\Omega} \omega^2(x, y) \, dy \leq M, \quad \forall x \in \Omega$.

• Local limit:

$$
\eta \mathcal{D}(\omega(D^*_T))^T + \sigma \mathcal{D}_\omega(\text{Tr}(\mathcal{D}_\omega^* u)) \rightarrow \mu \nabla \cdot \nabla + (\mu + \lambda) \nabla \nabla.
$$

(Du/Gunzburger/Lehoucq/Zhou 2012 J. Elasticity)
Finite element approximations

Variational formulations of PD and nonlocal models allow systematic analysis of Galerkin type numerical approximations

Some examples:

  If $V \equiv H^s_p$ with $s \in [0, 1)$, we have the error order estimate for the finite element approximation with piecewise polynomials of order $m$,

$$
\|u - u_h\|_{H^s} \leq C(\delta) h^{m+1-s} \|b\|_{m+1-2s} \quad \text{if } b \in H^{m+1-2s}.
$$

And the stiffness matrix condition number estimate,

$$
\text{cond}(A^0) \leq C_1(\delta) h^{-2s}, \quad (\delta : \text{horizon}, \ h : \text{mesh size}).
$$

For some common kernels, with $V \equiv L^2$, $\text{cond}(A^0) \leq c \min\{\delta^{-2}, h^{-2}\}$.

- Similar results for finite element approximations of volume constrained nonlocal diffusion equation (Du/Gunzburger/Lehoucq/Zhou 2012 SIAM Rev)
Adaptive FEM for nonlocal models

Nonlocal models potentially allow solutions that lack sufficient regularity, thus it is crucial to develop effective adaptive numerical solution algorithms:

\[
\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}
\]

Adaptive algorithm generates a sequence of nested mesh \( \mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{T}_2 \subset \ldots \)

Sequence of adaptive meshes

(Tian/Chen/Du 2011)

Robust error estimator: there are generic positive constants \( \{C_i\}_{i=1}^2 \):

\[
C_1 \eta^h \leq \|u - u_h\| \leq C_2 \eta^h.
\]

A popular type of estimator: residual type a posteriori error estimator
A residual-type a posteriori error estimator

For finite element solution $u_h$ of the nonlocal diffusion problem,
\[
\begin{cases}
-\mathcal{L}_d u = b, & x \in \Omega_s, \\
u = 0 & x \in \Omega_I.
\end{cases}
\]

Let $u_h$ be a Galerkin FE solution, error $e_h = u - u_h$, residual $R_h = b + \mathcal{L}_d u_h$.

Lemma
\[
\|e_h\|_V = \|R_h\|_{V^*}
\]
$V^*$: dual of $V_c$.

- True for 2nd order elliptic PDEs, but $H^{-1}$ norms not easily computable.
- Meanwhile, for nonlocal $\mathcal{L}_d$ with an integrable kernel, $V \equiv L^2(\Omega) \equiv V^*$.

Theorem [Du/Ju/Tian/Zhou 2011]
- For $\eta_h = M(\delta)\|R_h\|_{L^2}$ ($M(\delta)$: a constant determined by influence function and horizon $\delta$), is a reliable and efficient a posteriori error estimator: for some positive constants $C_1$ and $C_2$, independent of $\delta$ and $h$, $C_1 \eta_h \leq \|e_h\|_V \leq C_2 \eta_h$.
- For any given mesh, as $\delta \to 0$, the nonlocal residual $R_h$ converges weakly to its local counterpart (element-wise residual and flux jump across element.)
Convergence of AFEM

For $\mathcal{L}_d$ with a singular influence function such that $V \equiv H^s$ with $s \in (0, 1/2)$, we have

**Theorem** [Du/Tian/Zhao 2012] \[ \eta_h^\delta = h^s M(\delta) \| R_h \|_{L^2} \] is a reliable estimator: for a constant $C_3$ independent of $\delta$ and $h$, $\| e_h \|_V \leq C_3 \eta_h^\delta$.

**Adaptive algorithm**: pick marking parameter $\theta \in (0, 1]$, initial mesh $\mathcal{T}_0$, $k = 0$.

1. Solve for discrete solution $u_k$ over the mesh $\mathcal{T}_k$;
2. Estimate the error $\eta_k|T$ for each element $T \in \mathcal{T}_k$;
3. Mark a set $\mathcal{M}_k$ of $\mathcal{T}_k$ with minimal cardinality such that \[ \sum_{T \in \mathcal{M}_k} \eta_k^2|T \geq \theta \sum_{T \in \mathcal{T}_k} \eta_k^2|T \];
4. Refine $\mathcal{M}_k$ by a shape regular and nested local refinement to get $\mathcal{T}_{k+1}$;
5. Set $k := k + 1$ and go to step (1) until an error tolerance is met.

**Theorem** [Du/Tian/Zhao 2012] Let $\{ \mathcal{T}_k, u_k, \eta_k \}$ be the sequence of meshes, discrete solutions, and error estimators produced by the above algorithm, then there exists constants $\gamma > 0$ and $0 < \rho < 1$, such that

\[
\| u - u_{k+1} \|_V^2 + \gamma \eta_{k+1}^2 \leq \rho (\| u - u_k \|_V^2 + \gamma \eta_k^2).
\]

In short, the AFEM reduces error at each level of refinement and is convergent.
Numerical Experiments

Example: a 1-d nonlocal diffusion equation with an exact solution having a discontinuity inside mesh element, using discontinuous linear on uniformly refined meshes.

Figure: Exact solution and numerical solution.
Numerical Experiments

\[ \delta = 2 \]

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<th>( | e^h |_2 )</th>
<th>CR</th>
<th>( | e^h |_V )</th>
<th>CR</th>
<th>( \eta_\delta^h )</th>
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\[ \delta = 0.2 \]

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\[ \delta = 0.02 \]

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</table>

Table: Convergence results for different mesh sizes and \( \delta \)'s (CR: convergence rate)
Numerical Experiments

Example: a 2D nonlocal diffusion model with a solution having discontinuities along a line inside elements, discontinuous linear with uniform refinement

Figure: Exact error (left) and estimated error (right) ($\delta = 0.1$, $20 \times 20$ mesh).
Numerical Experiments

<table>
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<th>$\delta = 0.2$</th>
<th>[ | e^h |_2 ]</th>
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<th>[ \eta^h_\delta ]</th>
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Table: Convergence results for different mesh sizes and $\delta$’s (CR: convergence rate)
Numerical Experiments

Example: a 1-d nonlocal diffusion model with a discontinuous exact solution using discontinuous piecewise linear on uniformly/adaptively refined grids.

\[ \omega(x, y) = \delta^{-3/2} |x - y|^{-3/2} \Rightarrow V \equiv H^s \text{ with } s = 1/4. \]

Observed optimal rate of convergence:

\[ N^{-1/2+s} = N^{-1/4} \text{ uniform refinement vs. } N^{-2+s} = N^{-7/4} \text{ adaptive refinement.} \]

Figure: Convergence rate with uniform (left) and adaptive (right) refinements.
Numerical Experiments

Example: a 2-d nonlocal diffusion model with a discontinuous exact solution using discontinuous piecewise constant on uniformly/adaptively refined grids.

\[ \omega(x, y) = \delta^{-3/2} |x - y|^{-5/2} \Rightarrow V \equiv H^s \text{ with } s = 1/4. \]

Figure: Adaptively refined mesh and numerical solution.

Observed optimal rate of convergence:

\[ N^{-1/2 + s} = N^{-1/4} \text{ uniform refinement vs. } N^{-1 + s} = N^{-3/4} \text{ adaptive refinement.} \]
Comments on numerical methods for fixed $r = \delta/h$

As an illustration: 1d nonlocal model with a box kernel (Tian-Du 2012)

$$\mathcal{L}_b u(x) = \sigma_\delta \int_{-\delta}^{\delta} (u(x + s) - u(x)) \, ds$$

Patch test implies $c^2 = \sigma_\delta \delta^3 / 3$. Equivalently,

$$\mathcal{L}_b u(x) = \sigma_\delta \int_{0}^{\delta} (u(x + s) - 2u(x) + u(x - s)) \, ds$$

Finite difference/collocation, uniform mesh, $\alpha \in [0, 2]$

$$\mathcal{L}_b^\alpha u(x_k) = \sigma_\delta \sum_{j=1}^{r} \left( \frac{u(x_{k+j}) - 2u(x_k) + u(x_{k-j})}{(jh)^\alpha} \right) \int_{0}^{\delta} s^\alpha \, ds$$

Convergent schemes for $\delta$ fixed, $h \to 0$ (thus $r \to \infty$), that is good!

**Warning**: exercise caution for fixed $r$, $h \to 0$ (thus $\delta = rh \to 0$)

Eg. $r = 1$, convergent(!) but to a wrong equation unless $\alpha = 2$

$$\mathcal{L}^\alpha_b u(x_k) \to u''(x_k) \frac{\sigma_\delta \delta^3}{\alpha + 1} = \frac{3}{\alpha + 1} c^2 u''(x_k)$$
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Mathematical properties of the bond-based and state-based PD models, as well as nonlocal diffusion equations, are analyzed via a nonlocal vector calculus framework.

Error order, condition number, a posteriori error estimators and convergent adaptive algorithms are developed for some linear peridynamic and nonlocal problems.

Connections and differences between nonlocal and local models, variational and collocation methods are explored.

There are still huge gaps between practice and theory (linear vs. nonlinear, fixed or increasing horizon/mesh ratio, ...).
Comment: motivation for new math concepts

- Widely studied linear advection
\[ u_t + u_x = 0 \]
well-defined characteristics, finite speed of propagation,…

- Less studied linear nonlocal advection
\[ u_t + \int \rho(x, y)u(y)dy = 0 \]
(for a nonlinear model, see Du-Kamm-Lehoucq-Parks 2011 SIAP)

- Yet, there is a popular discrete analog (or, numerical scheme):
\[ u_t(x, t) + (u(x + h, t) - u(x, h))/h = 0 \]
For h small, nonzero, speed of propagation = \(\infty\) (classical sense)!
But a dominant finite traveling speed exists \(\Rightarrow\) new (broader) definition.

Thank you!