Results on the well-posedness of nonlinear peridynamics

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Joint work with Etienne Emmrich.

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Equation of motion

Classic equation in linear elasticity

$$\rho(\boldsymbol{x})\boldsymbol{u}_{tt}(\boldsymbol{x},t) = (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}(\boldsymbol{x},t) + \mu\,\mathrm{div}\,\mathrm{grad}\,\boldsymbol{u}(\boldsymbol{x},t) + \boldsymbol{b}(\boldsymbol{x},t)$$

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$\boldsymbol{u} \in \mathcal{C}^2$ in space vs. fracture (= discontinuity in space)

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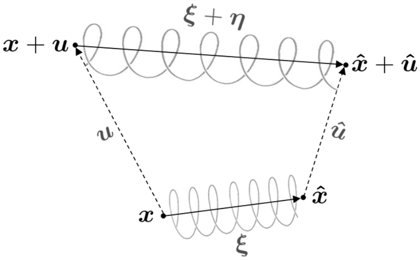
Solution: Silling 2000

$$\rho(\boldsymbol{x})\boldsymbol{u}_{tt}(\boldsymbol{x},t) = \int_{B_{\mathbb{R}^d}(\boldsymbol{x};\delta)\cap\Omega} \boldsymbol{f}(\hat{\boldsymbol{x}}-\boldsymbol{x}, \boldsymbol{u}(\hat{\boldsymbol{x}},t)-\boldsymbol{u}(\boldsymbol{x},t))d\hat{\boldsymbol{x}} + \boldsymbol{b}(\boldsymbol{x},t)$$

Notations

We set

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t), \quad \hat{\boldsymbol{u}} = \boldsymbol{u}(\hat{\boldsymbol{x}}, t)$$
$$\boldsymbol{\xi} = \hat{\boldsymbol{x}} - \boldsymbol{x},$$
$$\boldsymbol{\eta} = \hat{\boldsymbol{u}} - \boldsymbol{u},$$
$$\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \boldsymbol{f}(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{u}(\hat{\boldsymbol{x}}, t) - \boldsymbol{u}(\boldsymbol{x}, t)).$$



Physics gives us

1. $\boldsymbol{f}(-\boldsymbol{\xi}, -\boldsymbol{\eta}) = -\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta})$,
2. $(\boldsymbol{\xi} + \boldsymbol{\eta}) \times \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \boldsymbol{0}$,
3. $\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \Phi(\boldsymbol{\xi}, \boldsymbol{\eta})(\boldsymbol{\xi} + \boldsymbol{\eta})$.

Examples

A general class of pairwise force functions is

$$\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\eta}) = \Phi(|\boldsymbol{\xi}|, |\boldsymbol{\xi}+\boldsymbol{\eta}|)(\boldsymbol{\xi}+\boldsymbol{\eta}).$$

Bondstretch model:

$$\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\eta}) = c_{d,\delta} s(\boldsymbol{\xi},\boldsymbol{\eta}) \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{|\boldsymbol{\xi}+\boldsymbol{\eta}|}, \quad s(\boldsymbol{\xi},\boldsymbol{\eta}) = \frac{|\boldsymbol{\xi}+\boldsymbol{\eta}| - |\boldsymbol{\xi}|}{|\boldsymbol{\xi}|}, \quad c_{d,\delta} \sim \delta^{-(d+1)}$$

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Further examples:

$$\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\eta}) = c_{d,\delta}\big(|\boldsymbol{\xi}+\boldsymbol{\eta}| - |\boldsymbol{\xi}|\big)^2(\boldsymbol{\xi}+\boldsymbol{\eta})$$

$$\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\eta}) = a_{d,\delta}(|\boldsymbol{\xi}|)\left(|\boldsymbol{\xi}+\boldsymbol{\eta}|^2 - |\boldsymbol{\xi}|^2\right)(\boldsymbol{\xi}+\boldsymbol{\eta})$$

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Bondbreaking realized by multiplying $\boldsymbol{f}$ with

$$\mu(\boldsymbol{\xi},\boldsymbol{\eta},t) = \begin{cases} 1 & \text{if } s(\boldsymbol{\xi}, \boldsymbol{u}(\hat{\boldsymbol{x}},\tau) - \boldsymbol{u}(\boldsymbol{x},\tau)) \leq s_0 \ \ \forall \tau \leq t, \\ 0 & \text{else.} \end{cases}$$

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H. A. Erbay, A. Erkip & G. M. Muslu

$$\begin{cases} u_{tt}(x,t) = \int_{\mathbb{R}} \alpha(\hat{x}-x)g(u(\hat{x},t)-u(x,t))d\hat{x}, \quad x \in \mathbb{R},\ t>0, \\ u(x,0) = u_0, \quad u_t(x,0) = \dot{u}_0, \quad x \in \mathbb{R}. \end{cases}$$

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Identify $u : \mathbb{R} \to X$ by $[u(t)](x) := u(x,t)$.
Fixed-point for integral equation provides well-posedness for

Space X	Assumption
$\mathcal{C}_b(\mathbb{R})$	$\alpha \in L^1(\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R})$ with $g(0)=0$
$L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$	$\alpha \in L^1(\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R})$ with $g(0)=0$
$\mathcal{C}_b^1(\mathbb{R})$	$\alpha \in L^1(\mathbb{R})$ and $g \in \mathcal{C}^2(\mathbb{R})$ with $g(0)=0$
$W^{1,p}(\mathbb{R})$	$\alpha \in L^1(\mathbb{R})$ and $g \in \mathcal{C}^2(\mathbb{R})$ with $g(0)=0$
$W^{\sigma,2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ $\sigma > 0$	$\alpha \in L^1(\mathbb{R})$ and $g(\eta) = \eta^3$

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- Proofs: $u(t) = u_0 + \int_{t_0}^{t} g(\tau, u(\tau))d\tau$, Banach/Schauder

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We are looking at

$$u''(t) = g(t, u(t)), \quad t \in (0, T), \quad u(t_0) = u_0, \quad u'(t_0) = \dot{u}_0$$

in the infinite-dimensional Banach space X.

Ansatz

$$u''(t) = g(t, u(t)), \quad t \in (0, T), \quad u(t_0) = u_0, \quad u'(t_0) = \dot{u}_0$$

is equivalent to the fixed-point problem $u = Su$ with $S : \mathcal{A} \to \mathcal{A}$ and

$$(Sv)(t) = u_0 + (t - t_0)\dot{u}_0 + \int_{t_0}^{t} (t - \tau) g(\tau, v(\tau)) d\tau.$$

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Theorem (Local existence)

• $g : [0, T] \times \bar{B}_X(u_0; r) \to X$ *compact*
$\Rightarrow$ $\exists$ *solution* $u \in \mathcal{C}^2(I, X)$.

Proof would not work with reduction of order!

Well-posedness

Theorem (Local existence and uniqueness)

- $g : [0, T] \times \bar{B}_X(u_0; r) \to X$ *continuous and Lipschitz, i.e.*

$$\exists L > 0 \; \forall t \in [0, T], \; v, w \in \bar{B}_X(u_0; r)$$

$$\|g(t, v) - g(t, w)\| \leq L \|v - w\|$$

$\Rightarrow$ $\exists!$ *solution* $u \in \mathcal{C}^2(I, X)$.

Continuous dependence on the initial values

For a solution with $v_0 \in B_X(u_0; r)$, $\dot{v}_0 \in X$ there holds

$$\|u(t) - v(t)\| \leq e^{La^2/2} \left(\|u_0 - v_0\| + a \|\dot{u}_0 - \dot{v}_0\| \right)$$

Global theorems

Theorem (Global existence and uniqueness I)

- $g : J \times D \to X$ *continuous and locally Lipschitz on* D,
- g *locally bounded*

$\Rightarrow$ $\exists!$ *maximal solution* $u \in \mathcal{C}^2(J_{max}, X)$

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Theorem (Global existence and uniqueness II)

- $g : \mathbb{R} \times X \to X$ *continuous and locally Lipschitz on* X,
- $\|g(t, u(t))\| \leq M$

$\Rightarrow$ $\exists!$ *global solution* $u \in \mathcal{C}^2(\mathbb{R}, X)$

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Theorem (Global existence and uniqueness III)

- $g : [0, T] \times X \to X$ *continuous and Lipschitz*

$\Rightarrow$ $\exists!$ *global solution* $u \in \mathcal{C}^2([0, T], X)$

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Ansatz

$$\begin{cases} \rho(\boldsymbol{x})\boldsymbol{y}_{tt}(\boldsymbol{x},t) = \displaystyle\int_{B_{\mathbb{R}^d}(\boldsymbol{x};\delta)\cap\Omega} \boldsymbol{f}(\hat{\boldsymbol{x}}-\boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}},t)-\boldsymbol{y}(\boldsymbol{x},t))d\hat{\boldsymbol{x}} + \boldsymbol{b}(\boldsymbol{x},t) \\ \qquad =: (K\boldsymbol{y})(\boldsymbol{x},t) + \boldsymbol{b}(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega\times(0,T), \\ \boldsymbol{y}(\boldsymbol{x},0) = \boldsymbol{y_0}(\boldsymbol{x}), \quad \boldsymbol{y}_t(\boldsymbol{x},0) = \dot{\boldsymbol{y}}_{\boldsymbol{0}}(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega, \end{cases}$$

with $\boldsymbol{y} : \bar{\Omega}\times[0,T] \to \mathbb{R}^d, \quad \boldsymbol{y}(\boldsymbol{x},t) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x},t)$.

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with $\boldsymbol{y} : \bar{\Omega} \times [0,T] \to \mathbb{R}^d, \quad \boldsymbol{y}(\boldsymbol{x},t) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x},t)$.
Now identify $\boldsymbol{y} : [0,T] \to X$ by $[\boldsymbol{y}(t)](\boldsymbol{x}) := \boldsymbol{y}(\boldsymbol{x},t)$ and define

$$\boldsymbol{g} : [0,T] \times \bar{B}_X(\boldsymbol{y_0};r) \to X, \quad \boldsymbol{g}(t,\boldsymbol{v}) := \frac{K\boldsymbol{v}}{\rho} + \frac{\boldsymbol{b}(t)}{\rho},$$

then the nonlinear peridynamic initial value problem is rewritten as

$$\begin{cases} \boldsymbol{y}''(t) = \boldsymbol{g}(t,\boldsymbol{y}(t)) \\ \boldsymbol{y}(0) = \boldsymbol{y_0},\ \boldsymbol{y}'(0) = \dot{\boldsymbol{y}}_{\boldsymbol{0}} \end{cases}$$

Solutions $\boldsymbol{y} : [0,T] \to \mathcal{C}(\bar{\Omega})^d$

Theorem ($X = \mathcal{C}(\bar{\Omega})^d$)

• $\boldsymbol{f} : \bar{B}_{\mathbb{R}^d}(\mathbf{0};\delta) \times \bar{B}_{\mathbb{R}^d}(\mathbf{0};R) \to \mathbb{R}^d$ *continuous, Lipschitz i.e.*

$$\exists L_f \in L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta)) : |\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}_1) - \boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}_2)| \leq L_f(\boldsymbol{\xi})\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|$$

• $\boldsymbol{b} \in \mathcal{C}([0,T],X)$, $1/\rho \in \mathcal{C}(\bar{\Omega})$, $\boldsymbol{y_0}, \dot{\boldsymbol{y}}_{\mathbf{0}} \in X$
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Examples:

$$\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}) = c_{d,\delta}\left(|\boldsymbol{\zeta}| - |\boldsymbol{\xi}|\right)^2 \boldsymbol{\zeta}, \quad \boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}) = a_{d,\delta}\left(|\boldsymbol{\xi}|\right)\left(|\boldsymbol{\zeta}|^2 - |\boldsymbol{\xi}|^2\right)\boldsymbol{\zeta}$$

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Theorem ($X = \mathcal{C}(\bar{\Omega})^d$)

- $\boldsymbol{f} : \bar{B}_{\mathbb{R}^d}(\mathbf{0};\delta) \times \mathbb{R}^d \to \mathbb{R}^d$ *continuous, locally Lipschitz*
- $\boldsymbol{b} \in \mathcal{C}([0,\infty),X)$ *mit* $\|\boldsymbol{b}\|_{\mathcal{C}([0,\infty),X)} < \infty$, $1/\rho \in \mathcal{C}(\bar{\Omega})$, $\boldsymbol{y_0}, \dot{\boldsymbol{y}}_{\boldsymbol{0}} \in X$

$\Rightarrow$ $\exists!$ *maximal solution* $\boldsymbol{y} \in \mathcal{C}^2([0,T^*),X)$.

Solutions $\boldsymbol{y} : [0,T] \to L^p(\Omega)^d$

Theorem ($X = L^\infty(\Omega)^d$)

- $\boldsymbol{f} : \bar{B}_{\mathbb{R}^d}(\mathbf{0};\delta) \times \bar{B}_{\mathbb{R}^d}(\mathbf{0};R) \to \mathbb{R}^d$ *measurable, Lipschitz*
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Problem in L^p: no estimate of $|\boldsymbol{y_0}(\boldsymbol{x})| \leq C \, \|\boldsymbol{y_0}\|_{L^p(\Omega)^d}$, a.e.

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Theorem ($X = L^p(\Omega)^d$ für $1 \leq p \leq \infty$)

- $\boldsymbol{f} : B_{\mathbb{R}^d}(\mathbf{0};\delta) \times \mathbb{R}^d \to \mathbb{R}^d$ *measurable, Lipschitz*
- $\boldsymbol{b} \in \mathcal{C}([0,T],X)$, $1/\rho \in L^\infty(\Omega)$, $\boldsymbol{f}(\cdot,\mathbf{0}) \in L^1(B_{\mathbb{R}^d}(\mathbf{0};\delta))^d$, $\boldsymbol{y_0}, \dot{\boldsymbol{y}}_{\mathbf{0}} \in X$

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Compactness?

$\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = \Phi(|\boldsymbol{\xi}|, |\boldsymbol{\zeta}|)\boldsymbol{\zeta} \quad \Rightarrow \quad K$ not compact in general

$$
\begin{aligned}
(Kv)(x) &= \int_{x-\delta}^{x+\delta} \Phi(|\hat{x}-x|, |v(\hat{x})-v(x)|)\big(v(\hat{x})-v(x)\big)d\hat{x} \\
&= \int_{x-\delta}^{x+\delta} \Phi(|\hat{x}-x|, |v(\hat{x})-v(x)|)v(\hat{x})d\hat{x} \\
&\quad - \int_{x-\delta}^{x+\delta} \Phi(|\hat{x}-x|, |v(\hat{x})-v(x)|)d\hat{x}\, v(x) \\
&=: (Av)(x) - (Bv)(x)
\end{aligned}
$$

$f(\hat{x} - x, v(\hat{x}) - v(x), t)$

f continuous in time

- we can use the theorems proven in last section
- well-posedness if Lipschitz property is uniform in time, i.e.

$$\left| f(\xi, \zeta, t) - f(\xi, \tilde{\zeta}, t) \right| \leq L_f(\xi) \left| \zeta - \tilde{\zeta} \right|$$

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f measurable in time

- we need new Carathéodory theorems (done)
- well-posedness if

$$\left|\boldsymbol{f}(\boldsymbol{\xi}, \zeta, t) - \boldsymbol{f}(\boldsymbol{\xi}, \tilde{\zeta}, t)\right| \leq L_f(\boldsymbol{\xi}, t) \left|\zeta - \tilde{\zeta}\right|$$

Fracture modeling

Breaking bonds realized by multiplying f with

$$\mu(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}([0,t])) = \begin{cases} 1 & s(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}}, \tau) - \boldsymbol{y}(\boldsymbol{x}, \tau)) \leq s_0 \text{ for all } \tau \leq t, \\ 0 & \text{else.} \end{cases}$$

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Therefore we need theory for

$$u''(t) = g(t, u([t-T, t])) \text{ for } t \in (0, T)$$

with initial conditions

$$u(t) = \Phi(t) \text{ for } t \in [-T, 0], \quad u'(0) = \dot{u}_0.$$

Examples are

$$u''(t) = -u(t) + \int_{-T}^{0} h(t, \tau, u(t+\tau)) d\tau \quad \text{or} \quad u''(t) = \max_{t-T \leq \tau \leq t} u(\tau).$$

A nonlocal result

Theorem

Let $g : [0,T] \times \mathcal{C}([-T,0];X) \to X$ *be continuous and Lipschitz, i.e.* $\forall\, K > 0\ \exists\, L > 0$ *s.t.* $\forall\, t \in [0,T]$ *and* $v, w \in \mathcal{C}([-T,0];X)$ *with* $\|v\|_{\mathcal{C}}, \|w\|_{\mathcal{C}} \leq K$ *holds*

$$\|g(t,v) - g(t,w)\|_X \leq L\,\|v - w\|_{\mathcal{C}([-T,0];X)}\,.$$

$\Rightarrow\ \exists!$ *solution* $\boldsymbol{y} \in \mathcal{C}([-T,a],X) \cap \mathcal{C}^2((0,a],X), \quad a \in (0,T].$

Proof: find a fixed-point $u = Su$ for $S : \mathcal{A} \to \mathcal{A}$ with

$$\mathcal{A} := \bar{B}_{C([-T,a];X)}(\bar{\Phi}; r)$$

and

$$Sv(t) := \begin{cases} \Phi(t) & t \in [-T,0], \\ \Phi(0) + t\dot{u}_0 + \int_0^t (t-s)g(s,v_s)ds & t \in [0,a]. \end{cases}$$

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1st idea:

$$\mu(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}) = \operatorname{sgn}^+ \left(\min_{-T+t \leq \tau \leq t} \left(s_0 - s(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}}, \tau) - \boldsymbol{y}(\boldsymbol{x}, \tau)) \right) \right)$$

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Hard to show Lipschitz property of the minimum.
2nd idea:

$$\mu(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}) = 1 - \operatorname{sgn}^+ \left(\int_{t-T}^{t} \operatorname{sgn}^+ \left((s(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}}, \tau) - \boldsymbol{y}(\boldsymbol{x}, \tau)) - s_0 \right) d\tau \right)$$

sgn^+ is not continuous. BUT regularization makes it Lipschitz. Then there exists a unique solution (with fracture) for a very short time interval.

Contents

Example: $\boldsymbol{f}_{\delta}(\boldsymbol{\xi},\zeta) = \delta^{-(d+1)}\big(|\zeta| - |\boldsymbol{\xi}|\big)^2\zeta$

Its elastic energy

$$W_{\delta} = \frac{1}{2}\int_{B(\boldsymbol{x};\delta)} w_{\delta}(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x}))\, d\hat{\boldsymbol{x}}$$

with $w_{\delta}(\boldsymbol{\xi},\zeta) = \delta^{-(d+1)}\int^{|\zeta|}\big(z - |\boldsymbol{\xi}|\big)^2 z dz$ converges for $\delta \to 0$ towards

$$W_0 = \begin{cases} 3\,|\nabla \boldsymbol{y}|^4 - 8\,|\nabla \boldsymbol{y}|^2 - 4(\det \nabla \boldsymbol{y})^2 & d = 2, \\ 3\,|\nabla \boldsymbol{y}|^4 - 10\,|\nabla \boldsymbol{y}|^2 - 4\,|\mathrm{cof}\, \nabla \boldsymbol{y}|^2 & d = 3. \end{cases}$$

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Thus, the *limit* PDE is

$$\boldsymbol{y}_{tt}(\boldsymbol{x},t) = \operatorname{div} \Pi(\nabla \boldsymbol{y}(\boldsymbol{x},t)) + \boldsymbol{b}(\boldsymbol{x},t)$$

with the 1st Piola-Kirchhoff stress tensor

$$\boldsymbol{\Pi} = \frac{\partial W_0}{\partial(\nabla \boldsymbol{y})}.$$

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With $\boldsymbol{C} = (\nabla\boldsymbol{y})^T\nabla\boldsymbol{y}$ the 2nd Piola-Kirchhoff tensor is given by

$$\boldsymbol{P} = 2\frac{\partial W_0}{\partial \boldsymbol{C}} = \begin{cases} 2\boldsymbol{C} + \operatorname{tr}(\boldsymbol{C}) - 4 & d = 2, \\ 2\boldsymbol{C} + \operatorname{tr}(\boldsymbol{C}) - 5 & d = 3. \end{cases}$$

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In terms of the Green deformation tensor $\boldsymbol{X} = \frac{1}{2}(\boldsymbol{C} - \mathbf{1})$ it holds

$$\boldsymbol{P} = 4\boldsymbol{X} + 2\operatorname{tr}(\boldsymbol{X})$$

thus, $\boldsymbol{f}$ describes a Saint Venant–Kirchhoff material.

Open questions

- existence for fracture solutions
- limit of vanishing nonlocality (in prep.)
- existence theory for bondstretch model

Thanks.