Weierstrass Institute for
Applied Analysis and Stochastics

A vanishing viscosity approach in fracture mechanics

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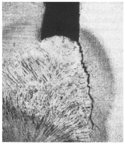
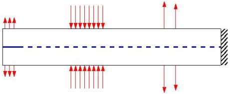
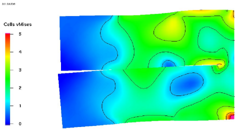
Nonlocal Models and Peridynamics, Nov. 5–7, 2012, TU Berlin

Introduction







Goal: Derive a rate independent model for crack propagation based on the Griffith criterion.

$$\mathrm{D}_u\mathcal{E}(u(t), s(t)) = \ell(t), \quad 0 \in \partial\mathcal{R}(\dot{s}(t)) + \mathrm{D}_s\mathcal{E}(u(t), s(t)).$$

Questions: The energy $\mathcal{E}$ is not convex in s $\Rightarrow$ The evolution might be dicontinuous.

Suitable jump criteria?

Hyperelstic material with polyconvex energy density:
$\mathrm{D}_s\mathcal{E}$ well defined?

Convergence of fully discretized models?

Contents

1 Fracture model and vanishing viscosity solutions (finite strains)

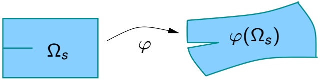
2 FE-approximation of vanishing viscosity solutions (small strains)

3 Numerical example

4 Summary

Energies and notation (2D)

$\varphi : \Omega_s \to \mathbb{R}^2$ deformation field
$W : \mathbb{R}^{2\times 2} \to [0, \infty]$ elastic energy density



Assumptions: $F = \nabla\varphi \in \mathbb{R}^{2\times 2}$

- $W(F) = \infty$ if $\det F \leq 0$ + coercivity.
- polyconvexity: $W(F) = g(F, \det F)$, g convex and lower semicontinuous.
- multiplicative stress control: $\forall F \in \mathbb{R}^{2\times 2}_+$: $\left|F^\top DW(F)\right| \leq c_1(W(F) + 1)$

Example: $W(F) = c_1\,|F|^2 + c_2(\det F)^2 - c_3 \log(\det F)$.

Admissible deformations $V(\Omega_s) = \{\, \varphi \in W^{1,p}(\Omega_s)\,;\; \varphi\big|_{\Gamma_D} = \varphi_D \,\}$

Elastic energy $\mathcal{E}(t, \varphi, s) = \int_{\Omega_s} W(\nabla\varphi)\,\mathrm{d}x - \int_{\Gamma_N} h(t)\cdot\varphi\,\mathrm{d}a$

Reduced energy $\mathcal{I}(t, s) = \inf_{\varphi\in V(\Omega_s)} \mathcal{E}(t, s, \varphi)$

Ball'77: Minimizers exist (not necessarily unique!)

Griffith criterion (1921)

The crack is stationary, if the (locally) released elastic energy is less than the energy dissipated to create the new crack surface.

Energy release rate:

$$\mathcal{G}(t,s) = -\partial_s \mathcal{I}(t,s) = -\partial_s \big(\min_{\varphi \in V(\Omega_s)} \mathcal{E}(t,s,\varphi) \big)$$

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Dissipated energy:

$$\mathcal{R}(s_{\mathrm{new}} - s_{\mathrm{old}}) = \begin{cases} \kappa(s_{\mathrm{new}} - s_{\mathrm{old}}) & \text{if } s_{\mathrm{new}} \geq s_{\mathrm{old}} \\ \infty & \text{else} \end{cases} .$$

$\kappa > 0$ fracture toughness

Evolution criterion:

$$\left.\begin{array}{ll} \text{local stability:} & \kappa \geq -\partial_s \mathcal{I}(t,s(t)), \\ \text{complementarity:} & \dot{s}(t)\big(\kappa + \partial_s \mathcal{I}(t,s(t))\big) = 0 \end{array}\right\} \Leftrightarrow \boxed{0 \in \partial \mathcal{R}(\dot{s}(t)) + \partial_s \mathcal{I}(t,s(t))}$$

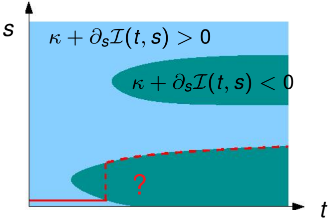
Problems: (a) Discontinuous solutions may occur. (b) $\partial_s \mathcal{I}$ well defined?

Why discontinuous solutions?

Evolution law:

Local stability: $\kappa \geq -\partial_s \mathcal{I}(t, s(t))$,

Complementarity: $\dot{s}(t)\big(\kappa + \partial_s \mathcal{I}(t, s(t))\big) = 0$.



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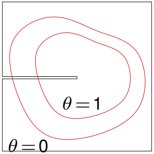
Formulation allowing for discontinuous solutions?

- Model based on global minimization of the total energy $\mathcal{I} + \mathcal{R}$
  $\rightarrow$ Global energetic formulation for rate independent processes (Mielke/Theil'99)
  $\rightarrow$ Francfort/Marigo-Model (arbitrary cracks), shape memory alloys, finite strain elastoplasticity, damage and delamination models,...
- Model based on viscous approximations
  General theory: Mielke/Efendiev '06, Rossi/Mielke/Savaré '08-12, Mielke/Zelik'10
  Cracks and damage: Lazzaroni/Toader 11, K./Mielke/Zanini 08-10, K./Schröder 10-12, K./Rossi/Zanini 12,

Energy release rate in the polyconvex case

Griffith formula with Eshelby tensor:

$$G(s,\varphi) := \int_{\Omega_s} \left(\nabla\varphi^\top DW(\nabla\varphi) - W(\nabla\varphi)\mathbb{I}\right) : (\mathbf{e}_1 \otimes \nabla\theta_s)\,\mathrm{d}x$$



Theorem (K./Mielke/Zanini)

$\mathcal{I}(\cdot,\cdot) \in C^{lip}([0,T]\times(0,L))$ *and*

$$\partial_s^+\mathcal{I}(t,s) = \min\{\,-G(s,\varphi)\,;\ \varphi \textit{ minimizes } \mathcal{E}(t,s,\cdot)\,\} \textit{ is lower semicontinuous,}$$

$$\partial_s^-\mathcal{I}(t,s) = \max\{\,-G(s,\varphi)\,;\ \varphi \textit{ minimizes } \mathcal{E}(t,s,\cdot)\,\} \textit{ is upper semicontinuous.}$$

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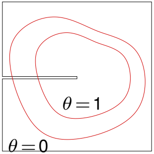

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Remarks:

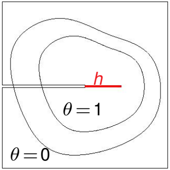
$[\partial_s^+\mathcal{I}(t,s), \partial_s^-\mathcal{I}(t,s)] \ =$ Clarke-differential $\partial_s^{\mathrm{Cl}}\mathcal{I}$ of the mapping $s \mapsto \mathcal{I}(t,s)$.

$\partial_s^{\mathrm{Cl}}\mathcal{I}(t,\cdot) : (0,L) \to \mathcal{P}(\mathbb{R})$ is upper semicontinuous as a set-valued mapping.

Conclusion:

If for all t, s the minimizers are unique, then $\mathcal{I} \in C^1([0,T]\times(0,L))$.

Comment on the proof



For $h > 0$ define a family of inner variations via

$$T_h : \Omega_s \to \Omega_{s+h}, \quad x \mapsto x + h\theta(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \tfrac{1}{h}\big(\mathcal{E}(t, s+h, \varphi_s \circ T_h^{-1}) - \mathcal{E}(t, s, \varphi_s)\big) \\ \geq \tfrac{1}{h}\big(\mathcal{I}(t, s+h) - \mathcal{I}(t, s)\big) \\ \geq \tfrac{1}{h}\big(\mathcal{E}(t, s+h, \varphi_{s+h}) - \mathcal{E}(t, s, \varphi_{s+h} \circ T_h)\big) \end{aligned}$$

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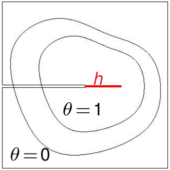

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$$\geq \tfrac{1}{h}\big(\mathcal{I}(t, s+h) - \mathcal{I}(t, s)\big)$$

$$\geq \tfrac{1}{h}\big(\mathcal{E}(t, s+h, \varphi_{s+h}) - \mathcal{E}(t, s, \varphi_{s+h} \circ T_h)\big)$$

↑

abstract convergence principle:

$\psi_n \rightharpoonup \psi$, $E(s, \psi_n) \to E(s, \psi)$, then $\partial_s E(s, \psi_n) \to \partial_s E(s, \psi)$

$$\Rightarrow \partial_s^+ \mathcal{I}(\sigma) \geq \min\{\, -G(s, \varphi_s)\,;\ \varphi_s \text{ minimizes } \mathcal{E}(t, s, \cdot)\,\}.$$

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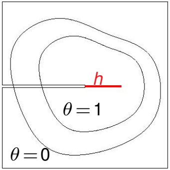

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abstract convergence principle:

$\psi_n \rightharpoonup \psi$, $E(s,\psi_n) \to E(s,\psi)$, then $\partial_s E(s,\psi_n) \to \partial_s E(s,\psi)$

$$\Rightarrow \partial_s^+ \mathcal{I}(\sigma) \geq \min\{\, -G(s,\varphi_s)\,;\ \varphi_s \text{ minimizes } \mathcal{E}(t,s,\cdot)\,\}.$$

$$\stackrel{h\to 0}{\to} -\int_{\Omega_s} \mathbb{E}(\nabla\varphi_s) : \nabla(\mathbf{e}_1 \otimes \nabla\theta_s)\,\mathrm{d}x = -G(s,\varphi_s)$$

Since φ_s was an arbitrary minimizer we may take the infimum:
$\min\{\, -G(s,\varphi_s)\,;\ \varphi_s \text{ minimizes } \mathcal{E}(t,s,\cdot)\,\} \geq \partial_s^+ \mathcal{I}(\sigma)$.

Time incremental, viscous evolution model

τ time step size, s_τ^i crack length at time $t_i = i\tau$, $\nu > 0$ viscosity

Time incremental minimization with viscosity term

Find $s_\tau^i \geq s_\tau^{i-1}$ such that

$$s_\tau^i \in \mathrm{Argmin}\left\{ \mathcal{I}(t_i, \sigma) + \tau \mathcal{R}\left(\frac{\sigma - s_\tau^{i-1}}{\tau} \right) + \frac{\nu}{2\tau} \left| \sigma - s_\tau^{i-1} \right|^2 ; \, \sigma \geq s_\tau^{i-1} \right\}.$$

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Minimizers satisfy

(a) local stability: $\kappa + \partial_s\mathcal{I}^+(t_i, s_\tau^i) + \frac{\nu}{\tau}(s_\tau^i - s_\tau^{i-1}) \geq 0$

(b) complementarity: $\Big(\kappa + \partial_s\mathcal{I}^+(t_i, s_\tau^i) + \nu\frac{s_\tau^i - s_\tau^{i-1}}{\tau}\Big)\frac{s_\tau^i - s_\tau^{i-1}}{\tau} = 0.$

or equivalently $0 \in \partial_s^+\mathcal{I}(t_i, s_\tau^i) + \partial\mathcal{R}\Big(\frac{s_\tau^i - s_\tau^{i-1}}{\tau}\Big) + \nu\frac{s_\tau^i - s_\tau^{i-1}}{\tau}.$

Estimates: $\sup_\tau \sqrt{\nu}\,\|\hat{s}_\tau'\|_{L^2(0,T)} < \infty, \quad \|\overline{s}_\tau - \hat{s}_\tau\|_{L^\infty(0,T)} \leq c(\tau/\nu)^{\frac{1}{2}}.$

Problem: $\partial_s\mathcal{I}^+$ is lower semicontinuous, only. (Recall: $\partial_s\mathcal{I}^+ \leq \partial_s\mathcal{I}^-$).

Theorem (K./Mielke/Zanini)

Let $\frac{\tau}{\nu} \to 0$. *There exists* $s \in BV([0,T],\mathbb{R})$ *and a subsequence* $\tau \searrow 0$ *with*

$$\hat{s}_\tau \overset{*}{\rightharpoonup} s \quad \text{in } BV([0,T],\mathbb{R}) \quad \text{and} \quad \hat{s}_\tau(t) \to s(t) \;\; \text{for every } t \in [0,T].$$

Moreover,

- **(a)** s *is non-decreasing,*
- **(b$^-$)** $\kappa + \partial_s \mathcal{I}^-(t, s(t)) \geq 0$ *for every* $t \in [0,T] \backslash J(s)$,
- **(c$^+$)** *if* $\kappa + \partial_s \mathcal{I}^+(t, s(t)) > 0$, *then* $t \in D(s)$ *and* $\dot{s}(t) = 0$,
- **(d$^+$)** $\forall\, t \in J(s), \forall s_* \in [s(t_-), s(t_+)]$ *we have* $\kappa + \partial_s \mathcal{I}^+(t, s_*) \leq 0$.

$J(s)$ jump set of s; $D(s)$ set of differentiable points of s.

Proof: A-priori estimates, Helley selection principle, continuity property of $\partial_s^{\mathrm{Cl}}\mathcal{I}$, change of variables ($s \leftrightarrow t$) to obtain (d$^+$).

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(d^+) $\forall\, t \in J(s), \forall s_* \in [s(t_-), s(t_+)]$ *we have* $\kappa + \partial_s \mathcal{I}^+(t,s_*) \leq 0$.

$J(s)$ jump set of s; $D(s)$ set of differentiable points of s.

Complementarity condition:
For every $t \in D(s)$ exists $g(t) \in [\partial_s^+ \mathcal{I}(t,s(t)), \partial_s^- \mathcal{I}(t,s(t))]$ with

$$\big(\kappa + g(t)\big)\dot{s}(t) = 0, \qquad 0 \in \partial\mathcal{R}(\dot{s}(t)) + g(t).$$

Proof of the jump condition (d)

Let $t_* \in J(s)$, $\quad s_0, s_1 \in (s(t_*-), s(t_*+))$ arbitrary

Continuity of $\hat{s}_\tau$: $\quad \exists\, t_\tau^0 < t_\tau^1$ with $t_\tau^{0,1} \overset{\tau\to 0}{\to} t_*$ and $\hat{s}_\tau(t_\tau^{0,1}) = s_{0,1}$.

Complementarity condition: $\quad \forall \psi \in L^2([s_0, s_1])$, $\psi \geq 0$ we have

$$0 \geq \int_{t_\tau^0}^{t_\tau^1} \Big(\kappa + \partial_s \mathcal{I}^+(\bar{t}_\tau(t), \bar{s}_\tau(t))\Big)\, \psi(\hat{s}_\tau(t))\, \hat{s}_\tau'(t)\, \mathrm{d}t.$$

Change of variables: $\quad \sigma = \hat{s}_\tau(t), \quad \tilde{t}_\tau(\sigma) = \min\{\, t \in [t_\tau^0, t_\tau^1]\,;\, \hat{s}_\tau(t) = \sigma\,\}$

$$0 \geq \int_{s_0}^{s_1} \Big(\kappa + \partial_s \mathcal{I}^+(\bar{t}_\tau(\tilde{t}_\tau(\sigma)), \bar{s}_\tau(\tilde{t}_\tau(\sigma)))\Big)\, \psi(\sigma)\, \mathrm{d}\sigma.$$

Note: $\quad \bar{t}_\tau(\tilde{t}_\tau(\sigma)) \to t_*, \quad \left|\bar{s}_\tau(\tilde{t}_\tau(\sigma)) - \sigma\right| = \left|\bar{s}_\tau(\tilde{t}_\tau(\sigma)) - \hat{s}_\tau(\tilde{t}_\tau(\sigma))\right| \leq c\sqrt{\tau/\nu} \to 0.$

By lsc. of $\partial_s^+ \mathcal{I}$ we conclude that $\quad \boxed{\forall \sigma \in [s_0, s_1] : \quad 0 \geq \kappa + \partial_s \mathcal{I}(t_*, \sigma).}$

Theorem (K./Mielke/Zanini)

There exists a nondecreasing $s \in BV([0,T],\mathbb{R})$ *satisfying*

(b⁺) $\kappa + \partial_s^+\mathcal{I}(t, s(t)) \geq 0$ *for every* $t \in [0,T] \backslash J(s)$,

(c⁺) *if* $\kappa + \partial_s^+\mathcal{I}(t, s(t)) > 0$, *then* $t \in D(s)$ *and* $\dot{s}(t) = 0$,

(d⁺) $\forall\, t \in J(s), \forall s_* \in [s(t_-), s(t_+)]$ *we have* $\kappa + \partial_s^+\mathcal{I}(t, s_*) \leq 0$.

Moreover, there exists a measurable map $\varphi : [0,T] \to V(\Omega_L)$ *such that for all* t $\varphi(t)$ *minimizes* $\mathcal{E}(t,\cdot,s(t))$ *and* $\partial_s^+\mathcal{I}(t,s(t)) = -G(s(t),\varphi(t))$.

Proof: $s_{\max}(t) = \max\{\, s(t)\,;\ s \text{ local energetic sol.}\,\}$ satisfies (a),(b^+)–(d^+).

Conclusion: For every $t \in D(s)$ the complementarity condition is satisfied:

$$\big(\kappa + \partial_s^+\mathcal{I}(t,s(t))\big)\dot{s}(t) = 0 \quad \text{and} \quad 0 \in \partial\mathcal{R}(\dot{s}(t)) + \partial_s^+\mathcal{I}(t,s(t)).$$

Open question: viscous approximation $\Rightarrow$ vanishing viscosity solutions with (b^+)?

Theorem (K./Mielke/Zanini)

Every special solution (i.e. with (b$^+$)) satisfies for every $t_0 < t_1$, $t_i \notin J(s)$,

$$\mathcal{I}(t_1, s(t_1)) + \int_{s(t_0)}^{s(t_1)} \kappa(\sigma)\mathrm{d}\sigma + \sum_{\hat{t}\in J(s)\cap(t_0,t_1)} \int_{s(\hat{t}-)}^{s(\hat{t}+)} -(\kappa(\sigma) + \partial_s^+\mathcal{I} + (\hat{t},\sigma))\mathrm{d}\sigma$$

$$= \mathcal{I}(t_0, s(t_0)) + \int_{t_0}^{t_1} \partial_t^- \mathcal{I}(t, s(t))\mathrm{d}\tau.$$

Proof: Switch to a parameterized formulation of the evolution problem, use a chain rule and (a), (b$^+$)–(d$^+$).

Vanishing viscosity solutions

(a) s nondecreasing,

(b) $\kappa + \partial_s \mathcal{I}(t, s(t)) \geq 0$ for all $t \in [0, T] \backslash J(s)$,

(c) if $\kappa + \partial_s \mathcal{I}(t, s(t)) > 0$, then $t \in D(s)$ and $\dot{s}(t) = 0$,

(d) $\forall\, t \in J(s)$, $\forall s_* \in [s(t_-), s(t_+)]$ it holds
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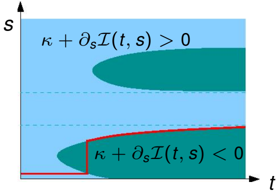

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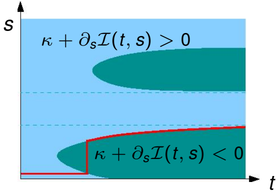


Global energetic solutions

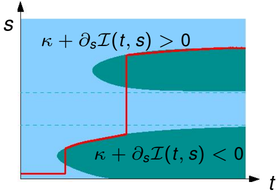
(S) **Global** stability

$$\mathcal{I}(t, s(t)) \leq \mathcal{I}(t, \tilde{s}) + \int_{s(t)}^{\tilde{s}} \kappa \,\mathrm{d}\sigma \qquad \forall \tilde{s} \geq s(t),$$

(E) Energy equality

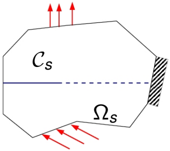
$$\mathcal{I}(t, s(t)) + \int_{s(0)}^{s(t)} \kappa \,\mathrm{d}\sigma = \mathcal{I}(0, s(0)) + \int_0^t \partial_t \mathcal{I}(\tau, s(\tau))\mathrm{d}\tau.$$

Jump condition: $\int_{s_-}^{s_+} \big(\partial_s \mathcal{I}(t, \sigma) + \kappa\big)\, \mathrm{d}\sigma = 0$



2 FE-approximation of vanishing viscosity solutions (small strains)

Assumptions:



- Small strain elasticity:

  $u:\Omega_s\to\mathbb{R}^2$ displacements, $\varepsilon(u)$ linearized strains

- Elastic energy density:

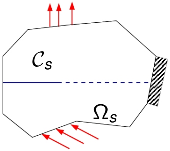
  $W(\varepsilon(u))=\frac{1}{2}\mathbf{C}\varepsilon(u):\varepsilon(u)$, $\mathbf{C}$ elasticity tensor.

- Non-interpenetration conditions on the crack $\mathcal{C}_s$:

$$K(\Omega_s)=\{\,v\in H^1(\Omega_s)\,;\;v\big|_{\Gamma_D}=0,\;\;[\![v]\!]\cdot\vec{n}\geq 0 \text{ on } \mathcal{C}_s\,\}$$

Notation in the small strain setting

Assumptions:



- Small strain elasticity:

  $u : \Omega_s \to \mathbb{R}^2$ displacements, $\varepsilon(u)$ linearized strains

- Elastic energy density:

  $W(\varepsilon(u)) = \frac{1}{2}\mathbf{C}\varepsilon(u) : \varepsilon(u)$, $\mathbf{C}$ elasticity tensor.

- Non-interpenetration conditions on the crack $\mathcal{C}_s$:

$$K(\Omega_s) = \{\, v \in H^1(\Omega_s)\,;\ v\big|_{\Gamma_D} = 0,\ \ [\![v]\!] \cdot \vec{n} \geq 0 \text{ on } \mathcal{C}_s \,\}$$

Deformation energy

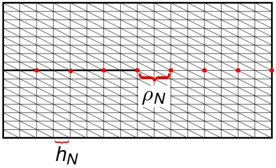
$$\mathcal{E}(t,s,v) = \int_{\Omega_s} W(\varepsilon(v))\,\mathrm{d}x - \int_{\Gamma_N} \ell(t) \cdot v\,\mathrm{d}a \quad \text{for } v \in K(\Omega_s).$$

$$u(t,s) = \operatorname{argmin}_{K(\Omega_s)} \mathcal{E}(t,s,\cdot) \qquad \text{unique minimizer.}$$

Reduced energy $\quad \mathcal{I}(t,s) = \inf_{v \in K(\Omega_s)} \mathcal{E}(t,s,v) = \mathcal{E}(t,s,u(t,s)).$

Minimizers unique $\Longrightarrow$ $\mathcal{I} \in C^1([0,T] \times (0,L)),\ \partial_s \mathcal{I}(t,\cdot) \in C^{\mathrm{Lip}}_{\mathrm{loc}}(0,L)$

Fully discretized viscous minimization problem

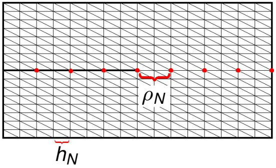


Parameters:

τ_N	time step size	ν_N	viscosity
h_N	mesh size	ρ_N	crack increment

Spaces: $K_s^N \subset K(\Omega_s)$ finite element space + contact conditions on $\mathcal{C}_s$
$Z^N = \{\sigma_N^1, \ldots, \sigma_N^{M_N}\}$ discrete crack lengths of distance ρ_N

Fully discretized viscous minimization problem



Parameters:

τ_N time step size

h_N mesh size

ν_N viscosity

ρ_N crack increment

Spaces: $K_s^N \subset K(\Omega_s)$ finite element space + contact conditions on $\mathcal{C}_s$

$Z^N = \{\sigma_N^1, \dots, \sigma_N^{M_N}\}$ discrete crack lengths of distance ρ_N

Fully discretized incremental minimization problem

$$\mathcal{I}_N(t,s) = \min\{\, \mathcal{E}(t,v,s)\,;\ v \in K_s^N \,\}$$

$$s_N^k \in \operatorname{Argmin}\{\, \mathcal{I}_N(t_N^k, \tilde{s}) + \tau_N \mathcal{R}_{\nu_N}\Big(\frac{\tilde{s} - s_N^{k-1}}{\tau_N}\Big)\,;\ \tilde{s} \in Z^N,\ \tilde{s} \geq s_N^{k-1}\,\}.$$

$\mathcal{R}_\nu(\eta) = \frac{\nu}{2}\eta^2 + \kappa\eta.$

Note: $\mathcal{I}_N(t,\cdot)$ piecewise constant in s.

Local stability and complementarity conditions

Incremental minimization problem:

$$s_N^k \in \operatorname{Argmin}\{\, \mathcal{I}_N(t_N^k, \tilde{s}) + \tau_N \mathcal{R}_{\nu_N}\Big(\frac{\tilde{s} - s_N^{k-1}}{\tau_N}\Big)\,;\ \tilde{s} \in Z^N,\ \tilde{s} \geq s_N^{k-1}\,\}.$$

The choice $\tilde{s} = s_N^k \pm \rho_N \in Z^N$ leads to

(a) local stability:

$$0 \leq \kappa + \nu_N \tfrac{s_N^k - s_N^{k-1}}{\tau_N} + \tfrac{1}{\rho_N}\left(\mathcal{I}_N(t_N, s_N^k + \rho_N) - \mathcal{I}_N(t_N, s_N^k)\right) + \tfrac{\nu_N \rho_N}{\tau_N}.$$

(b) complementarity condition:

$$\Big(\kappa + \nu_N \tfrac{s_N^k - s_N^{k-1}}{\tau_N} + \tfrac{1}{\rho_N}\left(\mathcal{I}_N(t_N^k, s_N^k) - \mathcal{I}_N(t_N^k, s_N^k - \rho_N)\right) - \tfrac{\nu_N \rho_N}{2\tau_N}\Big)\, \tfrac{s_N^k - s_N^{k-1}}{\rho_N} \leq 0.$$

Estimates: $\sup_N \sqrt{\nu_N}\, \|\hat{s}_N'\|_{L^2(0,T)} < \infty, \qquad \|\overline{s}_N - \hat{s}_N\|_{L^\infty(0,T)} \leq c(\tau_N/\nu_N)^{1/2}.$

$\overline{s}_N$ piecewise constant, $\hat{s}_N$ piecewise linear interpolation.

Assumption

A1 $\forall \delta, \mu > 0 \; \exists N_{\delta,\mu} \in \mathbb{N}$ such that $\forall N \geq N_{\delta,\mu}$, $s \in Z^N \cap [\delta, L-\delta]$ it holds

$$\left| \frac{1}{\rho_N} \big(\mathcal{I}_N(t, s+\rho_N) - \mathcal{I}_N(t,s) \big) - \partial_s \mathcal{I}(t,s) \right| \leq \mu.$$

Theorem (K./Schröder 10)

Assume **A1** *and that* $\frac{\rho\nu}{\tau} \to 0, \frac{\tau}{\nu} \to 0$.
Then there exits a vanishing viscosity solution $s \in BV([0,T])$ *and a subsequence with* $\hat{s}_N \stackrel{*}{\rightharpoonup} s$ *in* $BV([0,T])$ *and pointwise for all* t.
Moreover, $\overline{u}_N(t) \to u(t)$ *strongly in* $H^1(\Omega_L)$*, where* $u(t) = \operatorname{argmin} \mathcal{E}(t, \cdot, s(t))$.

Note: All BV-weak$*$-cluster points are vanishing viscosity solutions.

Questions: Sufficient conditions for **A1**?
Relation between mesh size h_N and crack increment ρ_N?
$\longrightarrow$ regularity of displacement fields

Regularity of minimizers with contact condition

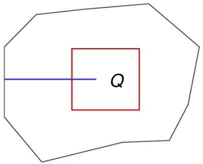
For $v \in K(\Omega_s) = \{ v \in H^1_{\Gamma_D}(\Omega_s)\,;\ [v] \cdot \mathbf{n} \geq 0 \text{ on } \mathcal{C}_s \}$:

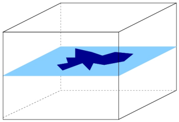
$$\mathcal{E}(v, s) = \int_{\Omega_s} \tfrac{1}{2} \mathbb{C}\varepsilon(v) : \varepsilon(v)\,\mathrm{d}x - \int_{\Gamma_N} \ell(t) \cdot v\,\mathrm{d}a$$

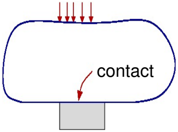
Theorem (K. 10)

Let be $u \in K(\Omega_s)$ a minimizer of $\mathcal{E}$ with respect to $K(\Omega_s)$. Then

$$u\big|_Q \in B^{\frac{3}{2}}_{2,\infty}(Q \cap \Omega_s) \subset H^{\frac{3}{2}-\delta}(\Omega_s \cap Q) \qquad \textit{for all } \delta > 0.$$







Comparison: Neumann conditions on the crack $\Rightarrow$

$$u = |x - x_s|^{\frac{1}{2}}\, v(\varphi) + \text{smooth terms} \quad \in B^{\frac{3}{2}}_{2,\infty}(\Omega_s \cap Q).$$

For $v \in K(\Omega_s) = \{\, v \in H^1_{\Gamma_D}(\Omega_s)\, ;\, [v] \cdot \mathbf{n} \geq 0 \text{ on } \mathcal{C}_s \,\}$:

$$\mathcal{E}(v, s) = \int_{\Omega_s} \tfrac{1}{2} \mathbb{C}\varepsilon(v) : \varepsilon(v)\, \mathrm{d}x - \int_{\Gamma_N} \ell(t) \cdot v \, \mathrm{d}a$$

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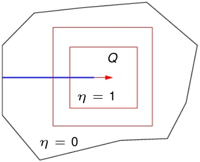
Proof: Derive estimates for finite differences of ∇u:

- tangential: $u_{he_1} := u(\cdot + h\eta(\cdot)\mathbf{e}_1)$, $h > 0$, is an admissible test for the variational inequality + uniform convexity of the elastic energy $\mathcal{E}$



$$\Longrightarrow \quad \sup_{h>0} h^{-\frac{1}{2}} \left\| u_h(\cdot + h\mathbf{e}_1) - u \right\|_{H^1(\Omega_s \cap Q)} \leq c$$

Regularity of minimizers with contact condition

For $v \in K(\Omega_s) = \{ v \in H^1_{\Gamma_D}(\Omega_s) ; [v] \cdot \mathbf{n} \geq 0 \text{ on } \mathcal{C}_s \}$:

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Theorem (K. 10)

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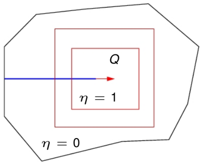
$$u\big|_Q \in B^{\frac{3}{2}}_{2,\infty}(Q \cap \Omega_s) \subset H^{\frac{3}{2}-\delta}(\Omega_s \cap Q) \quad \textit{for all } \delta > 0.$$

Proof: Derive estimates for finite differences of ∇u:

- normal: "solve" the variational inequality for the missing derivative $\partial_2 \nabla u$
  (variant of an argument by Frehse/Ebmeyer/Kassmann'02, Kassmann/Madych '07)



$$\Longrightarrow \quad \sup_{h>0, i\in\{1,2\}} h^{-\frac{1}{2}} \left\| u_{h\mathbf{e}_i} - u \right\|_{H^1(\Omega_s \cap Q)} \leq c$$

$$\Longrightarrow \quad u \in B^{\frac{3}{2}}_{2,\infty}(\Omega_s \cap Q).$$

Steklov regularization:

$$\mathcal{M}_h^i(w)(x) := \frac{1}{h}\int_0^h w(x + r\mathbf{e}_i)\,\mathrm{d}r, \quad x \in Q_{R/2}.$$

$$\mathcal{M}_h^i : L^2(Q_R) \to L^2(Q_{R/2}) \quad \text{linear, continuous with uniform bound,}$$

$$h\mathcal{M}_h^i\partial_i w = \triangle_h^i w = w(x + h\mathbf{e}_i) - w(x).$$

$$\int_{Q_{3R/2}} \left|\triangle_h^1 w\right|^2 \mathrm{d}x + \int_{Q_{R/2}} \left|\triangle_h^2 \mathcal{M}_h^1 w\right|^2 \mathrm{d}x \leq c\,|h|^{2s} \quad \Longrightarrow \quad \int_{Q_{R/2}} \left|\triangle_h^2 w\right|^2 \mathrm{d}x \leq c\,|h|^{2s}$$

Remark.

Original version by Ebmeyer/Frehse/Kassmann, Kassmann/Madych based on Fourier trafo.

By direct estimation of the integrals extendable to $p \neq 2$.

Condition A1 in case of contact conditions

Geometry: $\mathcal{T}_h$ regular, uniform triangulation of Ω_0 into triangles, $h = h_N$ mesh size; compatible with crack $\mathcal{C}_L$

Spaces:
$K_h(\Omega_L) = \{ v \in K(\Omega_L)\,;\, v\big|_\tau \text{ affine }, \tau \in \mathcal{T}_h \}$,
$K_h(\Omega_s) = K(\Omega_s) \cap K_h(\Omega_L)$,
$Z^N \subset \{$ nodes of $\mathcal{T}_h$ lying on the crack $\mathcal{C}_L\}$
crack increment $\rho_N \geq h_N = h$

Theorem (K./Schröder '10)

For all $s_N \in Z^N \cap (\epsilon, L-\epsilon)$ it holds $\|u(s_N) - u_N(s_N)\|_{H^1(\Omega_L)} \leq c_{\epsilon,\delta} h_N^{\frac{1}{2}-\delta}$,

$$\left| \frac{1}{\rho_N}\big(\mathcal{I}_N(t, s_N + \rho_N) - \mathcal{I}_N(t, s_N)\big) - \partial_s \mathcal{I}(t, s_N) \right| \leq c_{\epsilon,\delta}\big(\rho_N + h_N^{\frac{1}{2}-\delta}\rho_N^{-1}\big).$$

Possible choice for the convergence theorem: $\rho = \tau \approx h^{\frac{1}{4}},\, \nu = h^{\frac{1}{8}}$

Observe: Much coarser discretization for the crack than for the displacement field!

Proof: Falk approximation theorem for variational inequalities

Condition A1 in case of contact conditions

Geometry: $\mathcal{T}_h$ regular, uniform triangulation of Ω_0 into triangles, $h = h_N$ mesh size; compatible with crack $\mathcal{C}_L$

Spaces: $K_h(\Omega_L) = \{\, v \in K(\Omega_L)\,;\, v\big|_\tau \text{ affine }, \tau \in \mathcal{T}_h \,\}$,
$K_h(\Omega_s) = K(\Omega_s) \cap K_h(\Omega_L)$,
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Proof: Falk approximation theorem for variational inequalities + $H^{\frac{3}{2}-\delta}$-regularity of u:

$$\|u - u_N\|_{H^1(\Omega_s)} \leq c \inf_{v \in K_s^N}\left(\|u - v\|_{H^1(\Omega_s)} + \left(\|\mathcal{A}_s(u)\|_W + \|\ell(t)\|_{L^2(\Gamma_N)} \right)^{\frac{1}{2}} \|u - v\|_{W^*}^{\frac{1}{2}} \right),$$

where $W \subset V^* = (H^1_{\Gamma_D}(\Omega_s))^*$ is a dense subspace. Here: $W = (H^{\frac{1}{2}+\delta}_{\Gamma_D}(\Omega_s))^*$.

Theorem (K./Schröder '10)

No contact conditions on $\mathcal{C}_s$, mesh locally translation invariant parallel to the crack. Then **A1** *holds with*

$$\left| \frac{1}{\rho_N} \big(\mathcal{I}_N(t, s_N + \rho_N) - \mathcal{I}_N(t, s_N) \big) - \partial_s \mathcal{I}(t, s_N) \right| \leq c_\epsilon (\rho_N + h_N + h_N^2 \rho_N^{-1})$$

Possible choice for the convergence theorem: $\boxed{\rho_N = \tau_N = h_N, \nu_N = \sqrt{h_N}}$

Condition A1, improved estimates neglecting contact on the crack

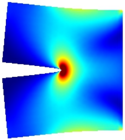
Theorem (K./Schröder '10)

No contact conditions on $\mathcal{C}_s$, mesh locally translation invariant parallel to the crack. Then **A1** *holds with*

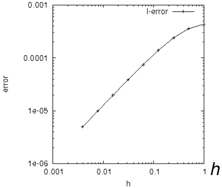
$$\left| \frac{1}{\rho_N} (\mathcal{I}_N(t, s_N + \rho_N) - \mathcal{I}_N(t, s_N)) - \partial_s \mathcal{I}(t, s_N) \right| \leq c_\epsilon (\rho_N + h_N + h_N^2 \rho_N^{-1})$$

Possible choice for the convergence theorem: $\rho_N = \tau_N = h_N, \nu_N = \sqrt{h_N}$

Example: Mode I crack



Error:



Observed convergence rate:

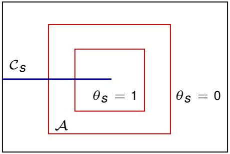
$$\left|\partial_s \mathcal{I}(s) - \tfrac{1}{h}(\mathcal{I}_h(s+h) - \mathcal{I}_h(s))\right| \leq ch^1$$

Comments on the proof

Proof.

- Regularity: $u \in B^{3/2}_{2,\infty}(\Omega_s)$, $u\big|_{\tilde{\mathcal{A}}} \in H^2(\Omega_s \cap \tilde{\mathcal{A}})$
- Local error estimates for FEM (Nitsche/Schatz'74):

$$\|u - u_h\|_{H^1(\mathcal{A})} \leq ch\,\|u\|_{H^2(\Omega_s\cap\tilde{\mathcal{A}})} + \|u - u_h\|_{L^2(\Omega_s)}$$
$$\leq ch\Big(\,\|u\|_{H^2(\Omega_s\cap\tilde{\mathcal{A}})} + \|u\|_{B^{3/2}_{2,\infty}(\Omega_S)}\Big)$$



Comments on the proof

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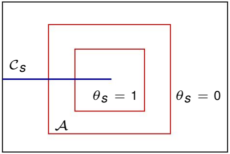
$$\begin{aligned} \|u - u_h\|_{H^1(\mathcal{A})} &\leq ch\,\|u\|_{H^2(\Omega_s\cap\tilde{\mathcal{A}})} + \|u - u_h\|_{L^2(\Omega_s)} \\ &\leq ch\Big(\,\|u\|_{H^2(\Omega_s\cap\tilde{\mathcal{A}})} + \|u\|_{B^{3/2}_{2,\infty}(\Omega_S)}\Big) \end{aligned}$$



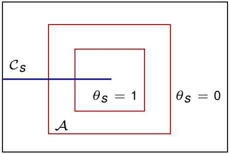
- Local character of the Griffith formula:

$$\partial_s\mathcal{I}(t,s) = -\int_{\mathcal{A}\cap\Omega_s} \mathbb{E}(\nabla u) : e_1 \otimes \nabla\theta_s \,\mathrm{d}x$$

Comments on the proof

Proof.

- Regularity: $u \in B^{3/2}_{2,\infty}(\Omega_s)$, $u\big|_{\tilde{\mathcal{A}}} \in H^2(\Omega_s \cap \tilde{\mathcal{A}})$
- Local error estimates for FEM (Nitsche/Schatz'74):



$$\begin{aligned}\|u-u_h\|_{H^1(\mathcal{A})} &\le ch\,\|u\|_{H^2(\Omega_s\cap\tilde{\mathcal{A}})} + \|u-u_h\|_{L^2(\Omega_s)}\\ &\le ch\Big(\|u\|_{H^2(\Omega_s\cap\tilde{\mathcal{A}})} + \|u\|_{B^{3/2}_{2,\infty}(\Omega_S)}\Big)\end{aligned}$$

- Local character of the Griffith formula:

$$\partial_s\mathcal{I}(t,s) = -\int_{\mathcal{A}\cap\Omega_s} \mathbb{E}(\nabla u) : e_1 \otimes \nabla\theta_s \,\mathrm{d}x$$

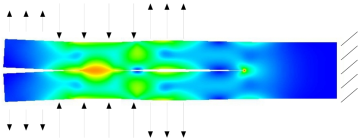
- Assumption on the mesh $\Longrightarrow$

$\exists$ projection $\mathbb{P}_{s_N} : V(\Omega_{s_N}) \to V^N(\Omega_{s_N})$, diffeomorphism $T_{s_N,\rho_N} : \Omega_{s_N} \to \Omega_{s_N+\rho_N}$ such that $\forall v \in V^N(\Omega_{s_N+\rho_N})$

$$\operatorname{supp}\left(\mathbb{P}_{s_N}(v\circ T_{s_N,\rho_N}) - v\circ T_{s_N,\rho_N}\right) \subset \mathcal{A}.$$

3 Numerical example

A numerical example



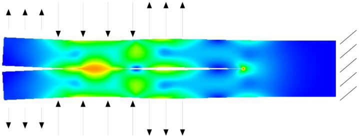
Computations: A. Schröder, HU Berlin

monotone loading: $\ell_1 \in L^2(\Gamma_N), \quad \ell(t) = t\,\ell_1.$

Deformation energy: $\mathcal{E}(t,s,v) = \int_{\Omega_s} \frac{1}{2}\mathbb{C}\varepsilon(v) : \varepsilon(v)\,\mathrm{d}x - \int_{\Gamma_N} t\,\ell_1 \cdot v\,\mathrm{d}a.$

density quadratic $\Rightarrow$ $\mathcal{I}(t,s) = t^2\mathcal{I}(1,s), \qquad \partial_s\mathcal{I}(t,s) = t^2\partial_s\mathcal{I}(1,s).$

A numerical example



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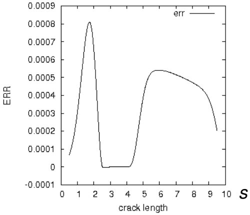
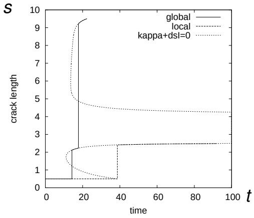
density quadratic $\Rightarrow$ $\mathcal{I}(t,s) = t^2\mathcal{I}(1,s), \qquad \partial_s\mathcal{I}(t,s) = t^2\partial_s\mathcal{I}(1,s).$

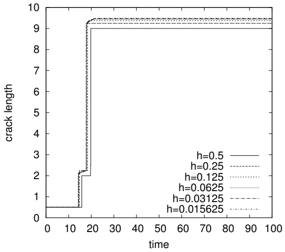
$-\partial_s\mathcal{I}(1,\cdot)$:



Solutions:

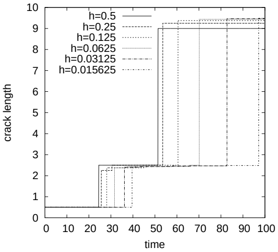


- Finite element discretization with continuous, piecewise bilinear functions on quadrilaterals, CG-like contact solver (Braess et al. ‘04), SOFAR (Scientific Object Oriented Finite Element Library for Application and Research)



Globale energetische Lösung

$\rho = h, \nu = 0, \tau = 0.01$



Viskositätslösung

$\rho = h, \nu = 0.013h^{0.5}, \tau = 0.1h$

$$s_N^k \in \mathrm{Argmin}\{\, \mathcal{I}_N(t_N^k, \tilde{s}) + \frac{\nu_N}{2\tau_N}(\tilde{s} - s_N^{k-1})^2 + \kappa(\tilde{s} - s_N^{k-1})\,;\, \tilde{s} \in Z^N,\, \tilde{s} \geq s_N^{k-1} \,\}.$$

Summary

- Vanishing viscosity approach + Griffith fracture criterion (finite strains)
- Convergence analysis for the approximation of vanishing viscosity solutions with FE-discretization in space (small strains)
  - with contact conditions: $\rho = \tau \approx h^{\frac{1}{4}}, \nu = h^{\frac{1}{8}}$
  - without contact conditions: $h = \rho = \tau, \nu = \sqrt{h}$.
- Similar analysis for the case with contact?
  Local error estimates (Nitsche/Schatz'74) for variational inequalities?
- Extension to the Lazzaroni/Toader model (single crack, free crack path)??
- Extension to damage models (collaboration with R. Rossi, C. Zanini)

References

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