Conditioning Analysis of Nonlocal Problems with Integrable and Singular Kernels

Burak Aksoylu

LSU

TOBB
UNIVERSITY OF
ECONOMICS AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

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Outline

1. Conditioning analysis of the stiffness matrix for integrable kernels
   - Summary of existing results
2. Nonlocal domain decomposition
   - Conditioning analysis of the Schur complement matrix
3. Conditioning analysis for singular kernels
   - Spectral analysis in fractional Sobolev spaces H^s
4. Crime(/Crack) Scene Investigation (CSI) of $\lambda^{\min}$ and $\lambda^{\max}$ by δ- and h-quantification in H^s

The bilinear forms with kernels from a certain family:

$$a(u,v) := \frac{1}{2}\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(\mathbf{x},\mathbf{x}')[u(\mathbf{x}')-u(\mathbf{x})][v(\mathbf{x}')-v(\mathbf{x})]\ d\mathbf{x}'\ d\mathbf{x}$$

$$b(u,v) := \frac{1}{2}\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(\mathbf{x},\mathbf{x}')\frac{[u(\mathbf{x}')-u(\mathbf{x})][v(\mathbf{x}')-v(\mathbf{x})]}{|\mathbf{x}'-\mathbf{x}|^{d+2s}}\ d\mathbf{x}'\ d\mathbf{x}$$

Properties of the kernel function $C(\mathbf{x},\mathbf{x}')$

- Radial; $C = C(|\mathbf{x}-\mathbf{x}'|)$.
- Anti-symmetric; $C(\mathbf{x},\mathbf{x}')\cdot[\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})] = -C(\mathbf{x}',\mathbf{x})\cdot[\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}')]$.
- Positive; $C(r)\geq 0$ for $r\in[0,\infty)$ and $C(r)>0$ for $r\in(0,\delta)$.
- Integrable; $C(r)r^{d-1}\in L^1_{loc}(0,\infty)$.

For **nonlocal characterization** of Sobolev spaces by Bourgain, Brezis, and Mironescu, we utilize mollifiers $\rho_\delta \in L^1_{loc}(0,\infty)$ and $\rho_\delta \geq 0$ with moment conditions:

$$\omega_d\int_0^\infty \rho_\delta(r)r^{d-1} = 1,\ \forall\delta>0,\quad \lim_{\delta\to 0}\int_{\delta_0}^\infty \rho_\delta(r)r^{d-1}dr = 0,\ \forall\delta_0>0.$$

Special choice of kernel functions

Consider $\gamma \in L^1_{loc}(0,\infty)$ with $\gamma \geq 0$, $\text{supp}(\gamma) \subset [0,2)$, $\gamma(r)r^{d-1} \in L^1_{loc}(0,\infty)$, and $\int_0^\infty \gamma(r) r^{d+1} dr = 1$.
Then, the sequence

$$\rho_\delta(r) := \frac{1}{\omega_d \delta^{d+2}} \gamma(r/\delta) r^2$$

satisfies the moment conditions for nonlocal characterization.
If we choose,

$$C(r) = \gamma(r/\delta),$$

then

$$\int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} \frac{|u(\mathbf{x}) - u(\mathbf{x}')|^2}{|\mathbf{x} - \mathbf{x}'|^2} \rho_\delta(|\mathbf{x} - \mathbf{x}'|) \; d\mathbf{x}' d\mathbf{x} = \frac{1}{\omega_d \delta^{d+2}} a(u,u).$$

Corollary of the nonlocal Poincaré inequality

For C as above $a(\cdot,\cdot)$ is coercive on V_M (also V_D) and V_N. Furthermore, there exists $\delta_0 = \delta_0(\overline{\overline{\Omega}}, \gamma)$ and $\underline{\lambda} = \underline{\lambda}(\overline{\overline{\Omega}}, \delta_0)$ such that for $0 < \delta < \delta_0$ and $u \in V_M, V_N$

$$\underline{\lambda}\delta^{d+2}\|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq a(u,u).$$

Theorem (spectral equivalence gives well-posedness and conditioning)

$$\underline{\lambda}(\overline{\overline{\Omega}}, \delta_0)\ \delta^{d+2} \leq \frac{a(u,u)}{\|u\|^2_{L_2(\overline{\overline{\Omega}})}} \leq \overline{\lambda}(\overline{\overline{\Omega}}, \gamma)\ \delta^d, \quad \delta \leq \delta_0, \quad u \in V_M, V_N.$$

The stiffness matrix K produced by the discretized $a(u,u)$ has the following condition number bound:

$$\kappa(K) \lesssim\ \delta^{-2}.$$

The upper bound is *sharp* in 1D

Choose the canonical kernel $C(|x-x'|) = \chi_\delta(|x-x'|)$ on $\overline{\overline{\Omega}} := [-1, 2]$ with the following piecewise constant function:

$$u(x) := \begin{cases} 1, & x \in [0, \delta] \\ 0, & \text{otherwise.} \end{cases}$$

The Rayleigh quotient becomes

$$\frac{a(u,u)}{\|u\|^2_{L^2(\overline{\overline{\Omega}})}} = \frac{\delta^2}{\delta} = \delta.$$

Conditioning for the nonradial kernel case

Du-Gunzburger-Lehoucq-Zhou 2012 studied this case.

Let $\gamma(\mathbf{x},\mathbf{x}') \geq 0, \mathbf{x}' \in \mathcal{H}_{\mathbf{x}}(\delta)$, $\gamma(\mathbf{x},\mathbf{x}') \geq \gamma_0, \mathbf{x}' \in \mathcal{H}_{\mathbf{x}}(\delta/2)$, and $\gamma(\mathbf{x},\mathbf{x}') = 0, \mathbf{x}' \notin \mathcal{H}_{\mathbf{x}}(\delta)$

Nonradial kernel bounded by radial functions

Let $s \in (0,1)$ and $\gamma_*, \gamma^*, \gamma_1, \gamma_2 > 0$.

$$\frac{\gamma_*}{|\mathbf{x}-\mathbf{x}'|^{d+2s}} \leq \gamma(\mathbf{x},\mathbf{x}') \leq \frac{\gamma^*}{|\mathbf{x}-\mathbf{x}'|^{d+2s}} \quad \Rightarrow \quad \kappa(K) \leq c\ h^{-2s}$$

$$\gamma_1 \leq \int_{\overline{\overline{\Omega}}\cap\mathcal{H}_{\mathbf{x}}(\delta)} \gamma(\mathbf{x},\mathbf{x}')d\mathbf{x}', \quad \int_{\overline{\overline{\Omega}}} \gamma^2(\mathbf{x},\mathbf{x}')d\mathbf{x}' \leq \gamma_2 \quad \Rightarrow \quad \kappa(K) \leq c,$$

where c is a generic constant which may depend on δ.

Conditioning in the $h \to 0$ regime

For our canonical kernel, Zhou-Du 2010 report:

$$\kappa(K) \quad \leq \quad c \;\; \min\{h^{-2}, \delta^{-2}\}$$

As $\delta \to 0$, the estimate recovers the classical local condition number.

Identifying δ-dependence is important for $h \ll \delta$

We used:
1D experiments: $\delta = 100h, 200h, 400h$ with $h = 1/8000$.
2D experiments: $\delta = 5h, 10h, 20h$ with $h = 1/200$.

$$\kappa(K) \leq c \; \delta^{-2}.$$

Important condition number implication

Condition number of the stiffness matrix depends (weakly) on the mesh size h but **bounded independently from h.**

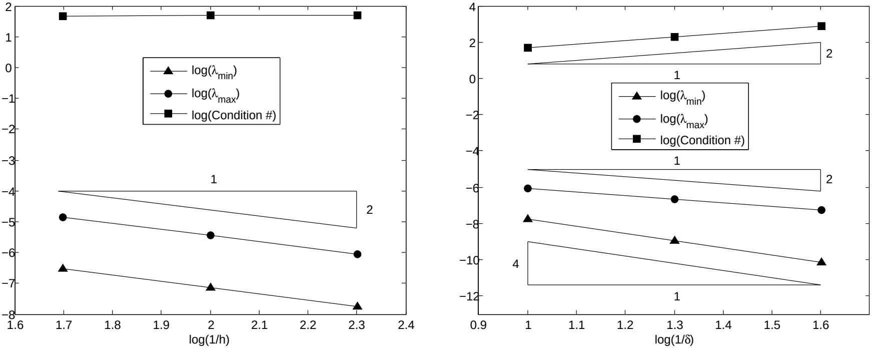
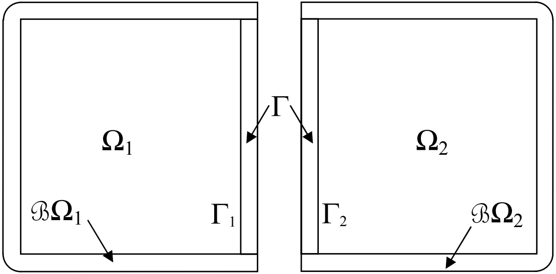


Figure: Condition number for K in 2D with canonical kernel. (Left) Fixed δ, varying h. (Right) Fixed h, varying δ. The condition number is weakly h-independent and varies with δ^{-2}.

Nonlocal nonoverlapping two-subdomain problem



The interface Γ is 2-dimensional. Define the overlapping subdomains $\Omega^{(i)}$:

$$\Omega^{(i)} := \Omega_i \cup \Gamma \cup \Gamma_i,$$

where Γ_i is the open line segment adjacent to Ω_i and Γ.

We define the spaces, $i = 1, 2$,

$$
\begin{aligned}
V^{(i)} &:= \left\{ v \in L_2(\overline{\overline{\Omega^{(i)}}}) \ : \ v|_{\mathcal{B}\Omega^{(i)}} = 0 \right\}, \\
V^{(i),0} &:= \left\{ v \in L_2(\overline{\overline{\Omega^{(i)}}}) \ : \ v|_{\mathcal{B}\Omega^{(i)} \cup \Gamma \cup \Gamma_i} = 0 \right\}, \\
\Lambda &:= \left\{ \mu \in L_2(\Gamma) : \mu = v|_\Gamma \text{ for some suitable } v \in L_{2,0}(\overline{\overline{\Omega}}) \right\}.
\end{aligned}
$$

Define a bilinear form: $a_{\Omega^{(i)}}(u, v) : V \times V \to \mathbb{R}$ as follows:

$$
\begin{aligned}
a_{\Omega^{(i)}}(u, v) \quad := \quad & -\int_{\Omega_i} \left\{ \int_{\Omega^{(i)} \cup \mathcal{B}\Omega^{(i)}} \chi_\delta(\mathbf{x} - \mathbf{x}') \ [u(\mathbf{x}') - u(\mathbf{x})] \ d\mathbf{x}' \right\} \ v(\mathbf{x}) \ d\mathbf{x} \\
& -\frac{1}{2} \int_\Gamma \left\{ \int_{\overline{\overline{\Omega}}} \chi_\delta(\mathbf{x} - \mathbf{x}') \ [u(\mathbf{x}') - u(\mathbf{x})] \ d\mathbf{x}' \right\} \ v(\mathbf{x}) \ d\mathbf{x}.
\end{aligned}
$$

Nonlocal domain decomposition equivalence

The two-domain weak formulation

Find $u^{(i)} \in V^{(i)}$, $i = 1, 2$:

$$a_{\Omega^{(i)}}(u^{(i)}, v_i) = (b, v_i)_{\Omega_i} \qquad \forall v_i \in V^{(i),0}, \qquad (1a)$$

$$u^{(1)} = u^{(2)} \qquad \text{on } \overline{\Gamma}, \qquad (1b)$$

$$\sum_{i=1,2} a_{\Omega^{(i)}}(u^{(i)}, \mathcal{R}^{(i)}\mu) = (b, \mu)_\Gamma + \sum_{i=1,2} (b, \mathcal{R}^{(i)}\mu)_{\Omega_i} \qquad \forall \mu \in \Lambda. \qquad (1c)$$

where $\mathcal{R}^{(i)}$ denotes any possible extension operator from $\Gamma \cup \Gamma_i$ to $V^{(i)}$.

Theorem

The single domain and two-subdomain weak (1) formulations are equivalent.

Conditioning of the Schur complement

Energy minimizing extension (analog of the local harmonic extension)

$E_i : \Gamma_h \subset L_2(\Gamma) \to V_h^{(i)}$ is the discrete energy minimizing extension into Ω_i

$$\begin{aligned} E_i(q)|_\Gamma &= q, \\ a(E_i(q), v) &= 0, \quad v \in V_h^{(i),0}. \end{aligned}$$

Energy minimizing property of $E_i(u_\Gamma)$, among $u \in V_h^{(i)}$ with $u|_\Gamma = u_\Gamma$:

$$a_i(E_i(u_\Gamma), E_i(u_\Gamma)) \le a_i(u, u).$$

$$\begin{aligned} s_i(u_\Gamma, u_\Gamma) &\le a_i(u, u) \le \overline{\lambda}\, \delta^d \, \|u\|^2_{L_2(\overline{\overline{\Omega^{(i)}}})}, \quad \forall u \in V_h^{(i)}, \text{ in particular, } u = u_\Gamma \\ s_i(u_\Gamma, u_\Gamma) &\le \overline{\lambda}\, \delta^d \, \|u\|^2_{L_2(\Gamma)}. \end{aligned}$$

For the lower bound:

$$\underline{\lambda}\, \delta^{d+2} \|u\|^2_{L_2(\Gamma)} \le \underline{\lambda}\, \delta^{d+2} \|E_i(u_\Gamma)\|^2_{L_2(\overline{\overline{\Omega^{(i)}}})} \le a_i(E_i(u_\Gamma), E_i(u_\Gamma)) = s_i(u_\Gamma, u_\Gamma).$$

Spectral equivalence for the Schur complement matrix

$$\underline{\lambda}\;\delta^{d+2} \leq \frac{s_i(q,q)}{\|q\|^2_{L_2(\Gamma)}} \leq \overline{\lambda}\;\delta^{d}, \quad q \in L_2(\Gamma).$$

The condition number of the Schur complement matrix $S_\Gamma := S^{(1)} + S^{(2)}$ has the following bound:

$$\kappa(S_\Gamma) \lesssim \delta^{-2}.$$

Condition number summary for $a(u, u)$

$$\underline{\lambda}_K \, \delta^{d+2} \le \frac{a(u,u)}{\|u\|^2_{L_2(\overline{\overline{\Omega}})}} \le \overline{\lambda}_K \, \delta^d, \qquad \kappa(K) \quad \lesssim \; \delta^{-2}$$

$$\underline{\lambda}_S \, \delta^{d+2} \le \frac{s(u_\Gamma,u_\Gamma)}{\|u_\Gamma\|^2_{L_2(\Gamma)}} \le \overline{\lambda}_S \, \delta^d, \qquad \kappa(S_\Gamma) \quad \lesssim \; \delta^{-2}$$

$$\underline{\lambda}_{sharp} \, \delta^{d+1} \le \frac{s(u_\Gamma,u_\Gamma)}{\|u_\Gamma\|^2_{L_2(\Gamma)}} \le \overline{\lambda}_{sharp} \, \delta^d, \quad \kappa_{sharp}(S_\Gamma) \quad \lesssim \; \delta^{-1}$$

$$k_1 \le \frac{\ell(u,u)}{\|u\|^2_{L_2(\Omega)}} \le k_2 \, h^{-2}, \quad \kappa(K_{local}) \quad \lesssim \; h^{-2}$$

$$k_3 \le \frac{s_{local}(u,u)}{\|u\|_\Gamma} \le k_4 \, h^{-1}, \quad \kappa(S_{local}) \quad \lesssim \; h^{-1}$$

In conditioning, δ somewhat plays the role of h probably due to intrinsic lengthscale. Laplace operator in a **nonlocal sauce**.

Related publications

B. A. and M. L. Parks, *Variational theory and domain decomposition for nonlocal problems.*, Applied Mathematics and Computation, 217 (2011), pp. 6498–6515.

B. A. and T. Mengesha, *Results on nonlocal boundary value problems*, Numerical Functional Analysis and Optimization, 31 (2010), pp. 1301–1317.

Minimum eigenvalue characterization

Since $|\mathbf{x}' - \mathbf{x}|^{d+2s} \leq \delta^{d+2s}$, then $\frac{a(u,u)}{\delta^{d+2s}} \leq b(u,u)$. By Corollary for the same family of kernels, we immediately have:

$$\underline{\lambda}\frac{\delta^{d+2}}{\delta^{d+2s}}\|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq b(u,u).$$

Hence, $\lambda^{\min} \sim \delta^{2-2s}h^d$. Unlike the $a(\cdot,\cdot)$ case,
$\lambda^{\min}(\delta, h, s, \Omega) = \mathbf{c(s)}\delta^{2-2s}h^d$.

*delta**s*	$\frac{5}{100}$	$\frac{15}{100}$	$\frac{25}{100}$	$\frac{45}{100}$	$\frac{55}{100}$	$\frac{75}{100}$	$\frac{90}{100}$
$\mathbf{2^{-5}}$	4.6958	5.2879	6.0426	8.3937	10.3647	19.0857	48.6472
$\mathbf{2^{-6}}$	4.9051	5.5059	6.2694	8.6397	10.6214	19.3669	48.9509
$\mathbf{2^{-7}}$	5.0307	5.6362	6.4045	8.7850	10.7722	19.5304	49.1258
$\mathbf{2^{-8}}$	5.1038	5.7119	6.4828	8.8688	10.8590	19.6238	49.2251
$\mathbf{2^{-9}}$	5.1456	5.7551	6.5274	8.9164	10.9082	19.6765	49.2809
$\mathbf{2^{-10}}$	5.1456	5.7551	6.5274	8.9164	10.9082	19.6765	49.2809

Numerical setup

Piecewise linear FEM with homogenous Dirichlet BD, 1D domain
$\Omega = (0, 1)$,
$h = 2^{-n}$, hence, the system size $N = 2^n - 1$.

h	$\frac{1}{2^{10}}$	$\frac{1}{2^{11}}$	$\frac{1}{2^{12}}$	$\frac{1}{2^{13}}$	$\frac{1}{2^{14}}$	$\frac{1}{2^{15}}$	$\frac{1}{2^{16}}$
N	1023	2047	4095	8191	16383	32767	65535

$\delta = 1/\{2^8, \ldots, 2^{12}\}$,

Shift function $c(s)$

We call $\mathbf{c(s)}$ the shift function in $\lambda^{\min}(\delta, h, s, \Omega) = \mathbf{c(s)}\delta^{2-2s}h^d$.
The plot is the same for both fixed mesh and fixed delta indicating that it only depends on s.

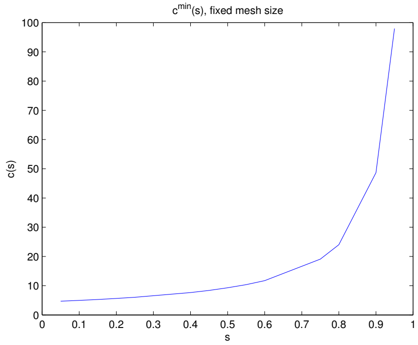


Figure: $\delta = 2^{-10}$ and $h = 2^{-12}$.

Numerical verification of $\lambda^{\min}$

Based on the theoretical result and omitting Ω dependence, we want to verify that

$$\lambda^{\min}(\delta, s, h) = c(s)\delta^{m(s)}h^{r(s)}$$

where $m(s) = 2 - 2s$ and $r(s) = 1$.
By fixing h and s, by using two values of δ; δ_1 and δ_2, we can extract $c(s_0)$:

$$\begin{aligned}
\lambda_1^{\min} := \lambda^{\min}(\delta_1, s_0, h_0) &= c(s_0)\delta_1^{m(s_0)}h_0^{r(s_0)} \\
\lambda_2^{\min} := \lambda^{\min}(\delta_2, s_0, h_0) &= c(s_0)\delta_2^{m(s_0)}h_0^{r(s_0)}
\end{aligned}$$

$$m(s_0) = \frac{\log \lambda_2^{\min} - \log \lambda_1^{\min}}{\log \delta_2 - \log \delta_1}.$$

Verification of $m(s) = 2 - 2s$

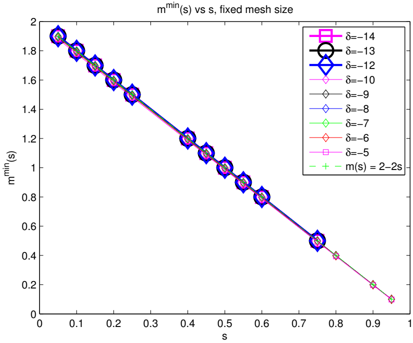


Figure: Plot of $m(s) = 2 - 2s$

Characterization of the shift function $c(s)$

To find an expression for $c(s_0)$, connect through $m(s_0)$:

$$(m(s_0)) = \frac{\log \lambda_2^{\min} - \log(c(s_0) h_0^{r(s_0)})}{\log \delta_2} = \frac{\log \lambda_1^{\min} - \log(c(s_0) h_0^{r(s_0)})}{\log \delta_1}$$

$$c(s_0) = e^{\frac{\log \delta_2 \log \lambda_1^{\min} - \log \delta_1 \log \lambda_2^{\min}}{\log \delta_2 - \log \delta_1}} / h_0^{r(s_0)}$$

Since we have $c(s)$ at hand, we can identify δ- and h-quantifications of $\lambda^{\min}$. Define

$$\mathbf{c}^{\min}(\mathbf{h}, \mathbf{s}) := \frac{\lambda^{\min}(\delta, s, h)}{c(s)\delta^{2-2s}}\Big|_{\delta=\delta_0}$$

$$\mathbf{c}^{\min}(\delta, \mathbf{s}) := \frac{\lambda^{\min}(\delta, s, h)}{c(s) h^d}\Big|_{h=h_0}$$

Plot of $c^{\min}(\delta, s) := \frac{\lambda^{\min}(\delta,h,s)}{c(s)h^d}$

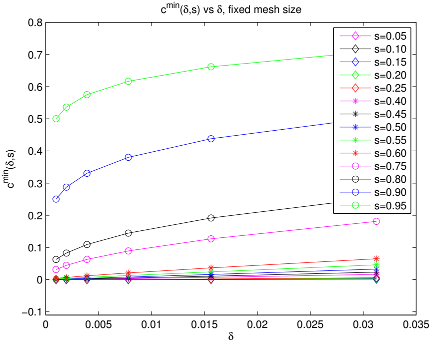


Figure: $c^{\min}(\delta, s)$ as a function of δ for each s.

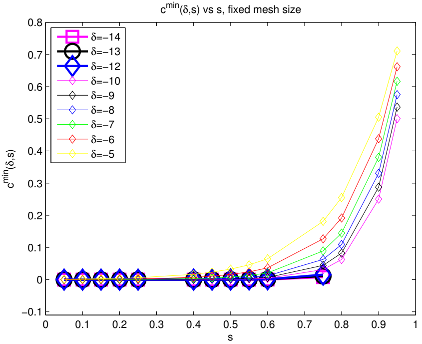


Figure: $c^{\min}(\delta, s)$ as a function of s for each δ.

Plot of $c_{shifted}(\delta, s) = c(s)c^{\min}(\delta, s)$

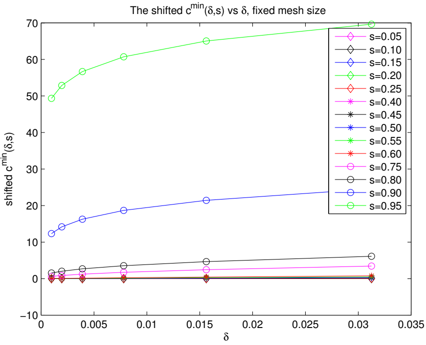


Figure: $c^{\min}(\delta, s)$ as a function of δ for each s.

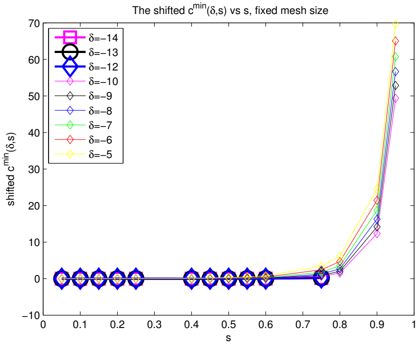


Figure: $c^{\min}(\delta, s)$ as a function of s for each δ.

3D log scale plot of $c^{\min}(\delta, s) := \frac{\lambda^{\min}(\delta, h_0, s)}{c(s) h_0^d}$, $h_0 = 2^{-12}$

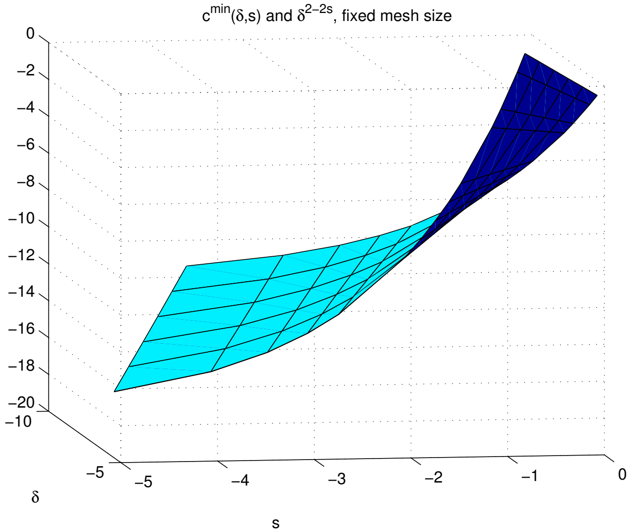
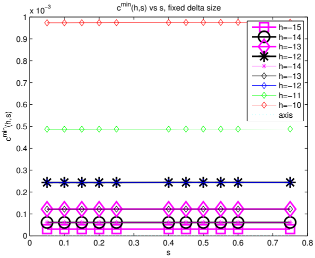
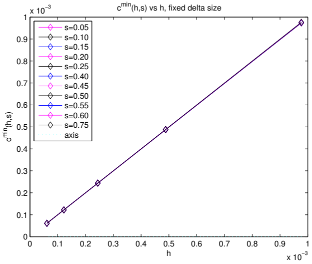
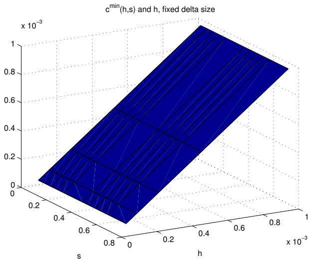


Figure: (Light blue) $c^{\min}(\delta, s) := \frac{\lambda^{\min}(\delta, h, s)}{c(s) h^d}$, (dark blue) $guess(\delta, s) = \delta^{2-2s}$.

h- and s-quantification of $c^{min}(h,s), \delta_0 = 2^{-8}$







Norm equivalence between $b(u,u)$ and $\|u\|^2_{H^s(\overline{\overline{\Omega}})}$

It is easy to prove that

$$|u|^2_{H^s(\overline{\overline{\Omega}})} \leq b(u,u) + 4|\overline{\overline{\Omega}}|\delta^{-(d+2s)}\|u\|^2_{L^2(\overline{\overline{\Omega}})}.$$

Du-Lehoucq-Gunzburger-Zhou 2012 gave a nonlocal Poincare inequality for $b(u,u)$.

$$c_{Pncr}\|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq b(u,u).$$

Hence,

$$\|u\|^2_{H^s(\overline{\overline{\Omega}})} \leq c(\delta, s, \Omega) b(u,u).$$

On the other hand, we trivially have

$$b(u,u) \leq 1\|u\|^2_{H^s(\overline{\overline{\Omega}})}.$$

Consequently, we have the norm equivalence

$$b(u,u) \sim \|u\|^2_{H^s(\overline{\overline{\Omega}})}, \quad c = c(\delta, s, \Omega).$$

Crime(/Crack) Scene Investigation (CSI)

Our strategy of finding δ- and h-quantification of $\lambda^{\max}$ is based on the norm equivalence $b(u,u) \sim \|u\|^2_{H^s(\overline{\overline{\Omega}})}$.

$$
\begin{aligned}
b(u,u) &\leq c_1(\delta,s,\Omega)\|u\|^2_{H^s(\overline{\overline{\Omega}})} \\
&\leq c_2(s,\kappa,k)c_1(\delta,s,\Omega)h^{-2s}\|u\|^2_{L^2(\overline{\overline{\Omega}})} \\
&\leq c_3(\kappa)c_2(s,\kappa,k)c_1(\delta,s,\Omega)h^{d-2s}\underline{u}^t\underline{u}.
\end{aligned}
$$

Since κ, $k=1$, and $d=1, \Omega=(0,1)$ are fixed, only concentrate on

$$
\lambda^{\max}(\delta,h,s) = c(\delta,s)h^{d-2s}.
$$

This quantification is compatible with the condition number estimate given by Du-Lehoucq-Gunzburger-Zhou 2012; $\kappa(K) \lesssim h^{-2s}$.

Power of h-quantification of $\lambda^{\max}(\delta_0, h, s)$

Lack of resolution in h value causes some loss of accuracy for the power of h. It still scales like $r(s) \sim 1 - 2s$. As h resolution increases scaling behavior becomes more accurate.

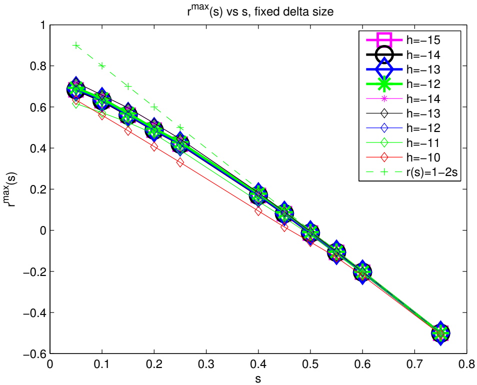
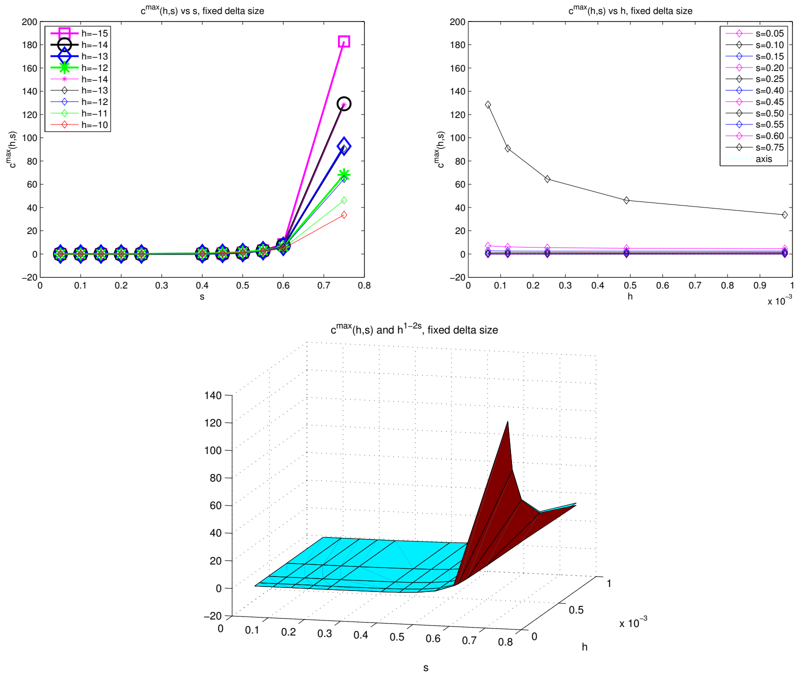


Figure: $\delta = 2^{-8}$ thin lines, $\delta = 2^{-10}$ thick lines.



c^max(h,s) vs s, fixed delta size
h=−15
h=−14
h=−13
h=−12
h=−14
h=−13
h=−12
h=−11
h=−10
c^max(h,s)
s
c^max(h,s) vs h, fixed delta size
s=0.05
s=0.10
s=0.15
s=0.20
s=0.25
s=0.40
s=0.45
s=0.50
s=0.55
s=0.60
s=0.75
axis
c^max(h,s)
h
x 10^−3
c^max(h,s) and h^1−2s, fixed delta size
s
h
x 10^−3

Crime scene of $\lambda^{\max}$ is more difficult

By fixing δ and s, we report

$$c^{\max}_{shifted}(\delta, s) = \frac{\lambda^{\max}(\delta, h, s)}{h^{d-2s}}|_{h=h_0}.$$

We have no prediction for the structure of $c^{\max}_{shifted}(\delta, s)$. For instance, a power function prediction $\delta^{m(s)}$ may become totally useless. If there is additional factor $c(s)$, without knowing it, we cannot plot $c(s)\delta^{m(s)}$.

Since we do not know if δ-quantification is a power function, we cannot identify a shift function. Dependences related to s and δ are already absorbed in $c^{\max}_{shifted}(\delta, s)$.

Plot of $c^{max}_{shifted}(\delta, s)$

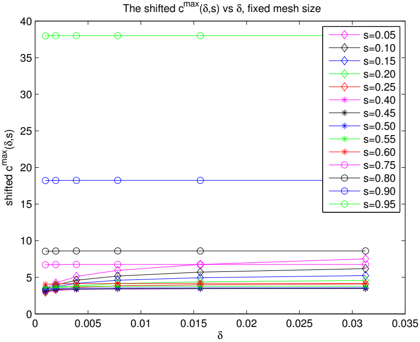


Figure: $c^{\max}(\delta, s)$ as a function of δ for each s.

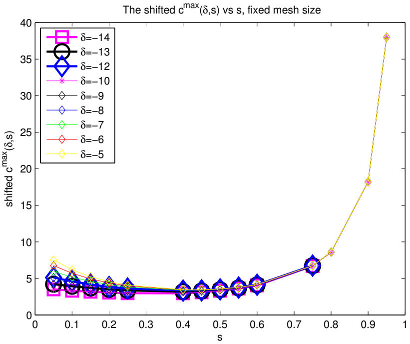


Figure: $c^{\max}(\delta, s)$ as a function of s for each δ. $h = 2^{-12}$ thin lines, $h = 2^{-16}$ thick lines.

3D plot of $c^{max}_{shifted}(\delta, s)$

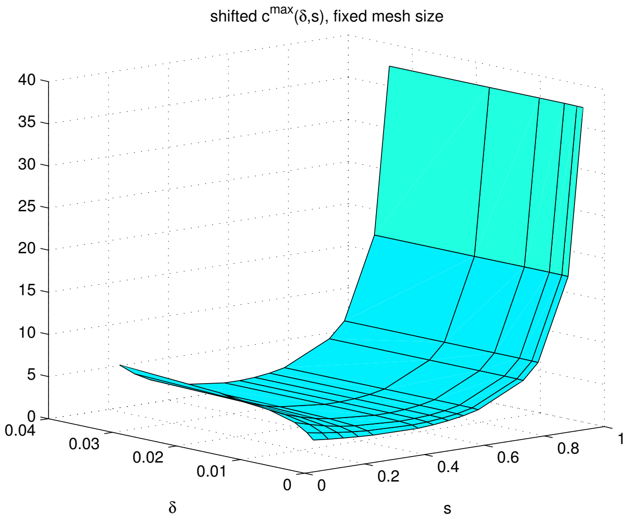


Figure: Fixing $h = 2^{-12}$, $c^{max}_{shifted}(\delta, s) := \frac{\lambda^{max}(\delta,h,s)}{h^{d-2s}}$, $guess(\delta, s) = unknown$.

Upper bound candidate

Lemma

Let Ω be a bounded set in $\mathbb{R}^d$ of class $C^{0,1}$ and $u \in H^1(\overline{\overline{\Omega}})$. Then,

$$b(u,u) \leq c(\Omega)\delta^{2(1-s)}\|u\|^2_{H^1(\overline{\overline{\Omega}})}.$$

Leads to:

$$\lambda^{\max} \sim \delta^{2-2s}h^{d-2}.$$

Summary of results

Utilizing a kernel function leading to uniformly bounded $\lambda^{\max}$ and using 1D Fourier analysis, Du-Zhou 2010 report:

$$\kappa(K) \leq c \min\{h^{-2s}\delta^{2s-2}, h^{-2}\}.$$

By spectral equivalence, Du-Lehoucq-Gunzburger-Zhou 2012 report:

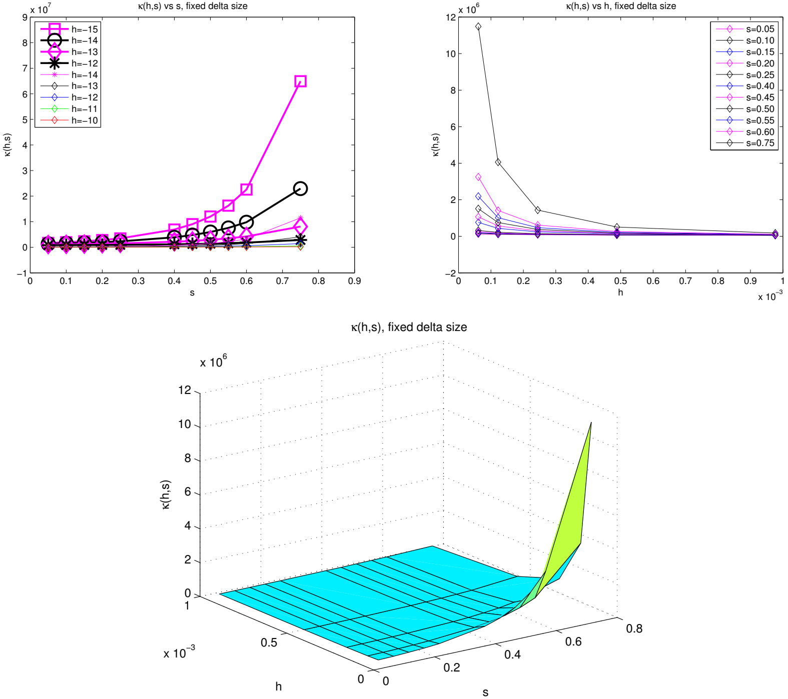
$$\kappa(K) \lesssim h^{-2s}.$$

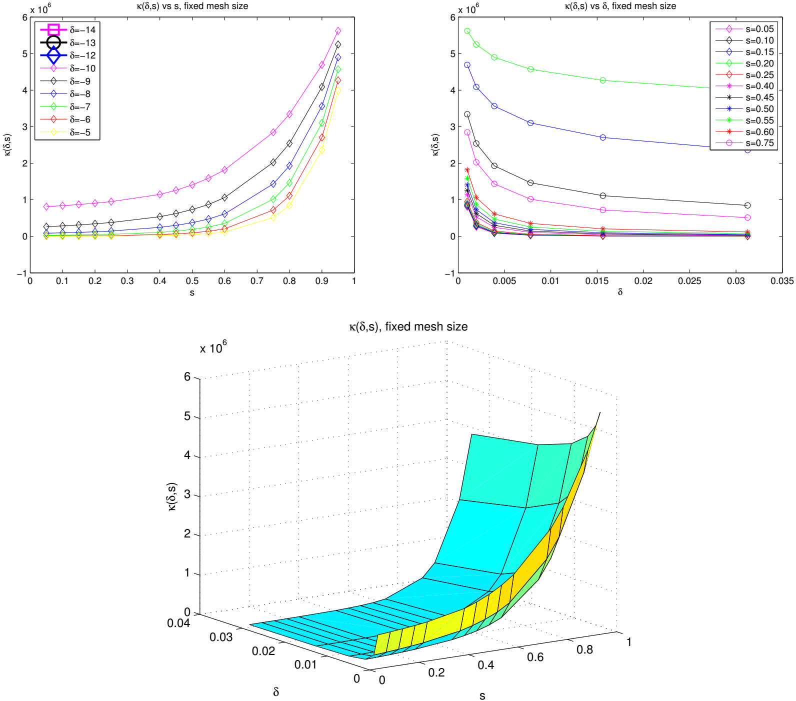
We report:

$$c(s)\delta^{2-2s}h^{d} \leq \frac{\underline{b}^t\underline{b}}{\underline{u}^t\underline{u}} \leq c(\delta, s)h^{d-2s}.$$

With the nonsharp theoretical result for $\lambda^{\max}$:

$$c(s)\delta^{2-2s}h^{d} \leq \frac{\underline{b}^t\underline{b}}{\underline{u}^t\underline{u}} \leq c\delta^{2-2s}h^{d-2}.$$


κ(h,s) vs s, fixed delta size
x 10^7
κ(h,s)
s
h=−15
h=−14
h=−13
h=−12
h=−14
h=−13
h=−12
h=−11
h=−10
κ(h,s) vs h, fixed delta size
x 10^6
κ(h,s)
h
x 10^−3
s=0.05
s=0.10
s=0.15
s=0.20
s=0.25
s=0.40
s=0.45
s=0.50
s=0.55
s=0.60
s=0.75
κ(h,s), fixed delta size
x 10^6
κ(h,s)
x 10^−3
h
s



κ(δ,s) vs s, fixed mesh size
κ(δ,s) vs δ, fixed mesh size
κ(δ,s), fixed mesh size

Conclusion

1. The estimate $\lambda^{\min}(\delta, h, s) = c(s)\delta^{2-2s}h$ is numerically sharp.
2. $\lambda^{\max}(\delta, h, s) = c(\delta, s)h^{1-2s}$ and further investigation is required to determine the δ-quantification of $c(\delta, s)$.
3. The theoretical bound for $\lambda^{\max}(\delta, h, s) \leq c\delta^{2-2s}h^{-2}$ is not sharp in the h-quantification.
4. As $h \to 0$, the condition number reaches its max near $s = 1$, so $\kappa(K) \lesssim h^{-2s}$ is numerically sharp.
5. As singularity degree increases $s \to 1$, the condition number gets larger.
6. As $\delta \to 0$ (closer to a local model), the condition number gets larger and reaches to its max near $s = 1$.
7. Police report: Fully identified the suspect for $\lambda^{\min}$ and provided a sketch of the suspect for $\lambda^{\max}$.