NONLOCAL DIFFUSION EQUATIONS

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Non-local diffusion.

The function J. Let $J : \mathbb{R}^N \to \mathbb{R}$, nonnegative, smooth with

$$\int_{\mathbb{R}^N} J(r)dr = 1.$$

Assume that is compactly supported and radially symmetric.

Non-local diffusion equation

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t).$$

Non-local diffusion.

In this model, $u(x,t)$ is the density of individuals in x at time t and $J(x-y)$ is the probability distribution of jumping from y to x. Then

$$(J * u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$$

is the rate at which the individuals are arriving to x from other places

$$-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$$

is the rate at which they are leaving from x to other places.

Non-local diffusion.

The non-local equation shares some properties with the classical heat equation

$$u_t = \Delta u.$$

Properties

- Existence, uniqueness and continuous dependence on the initial data.
- Maximum and comparison principles.
- Perturbations propagate with infinite speed. If u is a nonnegative and nontrivial solution, then $u(x,t) > 0$ for every $x \in \mathbb{R}^N$ and every $t > 0$.

Remark.

There is no regularizing effect for the non-local model.

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Newmann boundary conditions.

One of the boundary conditions that has been imposed to the heat equation is the *Neumann boundary condition*, $\partial u/\partial\eta(x,t) = 0$, $x \in \partial\Omega$.

Non-local Neumann model

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t) - u(x,t))dy$$

for $x \in \Omega$.

Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω.

Existence, uniqueness and a comparison principle

Theorem (Cortazar - Elgueta - R. - Wolanski)

For every $u_0 \in L^1(\Omega)$ there exists a unique solution u such that $u \in C([0,\infty); L^1(\Omega))$ and $u(x,0) = u_0(x)$.
Moreover the solutions satisfy the following comparison property:

$$\text{if } u_0(x) \le v_0(x) \text{ in } \Omega, \ \text{then } u(x,t) \le v(x,t) \text{ in } \Omega \times [0,\infty).$$

In addition the total mass in Ω is preserved

$$\int_\Omega u(y,t)\,dy = \int_\Omega u_0(y) dy.$$

Approximations

Now, our goal is to show that the Neumann problem for the heat equation, can be approximated by suitable nonlocal Neumann problems.
More precisely, for given J we consider the rescaled kernels

$$J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right),$$

with

$$C_1^{-1} = \frac{1}{2} \int_{B(0,d)} J(z) z_N^2 \, dz,$$

which is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it.

Approximations

Then, we consider the solution $u_\varepsilon(x,t)$ to

$$\begin{cases} (u_\varepsilon)_t(x,t) &= \dfrac{1}{\varepsilon^2}\displaystyle\int_\Omega J_\varepsilon(x-y)(u_\varepsilon(y,t)-u_\varepsilon(x,t))\,dy \\ u_\varepsilon(x,0) &= u_0(x). \end{cases}$$

Theorem (Cortazar - Elgueta - R. - Wolanski)

Let $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ be the solution to the heat equation with Neumann boundary conditions and u_ε be the solution to the nonlocal model. Then,

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\|u_\varepsilon(\cdot,t)-u(\cdot,t)\|_{L^\infty(\Omega)}=0.$$

Approximations

Idea of why the involved scaling is correct

Let us give an heuristic idea in one space dimension, with $\Omega = (0, 1)$, of why the scaling involved is the right one.

We have

$$u_t(x,t) = \frac{1}{\varepsilon^3} \int_0^1 J\left(\frac{x-y}{\varepsilon}\right)\left(u(y,t) - u(x,t)\right)dy$$

$$:= A_\varepsilon u(x,t).$$

Approximations

If $x \in (0,1)$ a Taylor expansion gives that for any fixed smooth u and ε small enough, the right hand side $A_\varepsilon u$ becomes

$$A_\varepsilon u(x) = \frac{1}{\varepsilon^3} \int_0^1 J\left(\frac{x-y}{\varepsilon}\right)(u(y) - u(x))\, dy$$

$$= \frac{1}{\varepsilon^2} \int_{\mathbb{R}} J(w)\,(u(x - \varepsilon w) - u(x))\, dw$$

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Approximations

$$= \frac{u_x(x)}{\varepsilon} \int_{\mathbb{R}} J(w)\, w \, dw + \frac{u_{xx}(x)}{2} \int_{\mathbb{R}} J(w)\, w^2 \, dw + O(\varepsilon)$$

As J is even

$$\int_{\mathbb{R}} J(w)\, w \, dw = 0$$

and hence,

$$A_\varepsilon u(x) \approx u_{xx}(x),$$

and we recover the Laplacian for $x \in (0, 1)$.

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Approximations

If $x = 0$ and ε small,

$$A_\varepsilon u(0) = \frac{1}{\varepsilon^3} \int_0^1 J\left(\frac{-y}{\varepsilon}\right) (u(y) - u(0))\, dy$$

$$= \frac{1}{\varepsilon^2} \int_{-\infty}^0 J(w) (-u(-\varepsilon w) + u(0))\, dw$$

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Approximations

$$= -\frac{u_x(0)}{\varepsilon}\int_{-\infty}^{0} J(w)\,w\,dw + O(1)$$

$$\approx \frac{C_2}{\varepsilon}u_x(0).$$

then

$$u_x(0) = 0$$

and we recover the boundary condition

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The $p-$Laplacian

The problem,

$$u_t(t,x) = \int_\Omega J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy,$$
$$u(x,0) = u_0(x).$$

is the analogous to the p-Laplacian

$$\begin{cases} u_t = \Delta_p u & \text{in } (0,T)\times\Omega, \\ |\nabla u|^{p-2}\nabla u\cdot\eta = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Approximations

For given $p \geq 1$ and J we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right), \quad C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p \, dz.$$

Theorem (Andreu - Mazon - R. - Toledo)

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $p \geq 1$. Assume $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$, $u_0 \in L^p(\Omega)$. Then,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon(t,.) - u(t,.)\|_{L^p(\Omega)} = 0.$$

Convective terms

$$\begin{cases} u_t(t,x) = (J * u - u)(t,x) + (G * (f(u)) - f(u))(t,x), \\ u(0,x) = u_0(x) \qquad (\text{ now } x \in \mathbb{R}^N \text{ !!}). \end{cases}$$

Theorem (Ignat - R.)

There exists a unique global solution

$$u \in C([0,\infty); L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty); \mathbb{R}^N).$$

Moreover, the following contraction property

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^N)}$$

holds for any $t \geq 0$. In addition, $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}$.

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Convective terms

Let us consider the rescaled problems

$$
\begin{cases}
(u_\varepsilon)_t(t,x) = \dfrac{1}{\varepsilon^{N+2}} \displaystyle\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right)(u_\varepsilon(t,y) - u_\varepsilon(t,x))\,dy \\
\qquad + \dfrac{1}{\varepsilon^{N+1}} \displaystyle\int_{\mathbb{R}^N} G\left(\frac{x-y}{\varepsilon}\right)(f(u_\varepsilon(t,y)) - f(u_\varepsilon(t,x)))\,dy, \\
u_\varepsilon(x,0) = u_0(x).
\end{cases}
$$

Note that the scaling of the diffusion, $1/\varepsilon^{N+2}$, is different from the scaling of the convective term, $1/\varepsilon^{N+1}$.

Convective terms

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Convective terms

Theorem (Ignat - R.)

We have

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon - v\|_{L^2(\mathbb{R}^N)} = 0,$$

where $v(t,x)$ is the unique solution to the local convection-diffusion problem

$$v_t(t,x) = \Delta v(t,x) + b \cdot \nabla f(v)(t,x),$$

with initial condition $v(x,0) = u_0(x)$ and $b = (b_1, ..., b_d)$ given by

$$b_j = \int_{\mathbb{R}^N} x_j \, G(x) \, dx, \qquad j = 1, ..., d.$$

Convective terms

Theorem (Ignat - R.)

Let $f(s) = s^q$ *with* $q > 1$ *and* $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then, for every* $p \in [1, \infty)$ *the solution* u *verifies*

$$\|u(t)\|_{L^p(\mathbb{R}^N)} \leq C(\|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}) \langle t \rangle^{-\frac{N}{2}(1-\frac{1}{p})}.$$

Convective terms

Theorem (Ignat - R.)

Let $f(s) = s^q$ with $q > (N+1)/N$ and let the initial condition u_0 belongs to $L^1(\mathbb{R}^N, 1+|x|) \cap L^\infty(\mathbb{R}^N)$. For any $p \in [2, \infty)$ the following holds

$$t^{-\frac{N}{2}(1-\frac{1}{p})} \|u(t) - MH(t)\|_{L^p(\mathbb{R}^N)} \leq C(J, G, p, d)\, \alpha_q(t),$$

where $M = \int_{\mathbb{R}^N} u_0(x)\,dx$, $H(t) = \frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac{d}{2}}}$, *and*

$$\alpha_q(t) = \begin{cases} \langle t\rangle^{-\frac{1}{2}} & \text{if} \quad q \geq (N+2)/N, \\ \langle t\rangle^{\frac{1-N(q-1)}{2}} & \text{if} \quad (N+1)/N < q < (N+2)/N. \end{cases}$$

Convective terms

The main idea for the proofs is to write the solution as

$$u(t) = S(t) * u_0 + \int_0^t S(t-s) * (G * (f(u)) - f(u))(s)\, ds,$$

with $S(t)$ the linear semigroup associated to

$$\begin{cases} w_t(t,x) = (J * w - w)(t,x), & t > 0,\ x \in \mathbb{R}^N, \\ w(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

Decay for the heat equation

For the heat equation we have an explicit representation formula for the solution in Fourier variables. In fact, from the equation

$$v_t(x,t) = \Delta v(x,t)$$

we obtain

$$\hat{v}_t(\xi,t) = -|\xi|^2 \hat{v}(\xi,t),$$

and hence the solution is given by,

$$\hat{v}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi).$$

From where it can be deduced that

$$\|v(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leq C\, t^{-d/2(1-1/q)}.$$

The convolution model

The asymptotic behavior as $t \to \infty$ for the nonlocal model

$$u_t(x,t) = (G * u - u)(x,t) = \int_{\mathbb{R}^d} G(x-y)u(y,t)\,dy - u(x,t),$$

is given by

Theorem The solutions verify

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le Ct^{-d/2}.$$

The convolution model

The proof of this fact is based on a explicit representation formula for the solution in Fourier variables. In fact, from the equation

$$u_t(x,t) = (G * u - u)(x,t),$$

we obtain

$$\hat{u}_t(\xi,t) = (\hat{G}(\xi) - 1)\hat{u}(\xi,t),$$

and hence the solution is given by,

$$\hat{u}(\xi,t) = e^{(\hat{G}(\xi)-1)t}\hat{u}_0(\xi).$$

From this explicit formula it can be obtained the decay in $L^\infty(\mathbb{R}^d)$ of the solutions. Just observe that

$$\hat{u}(\xi,t) = e^{(\hat{G}(\xi)-1)t}\hat{u_0}(\xi) \approx e^{-t}\hat{u_0}(\xi),$$

for ξ large and

$$\hat{u}(\xi,t) = e^{(\hat{G}(\xi)-1)t}\hat{u_0}(\xi) \approx e^{-|\xi|^2 t}\hat{u_0}(\xi),$$

for $\xi \approx 0$. Hence, one can obtain

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq Ct^{-d/2}.$$

This decay, together with the conservation of mass, gives the decay of the $L^q(\mathbb{R}^d)$-norms by interpolation. It holds,

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^d)} \leq C\, t^{-d/2(1-1/q)}.$$

Note that the asymptotic behavior is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a gaussian.

Non-local problems without a convolution

To begin our analysis, we first deal with a linear nonlocal diffusion operator of the form

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))\, dy.$$

Also consider

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\, dy.$$

Note that use of the Fourier transform is useless.

Energy estimates for the heat equation

Let us begin with the simpler case of the estimate for solutions to the heat equation in $L^2(\mathbb{R}^d)$-norm. Let

$$u_t = \Delta u.$$

If we multiply by u and integrate in $\mathbb{R}^d$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} \frac{1}{2}u^2(x,t)dx = -\int_{\mathbb{R}^d} |\nabla u(x,t)|^2 dx.$$

Now we use Sobolev's inequality, with $2^* = \frac{2d}{(d-2)}$,

$$\int_{\mathbb{R}^d} |\nabla u|^2(x,t)\,dx \geq C\left(\int_{\mathbb{R}^d} |u|^{2^*}(x,t)\,dx\right)^{2/2^*}$$

to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx \leq -C\left(\int_{\mathbb{R}^d} |u|^{2^*}(x,t)\,dx\right)^{2/2^*}.$$

Energy estimates for the heat equation

If we use interpolation and conservation of mass, that implies $\|u(t)\|_{L^1(\mathbb{R}^d)} \leq C$ for any $t > 0$, we have

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|u(t)\|^{\alpha}_{L^1(\mathbb{R}^d)} \|u(t)\|^{1-\alpha}_{L^{2^*}(\mathbb{R}^d)} \leq C \|u(t)\|^{1-\alpha}_{L^{2^*}(\mathbb{R}^d)}$$

with α determined by

$$\frac{1}{2} = \alpha + \frac{1-\alpha}{2^*}, \qquad \text{that is,} \qquad \alpha = \frac{2^* - 2}{2(2^* - 1)}.$$

Hence we get

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2(x,t)\, dx \leq -C \left(\int_{\mathbb{R}^d} u^2(x,t)\, dx \right)^{\frac{1}{1-\alpha}}$$

from where the decay estimate

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq C\, t^{-\frac{d}{2}\left(1-\frac{1}{2}\right)}, \qquad t > 0,$$

follows.

Energy estimates for the non-local equation

We want to mimic the steps for the nonlocal evolution problem

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))\, dy.$$

Hence, we multiply by u and integrate in $\mathbb{R}^d$ to obtain,

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} u^2(x,t)\, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))\, dy\, u(x,t)\, dx.$$

Energy estimates for the non-local equation

Now, we need to “integrate by parts”. We have

lemma

If J is symmetric, $J(x,y) = J(y,x)$ then it holds

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(\varphi(y)-\varphi(x))\psi(x)dydx$$
$$= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(\varphi(y)-\varphi(x))(\psi(y)-\psi(x))dydx.$$

Energy estimates for the non-local equation

If we use this lemma we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}u^2(x,t)dx = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(u(y,t)-u(x,t))^2\,dy\,dx.$$

Now we run into troubles since there is no analogous to Sobolev inequality. In fact, an inequality of the form

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(u(y,t)-u(x,t))^2\,dy\,dx \geq C\left(\int_{\mathbb{R}^d} u^q(x,t)\,dx\right)^{2/q}$$

can not hold for any $q > 2$.

Energy estimates for the non-local equation

Now the idea is to split the function u as the sum of two functions

$$u = v + w,$$

where on the function v (the "smooth"part of the solution) the nonlocal operator acts as a gradient and on the function w (the "rough"part) it does not increase its norm significatively. Therefore, we need to obtain estimates for the $L^p(\mathbb{R}^d)$-norm of the nonlocal operators.

Energy estimates for the non-local equation

Theorem Let $p \in [1, \infty)$ and $J(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ be a symmetric nonnegative function satisfying
HJ1) There exists a positive constant $C < \infty$ such that

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)\, dx \leq C.$$

HJ2) There exist positive constants c_1, c_2 and a function $a \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ satisfying

$$\sup_{x \in \mathbb{R}^d} |\nabla a(x)| < \infty$$

such that the set $B_x = \{y \in \mathbb{R}^d : |y - a(x)| \leq c_2\}$ verifies $B_x \subset \{y \in \mathbb{R}^d : J(x, y) > c_1\}$.

Energy estimates for the non-local equation

Theorem Then, for any function $u \in L^p(\mathbb{R}^d)$ there exist two functions v and w such that $u = v + w$ and

$$\|\nabla v\|^p_{L^p(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)} \leq C(J,p) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y)|u(x) - u(y)|^p \, dx \, dy.$$

Moreover, if $u \in L^q(\mathbb{R}^d)$ with $q \in [1, \infty]$ then the functions v and w satisfy

$$\|v\|_{L^q(\mathbb{R}^d)} \leq C(J,q) \|u\|_{L^q(\mathbb{R}^d)}$$

and

$$\|w\|_{L^q(\mathbb{R}^d)} \leq C(J,q) \|u\|_{L^q(\mathbb{R}^d)}.$$

Energy estimates for the non-local equation

We note that using the classical Sobolev's inequality

$$\|v\|_{L^{p^*}(\mathbb{R}^d)} \leq \|\nabla v\|_{L^p(\mathbb{R}^d)}$$

we get

$$\|v\|^p_{L^{p*}(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)} \leq C(J,p) \int_{\mathbb{R}^r} \int_{\mathbb{R}^d} J(x,y)|u(x)-u(y)|^p \, dx \, dy.$$

Energy estimates for the non-local equation

To simplify the notation let us denote by $\langle A_p u, u\rangle$ the following quantity,

$$\langle A_p u, u\rangle := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)|u(x)-u(y)|^p \, dx\, dy.$$

Corollary

$$\|u\|^p_{L^p(\mathbb{R}^d)} \leq C_1 \|u\|^{p(1-\alpha(p))}_{L^1(\mathbb{R}^d)} \langle A_p u, u\rangle^{\alpha(p)} + C_2 \langle A_p u, u\rangle,$$

where $\alpha(p)$ is given by

$$\alpha(p) = \frac{p^*}{p'(p^*-1)} = \frac{d(p-1)}{d(p-1)+p}.$$

Energy estimates for the non-local equation

Remark In the case of the local operator $B_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, using Sobolev's inequality and interpolation inequalities we have the following estimate

$$\|u\|^p_{L^p(\mathbb{R}^d)} \leq C_1 \|u\|^{p(1-\alpha(p))}_{L^1(\mathbb{R}^d)} \langle B_p u, u\rangle^{\alpha(p)}.$$

In the nonlocal case an extra term involving $\langle A_p u, u\rangle$ occurs.

Decay estimates for the non-local equation

Let us consider

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))\, dy + f(u)(x,t)$$

Theorem Let f be a locally Lipshitz function with $f(s)s \leq 0$.

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \leq C(q,d)\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{q})}$$

for all $q \in [1,\infty)$ and for all t sufficiently large.

Decay estimates for the non-local equation

Using these ideas we can also deal with the following nonlocal analogous to the $p-$laplacian evolution,

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\,dy.$$

Theorem Let $2 \le p < d$. For any $1 \le q < \infty$ the solution verifies

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le Ct^{-\left(\frac{d}{d(p-2)+p}\right)\left(1-\frac{1}{q}\right)}$$

for all t sufficiently large.

THANKS !!!!.

Thanks !!!