Optimal Packings of Lines

Matthew Fickus

Department of Mathematics and Statistics
Air Force Institute of Technology
Wright-Patterson Air Force Base, Ohio

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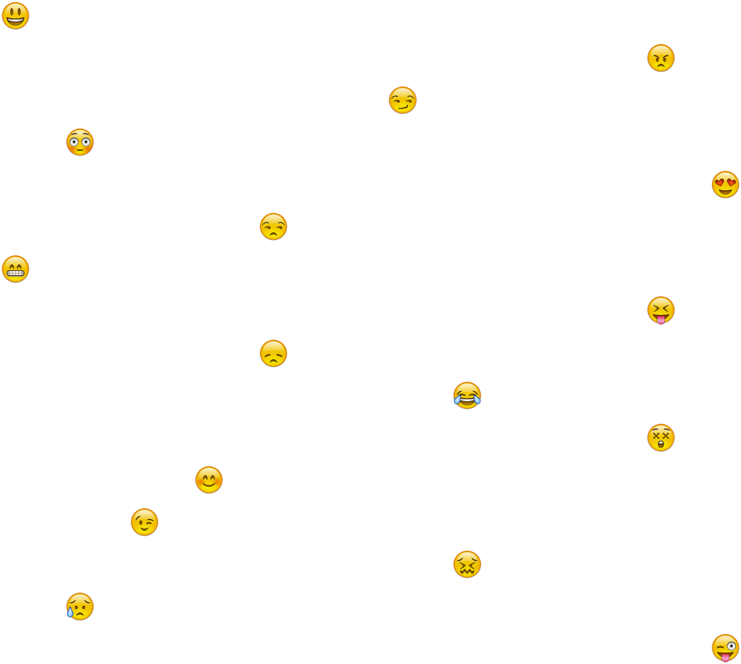
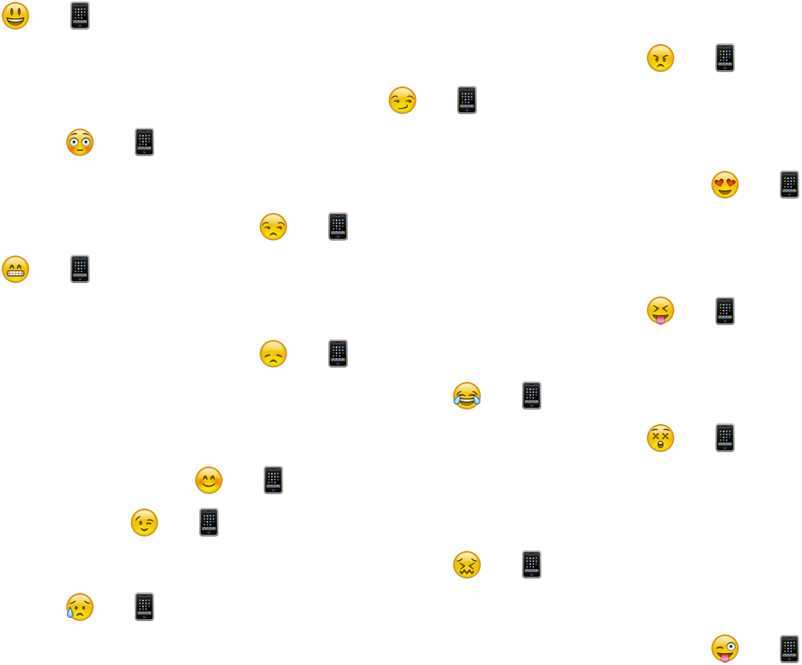
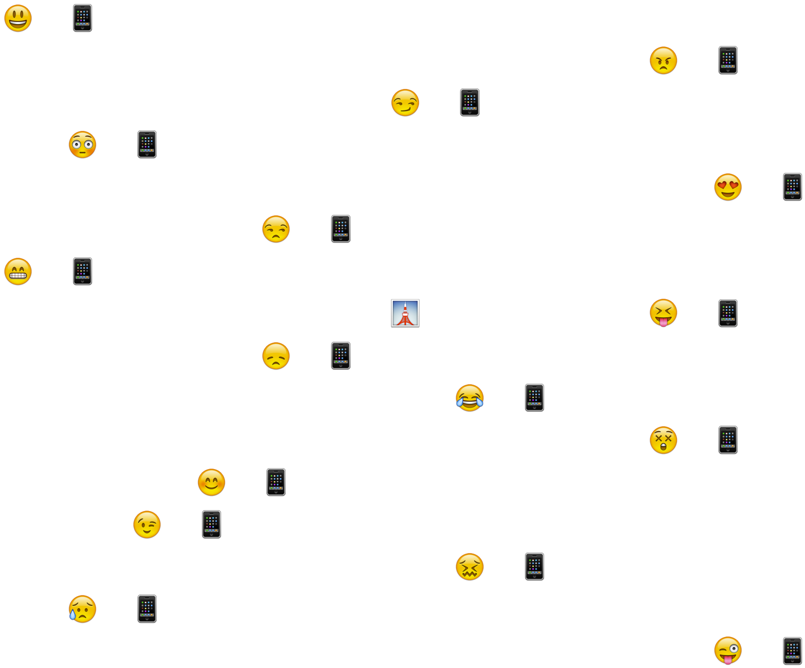
Motivation:

Wireless Communication

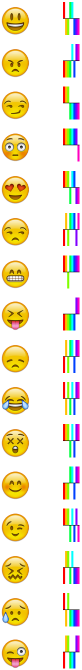


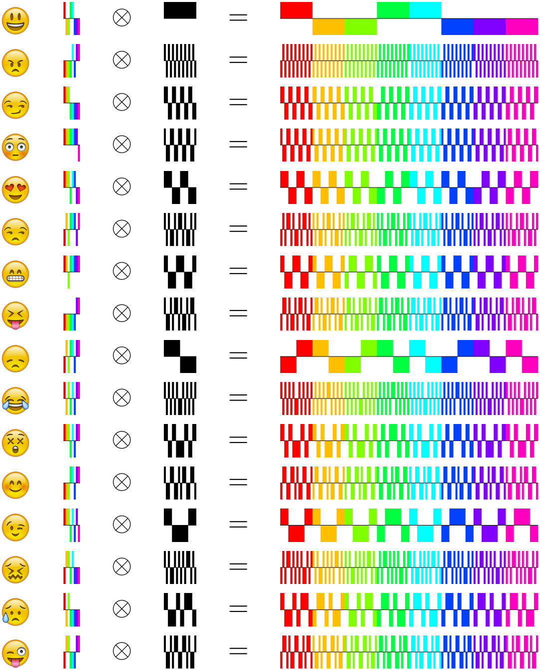


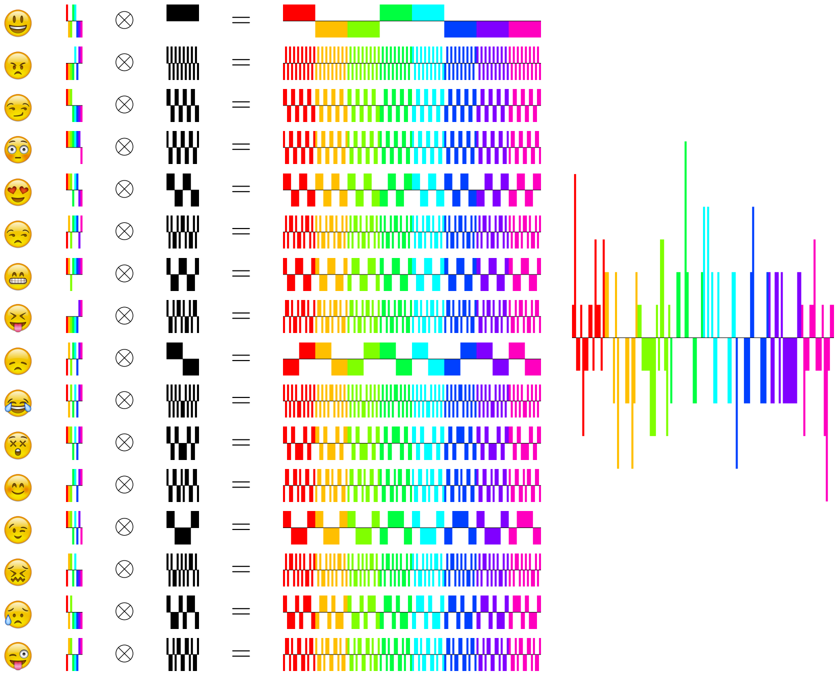


Real Application, Toy Example

- 16 transmitters
- Each transmits either a "$+1$", "-1" or "0" every second
- One receiver
- Receives one real scalar each second
- Transmitters are synchronized with receiver and each other
- Received signal is a superposition of transmitted signals
- Ignoring signal path issues, such as power loss, multipath
- Fixed codebook







Code Division Multiple Access (CDMA)

Received Signal: $\mathbf{\Phi x} \in \mathbb{R}^{16}$ where $\mathbf{x} \in \mathbb{R}^{16}$ contains the first ± 1 "bits" of each user and $\mathbf{\Phi} \in \mathbb{R}^{16\times 16}$ is the Hadamard matrix:

$$\mathbf{\Phi} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
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1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix}$$

Pros: Strong reception, decoding is straightforward:

$$\mathbf{\Phi x} = \mathbf{y} \quad \Longrightarrow \quad \mathbf{x} = \tfrac{1}{16}\mathbf{\Phi}^{\mathrm{T}}\mathbf{y} \;\; \text{since } \mathbf{\Phi} \textit{ has orthogonal columns.}$$

Cons: Is inefficient when only a few users are transmitting.

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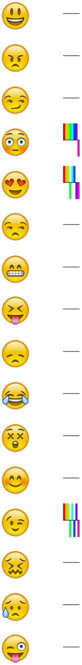

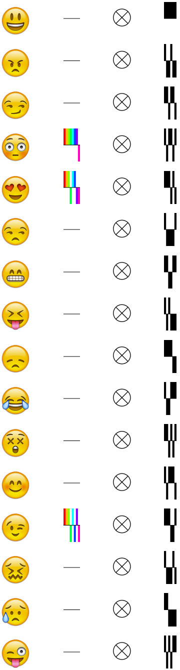

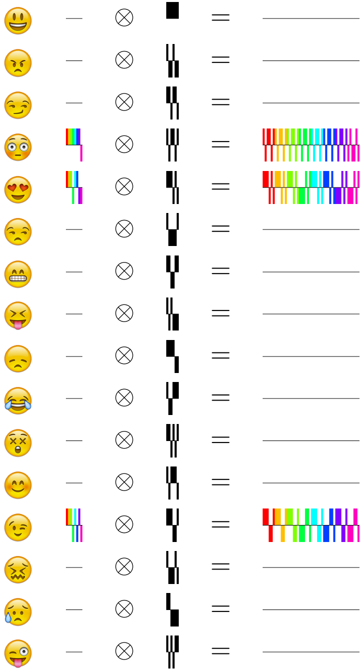

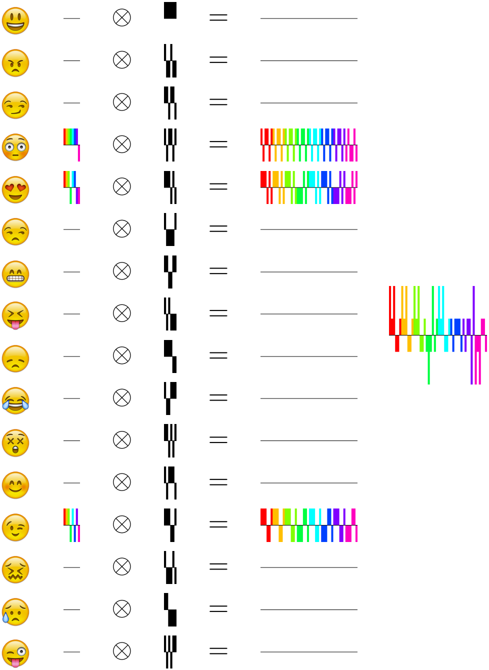

Overcomplete CDMA

Received Signal: $\mathbf{\Phi x} \in \mathbb{R}^6$ where $\mathbf{x} \in \mathbb{R}^{16}$ contains the first ± 1 "bits" of each user and $\mathbf{\Phi} \in \mathbb{R}^{6 \times 16}$ is the partial Hadamard matrix:

$$\mathbf{\Phi} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Pros: Strong reception, higher bit rates without reallocating codewords; (one bit every six seconds instead of every sixteen).

Cons: Decoding is impossible if more than 6 users are active at any time; ($\mathbf{\Phi x} = \mathbf{y}$ will have an infinite number of solutions.)
Even when possible, decoding is tough; interference is a problem.

Challenge: Given $M \leq N$, design an $M \times N$ matrix $\mathbf{\Phi}$ so that

- The entries of $\mathbf{\Phi}$ are ± 1 (or more generally, unimodular complex).
- The columns of $\mathbf{\Phi}$ are as "orthogonal as possible."
  (It turns out this minimizes interference and help reconstruction.)

Maximal Orthonormality

Packing Lines

- N = number of lines (e.g. $N = 16$)
- M = dimension of space (e.g. $M = 6$)
- $\{\varphi_n\}_{n=1}^N$ = unit vectors in $\mathbb{R}^M$ (e.g. normalized codewords)
- $\langle \varphi_n, \varphi_{n'} \rangle = \cos(\theta_{n,n'})$ where $\theta_{n,n'}$ is the angle between φ_n and $\varphi_{n'}$
- $|\langle \varphi_n, \varphi_{n'} \rangle|$ is the *interior* angle between $\text{span}\{\varphi_n\}$ and $\text{span}\{\varphi_{n'}\}$
- **Packing** lines is maximizing the minimum $\theta_{n,n'}$, namely minimizing

$$\max_{n \neq n'} |\langle \varphi_n, \varphi_{n'} \rangle|.$$

Frame Terminology and Notation

Definition: For any $M \leq N$ and any unit norm $\{\varphi_n\}_{n=1}^{N}$ in $\mathbb{C}^M$:

- The corresponding **synthesis operator** is the $M \times N$ matrix

$$\mathbf{\Phi} = \left[\begin{array}{ccc} \varphi_1 & \cdots & \varphi_N \end{array} \right].$$

- The **analysis operator** is its $N \times M$ adjoint (conjugate transpose)

$$\mathbf{\Phi}^* = \left[\begin{array}{c} \varphi_1^* \\ \vdots \\ \varphi_N^* \end{array} \right].$$

- The **Gram matrix** is the $N \times N$ matrix

$$\mathbf{\Phi}^*\mathbf{\Phi} = \left[\begin{array}{cccc} 1 & \langle\varphi_2, \varphi_1\rangle & \cdots & \langle\varphi_N, \varphi_1\rangle \\ \langle\varphi_1, \varphi_2\rangle & 1 & \cdots & \langle\varphi_N, \varphi_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varphi_1, \varphi_N\rangle & \langle\varphi_2, \varphi_N\rangle & \cdots & 1 \end{array} \right].$$

Distance from Orthonormality

Note: $\{\varphi_n\}_{n=1}^N$ is orthonormal if and only if $\mathbf{\Phi}^*\mathbf{\Phi} = \mathbf{I}$. Thus, for a given matrix norm $\|\cdot\|$ on the space of self-adjoint matrices, $\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|$ represents how orthonormal $\{\varphi_n\}_{n=1}^N$ is in terms of that norm:

- The **coherence** of $\{\varphi_n\}_{n=1}^N$ is

$$\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\max} = \max_{n \neq n'} |\langle \varphi_n, \varphi_{n'} \rangle|.$$

- The **frame potential** of $\{\varphi_n\}_{n=1}^N$ is

$$\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{Fro}}^2 = \sum_{n=1}^N \sum_{\substack{n'=1 \\ n' \neq n}}^N |\langle \varphi_n, \varphi_{n'} \rangle|^2.$$

- The **restricted isometry constant** of $\{\varphi_n\}_{n=1}^N$ is:

$$\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{RI}} = \max_{\substack{\mathcal{K} \subseteq [1,N] \\ |\mathcal{K}| = K}} \|\mathbf{\Phi}_{\mathcal{K}}^*\mathbf{\Phi}_{\mathcal{K}} - \mathbf{I}\|_2.$$

Unit Norm Tight Frames

Definition: $\{\varphi_n\}_{n=1}^N$ is a **tight frame** for $\mathbb{C}^M$ if $\exists A > 0$ such that

$$\mathbf{y} = \frac{1}{A}\sum_{n=1}^{N}\langle \mathbf{y}, \varphi_n\rangle \varphi_n = \frac{1}{A}\mathbf{\Phi\Phi}^*\mathbf{y}, \qquad \forall \mathbf{y} \in \mathbb{C}^M.$$

namely when the corresponding **frame operator** is $\mathbf{\Phi\Phi}^* = A\mathbf{I}$.

Note: $\{\varphi_n\}_{n=1}^N$ is a tight frame if and only if the rows of $\mathbf{\Phi}$ are orthogonal and have constant norm $\sqrt{A} > 0$.

Lemma: If $\{\varphi_n\}_{n=1}^N$ is a **unit norm tight frame** (a tight frame with $\|\varphi_n\|_2 = 1$ for all n) then A is necessarily the frame's **redundancy** N/M.

Proof: $MA = \mathrm{Tr}(A\mathbf{I}) = \mathrm{Tr}(\mathbf{\Phi\Phi}^*) = \mathrm{Tr}(\mathbf{\Phi}^*\mathbf{\Phi}) = \sum_{n=1}^{N} \|\varphi_n\|^2 = \sum_{n=1}^{N} 1 = N.$

Welch's Lower Bound on Frame Potential

Theorem: [Folklore, Welch 74] For unit norm vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{C}^M$,

$$\frac{N(N-M)}{M} \leq \|\boldsymbol{\Phi}^*\boldsymbol{\Phi} - \mathbf{I}\|_{\mathrm{Fro}}^2 = \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n' \neq n}}^{N} |\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle|^2,$$

with equality $\Leftrightarrow$ $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ is a unit norm tight frame (UNTF).

Proof:

$$\begin{aligned}
0 &\leq \|\boldsymbol{\Phi}\boldsymbol{\Phi}^* - \tfrac{N}{M}\mathbf{I}\|_{\mathrm{Fro}}^2 \\
&= \mathrm{Tr}[(\boldsymbol{\Phi}\boldsymbol{\Phi}^* - \tfrac{N}{M}\mathbf{I})^2] \\
&= \mathrm{Tr}[(\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2] - 2\tfrac{N}{M}\,\mathrm{Tr}(\boldsymbol{\Phi}\boldsymbol{\Phi}^*) + \tfrac{N^2}{M^2}\,\mathrm{Tr}\,\mathbf{I} \\
&= \mathrm{Tr}[(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^2] - 2\tfrac{N^2}{M} + \tfrac{N^2}{M} \\
&= \sum_{n=1}^{N} \sum_{n'=1}^{N} |\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle|^2 - \tfrac{N^2}{M} \\
&= \|\boldsymbol{\Phi}^*\boldsymbol{\Phi} - \mathbf{I}\|_{\mathrm{Fro}}^2 + N - \tfrac{N^2}{M}.
\end{aligned}$$

Welch's Lower Bound on Coherence

Theorem: [Rankin 56, Welch 74] For unit norm vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{C}^M$,

$$\sqrt{\frac{N-M}{M(N-1)}} \leq \|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\max} = \max_{n\neq n'} |\langle \varphi_n, \varphi_{n'}\rangle|,$$

with equality $\Leftrightarrow$ $\{\varphi_n\}_{n=1}^N$ is an **equiangular tight frame (ETF)**, i.e.

- The rows of $\mathbf{\Phi}$ are orthogonal and have constant norm,
- The columns of $\mathbf{\Phi}$ have unit norm,
- The dot products of distinct columns of $\mathbf{\Phi}$ have constant magnitude.

Proof: $$\frac{N(N-M)}{M} \leq \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n'\neq n}}^{N} |\langle \varphi_n, \varphi_{n'}\rangle|^2 \leq N(N-1) \max_{n\neq n'} |\langle \varphi_n, \varphi_{n'}\rangle|^2.$$

Equiangular Tight Frames

An Example of an ETF with $M = 6$, $N = 16$

$$\mathbf{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{\Phi}\mathbf{\Phi}^* = \frac{1}{6} \begin{bmatrix} 16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix}$$

$$\mathbf{\Phi}^*\mathbf{\Phi} = \frac{1}{6} \begin{bmatrix} 6 & -2 & 2 & 2 & 2 & -2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & -2 & -2 & 2 \\ -2 & 6 & 2 & 2 & -2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 \\ 2 & 2 & 6 & -2 & 2 & -2 & 2 & -2 & 2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 \\ 2 & 2 & -2 & 6 & -2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & -2 & 2 \\ 2 & -2 & 2 & -2 & 6 & -2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 2 & -2 & 2 & -2 & 6 & 2 & 2 & -2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 & 2 & 2 & 6 & -2 & -2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 \\ -2 & 2 & -2 & 2 & 2 & 2 & -2 & 6 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & -2 & -2 & 2 & 6 & -2 & 2 & 2 & 2 & -2 & 2 & -2 \\ 2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 & -2 & 6 & 2 & 2 & -2 & 2 & -2 & 2 \\ 2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 & 2 & 2 & 6 & -2 & 2 & -2 & 2 & -2 \\ 2 & 2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & -2 & 6 & -2 & 2 & -2 & 2 \\ 2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & -2 & 6 & -2 & 2 & 2 \\ -2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 & -2 & 2 & -2 & 2 & -2 & 6 & 2 & 2 \\ -2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & -2 & 2 & 2 & 6 & -2 \\ 2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & -2 & 2 & 2 & 2 & -2 & 6 \end{bmatrix}$$

Known ETF Constructions

So far, there are only two known ways of constructing **flexible** infinite families of ETFs, namely those that permit independent control of the orders of the sizes of M and N:

Method 1: Harmonic ETFs
[Turyn 65; Strohmer & Heath 03; Xia, *et al.* 05; Ding & Feng 07]

Take the character table of an abelian group, e.g. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; extract rows that correspond to a **difference set**, e.g.:

$-$	(0, 0, 0, 0)	(0, 0, 1, 0)	(1, 0, 0, 0)	(1, 0, 0, 1)	(1, 1, 0, 0)	(1, 1, 1, 1)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 1, 0)	(1, 0, 0, 0)	(1, 0, 0, 1)	(1, 1, 0, 0)	(1, 1, 1, 1)
(0, 0, 1, 0)	(0, 0, 1, 0)	(0, 0, 0, 0)	(1, 0, 1, 0)	(1, 0, 1, 1)	(1, 1, 1, 0)	(1, 1, 0, 1)
(1, 0, 0, 0)	(1, 0, 0, 0)	(1, 0, 1, 0)	(0, 0, 0, 0)	(0, 0, 0, 1)	(0, 1, 0, 0)	(0, 1, 1, 1)
(1, 0, 0, 1)	(1, 0, 0, 1)	(1, 0, 1, 1)	(0, 0, 0, 1)	(0, 0, 0, 0)	(0, 1, 0, 1)	(0, 1, 1, 0)
(1, 1, 0, 0)	(1, 1, 0, 0)	(1, 1, 1, 0)	(0, 1, 0, 0)	(0, 1, 0, 1)	(0, 0, 0, 0)	(0, 0, 1, 1)
(1, 1, 1, 1)	(1, 1, 1, 1)	(1, 1, 0, 1)	(0, 1, 1, 1)	(0, 1, 1, 0)	(0, 0, 1, 1)	(0, 0, 0, 0)

Method 2: Steiner/Kirkman ETFs
[Goethals & Seidel 70; F, Mixon & Tremain 12; Jasper, Mixon & F 13]

Steiner/Kirkman ETFs

Definition: A **(2,k,v)-Steiner system** is a set of v points $\mathcal{V}$ along with a collection $\mathcal{B}$ of b subsets (blocks) of $\mathcal{V}$ such that:

- every block contains exactly k points,
- every point is contained in exactly r blocks,
- any two distinct points are contained in exactly one block.

A Steiner system is **resolvable (Kirkman)** if its blocks $\mathcal{B}$ can be partitioned into subcollections so that each forms partition for $\mathcal{V}$.

Example: For when $k = 2$, $v = 4$, $r = 3$, $b = 6$ consider the $b \times v = 6 \times 4$ incidence matrix of all 2-subsets of $\{1, 2, 3, 4\}$:

$$\mathbf{B} = \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Example: Steiner ETF Construction

$$\text{“}+\text{''} = 1, \ \text{“}-\text{''} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{''} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1, \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ \color{red}{0} & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

"+" = 1, "−" = −1

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1, \ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{"}+\text{"} = 1,\ \text{"}-\text{"} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{"}\otimes\text{"} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction

$$\text{“}+\text{''} = 1, \ \text{“}-\text{''} = -1$$

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix} \text{“}\otimes\text{''} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Steiner ETF Construction ($\rightarrow$ Kirkman)

$$\text{“}+\text{”} = 1,\ \text{“}-\text{”} = -1$$

$$\left[\begin{array}{cccc} + & + & 0 & 0 \\ 0 & 0 & + & + \\ \hline + & 0 & + & 0 \\ 0 & + & 0 & + \\ \hline + & 0 & 0 & + \\ 0 & + & + & 0 \end{array}\right] \text{“}\otimes\text{”} \left[\begin{array}{cccc} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ \hline + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ \hline + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Kirkman ETF Construction

$$
\left[\begin{array}{cc|cc|cc}
+ & + & 0 & 0 & 0 & 0 \\
+ & - & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & + & + & 0 & 0 \\
0 & 0 & + & - & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & + & + \\
0 & 0 & 0 & 0 & + & -
\end{array}\right]
$$

$$
\times \left[\begin{array}{cccc|cccc|cccc|cccc}
+ & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ \hline
+ & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ \hline
+ & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\
0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
= \left[\begin{array}{cccccccccccccccc}
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{array}\right]
$$

Results on Steiner/Kirkman ETFs

Theorem: [F, Mixon & Tremain 12] Any $(2, k, v)$-Steiner system generates an ETF of size

$$M = b = \frac{v(v-1)}{k(k-1)}, \quad N = v(r+1) = v\left(\frac{v-1}{k-1} + 1\right).$$

The redundancy of this ETF is $\frac{N}{M} = k(1 + \frac{1}{r}) \approx k$.

Theorem: [Jasper, Mixon & F 13]

- If a Steiner system is resolvable its ETF can be unitarily transformed into a constant-amplitude Kirkman ETF.
- Every harmonic ETF generated by a McFarland difference set is a Kirkman ETF arising from a finite affine geometry.
- Every real constant-amplitude ETF corresponds to a complementary binary code that achieves equality in the **Grey-Rankin bound:**

$$2N \leq \frac{8\Delta(M-\Delta)}{M-(M-2\Delta)^2}, \quad \Delta := \min_{n \neq n'} \mathrm{d}(\mathbf{c}_n, \mathbf{c}_{n'}),$$

and vice versa.

Infinite Flexible Families of Steiner/Kirkman ETFs

Steiner ETFs from Finite Projective Geometries: For any $d \geq 2$ and prime power q there exists a Steiner ETF with

$$M = \frac{(q^j - 1)(q^{j+1} - 1)}{(q+1)(q-1)^2}, \quad N = \frac{q^{j+1} - 1}{q - 1}\left(\frac{q^j - 1}{q - 1} - 1\right).$$

These systems are only known to be resolvable for $d = 3$.

Kirkman ETFs from Finite Affine Geometries: For any $d \geq 1$ and prime power q there exists a Kirkman ETF with

$$M = q^d\left(\frac{q^{d+1} - 1}{q - 1}\right), \quad N = q^{d+1}\left(\frac{q^{d+1} - 1}{q - 1} + 1\right).$$

We've shown that every McFarland Harmonic ETF is one of these.

Kirkman ETFs from Denniston Designs: For $i \leq j$, $\exists$ an ETF with

$$M = (2^j + 1)(2^j + 1 - 2^{j-i}), \quad N = 2^i(2^j + 2)(2^j + 1 - 2^{j-i}).$$

Necessary Conditions on ETF Existence

Lemma: [Folklore] $\exists$ an $M \times N$ ETF $\Leftrightarrow$ $\exists$ an $(N-M) \times N$ ETF.

Lemma: [Gerzon 73, Tropp 05] If $\{\varphi_n\}_{n=1}^N$ are equiangular and not colinear in $\mathbb{F}^M$ then $N \le M^2$ when $\mathbb{F} = \mathbb{C}$ and $N \le \binom{M+1}{2}$ when $\mathbb{F} = \mathbb{R}$.

Proof: The vectors $\{\varphi_n \varphi_n^*\}_{n=1}^N$ are necessarily linearly independent in the M-dimensional real space of $M \times M$ self-adjoint matrices, and, in the real case, in the $\binom{M+1}{2}$-dimensional space of $M \times M$ real symmetric matrices.

Note: Generalizing this argument gives that *constant-amplitude* ETFs satisfy $N \le M^2 - M + 1$ when $\mathbb{F} = \mathbb{C}$ and $N \le \binom{M}{2} + 1$ when $\mathbb{F} = \mathbb{R}$.

Theorem: [Sustik, Tropp, Dhillon & Heath 07] Given $1 < M < N-1$ such that $N \ne 2M$, if there exists an N-vector ETF for $\mathbb{R}^M$ then

$$\sqrt{\frac{M(N-1)}{N-M}} \quad \text{and} \quad \sqrt{\frac{(N-M)(N-1)}{M}}$$

are necessarily odd integers.

Optimal Line Packing: Open Problems

For a given M and N, what are the optimal packings of N lines in $\mathbb{R}^M$ or $\mathbb{C}^M$, namely the sequences of unit norm vectors $\{\varphi_n\}_{n=1}^N$ which minimize:

$$\max_{n \neq n'} |\langle \varphi_n, \varphi_{n'} \rangle|$$

- overall? (including when ETFs of this size do not exist)
- which are also tight?
- which are also constant-amplitude?

Note: These problems are open in even very low-dimensional spaces. E.g., does there exists an ETF of 8 vectors in $\mathbb{C}^3$? Numerical evidence suggests no; in this case what are the optimal packings?

To begin to answer these hard questions, we seek a more basic understanding of a more general class of frames...

Unit Norm Tight Frames

Example: 3×5 UNTFs

(**Recall:** Unit vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{C}^M$ are a UNTF if $\mathbf{\Phi\Phi}^* = \frac{N}{M}\mathbf{I}$.)

$$\mathbf{\Phi} = \begin{bmatrix} \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) \\ \varphi_1(2) & \varphi_2(2) & \varphi_3(2) & \varphi_4(2) & \varphi_5(2) \\ \varphi_1(3) & \varphi_2(3) & \varphi_3(3) & \varphi_4(3) & \varphi_5(3) \end{bmatrix}$$

- $MN = 3 \cdot 5 = 15$ unknowns,
- $M = 3$ row-norm conditions,
- $\binom{M}{2} = 3$ row-orthogonality conditions,
- $N = 5$ column-norm conditions,
  (one of which is superfluous since $\mathrm{Tr}(\mathbf{\Phi}^*\mathbf{\Phi}) = \mathrm{Tr}(\mathbf{\Phi\Phi}^*)$),
- Modulo rotations in $\mathbb{R}^M$; $\mathrm{O}(M)$ has dimension $\binom{M}{2} = 3$.

Thus, real $M \times N$ UNTFs, modulo rotation, should have "dimension"

$$MN - M - 2\binom{M}{2} - N + 1 = (M-1)(N-M-1) = 2 \cdot 1 = 2.$$

Question: Can we move onto (and around in) the set of $M \times N$ UNTFs?

Open Problems on Unit Norm Tight Frames

What we know about the set of all $M \times N$ UNTFs:

- [Goyal, Vetterli & Thao 98] It's nonempty for any $M \leq N$.
- It's a real algebraic variety:
  the set of solutions to a system of real quadratic equations.
- [Benedetto & F 03; Casazza, F & Mixon 12]
  They are precisely the set of *local* minimizers of the frame potential.
- [Dykema & Strawn 06] It's a real manifold whenever M is relatively prime to N; otherwise it's a finite union of manifolds.
- [Cahill, F, Mixon, Poteet & Strawn 13]
  We have an explicit parametrization for it.
- [Cahill, Mixon & Strawn 13] It's connected.

What we don't know about this set:

- **Paulsen Problem:** How to project onto it.
- How to optimize over it, e.g., find the UNTF of minimal coherence.

Minimizing Frame Potential

Recall: For any unit norm vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{C}^M$, the corresponding **frame potential** $\|\mathbf{\Phi}^*\mathbf{\Phi}-\mathbf{I}\|_{\mathrm{Fro}}^2$ satisfies

$$\frac{N(N-M)}{M} \leq \|\mathbf{\Phi}^*\mathbf{\Phi}-\mathbf{I}\|_{\mathrm{Fro}}^2 = \sum_{n=1}^{N}\sum_{\substack{n'=1\\ n'\neq n}}^{N} |\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'}\rangle|^2,$$

with equality $\Leftrightarrow$ $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ is a UNTF.

Theorem: [Benedetto & F 03] The frame potential is the potential energy contained within $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ under the assumption that each $\boldsymbol{\varphi}_n$ exerts a "frame force" on each other $\boldsymbol{\varphi}_{n'}$ of $\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'}\rangle(\boldsymbol{\varphi}_n - \boldsymbol{\varphi}_{n'})$.

Moreover, every *local* minimizer of the frame potential is a UNTF, meaning every local minimizer is a global minimizer.

Minimizing Frame Potential (Numerically)

Theorem: [Casazza, F & Mixon 12] Let M and N are relatively prime, and let $\{\varphi_n^{(0)}\}_{n=1}^N$ be any unit norm vectors in $\mathbb{C}^M$ which satisfy

$$\|\mathbf{\Phi}_0\mathbf{\Phi}_0^* - \tfrac{N}{M}\mathbf{I}\|_{\mathrm{Fro}}^2 \leq \tfrac{2}{M^3}.$$

For any $k \geq 0$, move the unit norm vectors $\{\varphi_n^{(k)}\}_{n=1}^N$ along great circles

$$\varphi_n^{(k+1)} := \begin{cases} \cos(\|\psi_n^{(k)}\|t)\varphi_n^{(k)} - \sin(\|\psi_n^{(k)}\|t)\frac{\psi_n^{(k)}}{\|\psi_n^{(k)}\|}, & \psi_n^{(k)} \neq 0, \\ \varphi_n^{(k)}, & \psi_n^{(k)} = 0, \end{cases}$$

with "step size" $t = \frac{1}{4N}$ in the direction of the gradient of the frame potential (restricted to unit norm vectors):

$$\psi_n^{(k)} = \mathbf{\Phi}_k\mathbf{\Phi}_k^*\varphi_n^{(k)} - \langle \mathbf{\Phi}_k\mathbf{\Phi}_k^*\varphi_n^{(k)}, \varphi_n^{(k)}\rangle\varphi_n^{(k)}, \quad \forall n = 1, \ldots, N.$$

Then, $\mathbf{\Phi}_\infty := \lim_k \mathbf{\Phi}_k$ exists and is a UNTF satisfying

$$\|\mathbf{\Phi}_\infty - \mathbf{\Phi}_0\|_{\mathrm{Fro}} \leq 8M^{20}N^{8.5}\left\|\mathbf{\Phi}_0\mathbf{\Phi}_0^* - \tfrac{N}{M}\mathbf{I}\right\|_{\mathrm{Fro}}.$$

Minimizing Generalized Frame Potentials

Definition: For any $p \geq 1$, define the ***p*th frame potential** as:

$$\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{Fro}(p)} := \left(\sum_{n=1}^{N}\sum_{\substack{n'=1 \\ n' \neq n}}^{N} |\langle \varphi_n, \varphi_{n'} \rangle|^p\right)^{\frac{1}{p}}.$$

So, $\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{Fro}(\infty)} =$ coherence, $\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{Fro}(2)}^2 =$ frame potential.

Theorem: If an $M \times N$ ETF exists, then for every $p > 1$, the $M \times N$ ETFs are precisely the global miminizers of $\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{Fro}(p)}$ over all unit norm vectors $\{\varphi_n\}_{n=1}^{N}$ in $\mathbb{C}^M$.

Open Problems: For $p > 2$,

- If $\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{Fro}(p)}$ is sufficiently small, are we close to an ETF?
- What are the minimizers of $\|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{Fro}(p)}$ when $\nexists$ an $M \times N$ ETF?

Summary and Future Work

Take Home:

- Line packing has real-world applications (e.g., communications).
- Equiangular tight frames (ETFs) are optimal line packings.
- Only a few methods for constructing ETFs are known; all involve combinatorial design and/or finite geometry.
- ETFs only exist in certain circumstances; in many cases the question of there existence is open.
- ETFs are a special case of unit norm tight frames (UNTFs); much has been learned about UNTFs over the past decade.

Future Work:

- Compile a table of all known constructions of ETFs.
- Generalize analysis-based tools from UNTFs to the study of ETFs; numerically minimize pth potentials to gain insight.

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