Accumulative density

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ABSTRACT

In this paper we study a notion of density for subsets of $\mathbb{R}^2$ called accumulative density, which is similar to the density for sequences in the unit disc developed by Seip. Along the way we derive some new properties of Beurling density. Finally, we prove that the accumulative density and the Beurling density coincide.

Keywords: Accumulative density, Beurling density, Separated sequence

1. INTRODUCTION

Gabor and wavelet systems are among the most important systems used for signal processing purposes. For many years regular Gabor systems and classical wavelet systems have been extensively employed. Recently, also the more general irregular Gabor and wavelet systems were studied. In this context, density conditions have turned out to be an especially useful and elegant tool. Conceptually, they are used to deliver necessary conditions for a system to form a frame or a Riesz basis. The densities employed for Gabor systems are the lower and upper Beurling density, whereas for wavelet systems a version of the Beurling density adapted to the geometry of the affine group has been used.

Given a function $g \in L^2(\mathbb{R})$ and a subset $\Lambda \subset \mathbb{R}^2$, the Gabor system determined by g and Λ is the collection of time-frequency shifts of g defined by

$$\mathcal{G}(g,\Lambda) = \{e^{2\pi ibx} g(x-a) : (a,b) \in \Lambda\}.$$

This system forms a *frame* for $L^2(\mathbb{R})$, if there exist $0 < A \le B < \infty$ (the *frame bounds*) such that

$$A\,\|f\|_2^2 \le \sum_{(a,b)\in\Lambda} |\langle f, e^{2\pi ib\cdot} g(\cdot - a)\rangle|^2 \le B\,\|f\|_2^2 \quad \text{for all } f \in L^2(\mathbb{R}),$$

and it forms a *Riesz basis* for $L^2(\mathbb{R})$, if it is complete and there exist $0 < A \le B < \infty$ such that for all finite sequences of scalars $c = \{c_{(a,b)}\}_{(a,b)\in\Lambda}$,

$$A \sum_{(a,b)\in\Lambda} |c_{(a,b)}|^2 \le \| \sum_{(a,b)\in\Lambda} c_{(a,b)} e^{2\pi ib\cdot} g(\cdot - a)\|^2 \le B \sum_{(a,b)\in\Lambda} |c_{(a,b)}|^2.$$

Classical results employing density are mostly concerned with the case of rectangular lattices of the form $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$. Rieffel[13] proved as a corollary of deep results on von Neumann algebras, that if $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ is a complete subset of $L^2(\mathbb{R})$ then necessarily $ab \le 1$. After a number of developments by Daubechies,[4] Landau,[11] Janssen,[7] and Ramanathan and Steger[12] proved that all Gabor frames $\mathcal{G}(g,\Lambda)$, without restrictions on g or Λ, satisfy a certain *Homogeneous Approximation Property* (HAP), and deduced from this that if $\mathcal{G}(g,\Lambda)$ is a frame then the lower Beurling density of Λ satisfies $D_B^-(\Lambda) \ge 1$ (note that $D_B^-(a\mathbb{Z} \times b\mathbb{Z}) = \frac{1}{ab}$), and that if this frame is a Riesz basis then $D_B^-(\Lambda) = D_B^+(\Lambda) = 1$. For the special case $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, they were also able to recover the completeness result of Rieffel. Some corrections and extensions were obtained by Christensen, Deng,

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and Heil.[3] In brief, in terms of necessary conditions for Gabor frames there is a critical or Nyquist density for Λ separating frames from non-frames, and furthermore the Riesz bases sit exactly at this critical density. Recently, Balan, Casazza, Heil, and Landau[1] showed that there is a fundamental connection between density properties and the so-called excess of a Gabor frame, which is a manifestation of deeper implications of the HAP and related properties of localized frames. One remarkable consequence[2] for Gabor frames $\mathcal{G}(g,\Lambda)$ with frame bounds A, B is the fact that they satisfy the relation $A \leq \mathcal{D}^-(\Lambda)\|g\|_2^2 \leq \mathcal{D}^+(\Lambda)\|g\|_2^2 \leq B$. Using a completely different method an extension of this result to weighted Gabor systems has been derived by Kutyniok.[9]

Another essential class of systems are wavelet systems generated by a wavelet $\psi \in L^2(\mathbb{R})$ and a set of indices $\Lambda \in \mathbb{R}^+ \times \mathbb{R}$ of the form

$$\mathcal{W}(\psi,\Lambda) = \{a^{-1/2}\psi(\tfrac{x}{a} - b) : (a,b) \in \Lambda\}.$$

The set $\mathbb{R}^+ \times \mathbb{R}$ can be endowed with a group structure called the affine group. Recently the notion of Beurling density was adapted to the geometry of this group and properties of wavelet systems were studied using this newly developed notion of affine density.[6, 8, 16, 17] Several results from Gabor systems were shown to hold also for these systems. However, the main difference is that in contrast to Gabor systems, wavelet systems do not possess a critical or Nyquist density.

It is well-known that density theorems for Gabor frames $\mathcal{G}(g,\Lambda)$ generated by Gaussian functions g are related to density questions in the Bargmann-Fock spaces.[14] Seip[15] introduced a notion of density for Bergman-type spaces on the unit disk, and it is possible to derive some density results for wavelet frames $\mathcal{W}(\psi,\Lambda)$ generated by certain wavelets ψ from those results. In this paper we study the notion of density which corresponds to this density for subsets of $\mathbb{R}^2$. In contrast to the definition of Beurling density the points of the sequence under consideration are not equally weighted but equipped with different weights dependent on their distance to some reference point, which itself moves across the plane during the computation. The weights of those points contained in a particular box are then added. Due to this accumulative way of computation we call this density *accumulative* density. Surprisingly, it will turn out that this density coincides with the Beurling density, thus leading to a different way of computing the Beurling density of a sequence in $\mathbb{R}^2$. Moreover, this result gives us insight in the meaning of Beurling density, showing that we can view it also as some sort of weighted measure. The overall goal is to derive some relationship between the density employed by Seip and the affine density developed for wavelet systems by Heil, Kutyniok, Sun, and Zhou. Our conjecture is that they in fact are not only equivalent, but even coincide. The main result contained in this paper is the first step towards an extensive study of this relation.

The paper is organized as follows. In Section 2 we introduce both notions of density. First we recall the definition of Beurling density and state some well-known results together with some new observations. Subsection 2.2 then deals with the definition of accumulative density. In the last section we compute both densities for one special sequence in $\mathbb{R}^2$ (Subsection 3.1). The main result (Theorem 3.1) which shows that for each sequence in $\mathbb{R}^2$ the Beurling density coincides with the accumulative density is stated in Subsection 3.2.

2. NOTIONS OF DENSITY

Given a countable subset Λ of $\mathbb{R}^2$, we are interested in measures of the "density" of their points. First we will introduce the classical Beurling density and study several of its properties. Secondly, we examine the notion of density which corresponds to the density for the unit disc developed by Seip.[15] We remark that both notions of density can be easily extended to subsets of $\mathbb{R}^d$. But in this paper we will focus only on the 2-dimensional situation, since this is the case needed to study Gabor systems in $L^2(\mathbb{R})$.

2.1. Beurling density

Beurling density is a measure of the "average" number of points of a set that lie inside a unit cube. For $h > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we let $Q_h(x)$ denote the cube centered at x with side length $2h$, i.e.,

$$Q_h(x) = [x_1 - h, x_1 + h] \times [x_2 - h, x_2 + h]\,.$$

Definition 2.1. *Given a sequence* Λ *in* $\mathbb{R}^2$ *the* upper Beurling density *of* Λ *is defined by*

$$\mathcal{D}_B^+(\Lambda) = \limsup_{h\to\infty} \frac{1}{4h^2} \sup_{x\in\mathbb{R}^2} \#(\Lambda \cap Q_h(x)),$$

and the lower Beurling density *of* Λ *is*

$$\mathcal{D}_B^-(\Lambda) = \liminf_{h\to\infty} \frac{1}{4h^2} \inf_{x\in\mathbb{R}^2} \#(\Lambda \cap Q_h(x)).$$

If $\mathcal{D}_B^-(\Lambda) = \mathcal{D}_B^+(\Lambda)$, *then we say that* Λ *has* uniform Beurling density *and denote this density by* $\mathcal{D}_B(\Lambda)$.

There exist an extension[9] of this definition to weighted subsets of $\mathbb{R}^2$.

The following properties of sequences in $\mathbb{R}^2$ will become crucial later.

Definition 2.2. *Let* Λ *be a sequence in* $\mathbb{R}^2$.

1. Λ *is* uniformly separated, *if* $\inf_{\lambda_1,\lambda_2\in\Lambda,\lambda_1\neq\lambda_2} |\lambda_1 - \lambda_2| > 0$.
2. Λ *is* relatively uniformly separated, *if it is a finite union of uniformly separated sequences.*

We may ask whether we can also define density using a different shape than the shape of cubes. In fact, Landau[10] showed that the definition does not depend on the shape of the analyzing exhausting sequence of sets $\{Q_h\}_{h>0}$. In particular, he proved the following result, which we state here only for subsets of $\mathbb{R}^2$.

Lemma 2.3. *Let* Λ *be a uniformly separated sequence in* $\mathbb{R}^2$, *and let* I *be a compact set of measure* 1 *whose boundary has measure zero. Then*

$$\mathcal{D}_B^+(\Lambda) = \limsup_{h\to\infty} \frac{1}{h^2} \sup_{x\in\mathbb{R}^2} \#(\Lambda \cap x + hI)$$

and

$$\mathcal{D}_B^-(\Lambda) = \liminf_{h\to\infty} \frac{1}{h^2} \inf_{x\in\mathbb{R}^2} \#(\Lambda \cap x + hI).$$

The next result[9] shows that Beurling density is robust against perturbations.

Lemma 2.4. *Let* Λ *be a sequence in* $\mathbb{R}^2$ *and let* $\epsilon > 0$. *For each* $\Delta = \{\lambda + \delta_\lambda : \lambda \in \Lambda,\, \delta_\lambda \in [-\epsilon, \epsilon]\}$, *we have*

$$\mathcal{D}_B^-(\Lambda) = \mathcal{D}_B^-(\Delta) \quad \textit{and} \quad \mathcal{D}_B^+(\Lambda) = \mathcal{D}_B^+(\Delta).$$

In order to enlighten the properties of sequences having a finite upper and positive lower Beurling density, we state the following useful reinterpretation[3] of finite upper density.

Lemma 2.5. *Let* Λ *be a sequence in* $\mathbb{R}^2$. *Then the following conditions are equivalent.*

1. $\mathcal{D}_B^+(\Lambda) < \infty$.
2. *There exists* $h > 0$ *and* $N_h < \infty$ *such that*
$$\#(\Lambda \cap Q_h(x)) < N_h \quad \textit{for all } x \in \mathbb{R}^2.$$
3. *For each* $h > 0$, *there exists* $N_h < \infty$ *such that*
$$\#(\Lambda \cap Q_h(x)) < N_h \quad \textit{for all } x \in \mathbb{R}^2.$$
4. Λ *is relatively uniformly separated.*

We show that a similar result holds for the case of positive lower weighted density.

LEMMA 2.6. *Let Λ be a sequence in $\mathbb{R}^2$. Then the following conditions are equivalent.*

1. $\mathcal{D}_B^-(\Lambda) > 0$.
2. *There exist $h, N_h > 0$ such that $\#(\Lambda \cap Q_h(x)) > N_h$ for all $x \in \mathbb{R}^2$.*
3. *Λ contains a subsequence of positive uniform density.*

Proof. First, suppose that $\mathcal{D}_B^-(\Lambda) > 0$. Then there exists $h > 0$ such that $\inf_{x\in\mathbb{R}^2} \#(\Lambda \cap Q_h(x)) > 0$. This immediately implies 2.

Secondly, let $h, N_h > 0$ be such that $\#(\Lambda \cap Q_h(x)) > N_h$ for all $x \in \mathbb{R}^2$. Hence, in particular, each set $\Lambda \cap Q_h(x)$ contains at least one element. Thus, for each $k \in \mathbb{Z}^2$ there exists some $y_k \in \Lambda \cap Q_h(2hk)$. Since $(y_k)_{k\in\mathbb{Z}}$ is a perturbation of $(2h\mathbb{Z})^2$, Lemma 2.4 implies that $(y_k)_{k\in\mathbb{Z}}$ has uniform density equal to $\frac{1}{4h^2}$. This proves 2. $\Rightarrow$ 3.

Finally, suppose that Λ contains a subsequence Δ of positive uniform density. Since $\mathcal{D}_B^-(\Lambda) > \mathcal{D}_B^-(\Delta) > 0$, 3. implies 1. ☐

2.2. Accumulative density

For the definition of the accumulative density we will employ the supremum-norm $\|\cdot\|_\infty$ of elements in $\mathbb{R}^2$ defined by $\|x\|_\infty = \sup\{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$. For the sake of brevity we will write $|x| := \|x\|_\infty$ for $x \in \mathbb{R}^2$.

Fix $1 \geq \delta > 0$. For $h > 0$ and $x \in \mathbb{R}^2$, we let $\tilde{Q}_h(x)$ denote the cube centered at x with side length $2h$ without the cube centered at x with side length 2δ, i.e.,

$$\tilde{Q}_h(x) = Q_h(x) \backslash Q_\delta(x).$$

In contrast to the definition of Beurling density, to compute the accumulative density the points of the sequence Λ contained in some cube $\tilde{Q}_h(x)$ are weighted dependent on their $\|\cdot\|_\infty$-distance to x. The weights of those points λ are the reciprocal of the perimeter of the box $Q_{|\lambda-x|}(0)$, i.e., equal to $\frac{1}{8|\lambda-x|}$. In this sense the weights accumulate. The sum of these weights is then divided by the $\|\cdot\|_\infty$-distance of the point x to the boundary of the cube $\tilde{Q}_h(x)$, i.e., by h.

DEFINITION 2.7. *Let Λ be a sequence in $\mathbb{R}^2$ and let $\delta > 0$. Then the* upper accumulative density *of Λ is defined by*

$$\mathcal{D}_A^+(\Lambda) = \limsup_{h\to\infty} \frac{1}{h} \sup_{x\in\mathbb{R}^2} \sum_{\lambda\in\Lambda\cap\tilde{Q}_h(x)} \frac{1}{8|\lambda - x|},$$

and the lower accumulative density *of Λ is*

$$\mathcal{D}_A^-(\Lambda) = \liminf_{h\to\infty} \frac{1}{h} \inf_{x\in\mathbb{R}^2} \sum_{\lambda\in\Lambda\cap\tilde{Q}_h(x)} \frac{1}{8|\lambda - x|}.$$

If $\mathcal{D}_A^-(\Lambda) = \mathcal{D}_A^+(\Lambda)$, then we say that Λ has uniform accumulative density *and denote this density by $\mathcal{D}_A(\Lambda)$.*

3. COMPARISON OF THE BEURLING DENSITY WITH THE ACCUMULATIVE DENSITY

In this section we first present an example of a sequence in $\mathbb{R}^2$ and compute both the Beurling density and the accumulative density of it to enlighten the difference in the computation. Then we will state our main result of this paper, which shows that for any sequence in $\mathbb{R}^2$ both densities coincide.

3.1. Example

We consider the lattice $\Lambda = \mathbb{Z}^2$. It will turn out that the computation of the Beurling density is trivial, whereas the computation of the accumulative density requires some more work.

Certainly, the Beurling density is known to be uniform and equal to 1, but we give the computation here in order to show the difference with the accumulative density. For this, let $h > 0$ and $x \in \mathbb{R}^2$. Then we have $\#(\Lambda \cap Q_h(x)) = 4h^2$. Thus

$$\mathcal{D}_B(\Lambda) = \limsup_{h\to\infty} \frac{1}{4h^2} \sup_{x\in\mathbb{R}^2} \#(\Lambda \cap Q_h(x)) = \liminf_{h\to\infty} \frac{1}{4h^2} \inf_{x\in\mathbb{R}^2} \#(\Lambda \cap Q_h(x)) = 1.$$

To compute the accumulative density of Λ, let $x \in \mathbb{R}^2$. If $x \in \mathbb{Z}^2$, then we obtain

$$\sum_{\lambda\in\Lambda\cap\tilde{Q}_h(x)} \frac{1}{8|\lambda - x|} = \sum_{n=1}^{\lfloor h\rfloor} 8n\frac{1}{8n} = \lfloor h\rfloor.$$

Now suppose $x \notin \mathbb{Z}^2$. Then, for each point the distance $|\lambda - x|$ varies at most by 1. Further notice that due to the definition of $\tilde{Q}_h(x)$, this distance has δ as lower bound. We obtain

$$\sum_{n=1}^{\lfloor h\rfloor} \frac{n}{n+1} = \sum_{n=1}^{\lfloor h\rfloor} 8n\frac{1}{8(n+1)} \leq \sum_{\lambda\in\Lambda\cap\tilde{Q}_h(x)} \frac{1}{8|\lambda - x|} \leq 8\frac{1}{8\delta} + \sum_{n=2}^{\lfloor h\rfloor} 8n\frac{1}{8(n-1)} = \frac{1}{\delta} + \sum_{n=2}^{\lfloor h\rfloor} \frac{n}{n-1}.$$

Thus

$$1 = \limsup_{h\to\infty} \frac{1}{h}\left(\sum_{n=1}^{\lfloor h\rfloor} \frac{n}{n+1}\right) \leq \mathcal{D}_A^+(\Lambda) \leq \limsup_{h\to\infty} \frac{1}{h}\left(\frac{1}{\delta} + \sum_{n=2}^{\lfloor h\rfloor} \frac{n}{n-1}\right) = 1.$$

This proves $\mathcal{D}_A^+(\Lambda) = 1$. The lower accumulative density can be treated similarly, and we obtain that Λ has a uniform accumulative density $\mathcal{D}_A(\Lambda) = 1$.

This computation shows that for the lattice $\mathbb{Z}^2$ the Beurling density and the accumulative density coincide. However, in Theorem 3.1 we will prove that this is always the case, i.e., that the Beurling density and the accumulative density coincide for any sequence of points in $\mathbb{R}^2$.

3.2. Main result

Although both the Beurling density and the accumulative density seem to be different in spirit, we will prove that they are not only equivalent but actually coincide.

THEOREM 3.1. *Let Λ be a sequence in $\mathbb{R}^2$. Then we have*

$$\mathcal{D}_B^+(\Lambda) = \mathcal{D}_A^+(\Lambda) \quad \textit{and} \quad \mathcal{D}_B^-(\Lambda) = \mathcal{D}_A^-(\Lambda).$$

Proof. We only prove the claim concerning the upper densities. The other claim can be treated similarly. Notice that without loss of generality we can assume that $\delta = 1$.

In the first step we prove that $\mathcal{D}_B^+(\Lambda) \geq \mathcal{D}_A^+(\Lambda)$. For this, set $\alpha = \mathcal{D}_B^+(\Lambda)$, and let $\epsilon > 0$. By the definition of upper Beurling density there exists $h_0 > 1$ such that for all $h \geq h_0$ and $x \in \mathbb{R}^2$,

$$\frac{1}{4h^2}\#(\Lambda \cap Q_h(x)) \leq \alpha + \epsilon \quad \Longrightarrow \quad \#(\Lambda \cap Q_h(x)) \leq 4h^2(\alpha + \epsilon). \tag{1}$$

Now fix some $x \in \mathbb{R}^2$ and let $N \in \mathbb{N}$, $h > h_0$. We consider the cube $\tilde{Q}_{(2N+1)h}(x)$ and break it up into growing "annuli" $A_n := Q_{(2n+1)h}(x)\backslash Q_{(2n-1)h}(x)$, $n = 1, \ldots, N$ and $\tilde{Q}_h(x)$. For each of the sets A_n we can construct a disjoint covering with $8n$ boxes $Q_h(y)$, $y \in \mathbb{R}^2$. For any $\lambda \in A_n$, we have

$$\frac{1}{8|\lambda - x|} \leq \frac{1}{8(2n-1)h}. \tag{2}$$

If λ is an element of the inner annulus $Q_h(x)\backslash Q_1(x)$ (recall that $\delta = 1$), in the same way as before we obtain

$$\frac{1}{8|\lambda - x|} \leq \frac{1}{8}. \tag{3}$$

Combining (1),(2), and (3) yields

$$\sum_{\lambda \in \Lambda \cap \tilde{Q}_{(2N+1)h}(x)} \frac{1}{8|\lambda - x|} \leq 4h^2(\alpha + \epsilon) \left(\sum_{n=1}^{N} \frac{8n}{8(2n-1)h} + \frac{1}{8} \right) = (\alpha + \epsilon) \left(\sum_{n=1}^{N} \frac{4nh}{2n-1} + \frac{h^2}{2} \right).$$

Thus

$$\frac{1}{(2N+1)h} \sum_{\lambda \in \Lambda \cap \tilde{Q}_{(2N+1)h}(x)} \frac{1}{8|\lambda - x|} \leq (\alpha + \epsilon) \left(\frac{1}{N + \frac{1}{2}} \sum_{n=1}^{N} \frac{n}{n - \frac{1}{2}} + \frac{h}{4N+2} \right). \tag{4}$$

Obviously, there exists $N_0 \in \mathbb{N}$ such that

$$\frac{1}{N + \frac{1}{2}} \sum_{n=1}^{N} \frac{n}{n - \frac{1}{2}} + \frac{h}{4N+2} < 1 + \epsilon \quad \text{for all } N \geq N_0 \text{ and for all } h \leq 2h_0. \tag{5}$$

Now let $k > (2N_0 + 1)h_0$, $k \in \mathbb{N}$ be arbitrary. Then there exist $N \geq N_0$, $N \in \mathbb{N}$, and $2h_0 \geq h \geq h_0$ such that $k = (2N+1)h$. Hence considering $\tilde{Q}_k(x)$ and using (4) and (5) proves

$$\frac{1}{k} \sum_{\lambda \in \Lambda \cap \tilde{Q}_k(x)} \frac{1}{8|\lambda - x|} \leq (\alpha + \epsilon)(1 + \epsilon).$$

This in turn implies

$$\mathcal{D}_A^+(\Lambda) \leq \limsup_{k \to \infty} \frac{1}{k} \sup_{x \in \mathbb{R}^2} \sum_{\lambda \in \Lambda \cap \tilde{Q}_k(x)} \frac{1}{8|\lambda - x|} \leq (\alpha + \epsilon)(1 + \epsilon).$$

Since $\epsilon > 0$ was chosen arbitrarily, this shows $\mathcal{D}_A^+(\Lambda) \leq \alpha = \mathcal{D}_B^+(\Lambda)$.

In the second step we prove that $\mathcal{D}_B^+(\Lambda) \leq \mathcal{D}_A^+(\Lambda)$. Again we set $\alpha = \mathcal{D}_B^+(\Lambda)$, and let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ so that

$$\frac{1}{N + \frac{1}{2}} \sum_{n=1}^{N} \frac{n}{n + \frac{1}{2}} \geq 1 - \epsilon. \tag{6}$$

Further, let $\epsilon_1, \epsilon_2 > 0$ be chosen such that

$$(2N+1)^2(\epsilon_1 + \epsilon_2) - \epsilon_1 < \epsilon. \tag{7}$$

By (1), there exists $h_0 > 0$ such that every cube $\tilde{Q}_h(x)$, $x \in \mathbb{R}^2$ contains at most $4h^2(\alpha + \epsilon_1)$ elements of Λ. Since $\mathcal{D}_B^+(\Lambda) = \alpha$, there exist a sequence $(x_i)_{i \in \mathbb{N}} \subset \mathbb{R}^2$ and a sequence $(h_i)_{i \in \mathbb{N}} \subset \mathbb{R}^+$ with $h_i > (2N+1)h_0$ for all $i \in \mathbb{N}$ and $h_i \to \infty$ as $i \to \infty$ such that

$$\frac{1}{4h_i^2} \#(\Lambda \cap Q_{h_i}(x_i)) \geq \alpha - \epsilon_2 \quad \Longrightarrow \quad \#(\Lambda \cap Q_{h_i}(x_i)) \geq 4h_i^2(\alpha - \epsilon_2). \tag{8}$$

Fix $i \in \mathbb{N}$. Now consider the cube $\tilde{Q}_{h_i}(x_i)$ and set $h := \frac{h_i}{2N+1}$. Next we wish to partition this cube into $(2N+1)^2$ boxes $Q_h(y)$, $y \in \mathbb{R}^2$.

Our claim is that each of these squares contains at least $4h^2(\alpha - \epsilon)$ elements of Λ. First recall that each of these boxes contains at most $4h^2(\alpha + \epsilon_1)$ points of Λ. Thus there are at most $4h^2(\alpha + \epsilon_1)[(2N+1)^2 - 1]$ elements of Λ in $(2N+1)^2 - 1$ squares. But by (8) and the definition of h, the box $Q_{h_i}(x_i)$ and hence, in particular, $\tilde{Q}_{h_i}(x_i)$ contains at least

$$4h_i^2(\alpha - \epsilon_2) = 4h^2(2N+1)^2(\alpha - \epsilon_2)$$

points of Λ. Thus each box $Q_h(y)$ of the partition of $\tilde{Q}_{h_i}(x_i)$ contains at least

$$4h^2(2N+1)^2(\alpha-\epsilon_2) - 4h^2(\alpha+\epsilon_1)[(2N+1)^2-1] = 4h^2(\alpha+\epsilon_1-(2N+1)^2(\epsilon_1+\epsilon_2))$$

elements of Λ. Using the choice of ϵ_1, ϵ_2 (see (7)), we can conclude that each box $Q_h(y)$ of the partition of $\tilde{Q}_{h_i}(x_i)$ contains at least $4h^2(\alpha-\epsilon)$ elements of Λ.

We use again the partition into "annuli" from the first part. This leads to

$$\sum_{\lambda\in\Lambda\cap\tilde{Q}_{h_i}(x_i)} \frac{1}{8|\lambda-x_i|} \geq 4h^2(\alpha-\epsilon)\sum_{n=1}^{N}\frac{8n}{8(2n+1)h} \geq (\alpha-\epsilon)\sum_{n=1}^{N}\frac{2hn}{n+\frac{1}{2}}.$$

Employing (6) and the choice of h yield

$$\frac{1}{h_i}\sum_{\lambda\in\Lambda\cap\tilde{Q}_{h_i}(x_i)} \frac{1}{8|\lambda-x_i|} \geq (\alpha-\epsilon)\frac{1}{(2N+1)h}\sum_{n=1}^{N}\frac{2hn}{n+\frac{1}{2}} \geq (\alpha-\epsilon)\frac{1}{N+\frac{1}{2}}\sum_{n=1}^{N}\frac{n}{n+\frac{1}{2}} \geq (\alpha-\epsilon)(1-\epsilon).$$

Thus

$$\mathcal{D}_A^+(\Lambda) \geq \limsup_{k\to\infty}\frac{1}{k}\sup_{x\in\mathbb{R}^2}\sum_{\lambda\in\Lambda\cap\tilde{Q}_k(x)}\frac{1}{8|\lambda-x|} \geq \lim_{i\to\infty}\frac{1}{h_i}\sum_{\lambda\in\Lambda\cap\tilde{Q}_{h_i}(x_i)}\frac{1}{8|\lambda-x_i|} \geq (\alpha-\epsilon)(1-\epsilon).$$

Thus $\mathcal{D}_A^+(\Lambda) \geq \alpha = \mathcal{D}_B^+(\Lambda)$. □

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