Chapter 1
Fusion Frames

Peter G. Casazza and Gitta Kutyniok

Abstract Novel technological advances significantly increased the demand to model applications requiring distributed processing. Frames are however too restrictive for such applications, wherefore it was necessary to go beyond classical frame theory. Fusion frames, which can be regarded as frames of subspaces, do satisfy exactly those needs. They analyze signals by projecting them onto multidimensional subspaces, in contrast to frames which consider only one-dimensional projections. This chapter shall serve as an introduction to and a survey about this exciting area of research as well as a reference for the state-of-the-art of this research field.

Key words: Compressed sensing, distributed processing, fusion coherence, fusion frame, fusion frame potential, isoclinic subspaces, mutually unbiased bases, sparse fusion frames, spectral tetris, non-orthogonal fusion frames.

1.1 Introduction

In the 21st century, scientists face massive amounts of data, which can typically not be handled anymore with a single processing system. A seemingly unrelated problem arises in sensor networks when communication between any pair of sensors is not possible due to, for instance, low communication bandwidth. A yet different question is the design of erasure-resilient packet-based encoding when data is broken into packets for separate transmission.

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All these problems can be regarded as belonging to the field of distributed processing. However, they have an even more special structure in common, since each one can be regarded as a special case of the following mathematical framework: Given data and a collection of subspaces, project the data onto the subspaces, then process the data within each subspace, and finally 'fuse' the locally computed objects. The decomposition of the given data into the subspaces coincides with – relating to the initial three problems – the splitting into different processing systems, the local measurements of groups of close sensors, and the generation of packets. The distributed fusion models the reconstruction procedure, also enabling, for instance, an error analysis of resilience against erasures. This is however only possible if the data is decomposed in a *redundant* way, which forces the subspaces to be redundant.

Fusion frames provide a suitable mathematical framework to design and analyze such applications under distributed processing requirements. Interestingly, fusion frames are also a versatile tool for more theoretically oriented problems in mathematics, and we will see various examples for this throughout this chapter as well.

1.1.1 The Fusion Frame Framework

Let us now give a first half-formal introduction to fusion frames, utilizing another motivation as a guideline. One goal in frame theory is to construct large frames by fusing 'smaller' frames, and, in fact, this was the original reasoning for introducing fusion frames by the two authors in [21]. We will come back to the three signal processing applications in Subsection 1.1.3 and show in more detail how they fit into this framework.

Locality of frames can be modeled as frame sequences, i.e., frames for their closed linear span. Now assume we have a collection of frame sequences $(\varphi_{ij})_{j=1}^{J_i}$ in $\mathcal{H}^N$ with $i = 1, \ldots, M$, and set $\mathcal{W}_i := \text{span}\{\varphi_{ij} : j = 1, \ldots, J_i\}$ for each i. The key questions are whether the collection $(\varphi_{ij})_{i=1,j=1}^{M,J_i}$ forms a frame for $\mathcal{H}^N$ and, if yes, which frame properties does it have. The first question is easy to answer, since what is required is the spanning property of the family $(\mathcal{W}_i)_{i=1}^M$. The second question requires more thought. But it is intuitively clear that – besides the knowledge of the frame bounds of the frame sequences – it will depend solely on the structural properties of the family of subspaces $(\mathcal{W}_i)_{i=1}^M$. In fact, it can be proven that the crucial property are the constants associated with the ℓ_2-stability of the mapping

$$\mathcal{H}^N \ni x \mapsto (P_i(x))_{i=1}^M \in \mathbb{R}^{NM}, \tag{1.1}$$

where P_i denotes the orthogonal projection onto the subspace $\mathcal{W}_i$. A family of subspaces $(\mathcal{W}_i)_{i=1}^M$ satisfying such a stability condition is then called a *fusion frame*.

We would like to emphasize that (1.1) leads to the basic fusion frame definition. It can, for instance, be modified by considering weighted projections to allow flexibility in the significance of each subspace, hence of each locally constructed frame $(\varphi_{ij})_{j=1}^{J_i}$.

It should also be stressed that in [21], the introduced notion was coined 'frames of subspaces' for reasons which become clear in the sequel. Later, to avoid confusion with the term 'frames *for* subspaces' and to emphasize the local fusing of information, it was baptized 'fusion frames' in [22].

1.1.2 Fusion Frames versus Frames

The main distinction between frames and fusion frames lies in the fact that a frame $(\varphi_i)_{i=1}^M$ for $\mathcal{H}^N$ provides the following measurements of a signal $x \in \mathcal{H}^N$:

$$x \mapsto (\langle x, \varphi_i \rangle)_{i=1}^M \in \mathbb{R}^M.$$

A fusion frame $(\mathcal{W}_i)_{i=1}^M$ for $\mathcal{H}^N$ on the other hand analyzes the signal x by

$$x \mapsto (P_i(x))_{i=1}^M \in \mathbb{R}^{MN}.$$

Thus the *scalar* measurements of a frame are substituted by *vector* measurements, and consequently, the representation space of a frame is $\mathbb{R}^M$, whereas those of a fusion frame is $\mathbb{R}^{MN}$. This latter space can sometimes be reduced, and we refer to the next section for details.

A further natural question is whether the theory of fusion frames includes the theory of frames, which is indeed the case. In fact, – and the next section will provide more detailed information – a frame can be regarded as a collection of the one-dimensional subspaces its frame vectors generate. Taking the norms of the frame vectors as the aforementioned weights, it can be shown that this is a fusion frame with similar properties. Conversely, taking a fusion frame, one can fix an orthonormal basis in each subspace and then consider the union of these bases. This will form a frame, which can be regarded as being endowed with a particular substructure.

These two viewpoints indicate already at this point that fusion frame theory is much more difficult than frame theory. In fact, most results in this chapter will be solely stated for the case of the weights being equal to 1, and even in this situation many questions which are answered for frames remain open in the general situation of fusion frames.

1.1.3 Applications of Fusion Frames

The generality of the framework of fusion frames allows their application to various problems both of practical as well as theoretical nature – which then certainly require additional adaptations in the specific setting considered. We first highlight the three signal processing applications, which were already mentioned in the beginning.

- *Distributed Sensing.* Given a collection of small and inexpensive sensors scattered over a large area, the measurement each sensor produces of an incoming signal $x \in \mathcal{H}^N$ can be modeled as $\langle x, \varphi_i \rangle$, $\varphi_i \in \mathcal{H}^N$ being the specific characteristics of the sensor. Since due to, for instance, limited bandwidth and transmission power, sensors can only communicate locally, the recovery of the signal x can first only be performed among groups of sensors. Let $(\varphi_{ij})_{j=1}^{J_i}$ in $\mathcal{H}^N$ with $i = 1, \ldots, M$ be such a grouping. Then, setting $\mathcal{W}_i := \text{span}\{\varphi_{ij} : j = 1, \ldots, J_i\}$ for each i, local frame reconstruction leads to the collection of vectors $(P_i(x))_{i=1}^M$. This data is then passed on by special transmitters to a central processing station for joint processing. At this point, fusion frame theory kicks in and provides a means for performing and analyzing the reconstruction of the signal x. The modeling of sensor networks through fusion frames was considered in the series of papers [23, 38]. A similar local-global signal processing principle is applicable to modeling of human visual cortex as discussed in [44].
- *Parallel Processing.* If a frame is too large for efficient processing – from a computational complexity or a numerical stability standpoint –, one approach is to divide it into multiple small subsystems for simple and ideally parallel processing. Fusion frames allow a stable splitting into smaller frames and afterwards a stable recombining of the local outputs. Splitting of a large system into smaller subsystems for parallel processing was first considered in [3, 43].
- *Packet Encoding.* Transmission of data over a communication network, for instance the internet, is often achieved by first encoding it into a number of packets. By introducing redundancy in the encoding scheme, the communication scheme becomes resilient against corruption or even complete loss of transmitted packets. Fusion frames provide a means to achieve and analyze redundant subspace representations, where each packet carries one of the fusion frame projections. The use of fusion frames for packet encoding is considered in [4].

Fusion frames also arise in more theoretical problems as the next two examples show.

- *Kadison-Singer Problem.* The 1959 Kadison-Singer Problem [25] is one of the most famous unsolved problems in analysis today. One of the many equivalent formulations is the following question: Can a bounded frame be partitioned such that the spans of the partitions as a fusion frame lead to a ‘good’ lower fusion frame bound? Therefore, advances in the design of fusion frames will have direct impact in providing new angles for a renewed attack on the Kadison-Singer Problem.

- *Optimal Packings.* Fusion frame theory also bears close connections with Grassmannian packings. It was shown in [38], that the special class of Parseval fusion frames consisting of equi-distance and equi-dimensional subspaces are in fact optimal Grassmannian packings. Thus, novel methods for constructing such fusion frames simultaneously provide ways to construct optimal packings.

1.1.4 Related Approaches

Several approaches related to fusion frames have appeared in the literature. The concept of a frame-like collection of subspaces was first exploited in relation to domain decomposition techniques in papers by Bjørstad and Mandel [3] and Oswald [43]. In 2003, Fornasier introduced in [33] what he coined quasi-orthogonal decompositions. The framework of fusion frames was in fact developed at the same time by the two authors in [21] and contains those decompositions as a special case. It should also be mentioned that Sun introduced so-called G-frames in the series of papers [45, 46], which extend the definition of fusion frames by generalizing the utilized orthogonal projections to arbitrary operators. The generality of this notion is however not suitable for modeling distributed processing.

1.1.5 Outline

In Section 1.2, we introduce the basic notions and definitions of fusion frame theory, discuss the relation to frame theory, and present a reconstruction formula. Section 1.3 is concerned with the introduction and application of the fusion frame potential as a highly useful method for analyzing fusion frames. The construction of fusion frames is then the focus of Section 1.4. In this section, we present the spectral tetris algorithm as a versatile means to construct general fusion frames followed by a discussion on the construction of equi-isoclinic fusion frames and the construction of fusion frames for filter banks. Section 1.5 discusses the resilience of fusion frames against the impacts of additive noise, erasures, and perturbations. The relation of fusion frames with the novel paradigm of sparsity – optimally sparse fusion frames and sparse recovery from fusion frame measurements – is the topic of Section 1.6. We finish this chapter with the novel direction of non-orthogonal fusion frames presented in Section 1.7.

1.2 Basics of Fusion Frames

We start by making the intuitive view of fusion frames presented in the introduction mathematically precise. We then state a reconstruction formula for reconstructing signals from fusion frame measurements, which will also require the introduction of the fusion frame operator.

We should mention that fusion frames were initially introduced in the general setting of a Hilbert space. We are restricting ourselves here to the finite dimensional setting, which is of more interest for applications. It is worth noting that the level of difficulty is by far not diminished by this restriction.

1.2.1 What is a Fusion Frame?

Let us start by stating the mathematically precise definition of a fusion frame, which we already motivated in the introduction.

Definition 1. Let $(\mathcal{W}_i)_{i=1}^M$ be a family of subspaces in $\mathcal{H}^N$, and let $(w_i)_{i=1}^M \subseteq \mathbb{R}^+$ be a family of weights. Then $((\mathcal{W}_i, w_i))_{i=1}^M$ is a *fusion frame* for $\mathcal{H}^N$, if there exist constants $0 < A \le B < \infty$ such that

$$A\|x\|_2^2 \le \sum_{i=1}^M w_i^2 \|P_i(x)\|_2^2 \le B\|x\|_2^2 \quad \text{for all } x \in \mathcal{H}^N,$$

where P_i denotes the orthogonal projection onto $\mathcal{W}_i$ for each i. The constants A and B are called the *lower* and *upper fusion frame bound*, respectively. The family $((\mathcal{W}_i, w_i))_{i=1}^M$ is referred to as a *tight fusion frame*, if $A = B$ is possible. In this case we also refer to the fusion frames as an A-*tight fusion frame*. Moreover, it is called a *Parseval fusion frame*, if A and B can be chosen as $A = B = 1$. Finally, if $w_i = 1$ for all i, often the notation $(\mathcal{W}_i)_{i=1}^M$ is simply utilized.

To illustrate the notion of a fusion frame, we first present some illuminating examples, which also show the delicateness of constructing fusion frames.

Example 1.(a) Let $(e_i)_{i=1}^3$ be an ONB of $\mathbb{R}^3$, define subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ by $\mathcal{W}_1 = \text{span}\{e_1, e_2\}$ and $\mathcal{W}_2 = \text{span}\{e_2, e_3\}$ and let w_1 and w_2 be two weights. Then $((\mathcal{W}_i, w_i))_{i=1}^2$ is a fusion frame for $\mathbb{R}^3$ with optimal fusion frame bounds $\min\{w_1^2, w_2^2\}$ and $w_1^2 + w_2^2$. We omit the obvious proof, but mention that this example shows that even changing the weights does not always allow us to turn a fusion frame into a tight fusion frame.

(b) Let now $(\varphi_j)_{j=1}^J$ be a frame for $\mathcal{H}^N$ with bounds A and B. A natural question is whether $\{1, \ldots, J\}$ can be partitioned into subsets $J_1, \ldots, J_M$ such that the family of subspaces $\mathcal{W}_i = \text{span}\{\varphi_j : j \in J_i\}$, $i = 1, \ldots, M$, forms a fusion frame with 'good' fusion frame bounds – in the sense of their ratio being close to 1, since this ensures a low computational complexity of reconstruction. Remembering the sensor network application, we also seek to choose the partitioning such that $(\varphi_j)_{j \in J_i}$ possesses 'good' frame bounds. However, it was shown in [25] that the problem of dividing a frame into a finite number of subsets each of which has good lower frame bounds is equivalent to the still unsolved Kadison-Singer Problem, see Subsection 1.1.3. The next subsection will, however, present some computationally possible scenarios for deriving a fusion frame by partitioning a frame into subsets.

1.2.2 Fusion Frames versus Frames

One question when introducing a new notion is its relation to the previously considered classical notion, in this case to *frames*. Our first result shows that fusion frames can be regarded as a generalization of frames in the following sense.

Lemma 1. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$ with frame bounds A and B. Then $(\mathrm{span}\{\varphi_i\}, \|\varphi_i\|_2)_{i=1}^M$ constitutes a fusion frame for $\mathcal{H}^N$ with fusion frame bounds A and B.*

Proof. For all $x \in \mathcal{H}^N$, we have

$$\sum_{i=1}^M \|\varphi_i\|_2^2 \|P_i(x)\|_2^2 = \sum_{i=1}^M \|\varphi_i\|_2^2 \|\langle x, \tfrac{\varphi_i}{\|\varphi_i\|_2}\rangle \tfrac{\varphi_i}{\|\varphi_i\|_2}\|_2^2 = \sum_{i=1}^M |\langle x, \varphi_i\rangle|^2.$$

Applying the definitions of frames and fusion frames finishes the proof. □

On the other hand, if we choose any spanning set inside each subspace of a given fusion frame, the collection of these families of vectors forms a frame for $\mathcal{H}^N$. In this sense, a fusion frame might also be considered as a structured frame. Note though, that this viewpoint depends heavily on the selection of the subspace spanning sets. The next theorem states this local-global interaction in detail.

Theorem 1 ([21]). *Let $(\mathcal{W}_i)_{i=1}^M$ be a family of subspaces in $\mathcal{H}^N$, and let $(w_i)_{i=1}^M \subseteq \mathbb{R}^+$ be a family of weights. Further, let $(\varphi_{ij})_{j=1}^{J_i}$ be a frame for $\mathcal{W}_i$ with frame bounds A_i and B_i for each i, and set $A := \min_i A_i$ and $B := \max_i B_i$. Then the following conditions are equivalent.*

1. *$((\mathcal{W}_i, w_i))_{i=1}^M$ is a fusion frame for $\mathcal{H}^N$.*
2. *$(w_i\varphi_{ij})_{i=1,j=1}^{M,J_i}$ is a frame for $\mathcal{H}^N$.*

In particular, if $((\mathcal{W}_i, w_i))_{i=1}^M$ *is a fusion frame with fusion frame bounds* C *and* D*, then* $(w_i\varphi_{ij})_{i=1,j=1}^{M,J_i}$ *is a frame with bounds* AC *and* BD*. On the other hand, if* $(w_i\varphi_{ij})_{i=1,j=1}^{M,J_i}$ *is a frame with bounds* C *and* D*, then* $((\mathcal{W}_i, w_i))_{i=1}^M$ *is a fusion frame with fusion frame bounds* $\frac{C}{B}$ *and* $\frac{D}{A}$*.*

Proof. To prove the theorem, it is sufficient to prove the *in particular*-part. For this, first assume that $((\mathcal{W}_i, w_i))_{i=1}^M$ is a fusion frame with fusion frame bounds C and D. Then

$$\begin{aligned}\sum_{i=1}^M w_i^2 \sum_{j=1}^{J_i} |\langle x, \varphi_{ij}\rangle|^2 &= \sum_{i=1}^M w_i^2 \left[\sum_{j=1}^{J_i} |\langle P_i(x), \varphi_{ij}\rangle|^2\right] \\ &\le \sum_{i=1}^M w_i^2 B_i \|P_i(x)\|_2^2 \le BD\|x\|_2^2.\end{aligned}$$

The lower frame bound AC can be proved similarly.

Secondly, we assume that $(w_i\varphi_{ij})_{i=1,j=1}^{M,J_i}$ is a frame with bounds C and D. In this case, we obtain

$$\sum_{i=1}^M w_i^2 \|P_i(x)\|_2^2 \le \frac{1}{A} \sum_{i=1}^M w_i^2 \left[\sum_{j=1}^{J_i} |\langle P_i(x), \varphi_{ij}\rangle|^2\right] \le \frac{D}{A}\|x\|_2^2.$$

As before, the lower fusion frame bound $\frac{C}{B}$ can be shown using similar arguments. This finishes the proof. □

The following is an immediate consequence.

Corollary 1. *Let* $(\mathcal{W}_i)_{i=1}^M$ *be a family of subspaces in* $\mathcal{H}^N$*, and let* $(w_i)_{i=1}^M \subseteq \mathbb{R}^+$ *be a family of weights. Then* $((\mathcal{W}_i, w_i))_{i=1}^M$ *is a fusion frame for* $\mathcal{H}^N$ *if and only if the subspaces* $\mathcal{W}_i$ *span* $\mathcal{H}^N$*.*

Since tight fusion frames play a particularly important role due to their advantageous reconstruction properties (see Theorem 2), we state the special case of the previous result for tight fusion frames explicitly. It follows immediately from Theorem 1.

Corollary 2. *Let* $(\mathcal{W}_i)_{i=1}^M$ *be a family of subspaces in* $\mathcal{H}^N$*, and let* $(w_i)_{i=1}^M \subseteq \mathbb{R}^+$ *be a family of weights. Further, let* $(\varphi_{ij})_{j=1}^{J_i}$ *be an* A*-tight frame for* $\mathcal{W}_i$ *for each* i*. Then the following conditions are equivalent.*

1. $((\mathcal{W}_i, w_i))_{i=1}^M$ *is a* C*-tight fusion frame for* $\mathcal{H}^N$*.*
2. $(w_i\varphi_{ij})_{i=1,j=1}^{M,J_i}$ *is an* AC*-tight frame for* $\mathcal{H}^N$*.*

This result has an interesting consequence. Since redundancy is the crucial property of a fusion frame as well as of a frame, one might be interested in a quantitative way to measure it. In the situation of frames, the rather crude measure of the number of frame vectors divided by the dimension − which is the frame bound in case of a tight frame with normalized vectors − has recently been replaced by a more appropriate measure, see [5]. In the situation of fusion frames, this is still under investigation. However, as a first notion of redundancy in the situation of a tight fusion frame, we can choose its fusion frame bound as a measure. The following result computes its value.

Proposition 1. *Let $((\mathcal{W}_i, w_i))_{i=1}^M$ be an A-tight fusion frame for $\mathcal{H}^N$. Then we have*

$$A = \frac{\sum_{i=1}^M w_i^2 \dim \mathcal{W}_i}{N}.$$

Proof. Let $(e_{ij})_{j=1}^{\dim \mathcal{W}_i}$ be an orthonormal basis for $\mathcal{W}_i$ for each $1 \le i \le M$. By Corollary 2, the sequence $(w_i e_{ij})_{i=1,j=1}^{M,\dim \mathcal{W}_i}$ is an A-tight frame. Thus, we obtain

$$A = \frac{\sum_{i=1}^M \sum_{j=1}^{\dim \mathcal{W}_i} \|w_i e_{ij}\|^2}{N} = \frac{\sum_{i=1}^M w_i^2 \dim \mathcal{W}_i}{N}. \qquad \square$$

1.2.3 The Fusion Frame Operator

As discussed before, the fusion frame measurements of a signal $x \in \mathcal{H}^N$ are its (weighted) orthogonal projections onto the given family of subspaces. Consequently, given a fusion frame $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M$ for $\mathcal{H}^N$, we define the associated *analysis operator* $T_\mathcal{W}$ by

$$T_\mathcal{W} : \mathcal{H}^N \to \mathbb{R}^{MN}, \quad x \mapsto (w_i P_i(x))_{i=1}^M.$$

To reduce the dimension of the representation space $\mathbb{R}^{MN}$, we can select an orthonormal basis in each subspace $\mathcal{W}_i$, which we combine to an $N \times \dim \mathcal{W}_i$-matrix U_i. Then the analysis operator can be modified to $T_\mathcal{W}(x) = (w_i U_i^T(x))_{i=1}^M$. This approach was undertaken, for instance, in [38].

As is customary in frame theory, the synthesis operator is defined to be the adjoint of the analysis operator. Hence in this situation, the *synthesis operator* $T_\mathcal{W}^*$, has the form

$$T_\mathcal{W}^* : \mathbb{R}^{MN} \to \mathbb{R}^N, \quad (y_i)_{i=1}^M \mapsto \sum_{i=1}^M w_i P_i(y_i).$$

This leads to the following definition of an associated *fusion frame operator* $S_\mathcal{W}$:

$$S_\mathcal{W} = T_\mathcal{W}^* T_\mathcal{W} : \mathcal{H}^N \to \mathcal{H}^N, \quad x \mapsto \sum_{i=1}^M w_i^2 P_i(x).$$

1.2.4 Reconstruction Formula

Having introduced a fusion frame operator associated with each fusion frame, we expect it to lead to a reconstruction formula as in the frame theory case. Indeed, a similar result is true as the following theorem shows.

Theorem 2 ([21]). *Let $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M$ be a fusion frame for $\mathcal{H}^N$ with fusion frame bounds A and B and associated fusion frame operator $S_{\mathcal{W}}$. Then $S_{\mathcal{W}}$ is a positive, self-adjoint, invertible operator on $\mathcal{H}^N$ with $A\,Id \le S_{\mathcal{W}} \le B\,Id$. Moreover, we have the reconstruction formula*

$$x = \sum_{i=1}^M w_i^2 S_{\mathcal{W}}^{-1}(P_i(x)) \quad \textit{for all } x \in \mathcal{H}^N.$$

Note however, that this reconstruction formula – in contrast to the analogous one for frames – does not automatically lead to a 'dual fusion frame.' In fact, the appropriate definition of a dual fusion frame is still a topic of research.

Theorem 2 immediately implies that a fusion frame is tight if and only if $S_{\mathcal{W}} = A\,Id$, and in this situation the reconstruction formula takes the advantageous form

$$x = A^{-1} \sum_{i=1}^M w_i^2 (P_i(x)) \quad \text{for all } x \in \mathcal{H}^N.$$

This fact makes tight fusion frames particularly attractive for applications.

If practical constraints prevent the utilization or construction of an appropriate tight fusion frame, inverting the fusion frame operator can be still circumvented for reconstruction. Recalling the frame algorithm introduced in [49], we can generalize it to an iterative algorithm for reconstruction of signals from fusion frame measurements. The proof of the following result follows the arguments of the frame analog very closely; therefore, we omit it.

Proposition 2 ([22]). *Let $((\mathcal{W}_i, w_i))_{i=1}^M$ be a fusion frame in $\mathcal{H}^N$ with fusion frame operator $S_{\mathcal{W}}$ and fusion frame bounds A and B. Further, let $x \in \mathcal{H}^N$, and define the sequence $(x_n)_{n\in\mathbb{N}_0}$ by*

$$x_n = \begin{cases} 0, & n = 0, \\ x_{n-1} + \frac{2}{A+B} S_{\mathcal{W}}(x - x_{n-1}), & n \ge 1. \end{cases}$$

Then we have $x = \lim_{n\to\infty} x_n$ with the error estimate

$$\|x - x_n\| \le \left(\frac{B-A}{B+A}\right)^n \|x\|.$$

This algorithm enables reconstruction of a signal x from its fusion frame measurements $(w_i P_i(x))_{i=1}^M$, since $S_{\mathcal{W}}(x)$ – necessary for the algorithm – only requires the knowledge of those measurements and of the sequence of weights $(w_i)_{i=1}^M$.

1.3 Fusion Frame Potential

The frame potential, which was introduced in [2] (see also [49]), gives a quantitative estimate of the orthogonality of a system of vectors by measuring the total potential energy stored in the system under a certain force which encourages orthogonality. It was proven in [16] that, given a complete set of vectors, the minimizers of the associated frame potential are precisely the tight frames. This fact made the frame potential attractive for both theoretical results as well as for deriving large classes of tight frames. However, a slight drawback is the lack of an associated algorithm to actually construct such frames, wherefore these results are mostly used as existence results.

The question of whether a similar quantitative measure exists for fusion frames was answered in [14] by the introduction of a fusion frame potential. These results were significantly generalized and extended in [42]. In this section, we will present a selection of the most fundamental results of this theory.

Let us start by stating the definition of the fusion frame potential. Recalling that in the case of a frame $\Phi = (\varphi_i)_{i=1}^M$ its frame potential is defined by

$$FP(\Phi) = \sum_{i,j=1}^{M} |\langle \varphi_i, \varphi_j \rangle|^2,$$

it is not initially clear how this can be extended. The following definition from [14] presents a suitable candidate. Note that this includes the classical frame potential by Lemma 1.

Definition 2. Let $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M$ be a fusion frame for $\mathcal{H}^N$ with associated fusion frame operator $S_{\mathcal{W}}$. Then the associated *fusion frame potential* of $\mathcal{W}$ is defined by

$$FFP(\mathcal{W}) = \sum_{i,j=1}^{M} w_i^2 w_j^2 Tr[P_i P_j] = Tr[S_{\mathcal{W}}^2].$$

The following result is immediate.

Lemma 2. *Let $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M$ be a fusion frame for $\mathcal{H}^N$ with associated fusion frame operator $S_{\mathcal{W}}$, and let $(\lambda_i)_{i=1}^N$ be the eigenvalues of $S_{\mathcal{W}}$. Then*

$$FFP(\mathcal{W}) = \sum_{i=1}^{N} \lambda_i^2.$$

We next define the class of fusion frames over which we seek to minimize the fusion frame potential.

Definition 3. Letting $d = (d_i)_{i=1}^M$ be a sequence of positive integers and $w = (w_i)_{i=1}^M$ be a sequence of positive weights, we define the set

$$B_{M,N}(d) = \{((\mathcal{W}_i, v_i))_{i=1}^M : ((\mathcal{W}_i, v_i))_{i=1}^M \text{ is a fusion frame with } \dim \mathcal{W}_i = d_i \text{ for all } i = 1, 2, \ldots, M\}$$

and the two subsets

$$B_{M,N}(d,w) = \{((\mathcal{W}_i, v_i))_{i=1}^M \in B_{M,N}(d) \colon v_i = w_i \text{ for all } i = 1, 2, \ldots, M\},$$
$$B^1_{M,N}(d) = \{\mathcal{W} = ((\mathcal{W}_i, v_i))_{i=1}^M \in B_{M,N}(d) : Tr[S_{\mathcal{W}}] = \sum_{i=1}^{M} v_i^2 d_i = 1.\}$$

We first focus on the set $B^1_{M,N}(d)$, and start with a crucial property of the fusion frame potential of elements therein. In the following result, by $\|\cdot\|_F$ we denote the Frobenius norm.

Proposition 3 ([42]). *Let* $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M \in B^1_{M,N}(d)$*, then*

$$\left\| \frac{1}{N} Id - S_{\mathcal{W}} \right\|_F^2 = FFP(\mathcal{W}) - \frac{1}{N}.$$

Proof. Since $Tr[S_{\mathcal{W}}] = 1$ by definition of $B^1_{M,N}(d)$, a direct computation shows that

$$\left\| \frac{1}{N} Id - S_{\mathcal{W}} \right\|_F^2 = Tr\left[\frac{1}{N^2} Id - \frac{2}{N} S_{\mathcal{W}} + S_{\mathcal{W}}^2\right] = Tr[S_{\mathcal{W}}^2] - \frac{1}{N}.$$

The definition of $FFP(\mathcal{W})$ finishes the proof. □

This result implies that minimizing the fusion frame potential over the family of fusion frames of $B^1_{M,N}(d)$ is equivalent to minimizing the Frobenius distance between $S_{\mathcal{W}}$ and a multiple of the identity.

In this spirit the following result does not seem surprising, but it requires a technical proof which we omit here.

Theorem 3 ([42]). *Local minimizers of FFP over* $B^1_{M,N}(d)$ *are global minimizers, and they are tight fusion frames.*

We caution the reader that this theorem does not necessarily imply the existence of local minimizers, only that they are tight fusion frames *if* they exist. Lower bounds of FFP provide a means to show the existence of local minimizers. The following result is a direct consequence of Proposition 3.

Corollary 3. *Let* $\mathcal{W} \in B^1_{M,N}(d)$*. Then we have* $FFP(\mathcal{W}) \geq \frac{1}{N}$*. Moreover,* $FFP\ (\mathcal{W}) = \frac{1}{N}$ *if and only if* $\mathcal{W}$ *is a tight fusion frame for* $\mathcal{H}^N$*.*

We now turn to analyzing the fusion frame potential defined on $B_{M,N}(d, v)$. As a first step, we state a lower bound for FFP, which will also lead to a fundamental equality for tight fusion frames.

Proposition 4 ([42]). *Let* $d = (d_i)_{i=1}^M$ *be a sequence of positive integers and* $w = (w_i)_{i=1}^M$ *be a decreasing sequence of positive weights such that* $\sum_{i=1}^M w_i^2 d_i = 1$ *and* $\sum_{i=1}^M d_i \geq N$*, and let* $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M \in B_{M,N}(d, w)$*. Further, let* $j_0 \in \{1, \dots, M\}$ *be defined by*

$$j_0 = j_0(N, d, v) = \max_{1 \leq j \leq M} \left\{ j : \left(N - \sum_{i=1}^{j} d_i \right) w_j^2 > \sum_{i=j+1}^{M} w_i^2 d_i \right\},$$

and let $j_0 = 0$ *if the set is empty. If*

$$c := \frac{\sum_{i=j_0+1}^{M} w_i^2 d_i}{N - \sum_{i=1}^{j_0} d_i} < w_{j_0}^2,$$

then

$$FFP(\mathcal{W}) \geq \sum_{i=1}^{j_0} w_i^4 d_i + \left(N - \sum_{i=j_0+1}^{M} d_i \right) c^2. \tag{1.2}$$

Moreover, we have equality in (1.2) *if and only if the following two conditions are satisfied:*

(1) $P_i P_j = 0$ *for all* $1 \leq i \neq j \leq j_0$*,*
(2) $((\mathcal{W}_i, w_i))_{i=j_0+1}^M$ *is a tight fusion frame for* $\operatorname{span}\{\mathcal{W}_i : 1 \leq i \leq j_0\}^\perp$*.*

The main result from [42] is highly technical. Its statement utilizes the notion of admissible $(M + 1)$-tuples $(J_0, J_1, \dots J_M)$ with

$$J_r = \{1 \leq j_1 < j_2 < \cdots < j_r \leq N\},$$

and an associated partition

$$\lambda(J) = (j_r - r, \dots, j_1 - 1),$$

where $r \leq N$. Due to lack of space we are not able to go into more detail. We merely mention that an *admissible* $(M + 1)$-tuple is defined as one for which the Littlewood-Richardson coefficient of the associated partitions $\lambda(J_0), \dots, \lambda(J_M)$ is positive [34]. This allows us to phrase the following result.

Theorem 4 ([42]). *Let $d = (d_i)_{i=1}^M$ be a sequence of positive integers satisfying $\sum_i d_i \geq N$, let $w = (w_i)_{i=1}^M$ be a sequence of positive weights, and set $c = \sum_{i=1}^M w_i^2 d_i$. Then the following conditions are equivalent.*

(i) There exists a $\frac{c}{N}$-tight fusion frame in $B_{M,N}(d,w)$.
(ii) For every $1 \leq r \leq N-1$ and every admissible $(M+1)$-tuple $(J_0,\ldots,J_M)$,

$$\frac{r \cdot c}{N} \leq \sum_{i=1}^M w_i^2 \cdot \#(J_i \cap \{1,2,\ldots,d_i\}).$$

Finally, we mention that [42] also provides necessary and sufficient conditions for the existence of uniform tight fusion frames by exploiting the Horn-Klyachko inequalities. We refer to [42] for the statement and proof of this deep result.

1.4 Construction of Fusion Frames

Different applications might have different desiderata which a fusion frame is required to satisfy. In this chapter we present three approaches for constructing fusion frames: firstly, a construction procedure based on a given sequence of eigenvalues of the fusion frame operator; secondly, a construction which focusses on the angles between subspaces; and thirdly a construction which yields fusion frames with particular filter-bank-like properties.

1.4.1 Spectral Tetris Fusion Frame Constructions

Both from a theoretical standpoint and for applications, we often seek to construct fusion frames with a prescribed sequence of eigenvalues of the fusion frame operator. Examples are the analysis of streaming signals for which a fusion frame needs to be designed with respect to eigenbases of inverse noise covariance matrices with given associated eigenvalues, similar to water-filling principles for precoder design in wireless communication or face recognition in which significance-weighted bases of eigenfaces might be given.

Let us go back to frame theory for a moment to see how the development in this theory has had its impact on fusion frame theory. Although unit norm tight frames are the most useful frames in practice, until recently very few techniques for constructing such frames existed. In fact, the main methodology employed was to truncate harmonic frames, and a constructive method for obtaining all equal norm tight frames was available only for $\mathbb{R}^2$ [36]. For years, the field was relying on *existence proofs* given by frame potentials and majorization techniques [24]. A recent significant advance in frame construction occurred with the introduction of spectral tetris methods [17] (see [50]). In this paper, spectral tetris was used to both classify and construct all tight fusion frames which exist for equal dimensional subspaces and weights equal to one. Quickly afterwards, this was generalized to constructing fusion frames with prescribed fusion frame operators restricted to the case where the eigenvalues are ≥ 2 [11]. It was further generalized in [15] to construct fusion frames $(\mathcal{W}_i)_{i=1}^M$ for $\mathcal{H}^N$ with prescribed eigenvalues for the fusion frame operator and with prescribed dimensions for the subspaces. The results in [15] which include the case of eigenvalues smaller than two, are achieved by first extending the spectral tetris algorithm and changing the basic building blocks from adjusted 2×2 unitary matrices to adjusted $k \times k$ discrete Fourier transform matrices.

1.4.2 Constructing Tight Fusion Frames

We start with a result on the existence and construction of tight fusion frames $((\mathcal{W}_i, w_i))_{i=1}^M$ for $\mathcal{H}^N$ with $M \geq 2N$ for equal-dimensional subspaces.

The first result from [17] we present is a slightly technical result which will allow us to immediately construct new tight fusion frames from given ones. The associated procedures are given by the following definitions from [11], which for later use we state for more general non-equal dimensional subspaces.

Definition 4. Let $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M$ be an A-tight fusion frame for $\mathcal{H}^N$.

(a) If $\dim \mathcal{W}_i < N$ for all $i = 1 \ldots, M$ and $\bigcap_{i=1}^M \mathcal{W}_i = \{0\}$, then the *spatial complement of* $\mathcal{W}$ is defined as the fusion frame

$$((\mathcal{W}_i^\perp, w_i))_{i=1}^M.$$

(b) For $i = 1, 2, \ldots, M$, let $(e_{ij})_{j=1}^{m_i}$ be an orthonormal basis for $\mathcal{W}_i$, hence $(\frac{w_i}{\sqrt{A}} e_{ij})_{i=1,j=1}^{M\ ,\ m_i}$ is a Parseval frame for $\mathcal{H}^N$. Set $m = \sum_{i=1}^M m_i$, and let P denote the orthogonal projection which maps an orthonormal basis $(e'_{ij})_{i=1,j=1}^{M\ ,\ m_i}$ for a containing Hilbert space $\mathcal{H}^m$ onto the Parseval frame $(\frac{w_i}{\sqrt{A}} e_{ij})_{i=1,j=1}^{M,m_i}$ given by Naimark's Theorem (see [49]). Then the fusion frame

$$(\text{span}\{(Id - P)e_{ij}\}_{j=1}^{m_i}, \sqrt{A - w_i^2})_{i=1}^M$$

is called the *Naimark complement of* $\mathcal{W}$ *with respect to* $(e_{ij})_{i=1,j=1}^{M\ ,\ m_i}$.

We should mention that the Naimark complement of a fusion frame depends on the particular choice of initial orthonormal bases for the subspaces. If we do not need to make this dependence explicit, we also speak of *a Naimark complement of* $\mathcal{W}$.

We next quickly check whether in the case of tight fusion frames – our situation in this subsection – this indeed yields tight fusion frames.

Lemma 3. *Let $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M$ be a tight fusion frame for $\mathcal{H}^N$, not all of whose subspaces equal $\mathcal{H}^N$. Then both the spatial complement and each Naimark complement of $\mathcal{W}$ are tight fusion frames.*

Proof. To show the claim for the spatial complement, let $x \in \mathcal{H}^N$, denote the tight frame bound of $\mathcal{W}$ by A, and observe that

$$\sum_{i=1}^M w_i^2 \|(Id - P_i)(x)\|_2^2 = \sum_{i=1}^M w_i^2 (\|x\|_2^2 - \|P_i(x)\|_2^2) = \Big(\sum_{i=1}^M w_i^2 - A\Big)\|x\|_2^2.$$

Since $\sum_{i=1}^M \omega_i^2 - A = 0$ if and only if $\dim \mathcal{W}_i = N$ for all $1 \le i \le M$, we have that $((\mathcal{W}_i^\perp, w_i))_{i=1}^M$ is a tight fusion frame.

Turning to Naimark complements, since

$$\langle Pe_{ij}, Pe_{i\ell}\rangle = -\langle (Id - P)e_{ij}, (Id - P)e_{i\ell}\rangle,$$

for $j \neq \ell$, it follows that $((Id - P)e_{ij})_{j=1}^{m_i}$ is an orthogonal set. This implies that $(\mathrm{span}\{(Id - P)e_{ij}\}_{j=1}^{m_i}, \sqrt{1 - w_i^2})_{i=1}^M$ is a tight fusion frame. □

Armed with these definitions, we can now state and prove our first result from [17].

Proposition 5 ([17]). *Let N, M, and m be positive integers such that $1 < m < N$.*

(i) There exist tight fusion frames $((\mathcal{W}_i, w_i))_{i=1}^M$ for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, M$ if and only if tight fusion frames $((\mathcal{V}_i, v_i))_{i=1}^M$ for $\mathcal{H}^N$ with $\dim \mathcal{V}_i = N - m$ for all $i = 1, \ldots, M$ exist.

(ii) There exist tight fusion frames $((\mathcal{W}_i, w_i))_{i=1}^M$ for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, M$ if and only if tight fusion frames $((\mathcal{V}_i, v_i))_{i=1}^M$ for $\mathbb{R}^{Mm-N}$ with $\dim \mathcal{V}_i = (M-1)m - N$ for all $i = 1, \ldots, M$ exist.

Proof. Part (i) follows directly by taking the spatial complement and then using Lemma 3. Part (ii) follows from repeated spatial complement constructions followed by applications of Naimark complements and again application of Lemma 3. □

We now turn to the main theorem of this subsection (cf. [17]), which can be used to answer the question whether for a given a triple (M, m, N) of positive integers, a tight fusion frame (with weights equal to one) of M subspaces of equal dimension m exists for $\mathcal{H}^N$. The result is not merely an existence result but answers the question by explicitly constructing a fusion frame of the given parameters in most cases where one exists. Therefore, besides our previous construction of fusion frames from given ones through complement methods, we need a construction for fusion frames to begin with. Using Theorem 1, one way to construct a tight fusion frame with the parameters (M, m, N) is to construct a tight unit norm frame $(\varphi_{i,j})_{i=1,j=1}^{M,m}$ of Mm elements for $\mathcal{H}^N$, such that $(\varphi_{i,j})_{j=1}^{m}$ is an orthogonal sequence for all $i = 1, \ldots, M$. We can then define the desired tight fusion frame $(\mathcal{W}_i)_{i=1}^M$ by letting $\mathcal{W}_i$ be the span of $(\varphi_{i,j})_{j=1}^m$ for $i = 1, \ldots, M$.

The tool of choice to construct unit norm tight frames whose elements can be partitioned into sets of orthogonal vectors is the spectral tetris construction (see [50]). In general, fusion frame constructions involving spectral tetris work due to the fact that frames constructed via spectral tetris are sparse (cf. also Section 1.6). The sparsity property ensures that the constructed frames can be partitioned into sets of orthonormal vectors, the spans of which are the desired fusion frames.

Theorem 5 ([17]). *Let N, M, and m be positive integers such that $m \leq N$.*

(i) Suppose that $m|N$. Then there exist tight fusion frames $(\mathcal{W}_i)_{i=1}^M$ for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, M$ if and only if $M \geq \frac{N}{m}$.

(ii) Suppose that $m \not| N$. Then the following is true.

(a) If there exists a tight fusion frame $(\mathcal{W}_i)_{i=1}^M$ for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, M$, then $M \geq \lceil \frac{N}{m} \rceil + 1$.

(b) If $M \geq \lceil \frac{N}{m} \rceil + 2$, then tight fusion frames $(\mathcal{W}_i)_{i=1}^M$ for $\mathbb{C}^N$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, M$ do exist.

Proof (Sketch of proof). (i). Suppose that there exists a tight fusion frame $(\mathcal{W}_i)_{i=1}^M$ for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, M$. Then any collection of spanning sets for its subspaces consists of at least Mm vectors which span $\mathcal{H}^N$, thus $M \geq \frac{N}{m}$.

Conversely, assume that $M \geq \frac{N}{m}$ with $K := \frac{N}{m}$ being an integer by assumption. Let $(e_j)_{j=1}^K$ be an orthonormal basis for $\mathcal{H}^K$. There exists a unit norm tight frame $(\varphi_i)_{i=1}^M$ for $\mathcal{H}^K$ (see [49]). Now consider the m sets of orthonormal bases given by $(e_{i+(k-1)m})_{k=1}^K$ for $i = 1, \ldots, m$, and project the tight frame elements onto each of the generated spaces, leading to m unit norm tight frames $(\varphi_{ij})_{i=1}^M$ for $j = 0, \ldots, m-1$. Setting $\mathcal{W}_i = \text{span}\{\varphi_{ij} : j = 0, \ldots, m-1\}$, we obtain the required fusion frame.

(ii). If there exists a tight fusion frame $(\mathcal{W}_i)_{i=1}^M$ for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, M$, then $M \geq \frac{N}{m}$. Since m does not divide N, it follows that $M > \frac{N}{m}$. Hence, by Lemma 3, there exists a tight fusion frame $(\mathcal{V}_i)_{i=1}^M$ for $\mathcal{H}^{Mm-N}$ with $\dim \mathcal{V}_i = m$ for all $i = 1, \ldots, M$. Thus, there exist m orthonormal vectors in $\mathcal{H}^{Mm-N}$ implying that $m \leq Mm - N$. Hence, $M \geq \frac{N}{m} + 1$. The claim follows now from the fact that M is an integer.

(iii). This part of the proof uses the sparsity of frames generated by spectral tetris. For the arguments we refer to [17], and just remark that, first since spectral tetris can in general only be used to construct frames consisting of at least twice as many vectors as the dimension of the space, spatial complements have to be used. Second, the orthogonality relations of the frames constructed by spectral tetris then allow us to stack modulated copies of such frames, resulting in complex *Gabor fusion frames*. □

Theorem 5 leaves one case unanswered. Does a tight fusion frame of M subspaces of equal dimension m exist in $\mathbb{C}^N$ in the case that m does not divide N and $M = \lceil \frac{N}{m} \rceil + 1$? As it happens, the answer is *sometimes yes* and *sometimes no*. Which it is can be decided by repeatedly using Theorem 5 in conjunction with Proposition 5 for at most $m-1$ times and we again refer to [17] for the details. Also note that this result answers a non-trivial problem in operator theory, i.e., it classifies the triples (N, M, m) so that an N-dimensional Hilbert space has M rank m projections which sum to a multiple of the identity.

1.4.3 Spectral Tetris Constructions of General Fusion Frames

We next discuss a general construction introduced in [15], encompassing different eigenvalues of the fusion frame operator as well as different dimensions of the subspaces, therefore including [11] as a special case.

We start by introducing a so-called *reference fusion frame* for a given sequence of eigenvalues. This carefully constructed fusion frame – while having prescribed eigenvalues for its fusion frame operator – will have the striking property that the dimensions of its subspaces are in a certain sense 'maximal', allowing for a given sequence of dimensions to decide whether an associated fusion frame can be constructed using the generalized spectral tetris algorithm STC presented in Figure 1.1 (cf. [11]). This algorithm is a straightforward generalization of the original spectral tetris algorithm from the case of tight frames to the case of frames with prescribed spectrum for the frame operator; i.e. now the rows of the synthesis matrix that is being constructed square sum to the respective prescribed eigenvalues. We will say a tight fusion frame is *constructible via STC*, if there is a frame constructed by STC, whose vectors can be partitioned in such a way that the vectors in each set of the partition span the respective subspaces of the fusion frame.

The construction of the reference fusion frame for a prescribed sequence of eigenvalues is achieved by the following algorithm coined RFF (Figure 1.2). We will denote the reference fusion frame constructed for the sequence $(\lambda_j)_{j=1}^N$ via RFF by $RFF((\lambda_j)_{j=1}^N)$. In RFF and the following results of this section we restrict ourselves to the case of eigenvalues ≥ 2 and just want to mention that this restriction is dropped in [15], where the general case is handled by first extending the spectral tetris construction.

The main goal will now be to derive necessary and sufficient conditions for the constructibility of a fusion frame with prescribed eigenvalues of the fusion frame operator and prescribed dimensions of its subspaces via STC. This will

STC: SPECTRAL TETRIS CONSTRUCTION FOR PRESCRIBED EIGENVALUES

Parameters:

- Dimension: N.
- Number of frame vectors: M.
- Eigenvalues: $(\lambda_j)_{j=1}^N \subseteq [2,\infty)$ satisfying $\sum_{j=1}^N \lambda_j = M$.

Algorithm:

Set $k := 1$.
For $j = 1, \ldots, N$ do
    Repeat
        If $\lambda_j < 2$ and $\lambda_j \neq 1$ then
            $\varphi_k := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
            $\varphi_{k+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
            $k := k + 2$.
            $\lambda_j := 0$.
            $\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
        else
            $\varphi_k := e_j$.
            $k := k + 1$.
            $\lambda_j := \lambda_j - 1$.
        end;
    until $\lambda_j = 0$.
end;

Output:

- Unit norm $(\varphi_i)_{i=1}^M \subset \mathcal{H}^N$ with eigenvalues $(\lambda_j)_{j=1}^N$ for its frame operator.

Fig. 1.1 The STC for constructing a frame with prescribed spectrum of its frame operator.

require us to compare the dimensions of the subspaces of a reference fusion frame constructed by RFF with the prescribed sequence of dimensions.

We first need to recall the notion of majorization. Given a sequence $a = (a_n)_{n=1}^N \in \mathcal{H}^N$, we will denote the sequence obtained by rearranging the coordinates of a in decreasing order by $a^{\downarrow} \in \mathcal{H}^N$. For $(a_n)_{n=1}^N, (b_n)_{n=1}^N \in \mathcal{H}^N$, the sequence $(a_n)_{n=1}^N$ *majorizes* $(b_n)_{n=1}^N$, denoted by $(a_n) \succeq (b_n)$, provided that $\sum_{n=1}^m a_n^{\downarrow} \geq \sum_{n=1}^m b_n^{\downarrow}$ for all $m = 1, \ldots, N-1$ and $\sum_{n=1}^N a_n = \sum_{n=1}^N b_n$.

This notion will be the key ingredient for deriving a characterization of the constructability via spectral tetris of a fusion frame with prescribed eigenvalues and dimensions. We note that we will also use the notion of majorization between sequences of different lengths by agreeing to add zero entries to the shorter sequence in order to have sequences of the same length.

The proof of the following condition is constructive and we refer to [15] for how to iteratively construct the desired fusion frame starting from the reference fusion frame.

RFF (Reference Fusion Frame)

Parameters:

- Dimension: N.
- Eigenvalues: $(\lambda_j)_{j=1}^N \subseteq [2,\infty)$.

Algorithm:

1) Run STC for $(\lambda_j)_{j=1}^N$ and $M := \sum_{j=1}^N \lambda_j$ to obtain frame $(\varphi_i)_{i=1}^M$.
2) K :=maximal support size of the rows of the synthesis matrix of $(\varphi_i)_{i=1}^M$.
3) $S_i := \emptyset$ for $i = 1,\ldots,K$.
4) $k := 0$.
5) Repeat
6) $\quad k := k+1$.
7) $\quad j := \min\{1 \le r \le K\colon \operatorname{supp}\varphi_k \cap \operatorname{supp}\varphi_s = \emptyset \text{ for all } \varphi_s \in S_r\}$.
8) $\quad S_j := S_j \cup \{\varphi_k\}$.
9) until $k = M$.

Output:

- Fusion frame $(\mathcal{V}_i)_{i=1}^K$, where $\mathcal{V}_i = \operatorname{span} S_i$ for $i = 1,\ldots,K$.

Fig. 1.2 The RFF algorithm for constructing the reference fusion frame.

Theorem 6 ([15]). *Let M, N be positive integers with $M \ge 2N$, let $(\lambda_j)_{j=1}^N \subseteq [2,\infty)$, and let $(d_i)_{i=1}^D$ be a sequence of positive integers such that $\sum_{j=1}^N \lambda_j = \sum_{i=1}^D d_i = M$. Further, let $(\mathcal{V}_i)_{i=1}^K = RFF((\lambda_j)_{j=1}^N)$. If $(\dim \mathcal{V}_i) \succeq (d_i)$, then a fusion frame $(\mathcal{W}_i)_{i=1}^D$ for $\mathcal{H}^N$ such that $\dim \mathcal{W}_i = d_i$ for $i = 1,\ldots,D$ and whose fusion frame operator has the eigenvalues $(\lambda_j)_{j=1}^N$ can be constructed via STC.*

In the special case of tight fusion frames the majorization condition is also necessary for constructability via a partitioning into orthonormal sets of a frame constructed via STC.

Theorem 7 ([15]). *Let M, N be positive integers with $M \ge 2N$, and let $(d_i)_{i=1}^D$ be a sequence of positive integers such that $\sum_{i=1}^D d_i = M$. Further, let $(\mathcal{V}_i)_{i=1}^K = RFF((\lambda_j)_{j=1}^N)$ with $(\lambda_j)_{j=1}^N = (\frac{M}{N},\ldots,\frac{M}{N})$. Then the following conditions are equivalent.*

(i) A tight fusion frame $(\mathcal{W}_i)_{i=1}^M$ for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = d_i$ for $i = 1,\ldots,M$, is constructible via STC.
(ii) $(\dim \mathcal{V}_i) \succeq (d_i)$.

1.4.4 Equi-Isoclinic Fusion Frames

Equal norm equi-angular Parseval frames are highly useful for applications, in particular due to their optimal erasure resilience alongside an optimal condition number of the synthesis matrix. Examples include reconstruction without phase [1] and quantum state tomography [47].

The fusion frame analogue of this class of Parseval frames are fusion frames whose subspaces have equal chordal distances, or – as the stricter requirement – that the subspaces be equi-isoclinic [39]. The notion of *chordal distance* was introduced by Conway, Hardin and Sloane in [27], whereas the notion of equi-isoclinic subspaces was introduced by Lemmens and Seidel in [39], the later being further studied by Hoggar [37] and others [30, 31, 32, 35]. Similarly as in frame theory, also this analog class of fusion frames – with equal chordal distances as well as with equi-isoclinic subspaces – is optimally resilient against noise and erasures. For more details, we refer to the discussion in Subsection 1.5.2. At this point, to provide a first intuitive understanding, let us just mention that this class of fusion frames distributes the incoming energy most evenly to the fusion frame measurements.

As a prerequisite we first require the notion of principal angles.

Definition 5. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of $\mathcal{H}^N$ with $m := \dim \mathcal{W}_1 \leq \dim \mathcal{W}_2$. Then the *principal angles* $\theta_1, \theta_2, \ldots, \theta_m$ between $\mathcal{W}_1$ and $\mathcal{W}_2$ are defined as follows:

Let

$$\theta_1 = \min \left\{ \arccos \left(\frac{\langle x_1, x_2 \rangle}{\|x_1\|_2 \|x_2\|_2} \right) : x_i \in \mathcal{W}_i, i = 1, 2 \right\}$$

be the first principle angle, and let $x_i^{(1)} \in \mathcal{W}_i$, $i = 1, 2$ be chosen such that

$$\cos \theta_1 = \frac{\langle x_1^{(1)}, x_2^{(1)} \rangle}{\|x_1^{(1)}\|_2 \|x_2^{(1)}\|_2}.$$

Then, for any $1 \leq j \leq m$, the principle angle θ_j is defined recursively by

$$\theta_j = \min \left\{ \arccos \left(\frac{\langle x_1, x_2 \rangle}{\|x_1\|_2 \|x_2\|_2} \right) : x_i \in \mathcal{W}_i, x_i \perp x_i^{(\ell)} \, \forall 1 \leq \ell \leq j-1, i = 1, 2 \right\},$$

and letting $x_i^{(j)} \in \mathcal{W}_i$ with $x_i \perp x_i^{(\ell)}$ for all $1 \leq \ell \leq j-1$, $i = 1, 2$ be chosen such that

$$\cos \theta_j = \frac{\langle x_1^{(j)}, x_2^{(j)} \rangle}{\|x_1^{(j)}\|_2 \|x_2^{(j)}\|_2}.$$

Armed with this notion, we can now introduce the notion of chordal distance and isoclinicness.

Definition 6. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of $\mathcal{H}^N$ with $m := \dim \mathcal{W}_1 = \dim \mathcal{W}_2$ and denote by P_i the orthogonal projection onto $\mathcal{W}_i$, $i = 1, 2$. Further let $(\theta_j)_{j=1}^m$ denote the principal angles for this pair.

(a) The *chordal distance* $d_c(\mathcal{W}_1, \mathcal{W}_2)$ between $\mathcal{W}_1$ and $\mathcal{W}_2$ is given by

$$d_c^2(\mathcal{W}_1, \mathcal{W}_2) = m - Tr[P_1 P_2] = m - \sum_{j=1}^{m} \cos^2 \theta_j.$$

(b) The subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ are called *isoclinic*, if

$$\theta_{j_1} = \theta_{j_2} \quad \text{for all } 1 \le j_1, j_2 \le m.$$

Multiple subspaces are called *equi-isoclinic*, if they are pairwise isoclinic.

Part (b) of Definition 6 is an equivalent formulation of the standard definition. The main result of this subsection will be a construction of an equi-isoclinic fusion frame, meaning a fusion frame with equi-isoclinc subspaces. One main ingredient is the method of a Naimark complement (cf. Definition 4). As a first step – also as an interesting result by its own – we analyze the change of the principles angles under computing a Naimark complement. The proof is a straightforward computation, and we refer to [18] for the details.

Theorem 8 ([18]). *Let $((\mathcal{W}_i, w_i))_{i=1}^M$ be a Parseval fusion frame for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m$ for all $1 \le i \le M$, and let $((\mathcal{W}_i', \sqrt{1-w_i^2}))_{i=1}^M$ be a Naimark complement of it. For $1 \le i_1 \ne i_2 \le M$, we denote the principal angles for the pair of subspaces $\mathcal{W}_{i_1}, \mathcal{W}_{i_2}$ by $(\theta_j^{(i_1 i_2)})_{j=1}^m$. Then the principal angles for the pair $\mathcal{W}_{i_1}', \mathcal{W}_{i_2}'$ are*

$$\left(\arccos \left(\frac{w_{i_1}}{\sqrt{1-w_{i_1}^2}} \cdot \frac{w_{i_2}}{\sqrt{1-w_{i_2}^2}} \cdot \cos(\theta_j^{(i_1 i_2)}) \right) \right)_{j=1}^M.$$

Next, we utilize this result to provide a method to construct equi-isoclinic fusion frames, which was developed in [7].

Theorem 9 ([7]). *Let $(e_{ij})_{i=1,j=1}^{M\ ,\ N}$ be a union of M orthonormal bases for $\mathcal{H}^N$. Then $(\text{span}\{e_{ij} : j = 1, \ldots, N\}, \sqrt{1/M})_{i=1}^M$ is a Parseval fusion frame for $\mathcal{H}^N$, and we let $(\mathcal{W}_i', \sqrt{(M-1)/M})_{i=1}^M$ denote the Parseval fusion frame for $\mathbb{R}^{(M-1)N}$ derived as its Naimark complement with respect to $(e_{ij})_{i=1,j=1}^{M\ ,\ N}$. Then the following hold.*

(i) For all $i \in \{1, 2, \ldots, M\}$, we have

$$\text{span}\{\mathcal{W}_{i'}'\}_{i' \ne i} = \mathbb{R}^{(M-1)N}.$$

(ii) The principal angles for the pair $\mathcal{W}'_{i_1}, \mathcal{W}'_{i_2}$ *are given by*

$$\theta_j^{(i_1 i_2)} = \arccos\left(\frac{1}{M-1}\right).$$

Thus, $(\mathcal{W}'_i, \sqrt{(M-1)/M})_{i=1}^M$ *forms an equi-isoclinic Parseval fusion frame.*

Proof. The fact that $(\mathrm{span}\{e_{ij} : j = 1, \ldots, N\}, \sqrt{1/M})_{i=1}^M$ is a Parseval fusion frame for $\mathcal{H}^N$ is immediate. Let now $P : \mathbb{R}^{MN} \to \mathcal{H}^N$ denote the orthogonal projection given by Naimark's theorem, so that $e_{ij} = \sqrt{1/M} \cdot Pe'_{ij}$ for some orthonormal basis $(e'_{ij})_{i=1,j=1}^{M\ ,\ N}$ in $\mathbb{R}^{MN}$.

(i). Since, for a fixed i, the set $(e_{ij})_{j=1}^N$ is linearly independent, [6, Cor. 2.6] implies that

$$\mathcal{W}'_i = \mathrm{span}\{(Id - P)e_{i'j'} : i' \neq i\} \quad \text{for all } i = 1, \ldots, M.$$

This proves (i).

(ii). For this, let $i_1 \neq i_2 \in \{1, \ldots, M\}$. Note that the principles angles for the pair $\mathcal{W}_{i_1}, \mathcal{W}_{i_2}$ are all equal to 0. Hence, by Theorem 8, principal angles for the pair $\mathcal{W}'_{i_1}, \mathcal{W}'_{i_2}$ are given by

$$\arccos\left(\frac{\frac{1}{\sqrt{M}}}{\sqrt{1-(\frac{1}{\sqrt{M}})^2}}\frac{\frac{1}{\sqrt{M}}}{\sqrt{1-(\frac{1}{\sqrt{M}})^2}}\cos 0\right) = \arccos\left(\frac{1}{M-1}\right).$$

Thus, (ii) is also proved. □

We now present a particularly interesting special case of this result, namely, when the family $(e_{ij})_{i=1,j=1}^{M\ ,\ N}$ is chosen to be a family of *mutually unbiased bases*. We first define this notion.

Definition 7. A family of orthonormal sequences $\{e_{ij}\}_{i=1}^M$, $j = 1, \ldots, L$, in $\mathcal{H}^N$ is called *mutually unbiased*, if there exists a constant $c > 0$ such that

$$|\langle e_{i_1 j_1}, e_{i_2 j_2}\rangle| = c \quad \text{for all } j_1 \neq j_2$$

If $N = M$, then necessarily $c = \sqrt{1/N}$, and we refer to $\{e_{ij}\}_{i=1,j=1}^{M,\ L}$ as a *family of mutually unbiased bases.*

Now choosing $(e_{ij})_{i=1,j=1}^{M\ ,\ N}$ to be a family of mutually unbiased bases leads to the following special case of Theorem 9.

Corollary 4. *Let* $(e_{ij})_{i=1,j=1}^{M\ ,\ N}$ *be a family of mutually unbiased bases for* $\mathcal{H}^N$*. Then* $(\mathrm{span}\{e_{ij} : j = 1, \ldots, N\}, \sqrt{1/M})$ *is a Parseval fusion frame for* $\mathcal{H}^N$*, and we let* $(\mathcal{W}'_i, \sqrt{(M-1)/M})_{j=1}^M$ *denote the Parseval fusion frame for* $\mathbb{R}^{(M-1)N}$ *derived as its Naimark complement with respect to* $(e_{ij})_{i=1,j=1}^{M\ ,\ N}$.

Then $(\mathcal{W}_i', \sqrt{(M-1)/M})_{j=1}^M$ *is an equi-isoclinic fusion frame, and, moreover, the subspaces* $\mathcal{W}_i'$ *are spanned by mutually unbiased sequences.*

Since mutually unbiased bases are known to exist in all prime power dimensions p^r [48], this result implies the existence of Parseval fusion frames with $M \leq p^r + 1$ equi-isoclinic subspaces of dimension p^r, spanned by mutually unbiased basic sequences in $\mathbb{R}^{(M-1)p^r}$. If neither equi-distance nor equi-isoclinic Parseval fusion frames are realizable, a weaker version are families of subspaces with at most two different values, see [12].

Finally, we mention that a different class of equi-isoclinic fusion frames was recently introduced in [7] by using multiple copies of orthonormal bases.

1.4.5 Fusion Frame Filter Banks

In [26], the first efficiently implementable construction of fusion frames was derived. The main idea is to use specifically designed oversampled filter banks. A *filter* is a linear operator which computes the inner products of an input signal with all translates of a fixed function. In a *filter bank*, several filters are applied to the input, and each of the resulting signals is then downsampled. The problem in designing filter bank frames is to get them to satisfy the large number of conditions needed on the frame for the typical application. An important tool here is the *polyphase matrix*. The fundamental works on filter bank frames [8, 28] characterize translation-invariant frames in $\ell^2(\mathbb{Z})$ in terms of polyphase matrices. In particular, filter bank frames are characterized in [28] and [8] derives the optimal frame bounds of a filter bank frame in terms of the singular values of its polyphase matrix. In the paper [26], these characterizations are then subsequently utilized to construct filter bank fusion frame versions of discrete wavelet and Gabor transforms.

1.5 Robustness of Fusion Frames

Applications naturally call for robustness, which could mean resilience against noise and erasures or stability under perturbation. In this section we will give an introduction to several types of robustness properties of fusion frames.

1.5.1 Noise

One main advantage of redundancy is its property to provide resilience against noise and erasures. Theoretical guarantees for a given fusion frame are determined only in the situation of random signals, see [38]. We should mention that we focus on non-weighted fusion frames in this subsection.

1.5.1.1 Stochastic Signal Model

Let $(\mathcal{W}_i)_{i=1}^M$ be a fusion frame for $\mathbb{R}^N$ with bounds A and B, and for $i = 1, \ldots, M$, let m_i be the dimension of $\mathcal{W}_i$ and U_i be an $N \times m_i$-matrix whose columns form an orthonormal basis of $\mathcal{W}_i$ for $i = 1, \ldots, M$. Further, let $x \in \mathbb{R}^N$ be a zero-mean random vector with covariance matrix $E[xx^T] = R_{xx} = \sigma_x^2 Id$. The noisy fusion frame measurements can then be modeled as

$$z_i = U_i^T x + n_i, \quad i = 1, \ldots, M,$$

where $n_i \in \mathbb{R}^{m_i}$ is an additive white noise vector with zero mean and covariance matrix $E[n_i n_i^T] = \sigma_n^2 Id$, $i = 1, \ldots, M$. It is assumed that the noise vectors for different subspaces are mutually uncorrelated and that the signal vector x and the noise vectors n_i, $i = 1, \ldots, N$, are uncorrelated.

Setting

$$z = (z_1^T \;\; z_2^T \;\; \cdots \;\; z_M^T)^T \quad \text{and} \quad U = (U_1 \;\; U_2 \;\; \cdots \;\; U_M),$$

the composite covariance matrix between x and z can be written as

$$E\left[\begin{pmatrix} x \\ z \end{pmatrix} \begin{pmatrix} x^T & z^T \end{pmatrix}\right] = \begin{pmatrix} R_{xx} & R_{xz} \\ R_{zx} & R_{zz} \end{pmatrix},$$

where

$$R_{xz} = E[xz^T] = R_{xx} U$$

is the $M \times L$ ($L = \sum_{i=1}^M m_i$) cross-covariance matrix between x and z, $R_{zx} = R_{xz}^T$, and

$$R_{zz} = E[zz^T] = U^T R_{xx} U + \sigma_n^2 Id_L$$

is the $L \times L$ composite measurement covariance matrix. The linear MSE minimizer for estimating x from z is the Wiener filter or the LMMSE filter $F = R_{xz} R_{zz}^{-1}$, which estimates x by $\hat{x} = Fz$. Then the associated error covariance matrix R_{ee} is given by

$$R_{ee} = E[(x - \hat{x})(x - \hat{x})^T] = \Big(R_{xx}^{-1} + \frac{1}{\sigma_n^2} \sum_{i=1}^M P_i \Big)^{-1},$$

which is derived using the Sherman-Morrison-Woodbury formula. The MSE is obtained by taking the trace of R_{ee}.

A result from [38] shows that as in the frame case, a fusion frame is optimally resilient against noise if it is tight.

Theorem 10 ([38]). *Assuming the model previously introduced, the following conditions are equivalent.*

(i) The MSE is minimized.
(ii) The fusion frame is tight.

In this case, the MSE is given by

$$MSE = \frac{N\sigma_n^2\sigma_x^2}{\sigma_n^2 + \frac{\sigma_x^2 L}{N}}.$$

Proof. Since $R_{xx} = \sigma_x^2 Id$ and denoting the frame bounds by A and B, we obtain

$$\frac{N}{\frac{1}{\sigma_x^2} + \frac{B}{\sigma_n^2}} \leq (MSE = Tr[R_{ee}]) \leq \frac{N}{\frac{1}{\sigma_x^2} + \frac{A}{\sigma_n^2}}.$$

This implies that the lower bound will be achieved, provided that the fusion frame is tight. The explicit value of the MSE follows from here. □

1.5.2 Erasures

Similar to resilience against noise, redundancy is also beneficial for resilience against erasures. Again, we can distinguish between a deterministic and a stochastic signal model. The first case was analyzed in [4], whereas the second case was studied in [38]. As before, in this subsection we focus on non-weighted fusion frames.

1.5.2.1 Deterministic Signal Model

Let $\mathcal{W} = (\mathcal{W}_i)_{i=1}^M$ be a fusion frame for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, M$. Further, let $T_{\mathcal{W}}$ and $S_{\mathcal{W}}$ be the associated analysis and fusion frame operator, respectively.

The loss of a set of subspaces will be modeled deterministically in the following way. Given $K \subseteq \{1, \ldots, M\}$, the associated operator modeling erasures is defined by

$$E_K : \mathbb{R}^{MN} \to \mathbb{R}^{MN}, \quad E_K((x_i)_{i=1}^M)_j = \begin{cases} x_j : j \notin K, \\ 0 : j \in K. \end{cases}$$

The next ingredient of the model is the measure for the imposed error. In [4], the worst case measure was chosen, which in the case of k lost subspaces is defined by

$$e_k(\mathcal{W}) = \max\{\|Id - S_{\mathcal{W}}^{-1} T_{\mathcal{W}}^* E_K T_{\mathcal{W}}\| : K \subset \{1, \ldots, M\}, |K| = k\}.$$

We first state the result from [4] for one subspace erasure.

Theorem 11 ([4]). *Assuming the model previously introduced, the following conditions are equivalent.*

(i) The worst case error $e_1(\mathcal{W})$ is minimized.
(ii) The fusion frame $\mathcal{W}$ is a Parseval fusion frame.

Proof. Setting $D_K := Id - E_K$ for some $K \subset \{1, \ldots, M\}$ with $K = \{i_0\}$, we obtain

$$\|Id - S_{\mathcal{W}}^{-1} T_{\mathcal{W}}^* E_K T_{\mathcal{W}}\| = \|S_{\mathcal{W}}^{-1} T_{\mathcal{W}}^* D_K T_{\mathcal{W}}\| = \|S_{\mathcal{W}}^{-1} P_{i_0}\|$$

Hence, the quantity

$$e_1(\mathcal{W}) = \max\{\|S_{\mathcal{W}}^{-1} P_{i_0}\| : i_0 \in \{1, \ldots, M\}\}$$

needs to be minimized. This is achieved if and only if $S_{\mathcal{W}} = Id$, which is equivalent to $\mathcal{W}$ being a Parseval fusion frame. □

To analyze the situation of two subspace erasures, we now restrict ourselves to the class of fusion frames, already shown to behave optimally under one erasure and reduce the measure $e_2(\mathcal{W})$ accordingly. Then the following result is true; we refer to [4] for its lengthy proof.

Theorem 12 ([4]). *Assuming the model previously introduced, the following conditions are equivalent.*

(i) The worst case error $e_2(\mathcal{W})$ is minimized.
(ii) The fusion frame $\mathcal{W}$ is an equi-isoclinic fusion frame.

This shows the need to develop construction methodologies for equi-isoclinic fusion frames, and we refer the reader to Subsection 1.4.4 for details.

1.5.2.2 Stochastic Signal Model

We assume the model already detailed in Subsection 1.5.1.1. By Theorem 10, tight fusion frames are maximally robust against noise. Hence, from now on we restrict ourselves to tight fusion frames and study within this class which fusion frames are optimally resilient with respect to one, two, and more erasures. Also, we should mention that all erasures are considered equally important.

Again, the MSE shall be determined when the LMMSE filter F, as defined before, is applied to a measurement vector now with erasures. To model the erasures, let $K \subset \{1, 2, \ldots, M\}$ be the set of indices corresponding to the erased subspaces. Then, the measurements take the form

$$\tilde{z} = (Id - E)z,$$

where E is an $L \times L$ block-diagonal erasure matrix whose ith diagonal block is an $m_i \times m_i$ zero matrix, if $i \notin K$, or an $m_i \times m_i$ identity matrix, if $i \in K$.

The estimate of x is now given by

$$\tilde{x} = F\tilde{z},$$

with associated error covariance matrix

$$\widetilde{R}_{ee} = E\left[(x - \tilde{x})(x - \tilde{x})^T\right] = E\left[(x - F(Id - E)z)(x - F(Id - E)z)^T\right].$$

The MSE for this estimate can be written as

$$MSE = Tr[\widetilde{R}_{ee}] = MSE_0 + \overline{MSE},$$

where $MSE_0 = Tr[R_{ee}]$ and $\overline{MSE}$ is the extra MSE due to erasures given by

$$\overline{MSE} = \alpha^2 Tr\left[\sigma_x^2\left(\sum_{i\in\mathbb{S}} P_i\right)^2 + \sigma_n^2\left(\sum_{i\in\mathbb{S}} P_i\right)\right],$$

where $\alpha = \sigma_x^2/(A\sigma_x^2 + \sigma_n^2)$.

This leads to the following result from [38] for one subspace. We also refer to this paper for its proof.

Theorem 13 ([38]). *Assuming the model previously introduced and letting $(\mathcal{W}_i)_{i=1}^M$ be a tight fusion frame, the following conditions are equivalent.*

(i) The MSE due to the erasure of one subspace is minimized.
(ii) All subspaces $\mathcal{W}_i$ have the same dimension, i.e. $(\mathcal{W}_i)_{i=1}^M$ is an equi-dimensional *fusion frame.*

Recalling the definition of *chordal distance* $d_c(i,j)$ from Section 1.4.4, we can state the result for two and more erasures. As before, we now restrict to the class of fusion frames, already shown to behave optimally under noise and one erasure.

Theorem 14 ([38]). *Assuming the model previously introduced and letting $(\mathcal{W}_i)_{i=1}^M$ be a tight equi-dimensional fusion frame, the following conditions are equivalent.*

(i) The MSE due to the erasure of two subspaces is minimized.
(ii) The chordal distance between each pair of subspaces is the same and maximal, i.e. $(\mathcal{W}_i)_{i=1}^M$ is a maximal equi-distance *fusion frame.*

Finally, let $(\mathcal{W}_i)_{i=1}^M$ be an equi-dimensional, maximally equi-distance tight fusion frame. Then the MSE due to k subspace erasures, $3 \le k < N$, is constant.

As we already mentioned in the introduction, we will end this subsection with a brief remark on the relation of the previously discovered optimal family of fusion frames with Grassmannian packings. For this, we first state the following problem, which is typically referred to as the classical packing problem (see also [27]).

Classical Packing Problem: For given m, M, N, find a set of m-dimensional subspaces $(\mathcal{W}_i)_{i=1}^M$ in $\mathcal{H}^N$ such that $\min_{i\neq j} d_c(i,j)$ is as large as possible. In this case we call $(\mathcal{W}_i)_{i=1}^M$ an *optimal packing.*

A lower bound is given by the so-called *simplex bound*

$$\frac{m(N-m)M}{N(M-1)}.$$

Theorem 15 ([27]). *Each packing of m-dimensional subspaces $(\mathcal{W}_i)_{i=1}^M$ in $\mathcal{H}^N$ satisfies*

$$d_c^2(i,j) \leq \frac{m(N-m)}{N}\frac{M}{M-1}, \quad i,j = 1,\ldots,M.$$

Interestingly, there is a close connection between tight fusion frames and optimal packings given by the following theorem.

Theorem 16 ([38]). *Let $(\mathcal{W}_i)_{i=1}^M$ be a fusion frame of equi-dimensional subspaces with pairwise equal chordal distances d_c. Then, the fusion frame is tight if and only if d_c^2 equals the simplex bound.*

This shows that equi-distance tight fusion frames are optimal Grassmannian packings.

1.5.3 Perturbations

Perturbations are another common disturbance with respect to which one might seek resilience of a fusion frame. Several scenarios of perturbations of the subspaces can be envisioned. In [22], the following canonical Paley-Wiener-type definition was employed.

Definition 8. Let $(\mathcal{W}_i)_{i=1}^M$ and $(\mathcal{V}_i)_{i=1}^M$ be subspaces of $\mathcal{H}^N$ with associated orthogonal projections denoted by $(P_i)_{i=1}^M$ and $(Q_i)_{i=1}^M$, respectively. Further, let $(w_i)_{i=1}^M$ be positive weights, $0 \leq \lambda_1, \lambda_2 < 1$, and $\epsilon > 0$. If, for all $x \in \mathcal{H}^N$ and $1 \leq i \leq M$, we have

$$\|(P_i - Q_i)(x)\| \leq \lambda_1 \|P_i(x)\| + \lambda_2 \|Q_i(x)\| + \epsilon \|x\|,$$

then $((\mathcal{V}_i, w_i))_{i=1}^M$ is called a $(\lambda_1, \lambda_2, \epsilon)$-*perturbation* of $((\mathcal{W}_i, w_i))_{i=1}^M$.

Employing this definition, we obtain the following result about robustness of fusion frames under small perturbations of the associated subspaces. We wish to mention that a perturbation result using a different definition of perturbation can be derived by restricting [46, Thm. 3.1] to fusion frames, however without weights.

Proposition 6 ([22]). *Let* $((\mathcal{W}_i, w_i))_{i=1}^M$ *be a fusion frame for* $\mathcal{H}^N$ *with fusion frame bounds* A *and* B*. Further, let* $\lambda_1 \in [0,1)$ *and* $\epsilon > 0$ *be such that*

$$(1-\lambda_1)\sqrt{A} - \epsilon \left(\sum_{i=1}^M w_i^2 \right)^{1/2} > 0.$$

Moreover, let $((\mathcal{V}_i, w_i))_{i=1}^M$ *be a* $(\lambda_1, \lambda_2, \epsilon)$*-perturbation of* $((\mathcal{W}_i, w_i))_{i=1}^M$ *for some* $\lambda_2 \in [0,1)$*. Then* $((\mathcal{V}_i, w_i))_{i=1}^M$ *is a fusion frame with fusion frame bounds*

$$\left(\frac{(1-\lambda_1)\sqrt{A} - \epsilon \left(\sum_{i=1}^M w_i^2 \right)^{1/2}}{1+\lambda_2} \right)^2 \textit{ and } \left(\frac{\sqrt{B}(1+\lambda_1) + \epsilon \left(\sum_{i=1}^M w_i^2 \right)^{1/2}}{1-\lambda_2} \right)^2.$$

For the proof, we refer to [22]. An even more delicate problem is the perturbation of local frame vectors if we consider the full sensor network problem. The difficulty in this case is the possibility of frame vectors leaving the subspace and hence even changing the dimension of those. A collection of results in this direction can also be found in [22].

1.6 Fusion Frames and Sparsity

In this section we present two different types of results concerning sparsity properties of fusion frames. The first result concerns the construction of tight fusion frames consisting of optimally sparse vectors for efficient processing [20, 19], and the second analyzes the sparse recovery from underdetermined fusion frame measurements [9]. We refer at this point also to [51] for the theory of sparse recovery and Compressed Sensing.

1.6.1 Optimally Sparse Fusion Frames

Typically, data processing applications face low on-board computing power and/or small bandwidth budget. When the signal dimension is large, the decomposition of the signal into its fusion frame measurements requires a large number of additions and multiplications, which may be infeasible for on-board data processing. It would hence be a significant improvement, if the vectors of each orthonormal basis for the subspaces would contain very few non-zero entries, hence – phrasing it differently – be sparse in the standard unit vector basis, thereby ensuring low-complexity processing. In [19, 20], an algorithmic construction of optimally sparse tight fusion frames with prescribed fusion frame operators was indeed derived, which we will present and discuss in this subsection.

1.6.1.1 Sparseness Measure

As already elaborated upon before, we aim for sparsity of orthonormal bases for the subspaces with respect to the standard unit vector basis, which ensures low-complexity processing. Since we are interested in the performance of the whole fusion frame, the total number of non-zero entries seems to be a suitable sparsity measure. This viewpoint can also be slightly generalized by assuming that there exists a unitary transformation mapping the fusion frame into one having this 'sparsity' property. Taking these considerations into account, we are led to proclaim the following definition for a sparse fusion frame, which then reduces to the notion of a sparse frame.

Definition 9. Let $(\mathcal{W}_i)_{i=1}^M$ be a fusion frame for $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m_i$ for all $i = 1, \ldots, M$ and let $(e_j)_{j=1}^N$ be an orthonormal basis for $\mathcal{H}^N$. If for each $i \in \{1, \ldots, M\}$ there exists an orthonormal basis $(\varphi_{i,\ell})_{\ell=1}^{m_i}$ for $\mathcal{W}_i$ with the property that for each $\ell = 1, \ldots, m_i$ there is a subset $J_{i,\ell} \subset \{1, \ldots, N\}$ such that

$$\varphi_{i,\ell} \in \operatorname{span}\{e_j : j \in J_{i,\ell}\} \text{ and } \sum_{i=1}^{M} \sum_{\ell=1}^{m_i} |J_{i,\ell}| = k,$$

we refer to $(\varphi_{i,\ell})_{i=1,\ell=1}^{M,m_i}$ as an *associated k-sparse frame*. The fusion frame $(\mathcal{W}_i)_{i=1}^M$ is called *k-sparse* with respect to $(e_j)_{j=1}^N$, if it has an associated k-sparse frame and if, for any associated j-sparse frame, we have $k \leq j$.

1.6.1.2 Optimality and Maximally Achievable Sparsity

We now have the necessary machinery at hand to introduce a notion of an *optimally* sparse fusion frame. Optimality will typically be considered within a particular class of fusion frames, e.g., in the class of tight ones.

Definition 10. Let $\mathcal{FF}$ be a class of fusion frames for $\mathcal{H}^N$, let $(\mathcal{W}_i)_{i=1}^M \in \mathcal{FF}$, and let $(e_j)_{j=1}^N$ be an orthonormal basis for $\mathcal{H}^N$. Then $(\mathcal{W}_i)_{i=1}^M$ is called *optimally sparse in $\mathcal{FF}$ with respect to $(e_j)_{j=1}^N$*, if $(\mathcal{W}_i)_{i=1}^M$ is k_1-sparse with respect to $(e_j)_{j=1}^N$ and there does not exist a fusion frame $(\mathcal{V}_i)_{i=1}^K \in \mathcal{FF}$ which is k_2-sparse with respect to $(e_j)_{j=1}^N$ with $k_2 < k_1$.

Let N, M, m be positive integers. Then the class of tight fusion frames $(\mathcal{W}_i)_{i=1}^M$ in $\mathcal{H}^N$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, M$ will be denoted by $\mathcal{FF}(M, m, N)$.

In the case $\frac{Mm}{N} \geq 2$ and $\lfloor \frac{Mm}{N} \rfloor \leq M-3$ we know that $\mathcal{FF}(M,m,N)$ is not empty and moreover, can construct a tight fusion frame in this class using the algorithm STFF introduced in Figure 1.3 (see [11]). STFF can be used to construct fusion frames of equal dimensional subspaces with certain prescribed eigenvalues for the fusion frame operator. We want to use STFF to construct tight fusion frames, i.e. we apply STFF for the constant sequence of eigenvalues $\lambda_j = \frac{Mm}{N}$ for all $j = 1, \ldots, N$, and will refer to the constructed fusion frame as STFF(M,m,N). The following result shows that STFF(M,m,N) is optimally sparse in the class $\mathcal{FF}(M,m,N)$. It is a consequence of [19, Thm. 4.4], the analogous result for frames.

STFF (Spectral Tetris for Fusion Frames)

Parameters:

- Dimension: N.
- Number of subspaces: M.
- Dimension of subspaces: m.
- Eigenvalues: $(\lambda_j)_{j=1}^N \subseteq [2,\infty)$ satisfying $\sum_{j=1}^N \lambda_j = Mm$ and $\lfloor \lambda_j \rfloor \leq M-3$ for all $j = 1, \ldots, N$.

Algorithm:

Set $k := 1$.
For $j = 1, \ldots, N$ do
  Repeat
    If $\lambda_j < 2$ and $\lambda_j \neq 1$ then
      $\varphi_k := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
      $\varphi_{k+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
      $k := k+2$.
      $\lambda_j := 0$.
      $\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
    else
      $\varphi_k := e_j$.
      $k := k+1$.
      $\lambda_j := \lambda_j - 1$.
    end;
  until $\lambda_j = 0$.
end;

Output:

- Fusion frame $(\mathcal{W}_i)_{i=1}^M$ with $\mathcal{W}_i := \text{span}\{\varphi_{i+kM} : k = 0, \ldots, m-1\}$.

Fig. 1.3 The STFF algorithm for constructing a fusion frame.

Theorem 17 ([20]). *Let N, M, and m be positive integers such that $\frac{Mm}{N} \geq 2$ and $\lfloor \frac{Mm}{N} \rfloor \leq M-3$. Then the tight fusion frame STFF(M,m,N) is optimally* *sparse in the class $\mathcal{FF}(M,m,N)$ with respect to the standard unit vector basis.*

In particular, this tight fusion frame is $mM + 2(N - \gcd(Mm, N))$-sparse with respect to the standard unit vector basis.

1.6.2 Compressed Sensing and Fusion frames

One possible application of fusion frames is music segmentation, in which each note is not characterized by a single frequency but by the subspace spanned by the fundamental frequency of the instrument and its harmonics. Depending on the type of instrument, certain harmonics might or might not be present in the subspace. The overlapping subspaces from distinct instruments can be appropriately modeled by fusion frames. A canonical question is whether from receiving linear combinations of a collection of signals, each being in one of the subspaces, these signals can be extracted; preferably from as few linear combinations – the measurements – as possible.

This leads to the fundamental question of sparse recovery from fusion frame measurements, which can also be interpreted as structured sparse measurements. In this subsection, we will discuss the answer to this question given in [9], in which sparse recovery results in terms of coherence and RIP-type conditions as well as an average case analysis is provided. In this subsection, due to lack of space, we only focus on the first two.

1.6.2.1 Sparse Recovery from Underdetermined Fusion Frame Measurements

The just described scenario can be modeled in the following way. Let $(\mathcal{W}_i)_{i=1}^M$ be a fusion frame for $\mathcal{H}^N$, and let

$$x^0 = (x_i^0)_{i=1}^M \in \mathcal{H} := \{(x_i)_{i=1}^M : x_i \in \mathcal{W}_i \text{ for all } i = 1, \dots, M\} \subseteq \mathbb{R}^{MN}.$$

Now assume that we only observe n linear combinations of those vectors; i.e., there exist some scalars a_{ji} satisfying $\|(a_{ji})_{j=1}^n\|_2 = 1$ for all $i = 1, \dots, M$ such that we observe

$$y = (y_j)_{j=1}^n = \left(\sum_{i=1}^M a_{ji} x_i^0 \right)_{j=1}^n .$$

We first notice that this equation can be rewritten as

$$y = A_I x^0, \quad \text{where } A_I = (a_{ji} Id_N)_{1 \le j \le n,\, 1 \le i \le M},$$

i.e., A_I is the matrix consisting of the blocks $a_{ij} Id_M$.

We now aim to recover x^0 from those measurements. Since typically only a few subspaces will contain a signal, it is instructive to impose sparsity conditions as follows; we encourage the reader to compare this with the classical definition of sparsity in [51].

Definition 11. Let $x \in \mathcal{H}$. Then x is called *k-sparse*, if

$$\|x\|_0 := \sum_{i=1}^{M} \|x_i\|_0 \leq k.$$

The initial minimization problem to consider would hence be

$$\hat{x} = \mathrm{argmin}_{x \in \mathcal{H}} \|x\|_0 \quad \text{subject to } A_I x = y.$$

¿From the theory of Compressed Sensing, we know that this minimization is NP-hard. A means to circumvent this problem is to consider the associated ℓ_1 minimization problem. In this case, the suitable ℓ_1 norm on $\mathcal{H}$ is a mixed $\ell_{2,1}$ norm defined by

$$\left\|(x_i)_{i=1}^M\right\|_{2,1} := \sum_{i=1}^{M} \|x_i\|_2 \quad \text{for any } (x_i)_{i=1}^M \in \mathcal{H}.$$

This leads to the investigation of the following minimization problem,

$$\hat{x} = \mathrm{argmin}_{x \in \mathcal{H}} \|x\|_{2,1} \quad \text{subject to } A_I x = y.$$

Taking the special structure of $x \in \mathcal{H}$ into account, we can rewrite this minimization problem as

$$\hat{x} = \mathrm{argmin}_{x \in \mathcal{H}} \|x\|_{2,1} \quad \text{subject to } A_P x = y,$$

where

$$A_P = (a_{ji} P_i)_{1 \leq i \leq M,\, 1 \leq j \leq n}. \tag{1.3}$$

This problem is still difficult to implement, since minimization runs over $\mathcal{H}$. To come to the final utilizable form, let $m_i = \dim \mathcal{W}_i$ and U_i be an $N \times m_i$-matrix whose columns form an ONB of $\mathcal{W}_i$. This leads to the following two problems – one being equivalent to the previous ℓ_0 minimization problem, the other being equivalent to the just stated ℓ_1 minimization problem – which now merely use matrix-only notation:

$$(P_0) \quad \hat{c} = \mathrm{argmin}_c \|c\|_0 \text{ subject to } Y = AU(c)$$

and

$$(P_1) \quad \hat{c} = \mathrm{argmin}_c \|c\|_{2,1} \text{ subject to } Y = AU(c),$$

in which $A = (a_{ij}) \in \mathbb{R}^{n \times M}$, $j \in \mathbb{R}^{m_j}$, and $y_i \in \mathbb{R}^N$, and

$$U(c) = \begin{pmatrix} \frac{c_1^T U_1^T}{\vdots} \\ \hline c_M^T U_M^T \end{pmatrix} \in \mathbb{R}^{M \times N}, \; Y = \begin{pmatrix} \frac{y_1^T}{\vdots} \\ \hline y_n^T \end{pmatrix} \in \mathbb{R}^{n \times N}.$$

1.6.2.2 Coherence Results

A typically exploited measure for the coherence of the measurement matrix is its mutual coherence. In [9], the following notion adapted to fusion frame measurements was introduced.

Definition 12. The *fusion coherence* of a matrix $A \in \mathbb{R}^{n \times M}$ with normalized columns $(a_i = a_{\cdot,i})_{i=1}^M$ and a fusion frame $(\mathcal{W}_i)_{i=1}^M$ for $\mathbb{R}^N$ is given by

$$\mu_f(A, (\mathcal{W}_i)_{i=1}^M) = \max_{j \neq k} \left[|\langle a_j, a_k \rangle| \cdot \|P_j P_k\|_2 \right].$$

The reader should note that $\|P_j P_k\|_2 = |\lambda_{\max}(P_j P_k)|^{1/2}$ equals the largest absolute value of the cosines of the principle angles between $\mathcal{W}_j$ and $\mathcal{W}_k$.

This new notion now enables us to phrase the first main result about sparse recovery. Its proof follows some of the arguments of the proof of the analogous 'frame result' in [29] with increased technical difficulty; therefore, we refer the reader to the original paper [9].

Theorem 18 ([9]). *Let $A \in \mathbb{R}^{n \times M}$ have normalized columns $(a_i)_{i=1}^M$, let $(\mathcal{W}_i)_{i=1}^M$ be a fusion frame in $\mathbb{R}^N$, and let $Y \in \mathbb{R}^{n \times N}$. If there exists a solution c^0 of the system $Y = AU(c)$ satisfying*

$$\|c^0\|_0 < \frac{1}{2}(1 + \mu_f(A, (\mathcal{W}_i)_{i=1}^M)^{-1}),$$

then this solution is the unique solution of (P_0) as well as of (P_1).

This result generalizes the classical sparse recovery result from [29] by letting $N = 1$, since in this case $P_i = 1$ for all $i = 1, \ldots, M$.

1.6.2.3 RIP Results

The RIP property, which complements the mutual coherence conditions, was also adapted to the fusion frame setting in [9] in the following way.

Definition 13. Let $A \in \mathbb{R}^{n \times M}$ and $(\mathcal{W}_i)_{i=1}^M$ be a fusion frame for $\mathcal{H}^N$. Then the *fusion restricted isometry constant* δ_k is the smallest constant such that

$$(1 - \delta_k)\|z\|_2^2 \leq \|A_P z\|_2^2 \leq (1 + \delta_k)\|z\|_2^2$$

for all $z \in \mathbb{R}^{NM}$ of sparsity $\|z\|_0 \leq k$, where A_P is defined as in (1.3).

The definition of the *restricted isometry constant* in [13] is a special case of Definition 13 with $N = 1$ and $\dim \mathcal{W}_i = 1$ for $i = 1, \ldots, M$. Again, we refer to [9] for the proof of the following theorem.

Theorem 19 ([9]). *Let $(A, (\mathcal{W}_i)_{i=1}^M)$ have the fusion frame restricted isometry constant $\delta_{2k} < 1/3$. Then (P_1) recovers all k-sparse c from $Y = AU(c)$.*

1.7 Non-Orthogonal Fusion Frames

Until recently, fusion frame theory has mainly focused on the construction of fusion frames with specified properties. However, in practice, we might not have the freedom to choose the 'best fusion frame', since it is often given by the application. One example is the application to modeling of sensor networks (cf. Subsection 1.1.3), in which each sensor spans a fixed subspace $\mathcal{W}$ of $\mathcal{H}^N$ generated by the spatial reversal and the translates of the sensor's impulse response function [40, 41].

Although in such applications selection or manipulation of the subspaces is not possible, sometimes there is the freedom to choose the measuring procedure, i.e., the operators mapping the signal onto each element from the family of subspaces. Let us consider again the example of distributed sensing. At the first stage, each sensor in an particular area measures the scalar $\langle x, \varphi_i \rangle$ of an incoming signal $x \in \mathcal{H}^N$, where $\varphi_i \in \mathcal{H}^N$ depend on the characteristics of the respective sensor for all $i \in I$, say. Now, assume that $\mathcal{W} = \text{span}\{\varphi_i : i \in I\}$. Instead of combining the scalars $\langle x, \varphi_i \rangle$ to obtain the orthogonal projection of x onto $\mathcal{W}$, also $P(x)$, where P is a non-orthogonal projection onto $\mathcal{W}$, could be computed. In such cases, one objective is sparsity of the fusion frame operator, which ensures, despite the fact that tightness might not be achievable, an efficient reconstruction algorithm. Particularly desirable would be if the fusion frame operator is a multiple of the identity or at least a diagonal operator.

Another problem is the limited availability of tight fusion frames (cf. [50]). The effectiveness of fusion frame applications in distributed systems is heavily dependent on the end fusion process. This in turn depends upon the efficiency of the inversion of the fusion frame operator. Tight fusion frames take care of this problem because the frame operator is a multiple of the identity and hence its inverse operator is also a multiple of the identity. But tight fusion frames do not exist in situations. The idea here is to use non-orthogonal projections which will result in much larger classes of fusion frames with the (non-orthogonal) fusion frame operator equal to a multiple of the identity.

To tackle these problems, the theory of non-orthogonal fusion frames was recently introduced in [10]. The main idea is to replace the orthogonal projections in the definition of a fusion frame with general projections, i.e., with linear operators Q from $\mathcal{H}^N$ onto a subspace $\mathcal{W}$ of $\mathcal{H}^N$ which satisfy $Q = Q^2$. Recall that in this case, the adjoint Q^* is presumably also a non-orthogonal projection onto $\mathcal{N}(Q)^\perp$ with $\mathcal{N}(Q) \oplus \mathcal{W} = \mathcal{H}^N$, where $\mathcal{N}(Q) = \{x \in \mathcal{H}^N : Qx = 0\}$. This yields the following definition, which generalizes the classical notion of a fusion frame.

Definition 14. Let $(\mathcal{W}_i)_{i=1}^M$ be a family of subspaces in $\mathcal{H}^N$, and let $(w_i)_{i=1}^M \subseteq \mathbb{R}^+$ be a family of weights. For each $i = 1, 2, \ldots, M$ let Q_i be a (orthogonal or non-orthogonal) projection onto $\mathcal{W}_i$. Then $((Q_i, w_i))_{i=1}^M$ is a *non-orthogonal fusion frame* for $\mathcal{H}^N$, if there exist constants $0 < A \le B < \infty$ such that

$$A\|x\|_2^2 \le \sum_{i=1}^M w_i^2 \|Q_i(x)\|_2^2 \le B\|x\|_2^2 \quad \text{for all } x \in \mathcal{H}^N.$$

The constants A and B are called the *lower* and *upper fusion frame bound*, respectively.

Letting $\mathcal{W} = ((Q_i, w_i))_{i=1}^M$ be a non-orthogonal fusion frame for $\mathcal{H}^N$, the associated *analysis operator* $T_\mathcal{W}$ is defined by

$$T_\mathcal{W} : \mathcal{H}^N \to \mathbb{R}^{MN}, \quad x \mapsto (w_i Q_i(x))_{i=1}^M,$$

and the *synthesis operator* $T_\mathcal{W}^*$, has the form

$$T_\mathcal{W}^* : \mathbb{R}^{MN} \to \mathbb{R}^N, \quad (y_i)_{i=1}^M \mapsto \sum_{i=1}^M w_i Q_i^*(y_i).$$

The *non-orthogonal fusion frame operator* $S_\mathcal{W}$ is then given by

$$S_\mathcal{W} = T_\mathcal{W}^* T_\mathcal{W} : \mathcal{H}^N \to \mathcal{H}^N, \quad x \mapsto \sum_{i=1}^M w_i^2 Q_i^* Q_i(x).$$

Similar to Theorem 2, we have the following result.

Theorem 20 ([10]). *Let $\mathcal{W} = ((Q_i, w_i))_{i=1}^M$ be a non-orthogonal fusion frame for $\mathcal{H}^N$ with fusion frame bounds A and B and associated non-orthogonal fusion frame operator $S_\mathcal{W}$. Then $S_\mathcal{W}$ is a positive, self-adjoint, invertible operator on $\mathcal{H}^N$ with $A\,Id \le S_\mathcal{W} \le B\,Id$. Moreover, we have the reconstruction formula*

$$x = \sum_{i=1}^M w_i^2 S_\mathcal{W}^{-1}(Q_i^* Q_i(x)) \quad \text{for all } x \in \mathcal{H}^N.$$

We now focus on the second problem, when we have the freedom to choose the subspaces as well as the projections. Surprisingly, this additional freedom enables the construction of tight (non-orthogonal) fusion frames in almost all situations as the next result shows.

Theorem 21 ([10]). *Let $m_i \le \frac{N}{2}$ for all $i = 1, 2, \ldots, M$ satisfy $\sum_{i=1}^{M} m_i \ge N$. Then there exists a tight non-orthogonal fusion frame $((Q_i, w_i))_{i=1}^{M}$ for $\mathbb{R}^N$ such that rank$(Q_i) = m_i$ for all $i = 1, ..., M$.*

This result shows that if the dimensions of subspaces are less than or equal to half the dimension of the ambient space, there always exists a tight non-orthogonal fusion frame. The proof in fact shows that the weights can even be chosen to be equal to 1. Thus, non-orthogonality allows a much larger class of tight fusion frames.

To prove this result, we first require a particular classification of positive, self-adjoint operators by projections. In order to build up some intuition, let $T : \mathbb{R}^N \to \mathbb{R}^N$ be a positive, self adjoint operator. The goal is to classify the set

$$\Omega(T) = \{Q : Q^2 = Q,\ Q^*Q = T\}.$$

We first observe that, by the spectral theorem, T can be written as

$$T = \sum_{i=1}^{M} \lambda_i P_i,$$

where the λ_i is the ith eigenvalue of T and P_i is the orthogonal projection onto the space generated by the ith eigenvector of T. Hence $Q \in \Omega(T)$ if and only if the eigenvalues and eigenvectors of Q^*Q coincide with those of T. Noting that $Q \in \Omega(T)$ implies $\ker(Q) = \mathrm{im}(T)^\perp$ and recalling that a projection is uniquely determined by its kernel and its image it suffices to consider the set

$$\tilde{\Omega}(T) = \{\mathrm{im}(Q) : Q \in \Omega(T)\}.$$

Moreover, observe that since the only projection of rank N is the identity, we may assume rank$(T) < N$.

The next result states the classification of $\tilde{\Omega}(T)$ (and hence $\Omega(T)$) which we require for the proof of Theorem 21. Although the proof is fairly elementary we refer the reader to the complete argument in [10].

Theorem 22. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a positive, self-adjoint operator of rank $k \le \frac{N}{2}$. Let $(\lambda_j)_{j=1}^{k}$ be the nonzero eigenvalues of T and suppose $\lambda_j \ge 1$ for $j = 1, ..., k$ and suppose $(e_j)_{j=1}^{k}$ is an orthonormal basis of $\mathrm{im}(T)$ consisting of eigenvectors of T associated to the eigenvalues $(\lambda_j)_{j=1}^{k}$. Then*

$$\tilde{\Omega}(T) = \left\{ \mathit{span} \left\{ \frac{1}{\sqrt{\lambda_j}} e_j + \sqrt{\frac{\lambda_j - 1}{\lambda_j}} e_{j+k} \right\}_{j=1}^{k} : (e_j)_{j=1}^{2k} \text{ is orthonormal} \right\}.$$

Let $T : \mathbb{R}^N \to \mathbb{R}^N$ be a positive, self-adjoint operator. Applying Theorem 22 to $\frac{1}{\lambda_k} T$, where λ_k is the smallest non-zero eigenvalue of T and setting $v = \sqrt{\lambda_k}$, yields the following corollary.

Corollary 5. *Let $T : \mathbb{R}^N \to \mathbb{R}^N$ be a positive, self-adjoint operator of rank $\leq \frac{N}{2}$. Then there exists a projection Q and a weight v so that $T = v^2 Q^* Q$.*

Having these prerequisites, we can now prove Theorem 21.

Proof (Proof of Theorem 21). Let $(e_j)_{j=1}^N$ be an orthonormal basis of $\mathbb{R}^N$, and let $(\mathcal{W}_i)_{i=1}^M$ be a family of subspaces of $\mathcal{H}^N$ such that

(a) $\mathcal{W}_i = \text{span}\{e_j\}_{j\in J_i}$ with $|J_i| = m_i$ for each $i = 1, ..., M$.
(b) $\mathcal{W}_1 + \ldots + \mathcal{W}_M = \mathcal{H}^N$.

Also, let P_i denote the orthogonal projection onto $\mathcal{W}_i$, and set $S = \sum_{i=1}^M P_i$.
Notice that

$$Id = S^{-1}S = \sum_{i=1}^M S^{-1}P_i.$$

Since each projection P_i is diagonal with respect to $(e_j)_{j=1}^N$, the operator S^{-1} commutes with P_i for each $i = 1, \ldots, M$. Hence, for all $i = 1, ..., M$, $S^{-1}P_i$ is positive and self-adjoint. Now, letting γ denote the smallest nonzero eigenvalue of all $S^{-1}P_i$, $i = 1, \ldots, M$, the operator $\frac{1}{\gamma}S^{-1}P_i$ satisfies the hypotheses of Theorem 22. Thus, there exists a projection Q_i so that

$$Q_i^* Q_i = \frac{1}{\gamma} S^{-1} P_i,$$

leading to

$$\sum_{i=1}^M Q_i^* Q_i = \frac{1}{\gamma} Id.$$

The theorem is proved. □

If we are willing to extend the framework even further and allow *two* projections onto each subspace, it can be shown that Parseval non-orthogonal fusion frames can be constructed for any sequence of dimensions of the subspaces [10].

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