Decompositions of frames and a new frame identity

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ABSTRACT

We analyze a fundamental question in Hilbert space frame theory: What is the *optimal* decomposition of a Parseval frame? We will see that this question impacts several famous unsolved problems in different areas of mathematics. As a step towards the solution of this question, we give a new identity which holds for all Parseval frames.

Keywords: frame, Riesz basis, Bessel sequence

1. INTRODUCTION

Let $\mathbb{H}$ be a Hilbert space and $f_i \in \mathbb{H}$ for $i \in I$. Let $\mathbf{K} = \overline{\mathrm{span}}\ \{\mathrm{f_i}\}_{\mathrm{i\in I}}$. A number $A > 0$ (respectively, $B > 0$) is called a *lower* (respectively, *upper*) *frame bound* for $\mathcal{F} = \{f_i\}_{i\in I}$ if for all $f \in \mathbf{K}$

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2,$$

respectively,

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2.$$

If $B < \infty$ we call $\mathcal{F} = \{f_i\}_{i\in I}$ a *Bessel sequence with Bessel bound B*. If $0 < A \leq B < \infty$, then $\{f_i\}_{i\in I}$ is a *frame* for $\mathbf{K}$. If $\mathbf{K} \neq \mathbb{H}$ we call $\{f_i\}_{i\in I}$ a *frame sequence* in $\mathbb{H}$. The largest A and the smallest B satisfying the above inequalities are called the *optimal* lower and upper frame bound and will be denoted $A(\mathcal{F})$ and $B(\mathcal{F})$ respectively. If $A = B = \lambda$ we call this a λ-*tight frame* and if $\lambda = 1$ it is called a *Parseval frame*. If all the frame elements have the same norm we call this an *equal-norm* frame and if the frame elements have norm 1 it is called a *unit-norm frame*. If $\{f_i\}_{i\in I}$ is a Bessel sequence, the *synthesis operator* for $\{f_i\}_{i\in I}$ is the bounded linear operator $T : \ell_2(I) \to \mathbb{H}$ given by $T(e_i) = f_i$ for all $i \in I$ where $\{e_i\}_{i\in I}$ is the unit vector basis of $\ell_2(I)$. The *analysis operator* for $\{f_i\}_{i\in I}$ is T^* and satisfies:

$$T^*(f) = \sum_{i\in I} \langle f, f_i\rangle e_i.$$

Hence,

$$\|T^*(f)\|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

The *frame operator* for the frame is the positive, self-adjoint invertible operator $S = TT^* : \mathbb{H} \to \mathbb{H}$ satisfying

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i, \quad \text{for all } f \in \mathbb{H}.$$

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Finally, the family $\{f_i\}_{i\in I}$ is a *Riesz basic sequence* in $\mathbb{H}$ with Riesz basis bounds A, B if for all sequences of scalars $\{a_i\}_{i\in I}$ we have

$$A\left(\sum_{i\in I}|a_i|^2\right)^{1/2} \le \|\sum_{i\in I} a_i f_i\| \le B\left(\sum_{i\in I}|a_i|^2\right)^{1/2}.$$

If $\{f_i\}_{i\in I}$ also spans $\mathbb{H}$ it is called a *Riesz basis* for $\mathbb{H}$. For the fundamentals of frame theory we refer the reader to Christensen.[14]

A fundamental question in frame theory involves understanding the behavior of subsets of a Parseval frame. In particular, what is the *optimal decomposition* of a Parseval frame into subsets? To make this precise, if $\mathcal{F} = \{f_i\}_{i\in I}$ is a Bessel sequence and $J \subset I$, we write $A(\mathcal{F}, J)$ (or just $A(J)$ if no confusion will arise) for the optimal lower frame bound of $\{f_i\}_{i\in J}$. We now formulate the problems we are interested in.

Problem 1.1. *Let $\{f_i\}_{i\in I}$ be a Bessel sequence and fix $L \in \mathbb{N}$. Compute the value of*

$$A(L) =: \sup\{\min\ A(J_j) : \{J_j\}_{j=1}^{L} \textit{ is a partition of } I \textit{ into non-empty sets}\}.$$

We are looking for the "optimal" division of our family into L subsets where optimal means the smallest lower frame bound is a maximum over all divisions of the family into L subsets. We are particularly interested in the unit norm tight frame case. Understanding this case is equivalent to understanding the case of equal norm Parseval frames.

A related problem is:

Problem 1.2. *Given a frame $\{f_i\}_{i\in I}$ for $\mathbb{H}$, identify the subsets $J \subset I$ so that $\{f_i\}_{i\in J}$ is a frame for $\mathbb{H}$?*

Finally, we have

Problem 1.3. *Given a frame $\{f_i\}_{i\in I}$ for $\mathbb{H}$, when does there exist a subset $J \subset I$ so that both $\{f_i\}_{i\in J}$ and $\{f_i\}_{i\in J^c}$ are frames for $\mathbb{H}$. And in this case, what is the optimal choice of J?*

To give an indication of the difficulties involved in the above problem, we start with a simple example. Let $\mathbb{H}_{2N}$ be an $2N$-dimensional Hilbert space with unit vectors $\{e_i\}_{i=1}^{2N}$. If $I - P$ is the orthogonal projection of $\mathbb{H}_{2N}$ onto the one dimensional subspace spanned by $\sum_{i=1}^{2N} e_i$ then

$$Pe_j = e_j - \frac{1}{2N}\sum_{i=1}^{2N} e_i,$$

and $\{Pe_j\}_{j=1}^{2N}$ is a Parseval frame sequence in $\mathbb{H}_{2N}$. Let $J = \{1\}$ partition $\{1, 2, \cdots, 2N\}$ into two subsets J, J^c. We estimate the lower frame bound of $\{Pe_j\}_{j\in J^c}$ by first noting that $\sum_{j=1}^{2N} Pe_j = 0$. So $Pe_1 \in \text{span}\ \{Pe_j\}_{j\in J^c}$. Also,

$$\|Pe_1\|^2 = \langle Pe_1, Pe_1\rangle = \langle e_1, Pe_1\rangle = 1 - \frac{1}{2N} = \frac{2N-1}{2N}.$$

Hence, $f = \sqrt{\frac{2N}{2N-1}}Pe_1$ is a norm one vector in the span of $\{f_i\}_{i\in J^c}$. Then

$$\begin{aligned}
\sum_{j\in J^c}|\langle f, Pe_j\rangle|^2 &= \frac{2N}{2N-1}\sum_{j=2}^{2N}|\langle Pe_1, Pe_j\rangle|^2 \\
&= \frac{2N}{2N-1}\sum_{j=2}^{2N}|\langle e_1, Pe_j\rangle|^2 \\
&= \frac{2N}{2N-1}\sum_{j=2}^{2N}\frac{1}{(2N)^2} = \frac{1}{2N}.
\end{aligned}$$

It follows that $A(J^c) \leq \frac{1}{2N}$.

On the other hand, a significantly better decomposition of this Parseval frame would be to let $J = \{2, 4, \cdots, 2N\}$. Letting T_J be the synthesis operator $T_J e_{2j} = Pe_{2j}$ we have for all sequences of scalars $\{a_{2j}\}_{j=1}^N$

$$
\begin{aligned}
\|T(\{a_{2j}\}_{j=1}^N)\| &= \|\sum_{j=1}^N a_{2j}e_{2j} - \frac{\sum_{j=1}^N a_{2j}}{2N}\sum_{j=1}^{2N} e_j\| \\
&\geq \|\sum_{j=1}^N a_{2j}e_{2j}\| - \|\frac{\sum_{j=1}^N a_{2j}}{2N}\sum_{j=1}^{2N} e_j\| \\
&= \sqrt{\sum_{j=1}^N |a_{2j}|^2} - \frac{|\sum_{j=1}^N a_{2j}|}{2N}\sqrt{2N} \\
&\geq \sqrt{\sum_{j=1}^N |a_{2j}|^2} - \sqrt{\frac{1}{2}}\sqrt{\sum_{j=1}^N |a_{2j}|^2} \\
&= (1 - \sqrt{\frac{1}{2}})\sqrt{\sum_{j=1}^N |a_{2j}|^2}.
\end{aligned}
$$

It follows that for all $f \in span\ \{Pe_{2j}\}_{j=1}^N$ we have

$$\|T^*(f)\|^2 = \sum_{j=1}^N |\langle f, Pe_{2j}\rangle|^2 \geq (1 - \sqrt{\frac{1}{2}})\|f\|^2.$$

We have a similar calculation for $\{Pe_j\}_{j\in J^c}$. So this decomposition yields (by symmetry)

$$A(J) = A(J^c) \geq (1 - \sqrt{\frac{1}{2}})^2.$$

2. AN ANALYSIS OF THE PROBLEMS

In this section we will relate the problems from Section 1 to several unsolved problems in various areas of research. A famous unsolved problem in Banach space theory is the Strong Bourgain-Tzafriri Conjecture.

Conjecture 2.1 (Strong B-T). *There is a universal constant $c > 0$ so that for every $B > 0$ there is a natural number $M = M(B)$ so that if $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for all $1 \leq i \leq n$ and $\|T\| \leq B$, then there is a partition $\{I_j\}_{j=1}^M$ of $\{1, 2, \cdots, n\}$ so that for each $1 \leq j \leq M$ and all choices of scalars $\{a_i\}_{i\in I_j}$ we have:*

$$\|\sum_{i\in I_j} a_i Te_i\|^2 \geq c\sum_{i\in I_j} |a_i|^2.$$

It is known that the 1959 Kadison-Singer Problem in C^*-Algebra Theory[20] is equivalent to the *Paving Conjecture* in Operator Theory[1] which in turn has been shown to be equivalent to the Strong Bourgain-Tzafriri Conjecture.[13] There is also a *Weak Bourgain-Tzafriri Conjecture*. It is stated exactly like Conjecture 2.1 except that the Riesz basis bound is not *universal* but instead is a function of the Bessel bound. It is clear that Strong BT implies Weak BT but the converse is an open problem. It is also known that the Weak BT is equivalent to the Feichtinger Conjecture in frame theory.[9]

Conjecture 2.2 (Feichtinger Conjecture). *Can every unit norm frame be written as the finite union of Riesz basic sequences?*

Feichtinger observed that this conjecture is true for all of the Gabor frames he was working with. This led him to ask the problem for all Gabor frames and eventually for all frames. This conjecture is still open for general Gabor frames but has been shown to hold for rational lattice Gabor frames[9] and for Gabor frames whose window function has some form of rapid decay (see[6–8, 16–18] for results on the Feichtinger Conjecture). By adding an orthonormal basis to a general unit norm Bessel sequence we will obtain a unit norm frame which has the property that it can be written as a finite union of Riesz basic sequences if and only if the original Bessel sequence can be written this way. That is, the Feichtinger Conjecture is equivalent to the Bessel Feichtinger Conjecture.

Conjecture 2.3 (Bessel Feichtinger Conjecture). *Can every unit norm Bessel sequence be written as a finite union of Riesz basic sequences?*

Recall that a family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$ is called *ω-independent* if whenever $\{a_i\}_{i\in I}$ is a family of scalars and $\sum_{i\in I} a_i f_i = 0$ then $a_i = 0$ for all $i \in I$. In order for the Feichtinger conjecture to have a positive solution, it is necessary that we be able to write a unit norm Bessel sequence as a finite union of ω-independent sets. This weaker statement is an open question at this time. However, it was shown in[9] that every Bessel sequence can be written as a finite union of linearly independent sets with the number of sets less and or equal to the smallest integer which is greater than or equal to the Bessel bound.

Let us see how our Problem 1 relates to these conjectures. For the finite dimensional case, if $\{f_i\}_{i\in I}$ is a unit norm Bessel sequence in $\mathbb{H}_N$, we can easily extend it to a unit norm tight frame by just taking each f_i and extending it to an orthonormal basis for $\mathbb{H}_N$ to get a family $\{f_{ij}\}_{i\in I, j=1}^{N}$. This family is a union of M-orthonormal bases for $\mathbb{H}_N$ and so is a unit norm tight frame with tight frame bound M. This is not good for the above problems since decomposing a Bessel sequence into Riesz basic sequences is a function of the Bessel bound and we may have significantly increased the Bessel bound of our family. However, with a little more care, we can extend a unit norm Bessel family to make it a tight frame without disturbing the Bessel bound very much. To do this we need a result of Casazza and Leon.[11]

Theorem 2.4. *Let S be a positive self-adjoint operator on an N-dimensional Hilbert space $\mathbb{H}_N$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of S. Fix $M \geq N$ and real numbers $a_1 \geq a_2 \geq \cdots \geq a_M \geq 0$. The following are equivalent:*

(1) There is a frame $\{g_j\}_{j=1}^{M}$ for $\mathbb{H}_N$ with frame operator S and $\|g_j\| = a_j$ for all $j = 1, 2, \cdots, M$.

(2) For every $1 \leq k \leq N$ we have

$$\sum_{j=1}^{k} a_j^2 \leq \sum_{j=1}^{k} \lambda_j,$$

and

$$\sum_{j=1}^{M} a_j^2 = \sum_{j=1}^{N} \lambda_j.$$

Now we are ready to produce efficient extensions of unit norm Bessel sequences to unit norm tight frames.

Proposition 2.5. *Let $\{f_i\}_{i\in I}$ be a unit norm Bessel sequence in $\mathbb{H}$ with Bessel bound B. Then there is a unit norm family $\{g_j\}_{j\in J}$ so that $\{f_i\}_{i\in I} \cup \{g_j\}_{j\in J}$ is a unit norm tight frame with tight frame bound $\lambda \leq B + 2$.*

Proof. We will do the finite dimensional case. The infinite dimensional case follows by a similar arguement using the results of.[10] Let $\{f_i\}_{i=1}^{M}$ be a unit norm Bessel sequence with Bessel bound B in an N-dimensional Hilbert space $\mathbb{H}_N$. Let S be the frame operator for this family and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$ be the eigenvalues with respective eigenvectors $\{e_i\}_{i=1}^{N}$ for the positive self-adjoint operator S. So $B = \lambda_1$. If we consider $N(\lambda_1 + 1 + \epsilon) - M$, we see that this equals $N\lambda_1 + N - M$ if $\epsilon = 0$ and it equals $N\lambda_1 + 2N - M$ if $\epsilon = 1$. In particular, there is an $0 \leq \epsilon \leq 1$ so that $N(\lambda_1 + 1 + \epsilon) - M = K \geq N$ where $K \in \mathbb{N}$. Now let S_0 be the positive self adjoint operator on $\mathbb{H}_N$ given by

$$S_0\left(\sum_{i=1}^{N} c_i e_i\right) = \sum_{i=1}^{N} [(\lambda_1 + 1 + \epsilon) - \lambda_i] c_i e_i.$$

So S_0 is a positive self-adjoint operator on $\mathbb{H}_N$ with eigenvectors $\{e_i\}_{i=1}^N$ having respective eigenvalues $\{\lambda_1+1+\epsilon-\lambda_i\}_{i=1}^N$ (which now are in decreasing order). Since each of these eigenvalues is greater than 1, letting $a_i=1$ for $i=1,2,\cdots,K$ we immediately have the first inequality given in (2) of Theorem 2.4. Also,

$$\sum_{j=1}^N [(\lambda_1+1+\epsilon)-\lambda_i] = N(\lambda_1+1+\epsilon) - \sum_{i=1}^N \lambda_i = N(\lambda_1+1+\epsilon) - M = K.$$

The last equality above follows from the fact that

$$\sum_{i=1}^M \|f_i\|^2 = M = \sum_{i=1}^N \lambda_i.$$

Applying Theorem 2.4, there is a family of unit norm vectors $\{g_i\}_{i=1}^K$ in $\mathbb{H}_N$ having S_0 for its frame operator. It follows that $\{f_i\}_{i=1}^M \cup \{g_i\}_{i=1}^K$ is a unit norm frame for $\mathbb{H}_N$ having frame operator $S+S_0$. But $S+S_0$ has eigenvectors $\{e_i\}_{i=1}^N$ with respective eigenvalues

$$[(\lambda_1+1+\epsilon)-\lambda_i]+\lambda_i = \lambda_1+1+\epsilon =: \lambda.$$

So our unit norm frame is tight with tight frame bound $\lambda \le \lambda_1+2$. □

Proposition 2.5 says that even for quantative calculations, studying the behavior of subsets of equal norm tight frames is equivalent to studying the behavior of subsets of equal norm Bessel sequences. This was what let us to Problem 1.1 in the first place.

Now let us look at Problem 1.2. Our next Proposition gives one method of identifying which subsets of a frame might also give a frame for the space. This proposition is a generalization of Lemma 3.7 of.[6]

PROPOSITION 2.6. *Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space $\mathbb{H}$ with frame bounds A, B. Let $J \subset I$ so that $\{f_i\}_{i\in J}$ has Bessel bound $B(J) < A$. Then $\{f_i\}_{i\in J^c}$ is a frame for $\mathbb{H}$.*

Proof. Since $\{f_i\}_{i\in J^c}$ has B as a Bessel bound, we only need to check its lower frame bound. For this we just compute for any $f \in \mathbb{H}$:

$$\begin{aligned}\sum_{i\in J^c} |\langle f, f_i\rangle|^2 &= \sum_{i\in I} |\langle f, f_i\rangle|^2 - \sum_{i\in J} |\langle f, f_i\rangle|^2 \\ &\ge A\|f\|^2 - B(J)\|f\|^2 \\ &= (A - B(J))\|f\|^2.\end{aligned}$$

Since $A - B(J) > 0$, we have the required lower frame bound. □

For the Parseval frame case we have a stronger result (see,[6] Lemma 3.7).

COROLLARY 2.7. *Let $\{f_i\}_{i\in I}$ be a Parseval frame for $\mathbb{H}$ and $J \subset I$. In order for $\{f_i\}_{i\in J}$ to be a frame for $\mathbb{H}$ it is necessary and sufficient that $B(J^c) < 1$. In this case, the optimal lower frame bound for $\{f_i\}_{i\in J}$ is $1 - B(J^c)$.*

Proof. For any $f \in \mathbb{H}$ we have

$$\begin{aligned}\sum_{i\in J} |\langle f, f_i\rangle|^2 &= \sum_{i\in I} |\langle f, f_i\rangle|^2 - \sum_{i\in J^c} |\langle f, f_i\rangle|^2 \\ &\ge \|f\|^2 - B(J^c)\|f\|^2 \\ &= (1 - B(J^c))\|f\|^2.\end{aligned}$$

Since the inequality above is optimal, the corollary follows. □

A deep study of the subsets of a Parseval frame which still form a frame for the space was done in a sequence of papers.[4–7] In[6] a very general theory is developed for studying this problem in terms of localization. In[7] the authors studied Gabor frames indexed by arbitrary subsets of the plane and related this question to the density of the set.

Although these results technically give an answer to our problem, they are difficult to apply in concrete cases since we do not know yet how to identify those subsets of the frame $\{f_i\}_{i\in J}$ for which $B(J^c) < 1$. This is the cruxt of the analysis given in.[6] So much work still has to be done in this direction.

Finally, let us look at Problem 1.3. In order to divide a frame into two frames for the whole space, we first need to be able to divide the frame into two spanning sets. This already is a deep question answered by the Rado-Horn Theorem[19, 21] (see also[12]).

THEOREM 2.8 (RADO-HORN THEOREM). *Let I be a countable index set and let $\{f_i\}_{i\in I}$ be a collection of vectors in a vector space. There is a partition $\{I_j\}_{j=1}^{M}$ of I such that each $\{f_i\}_{i\in I_j}$ is linearly independent if and only if for all finite $J \subset I$ we have*

$$\frac{|J|}{dim\ span\{f_i\}_{i\in J}} \le M.$$

Recently, Casazza, Kutyniok and Speegle gave a redundant version of the Rado-Horn Theorem.[12] Using Theorem 2.8, Casazza, Christensen, Lindner and Vershynin[9] showed that every unit norm Bessel sequence with Bessel bound B can be written as a union of $\lceil B\rceil$ linearly independent sets where $\lceil B\rceil$ is the smallest integer greater than or equal to B. For equal norm Parseval frames this yields an exact decomposition. Since the proof for this case is simple, we include it for completeness.

PROPOSITION 2.9. *Let $\{f_i\}_{i=1}^{KN}$ be a equal norm Parseval frame for $\mathbb{H}_N$. Then $\{f_i\}_{i=1}^{KN}$ can be divided into K linearly independent sets each spanning $\mathbb{H}_N$.*

Proof. Let $J \subset \{1, 2, \cdots, KN\}$. Let P_J be the orthogonal projection of $\mathbb{H}_N$ onto $span_{i\in J}\{f_i\}$. Since

$$\sum_{i=1}^{KN} \|f_i\|^2 = KN\|f_1\|^2 = N,$$

it follows that $\|f_i\| = \sqrt{\frac{1}{K}}$. We now have

$$\dim(span_{i\in J}\{f_i\}) = \sum_{i=1}^{KN} \|P_J f_i\|^2 \ge \sum_{i\in J} \|P_J f_i\|^2 = \sum_{i\in J} \|f_i\|^2 = \frac{|J|}{K}.$$

Hence,

$$\frac{|J|}{\dim span_{i\in J}\{f_i\}} \le K.$$

By the Rado-Horn theorem, we can partition $\{1, 2, \cdots, KN\}$ into linearly independent sets $\{I_j\}_{j=1}^{K}$. Since these sets are linearly independent in $\mathbb{H}_N$ we have that $|I_j| \le N$ for all $1 \le j \le K$. Since $\sum_{j=1}^{K} |I_j| = KN$ it follows that $|I_j| = N$ for all $1 \le j \le K$ and so each family $\{f_i\}_{i\in I_j}$ must span $\mathbb{H}_N$. $\square$

The problem with Proposition 2.9 is that it does not give any quantative estimates for the frame bounds for our subsets. What is needed here is a quantative version of the Rado-Horn Theorem which carries such estimates with it. Although there are numerous proofs of the Rado-Horn Theorem available, they are all quite complicated and quantifying any of them at this time does not look promising. Even in the case of equal norm frames for $\mathbb{H}_N$ with $2N$-elements we do not have any quantative estimates at all for the frame bounds of the two subsets.

3. A NEW FRAME IDENTITY

This section is an announcement for the paper.[3] In[2] the authors were working on a fundamental problem in signal reconstruction: Can signal reconstruction be done without using noisy phase or its estimation? Reconstruction with noisy phase can be a critical problem in speech recognition technology. For many years Engineers believed that speech recognition should be independent of phase. By constructing new classes of Parseval frames for a Hilbert space, in[2] it was shown that signal reconstruction can be done without noisy phase or its estimation thus verifying the longstanding conjecture of the signal processing community.

By way of example, let us consider the Ephraim-Malah noise reduction method of speech signals.[15] Let $\{x(t) : t = 1, 2, \cdots, T\}$ be our samples of a speech signal. We transform these into the time-frequency domain by:

$$K(k,\omega) = \sum_{t=0}^{M-1} g(t)x(t+kN)e^{-2\pi i\omega\frac{t}{M}},$$

where $k = 0, 1, \cdots, \frac{T-M}{N}$, $\omega \in \{0, 1, \cdots, M-1\}$, g is the analysis window and M, N are respectively the window size and the time step. Next, a complicated nonlinear transformation is applied to $|X(k,\omega)|$ to produce an estimate of the short-time spectral amplitude

$$Y(k,\omega) = \frac{\sqrt{\pi}}{2}\frac{\sqrt{v(k,\omega)}}{\gamma(k,\omega)}e^{-\frac{v(k,\omega}{2})}\times$$

$$[(1+v(k,\omega))I_0(\frac{v(k,\omega)}{2}) + v(k,\omega)I_1(\frac{v(k,\omega)}{2})]|X(k,\omega)|,$$

where I_0, I_1 are modified Bessel functions of zero and first order, and $v(k,\omega), \gamma(k,\omega)$ are estimates of certain signal-to-noise ratios. The speech signal windowed Fourier coefficients are now estimated by:

$$\hat{X}(k,\omega) = Y(k,\omega)\frac{X(k,\omega)}{|X(k,\omega)|},$$

and then are transformed back into the time domain through an overlapping procedure

$$\hat{x}(t) = \sum_k \sum_{\omega=0}^{M-1} \hat{X}(k,\omega)e^{2\pi i\omega\frac{t-kN}{M}}h(t-kN),$$

where h is the synthesis window. This example illustrates a common feature of most signal enhancement algorithms: the nonlinear estimation in the representation domain modifies only the amplitude of the transformed signal while keeping its noisy phase. In some applications, such as speech recognition, reconstruction with noisy phase is a critical problem. The optimal solution to this problem would occur if we could perform reconstruction into the input domain without using phase.

In,[2] the problem of reconstruction with noisy phase was solved by constructing new classes of Parseval frames.

THEOREM 3.1. *A generic real (respectively, complex) Parseval frame $\{f_i\}_{i\in I}$ in an N-dimensional Hilbert space $\mathbb{H}_N$ with $2N-1$ elements (resp. $4N-2$ elements) has the property that for all $f \in \mathbb{H}_N$ the mapping*

$$\pm f \longmapsto \{|\langle f, f_i\rangle|\}_{i\in I}$$

is one-to-one. Theorem 3.1 allows us to recover our signal f directly from the absolute values of its frame coefficients without needing the phases of the coefficients.

The next part of the project of reconstruction without noisy phase involves producing efficient algorithms for doing the reconstruction. While looking for such algorithms, the authors of[3] were led to consider the following problem. Given a Parseval frame $\{f_i\}_{i\in I}$, let P be the orthogonal projection of $\ell_2(I)$ onto the range of the analysis operator for the frame. If $f \in \mathbb{H}$ let $x = \{x(i\}_{i\in I} = (\langle f, f_i\rangle)_{i\in I} \in \ell_2(I)$. For any subset $J \subset I$, let

$$x_J(i) = \begin{cases} x(i) & : \; i \in J^c, \\ -x(i) & : \; i \in J. \end{cases}$$

Now, since $Px = x$ we have that

$$(I-P)(x_J + x) = (I-P)(x_J - x).$$

Using the fact that

$$x_J = x - 2\sum_{i\in J}\langle f, f_i\rangle e_i,$$

it was shown that:

$$\|(I-P)(x_J - x)\|^2 = 4\left(\sum_{i\in J}|\langle f, f_i\rangle|^2 - \|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2\right).$$

A similar computation for $(I-P)(x_J + x)$ yields the same equality but for J^c leading to the following identity:

THEOREM 3.2. *Let $\{f_i\}_{i\in I}$ be a Parseval frame for $\mathbb{H}$. For any $J \subset I$ and all $f \in \mathbb{H}$ we have:*

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 - \|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2 = \sum_{i\in J^c}|\langle f, f_i\rangle|^2 - \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2.$$

The identity in Theorem 3.2 is quite surprising in that the corresponding terms on the two sides are not comparable to one another yet they cancel each out identically. For example, if J is the empty set then

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 = 0 = \|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2.$$

For the right hand side we have

$$\sum_{i\in J^c}|\langle f, f_i\rangle|^2 = \|f\|^2 = \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2,$$

and so the right hand side is zero also despite the fact that the corresponding terms on the two sides of the identity are not comparable. If we let $J = \{i_0\}$ be a single element set then both terms on the left hand side of the identity may be nearly zero and both terms on the right hand side may be both nearly $\|f\|^2$ but again they cancel out exactly to produce the identity.

In[3] there is an operator theoretic proof of the identity in Theorem 3.2. This allows the identity to be generalized in several directions. We will discuss just one of these here - the generalization to all frames. Recall that if $\{f_i\}_{i\in I}$ is frame then the *canonical dual frame* is $\{S^{-1}f_i\}_{i\in I}$. The canonical dual frame gives reconstruction for vectors in the space. That is, for all $f \in \mathbb{H}$ we have

$$f = \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i.$$

Now we can state the form of Theorem 3.2 for all frames.

THEOREM 3.3. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathbb{H}$ with canonical dual frame $\{\tilde{f}_i\}_{i\in I}$. If $J \subset I$ and S_J is the frame operator for $\{f_i\}_{i\in J}$ then for all $f \in \mathbb{H}$ we have*

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 - \sum_{i\in J^c}|\langle f, f_i\rangle|^2 = \sum_{i\in I}|\langle S_J f, \tilde{f}_i\rangle|^2 - \sum_{i\in J^c}|\langle S_{J^c} f, \tilde{f}_i\rangle|^2.$$

In the case that $\{f_i\}_{i\in I}$ is a Parseval frame, then $S = I$ so $\tilde{f}_i = f_i$ and

$$\sum_{i\in I}|\langle S_J f, \tilde{f}_i\rangle|^2 = \sum_{i\in I}|\langle S_J f, f_i\rangle|^2 = \|S_J f\|^2 = \|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2.$$

With the corresponding equality for J^c we see that our identity for Parseval frames follows from the identity in Theorem 3.3. In[3] there is also a "weighted" version of these identities.

For a Parseval frame $\{f_i\}_{i\in I}$, if $J \subset I$ and $f \in \mathbb{H}$ then

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 + \sum_{i\in J^c}|\langle f, f_i\rangle|^2 = \|f\|^2.$$

So one of these two quantities is greater than or equal to $\frac{1}{2}\|f\|^2$. It follows that

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 + \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2 = \sum_{i\in J^c}|\langle f, f_i\rangle|^2 + \|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2 \geq \frac{1}{2}\|f\|^2.$$

However, in[3] it is shown that this inequality is actually much larger than this. That is, the right-hand-side is actually $\frac{3}{4}\|f\|^2$.

Finally, let us look at the terms on each side of our identity for our Parseval frame $\{f_i\}_{i\in I}$. Since this is a Parseval frame we have for all sequences of scalars $\{a_i\}_{i\in I}$

$$\|\sum_{i\in I} a_i f_i\|^2 \leq \sum_{i\in I} |a_i|^2.$$

It follows that for all $J \subset I$ and all $f \in \mathbb{H}$ we have

$$\|\sum_{i\in J} \langle f, f_i\rangle f_i\|^2 \leq \sum_{i\in J} |\langle f, f_i\rangle|^2.$$

This shows that both sides of the identity in Theorem 3.2 are always positive. However, in general, the two terms above can be arbitrarily far apart. The optimal inequality relating these two terms comes from the frame operator S_J for $\{f_i\}_{i\in J}$ and shows for all $f \in \mathbb{H}$ we have

$$\sum_{i\in J} |\langle f, f_i\rangle|^2 \leq \|S_J^{-1}\| \|\sum_{i\in J} \langle f, f_i\rangle\|^2.$$

For finite frames, $\|S_J^{-1}\|$ can be arbitrarily large while for infinite frames it may equal infinity.

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