Time-frequency analysis on locally compact groups

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Introduction

In signal analysis it is important to know not only which different frequencies are present in a signal but also when those frequencies exist. In this context time-frequency analysis turned out to be of great importance especially for applications. Usually time-frequency analysis is investigated for functions on $\mathbb{R}$. Now recently also other settings have been considered. Working with discrete signals, time-frequency analysis has to be studied on $\mathbb{Z}$ (see [**AGT91, CM80b, LM94, Pol88**]). However, numerical implementations can only work with finite signals. These are naturally identified with discrete, periodic signals. Hence, for this, time-frequency analysis on finite cyclic groups is desirable [**CQ93, NR96, RW90**]. In image processing time-frequency analysis on $\mathbb{R}^2$ or, more generally, on $\mathbb{R}^d$, $d \in \mathbb{N}$, (compare [**RS97**]) and, in its discrete version, on $\mathbb{Z}^d$ and on finite abelian groups (see [**AGZ91, Dau88**]) is the appropriate setting.

Now time-frequency analysis admits a quite natural generalization to locally compact abelian groups. Such a unified approach seems to be useful, since it emphasizes in a clear way the basic features of time-frequency analysis and since it includes all cases important for applications. Although a lot of the basic results also hold for locally compact abelian groups, the general theory for these groups is considerably more complicated. The generalization to even non-abelian groups could often be done only for special classes of groups.

This thesis comprises four chapters. After having introduced essential definitions and basic information in the first chapter, we present in Chapter 2 results on ambiguity functions and Wigner distributions on locally compact abelian groups and some examples. The third chapter deals with the generalization of the Zak transform to locally compact abelian groups and certain non-abelian groups. The last chapter is concerned with the question of linear independence of time-frequency shifts extended to the situation of locally compact abelian groups.

In order to give a detailed survey of the contents of the chapters, we highlight our motivation and main results now.

The second chapter deals with ambiguity functions and Wigner distributions on locally compact abelian groups. In 1932, the Wigner distribution on $\mathbb{R}$ was introduced by Wigner [**Wig32**] in connection with quantum mechanics. For $f, g \in L^2(\mathbb{R})$, the *Wigner distribution* is the function $W_{f,g}$ on $\mathbb{R}\times\mathbb{R}$ defined by

$$W_{f,g}(y,x) = \int_{\mathbb{R}} \overline{f\left(x - \frac{t}{2}\right)} g\left(x + \frac{t}{2}\right) e^{-2\pi iyt}\, dt.$$

The concept was reintroduced by Ville [**Vil48**] in signal analysis some 15 years later. A mathematical basis for this new signal transform has been developed by De Bruijn [**Bru67**]. Some years later, Claasen and Mecklenbräuker [**CM80a, CM80b, CM80c**] developed a comprehensive approach and originated many new ideas and procedures suited to the time-frequency situation. Recently the Wigner distribution has also gained a lot of attention in synthesis of speech (compare [**AT87**]) and in optics [**Bas78, BBL80**].

In the early 1950s, Woodward introduced the ambiguity function on $\mathbb{R}$ for radar analysis [**Woo53**]. The *ambiguity function* is the function $A_{f,g}$ defined on $\mathbb{R} \times \mathbb{R}$ by

$$A_{f,g}(x,y) = \int_{\mathbb{R}} \overline{f\left(t - \frac{x}{2}\right)} g\left(t + \frac{x}{2}\right) e^{2\pi iyt}\, dt,$$

where $f, g \in L^2(\mathbb{R})$. Since the fundamental work of Wilcox [**Wil60**], the ambiguity function has been widely used in the context of radar and sonar (compare [**Pap77, BC67**]). Its properties are very well understood (see [**AT84, AT85, Jam99, Lie90**]).

One important connection between the Wigner distribution and the ambiguity function is that these time-frequency representations are related by the Plancherel transform. This fact implies that the Wigner distribution and the ambiguity function are indeed different signal representations. This connection is also of importance for the description of the general class of time-frequency representations by characteristic functions [**Coh95**, Chapter 9]. Furthermore, if the Wigner distribution or the ambiguity function of a signal is known, the other can be easily computed by the Plancherel transform.

In the last 20 years many papers have been published using the theory of group representations to examine the Wigner distribution and the ambiguity function on $\mathbb{R}$ [**AT85, AT87, Sch86**]. Further, there have been a lot of attempts to define both time-frequency representations on other groups, for example, on $\mathbb{Z}$, on finite abelian groups or, more generally, on compactly generated locally compact abelian Lie groups [**AT98, CM80b, FS98, Pol88**].

In the present work we generalize the notion of Wigner distribution and ambiguity function to locally compact abelian groups and study properties of them. Let G be a locally compact abelian group, H an open subgroup of G and $\Phi : G \to H$ a topological isomorphism. Further, let $f, g \in L^2(G)$. Then we define the *ambiguity function of f and g* on $G \times \widehat{G}$ by

$$A_{f,g}(x,\omega) = \int_G \overline{f(t\Phi(x^{-1}))} g(t\Phi(x))\omega(t)\, dt.$$

Moreover, the *Wigner distribution of f and g* on $\widehat{G} \times G$ is defined by

$$W_{f,g}(\omega, x) = \int_G \overline{f(x\Phi(t^{-1}))} g(x\Phi(t)) \overline{\omega(t)} \, dt.$$

In Section 2.2 we focus on the ambiguity function. The Wigner distribution is treated in Section 2.3. The program of the two sections is mainly the same. After defining the transform we examine its elementary properties. It turns out that all basic properties which hold in the real case remain true in the general situation. Then we prove that the transform is continuous and vanishes at infinity if and only if the group is 2-root compact (Theorem 2.2.10 and Theorem 2.3.6). Furthermore, we exhibit necessary and sufficient conditions for both transforms to be square-integrable (Theorem 2.2.14, Theorem 2.2.16, Corollary 2.3.8 and Corollary 2.3.9). In particular, this yields the following.

THEOREM. *Let G be a σ-compact locally compact abelian group. Suppose that $G^{(2)}$ is open. Then the following conditions are equivalent.*

(i) $A_{f,g} \in C_0(G \times \widehat{G})$ *for all* $f, g \in L^2(G)$.
(ii) G *is 2-root compact.*
(iii) $\ker \varphi$ *is compact.*
(iv) $A_{f,g} \in L^2(G \times \widehat{G})$ *for all* $f, g \in L^2(G)$.

The same holds for $W_{f,g}$ instead of $A_{f,g}$.

Finally, we give necessary and sufficient conditions (Theorem 2.2.23, Corollary 2.3.13 and Corollary 2.3.19) for the transform to be injective in the following sense. For all $f, g \in L^2(G)$, $A_{f,f} = A_{g,g}$ ($W_{f,f} = W_{g,g}$) holds if and only if there exists $\lambda \in \mathbb{T}$ such that $f = \lambda g$.

In Section 2.4 we then prove the following.

THEOREM. *Let G be a 2-root compact, second countable locally compact abelian group. Suppose that $G^{(2)}$ is open. Then, for all $f, g \in L^2(G)$,*

$$\widehat{A_{f,g}} = W_{f,g} \quad \text{in } L^2(\widehat{G} \times G).$$

At the end of this chapter we discuss several examples (compactly generated locally compact abelian Lie groups, discrete abelian groups, p-adic group, where p is a prime, etc.) and study properties of their ambiguity functions and Wigner distributions.

The purpose of the third chapter is to define the Zak transform on locally compact abelian groups and on certain locally compact groups and to examine its properties on these groups. The Zak transform on $\mathbb{R}$ was introduced in 1950 by Gelfand [**Gel50**] and it was rediscovered by Weil [**Wei64**] and independently by Zak [**Zak67**] in 1967 who used it to construct a quantum mechanical representation for the description of the motion of a Bloch electron in the presence of a magnetic or electric field. For $f \in L^2(\mathbb{R})$, the *Zak* *transform* is the function $Zf : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by

$$Zf(x,y) = \sum_{k=-\infty}^{\infty} f(x+k)e^{2\pi iyk}.$$

It later became a major tool in the analysis of Gabor systems, since it turned out to be highly efficient, for example, for integer oversampling. An important property of the Zak transform on $\mathbb{R}$ is the fact that it is quasi-periodic. Hence it is uniquely determined by its values on $[0,1) \times [0,1)$. Further, it is known that $Z : L^2(\mathbb{R}) \to L^2([0,1)\times[0,1))$ is a Hilbert space isomorphism. Another striking property is that the Zak transform Zf has a zero whenever Zf is continuous on $\mathbb{R}^2$, where $f \in L^2(\mathbb{R})$. This was independently shown by Zak [**BZ81**] and Janssen [**Jan82**]. A review of the Zak transform and of applications to signal analysis can be found in the survey article of Janssen [**Jan88**].

In 1964, Weil [**Wei64**] already introduced the Zak transform for the general setting of locally compact abelian groups. In the last 10 years this has been rediscovered in engineering for such groups as $\mathbb{Z}$ or finite cyclic groups [**AGT91, AGT92, Hei89**]. But until now there only exist a few papers dealing with the Zak transform on general locally compact abelian groups ([**FS98**, Chapter 6] and [**KK98**]).

Now in the first section of Chapter 3 we treat the Zak transform on locally compact abelian groups. Given a locally compact abelian group G and a uniform lattice K in G, the *Zak transform associated with K of $f \in L^2(G)$* can be defined (almost everywhere) on $G \times \widehat{G}$ by

$$Zf(x,\omega) = \sum_{k \in K} f(xk)\omega(k).$$

Also in this case the transform is quasi-periodic (Lemma 3.1.6). Hence the Zak transform Zf, $f \in L^2(G)$, is uniquely determined by its values on a fundamental domain $S\times\Omega$ (see Definition 3.1.1), which replaces the set $[0,1)\times[0,1)$ for $G = \mathbb{R}$. Then we prove that, if, in addition, G is second countable, the map $Z : L^2(G) \to L^2(S \times \Omega)$ is a Hilbert space isomorphism (Theorem 3.1.7). In the next subsection we study zeros of the Zak transform. In particular, we prove the following theorem.

THEOREM. *Let G be a compactly generated locally compact abelian group with non-compact connected component of the identity, let K be a uniform lattice in G and let Z denote the associated Zak transform. Let $f \in L^2(G)$ and suppose that Zf is continuous on $G \times \widehat{G}$. Then Zf has a zero.*

This is joint work with Kaniuth [**KK98**]. Moreover, we use the Zak transform to study properties of Gabor frames (Theorem 3.1.17).

The main purpose of the second part is to deal with locally compact groups. For a class of locally compact groups, which includes all connected and simply connected 2-step nilpotent Lie groups, we introduce a definition of the

Zak transform (Definition 3.2.3). We prove that this Zak transform is quasi-periodic (Proposition 3.2.6). Further, we show that it is a Hilbert space isomorphism (Theorem 3.2.10) when restricted to a fundamental domain. Finally, we study several examples of locally compact groups, for example, the connected and simply connected 2-step nilpotent Lie groups, with respect to properties of their Zak transforms.

The fourth chapter is concerned with the problem of linear independence of time-frequency shifts under a generalized Schrödinger representation. In Gabor analysis, a function $g \in L^2(\mathbb{R})$ is analyzed by examining the set of inner products $\{\langle g, \rho_{\mathbb{R}}(x,y,1)f\rangle : (x,y) \in \Lambda\}$, where $f \in L^2(\mathbb{R})$ and $\Lambda \subseteq \mathbb{R}^2$ are fixed and $\rho_{\mathbb{R}}$ denotes the Schrödinger representation of the Heisenberg group associated with $\mathbb{R}$, $H(\mathbb{R})$. It is natural to ask whether g can be reconstructed from this collection of inner products. To obtain less redundant reconstruction Λ could be chosen to be a discrete subset of $\mathbb{R}^2$. If we want to be able to reconstruct g completely, the set $S(\Lambda, f) := \{\rho_{\mathbb{R}}(x,y,1)f : (x,y) \in \Lambda\}$ has to be a complete subset of $L^2(\mathbb{R})$. In order to obtain a stable reconstruction, the set $S(\Lambda, f)$ has to form a frame. Now an important problem in time-frequency analysis is the implementation of frames [**Chr96**]. In fact, any practical implementation has to be finite. Since any finite collection of linearly independent vectors is a Riesz basis for its linear span, the following question arises: When is $S(\Lambda, f)$ linearly independent?

This problem has been investigated by Heil, Ramanathan and Topiwala in [**HRT96**]. They conjectured that $S(\Lambda, f)$ is linearly independent for any finite subset Λ of $\mathbb{R}^2$ and for any function $f \in L^2(\mathbb{R})$. They did not succeed in proving this, but instead they proved several partial results. In particular, they proved that such a set is linearly independent, if Λ is a finite subset of a unit lattice in $\mathbb{R}\times\mathbb{R}$ [**HRT96**, Proposition 2]. The case when Λ is a lattice in $\mathbb{R}\times\mathbb{R}$ is especially important for questions concerning frames. Later Linnell [**Lin99**] extended this result to finite subsets Λ of discrete subgroups of $\mathbb{R} \times \mathbb{R}$. This is proven by methods from different areas in mathematics, in particular, von Neumann algebra theory. In Section 4.2 we present a proof of a slightly weaker result (Theorem 4.2.3) than Linnell's, but much stronger than Proposition 2 of [**HRT96**].

In [**HRT96**], it was mentioned that "It would be of great interest to understand the problem of independence more generally,...". Now this problem admits a natural generalization to locally compact abelian groups. Let G be a locally compact abelian group. The *Heisenberg group associated with* G, $H(G)$, is the set $G \times \widehat{G} \times \mathbb{T}$ endowed with the multiplication given by

$$(x,\omega,z)(x',\omega',z') = (xx',\omega\omega',zz'\omega'(x)).$$

Then the *Schrödinger representation* $\rho_G : H(G) \to \mathcal{U}(L^2(G))$ is defined by

$$(\rho_G(x,\omega,z)f)(t) = z\,\omega(t)f(xt).$$

In this setting we conjecture the following to be true.

CONJECTURE. *Let G be a locally compact abelian group, let Λ be a finite subset of $H(G)$ and let $H(G)^c$ denote the set of compact elements in $H(G)$. Then the following conditions are equivalent.*

(I) *The subset $\{\rho_G(x,\omega,z)f : (x,\omega,z) \in \Lambda\}$ of $L^2(G)$ is linearly independent for all $f \in L^2(G)$, $f \neq 0$.*
(II) *The elements $(x,\omega,z)H(G)^c$, $(x,\omega,z) \in \Lambda$, are pairwise different.*

Theorem 4.3.4 proves, in particular, the implication (I) $\Rightarrow$ (II) of the conjecture. We did not succeed to prove the converse implication. This remains open even for $G = \mathbb{R}$ (compare [**HRT96**]). But we are going to establish the converse for several choices of Λ, which are especially important for applications. Let G be a locally compact abelian group, K a uniform lattice in G and let $q : H(G) \to H(G)/\mathbb{T}$ denote the quotient map. Then we consider finite subsets $\Lambda \subseteq H(G)$ such that $q(\Lambda) \subseteq K \times A(K,\widehat{G})$ (Theorem 4.3.8), where $A(K,\widehat{G})$ denotes the annihilator of K in $\widehat{G}$. Further, we study the case that $\Phi(q(\Lambda)) \subseteq K \times A(K,\widehat{G})$ (Corollary 4.3.11), where $\Phi : G \times \widehat{G} \to G \times \widehat{G}$ is a metaplectic transform (see Definition 4.3.9). Moreover, Theorem 4.3.15 treats finite subsets $\Lambda \subseteq H(G)$ such that $q(\Lambda) \subseteq G \times \widehat{G}$ is a collinear set (see Definition 4.3.12).

For the proofs of Theorem 4.2.3, Corollary 4.3.11 and Theorem 4.3.15 we need information about the structure of topological automorphisms of the Heisenberg group associated with a locally compact abelian group. This is treated in Section 4.1. There we give a characterization of these automorphisms (Proposition 4.1.3 and Corollary 4.1.4).

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I dedicate this work to my parents Hildegard and Norbert Kutyniok.

CHAPTER 1

Preliminaries and notation

Let G be a locally compact group with neutral element e. If $G = H_1 \times \ldots \times H_n$, $n \in \mathbb{N}$, then let $x_i \in H_i$ denote the i^{th} component of $x \in G$ for all $1 \leq i \leq n$. For $M \subseteq G$, the interior of M is denoted by M°.

An element $x \in G$ is said to be *compact*, if the smallest closed subgroup of G containing x is compact. Let G^c denote the set of compact elements in G. When G is abelian, G^c is a closed subgroup of G ([**HR63**, Theorem 9.10]). Further, G_0 denotes the connected component of the identity in G.

Let $C(G)$ denote the space of continuous functions on G, $C_c(G)$ the space of continuous functions with compact support on G and $C_0(G)$ the space of continuous functions on G which vanish at infinity. Further, let $L^1(G)$ and $L^2(G)$ denote the space of integrable and square-integrable functions on G, respectively.

For a function f on G, one defines $L_y f(x) = f(y^{-1}x)$ and $f^*(x) = \overline{f(x^{-1})}$ for all $x, y \in G$. Let $*$ denote the convolution product. For $M \subseteq G$, the characteristic function of M is denoted by χ_M.

Now let G be a locally compact abelian group. The dual group is denoted by $\widehat{G}$ with unit element 1_G (or just 1).

Remark 1.0.1. Recall that $\mathbb{R}$ can be identified with $\widehat{\mathbb{R}}$ via the topological isomorphism $y \mapsto e^{2\pi i y \cdot}$. Further, we have $\widehat{\mathbb{Z}} = \mathbb{T}$, since $z \mapsto z^{\cdot}$, $\mathbb{T} \to \widehat{\mathbb{Z}}$, is a topological isomorphism. Hence, by the Pontryagin duality theorem, we obtain $\widehat{\mathbb{T}} = \mathbb{Z}$. Moreover, the dual group of a finite abelian group is the group itself.

A subgroup K of G will be called a *uniform lattice*, if it is discrete and cocompact. The subgroup

$$A(K, \widehat{G}) = \{\omega \in \widehat{G} : \omega(k) = 1 \text{ for all } k \in K\}$$

is called the *annihilator of* K *in* $\widehat{G}$.

Remark 1.0.2. Let K be a uniform lattice in G. Then, since $A(K, \widehat{G}) = \widehat{G/K}$ and $\widehat{G}/A(K, \widehat{G}) = \widehat{K}$ and since the dual of a compact abelian group is discrete and vice versa, the subgroup $A(K, \widehat{G})$ is a uniform lattice in $\widehat{G}$.

In the following $G^{(2)}$ is the subgroup of G defined by $G^{(2)} = \{x^2 : x \in G\}$. Further, let the Fourier transform $\hat{}: L^1(G) \to C_0(\widehat{G})$, $f \mapsto \widehat{f}$, be defined by

$$\widehat{f}(\omega) = \int_G f(t)\overline{\omega(t)}\, dt.$$

As a general reference to duality theory of locally compact abelian groups we mention [**HR63**].

The classical Heisenberg group is the set $\mathbb{R}^3$ endowed with the group multiplication defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)).$$

However, the so-called reduced Heisenberg group is the more natural candidate for generalizations. It consists of the set $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$ with multiplication defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', zz'e^{2\pi i x y'}).$$

It is isomorphic to the classical Heisenberg group modulo $\mathbb{Z}$ ($\mathbb{Z}$ as a discrete subgroup of the center). In this work we refer to the reduced Heisenberg group as the *Heisenberg group associated with* $\mathbb{R}$ and denote it by $H(\mathbb{R})$. The irreducible unitary representations of this group are all well-known. The most common one for applications, dealt with in Gabor analysis, is the so-called *Schrödinger representation* $\rho_{\mathbb{R}}$ given by

$$\rho_{\mathbb{R}} : H(\mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R})), \quad (\rho_{\mathbb{R}}(x, y, z)f)(t) = z\, e^{2\pi i y t} f(x + t).$$

Here $\mathcal{U}(\mathcal{H})$ denotes the space of unitary operators on a Hilbert space $\mathcal{H}$.

A possible generalization is the following one (compare [**Fol89**]). The *Heisenberg group associated with* G, $H(G)$, is the set $G \times \widehat{G} \times \mathbb{T}$ with multiplication defined by

$$(x, \omega, z)(x', \omega', z') = (xx', \omega\omega', zz'\omega'(x)).$$

Moreover, then the *Schrödinger representation*, which is an irreducible unitary representation of $H(G)$, is defined by

$$\rho_G : H(G) \to \mathcal{U}(L^2(G)), \quad (\rho_G(x, \omega, z)f)(t) = z\, \omega(t) f(xt).$$

For further information concerning Heisenberg groups we mention [**Fol89**].

Finally, let $\delta_{\cdot,\cdot}$ be the Kronecker symbol. Let S be some set. Then Id_S denotes the identity operator. Furthermore, for a map $\varphi : X \to X_1 \times \ldots \times X_r$, $r \in \mathbb{N}$, we shall always denote by $\varphi_1, \ldots, \varphi_r$ the components of φ, i.e. $\varphi(x) = (\varphi_1(x), \ldots, \varphi_r(x))$ for all $x \in X$.

CHAPTER 2

Ambiguity function and Wigner distribution

In this chapter we generalize the definition of ambiguity function and Wigner distribution from $\mathbb{R}$ to locally compact abelian groups and examine their properties.

In the first section we introduce some notation and establish a result which is a helpful tool subsequently in this chapter.

In the second section we generalize the notion of ambiguity function to locally compact abelian groups. We check whether the basic facts, known for the classical ambiguity function on $\mathbb{R}$, also hold in the general case. Furthermore, we establish results concerning the behaviour at infinity and the square-integrability of ambiguity functions. Finally, we investigate the ambiguity function whether it is injective in a certain sense.

The third section is concerned with the generalization of the classical Wigner distribution. This section is organized almost the same way as the previous one.

In the fourth section we present the main result of this chapter. We prove that, under certain conditions, the Wigner distribution is the Plancherel transform of the ambiguity function.

The purpose of the fifth section is to study ambiguity functions and Wigner distributions of different groups and properties of them.

2.1. Preliminaries

We first have to introduce some terminology, which is needed in the following.

The definition of both an ambiguity function and a Wigner distribution of a locally compact abelian group G requires the existence of an open subgroup of G and a topological isomorphism of G onto this subgroup. Thus, for the remainder of this chapter, let H denote an open subgroup of G and Φ a topological isomorphism $\Phi : G \to H$. Furthermore, during this chapter let φ denote the map $x \mapsto x^2$, $G \to G$.

Further, we give a lemma which will be needed in this chapter several times.

LEMMA 2.1.1. (i) *There exists a positive constant c such that*

$$\int_G |f(\Phi(t))|\, dt \leq c \int_G |f(t)|\, dt \quad \textit{for all } f \in L^1(G).$$

(ii) *Let the Haar measure on H be induced by the Haar measure on G. Then there exists a positive constant d such that*

$$\int_G f(t)\,dt = d\int_H f(\Phi^{-1}(t))\,dt \quad \textit{for all } f \in L^1(G).$$

PROOF. Let I_G denote the Haar integral on G. To show (i), we consider the linear functional

$$J : C_c(H) \to \mathbb{C}, \quad J(f) := I_G(f \circ \Phi).$$

Then, for all $y \in H$ and $f \in C_c(H)$,

$$\begin{aligned} J(L_y f) &= \int_G f(y^{-1}\,\Phi(t))\,dt \\ &= \int_G f((\Phi \circ \Phi^{-1})(y^{-1})\Phi(t))\,dt \\ &= \int_G f(\Phi(\Phi^{-1}(y^{-1})\,t))\,dt \\ &= \int_G f(\Phi(t))\,dt \\ &= J(f), \end{aligned}$$

since the Haar integral I_G is translation-invariant. This proves that J is translation-invariant. Now let I_H denote the Haar integral on H induced by I_G. Since Haar integrals are unique up to a positive multiplicative constant ([**HR63**, Theorem 15.5]), there exists a positive constant c such that

$$cI_H(f) = J(f) \quad \text{for all } f \in L^1(H).$$

Let $f \in L^1(G)$. Then

$$\begin{aligned} \int_G |f(\Phi(t))|\,dt &= I_G(|f \circ \Phi|) \\ &= I_G((|f||_H) \circ \Phi) \\ &= J(|f||_H) \\ &= cI_H(|f||_H) \\ &= cI_G(|f||_H) \\ &\leq cI_G(|f|) \\ &= c\int_G |f(t)|\,dt \end{aligned}$$

This proves (i).

The claim in (ii) follows immediately from the fact that $\Phi : G \to H$ is a topological isomorphism. □

2.2. The ambiguity function on locally compact abelian groups

The purpose of this section is to generalize the notion of the classical ambiguity function to locally compact abelian groups and to exhibit important properties of it.

2.2.1. Definition and some basic facts. The classical ambiguity function of two signals $f, g \in L^2(\mathbb{R})$ is given by

$$A_{f,g}(x,y) = \int_{\mathbb{R}} \overline{f\left(t - \frac{x}{2}\right)} g\left(t + \frac{x}{2}\right) e^{2\pi i y t}\, dt \quad ((x,y) \in \mathbb{R} \times \mathbb{R}).$$

We may rewrite this expression in the following way.

$$\begin{aligned} A_{f,g}(x,y) &\overset{t \mapsto t + \frac{x}{2}}{=} e^{2\pi i y \frac{x}{2}} \int_{\mathbb{R}} \overline{f(t)} g(t+x) e^{2\pi i y t}\, dt \\ &= e^{2\pi i y \frac{x}{2}} \langle \rho_{\mathbb{R}}(x,y,1) g, f \rangle, \end{aligned}$$

where $\rho_{\mathbb{R}}$ denotes the classical Schrödinger representation (compare Chapter 1). Now it seems to be natural to generalize this definition to locally compact abelian groups G by setting

$$A_{f,g}(x,\omega) := \langle \rho_G(x,\omega,1) g, f \rangle \quad (f, g \in L^2(G),\ (x,\omega) \in G \times \widehat{G}),$$

as it was done, for example, in [**FS98**, Subsection 7.6.1] for compactly generated locally compact abelian Lie groups. But this definition will cause difficulties, if we want to generalize the classical notion of Wigner distribution in an analogous way and if, in addition, we want $\widehat{A_{f,g}} = W_{f,g}$ to be satisfied for all $f, g \in L^2(G)$. The reason for this is that the phase factor $e^{2\pi i y \frac{x}{2}}$ has been ignored. To fulfill the equation $\widehat{A_{f,g}} = W_{f,g}$ in the situation above at least for all 2-root compact, second countable locally compact abelian groups such that $G^{(2)}$ is open, $W_{f,g}$ has to be defined by

$$W_{f,g}(\omega, x) = \overline{f(x)} \int_G g(xt) \overline{\omega(t)}\, dt \quad ((\omega, x) \in \widehat{G} \times G).$$

To see this one has to follow the steps of the proof of Theorem 2.4.4. (We don't give the proof here, since this definition of ambiguity function and Wigner distribution is not used in the following.) Obviously, this definition of $W_{f,g}$ is not a canonical generalization of the classical notion of the Wigner distribution. Rather it is a generalization of the Rihaczek distribution [**Pou96**, Table 12.9].

To generalize the notion of ambiguity function in a more suitable way, let H be an open subgroup of G and $\Phi : G \to H$ a topological isomorphism (as said before in the beginning of this chapter). In Section 2.3 it will become clear, why H has to be open. There it shall be shown that otherwise the corresponding Wigner distribution is not defined everywhere. However, the openness of H is not a strong restriction as it is shown in Proposition 2.2.2. We define the ambiguity function in the general case as follows.

DEFINITION 2.2.1. Let G be a locally compact abelian group. For $f, g \in L^2(G)$, the *ambiguity function of f and g* on $G \times \widehat{G}$ (which depends on H and Φ) is defined by

$$A_{f,g}(x,\omega) := \int_G \overline{f(t\Phi(x^{-1}))}g(t\Phi(x))\omega(t)\,dt \quad ((x,\omega) \in G \times \widehat{G}).$$

Further, we denote $A_{f,f}$ by A_f.

It remains to be checked that this function is defined everywhere. But this follows immediately from

$$A_{f,g}(x,\omega) = \langle L_{\Phi(x^{-1})}g \cdot \omega, L_{\Phi(x)}f \rangle \text{ for all } f, g \in L^2(G), (x,\omega) \in G \times \widehat{G}.$$

In addition, it should be examined, whether this definition is indeed a generalization of the classical ambiguity function on $L^2(\mathbb{R})$. For this, we choose H to be $\mathbb{R}$ itself and define $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(x) = \frac{x}{2}$. Then the general definition reduces to the classical definition. Recall that all topological isomorphisms $\Phi : \mathbb{R} \to H$ are of the form $\Phi : \mathbb{R} \to \mathbb{R}$, $x \mapsto \alpha x$, $\alpha \in \mathbb{R}^*$.

First, we study for which groups G the subgroup H is automatically open. Indeed, this is the case for a large class of groups G.

PROPOSITION 2.2.2. *Let G be a locally compact abelian Lie group, H a closed subgroup of G and $\Phi : G \to H$ a topological isomorphism. Then H is open.*

PROOF. Since Φ is a topological isomorphism, we have $\Phi(G_0) = H_0$. Moreover, there exists a discrete group D such that $G/H_0 = (G/H_0)_0 \times D$. This implies

$$\dim G_0 = \dim H_0 + \dim(G/H_0)_0 = \dim G_0 + \dim(G/H_0)_0.$$

Hence $(G/H_0)_0$ is trivial. Therefore, G/H_0 is discrete. Thus H_0 is open in G. □

The next remark shows that H is not always open.

REMARK 2.2.3. Let G be a locally compact abelian group, H a closed subgroup of G and $\Phi : G \to H$ a topological isomorphism. Then it does not follow in general that H is open.

For this, let $G := \prod_{i=1}^{\infty} \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the additive group of residues mod 2 endowed with the product topology. Then define H by

$$H = \prod_{i=1}^{\infty}(\{0\} \times \mathbb{Z}_2) < G$$

and define $\Phi : G \to H$ by

$$\Phi((x_n)_{n\in\mathbb{N}}) = (y_n)_{n\in\mathbb{N}}, \quad \text{where } y_n = \begin{cases} x_{\frac{n}{2}} & : \quad n \text{ is even}, \\ 0 & : \quad n \text{ is odd}. \end{cases}$$

Obviously, Φ is a topological isomorphism, but H is not open in G.

Next we give some different points of view of the ambiguity function, which will be very helpful in the following. The idea to regard the ambiguity function as a matrix coefficient and use its properties already appears in [**AT85**, Theorem 3.4] for $G = \mathbb{R}$.

REMARK 2.2.4. Let G be a locally compact abelian group. For all $f, g \in L^2(G)$ and $(x, \omega) \in G \times \widehat{G}$, the following hold.

(i) $A_{f,g}(x,\omega) = \omega(\Phi(x)) \int_G \overline{f(t)} g(t\,\Phi(x^2))\omega(t)\,dt.$
(ii) $A_{f,g}(x,\omega) = \omega(\Phi(x))((f^* \cdot \overline{\omega}) * g)(\Phi(x^2)).$
(iii) $A_{f,g}(x,\omega) = \omega(\Phi(x))\langle \rho_G(\Phi(x^2), \omega, 1)g, f\rangle.$

These equations follow immediately from the definition of $A_{f,g}$.

We want to check whether the simple basic properties of the classical ambiguity function remain true in the general case.

PROPOSITION 2.2.5. *Let G be a locally compact abelian group. For all $f, g \in L^2(G)$ and $(x, \omega) \in G \times \widehat{G}$, the following hold.*

(i) $|A_{f,g}(x,\omega)| \leq \|f\|_2 \|g\|_2.$
(ii) $\overline{A_{f,g}(x,\omega)} = A_{g,f}(x^{-1}, \overline{\omega}).$
(iii) $A_f(e,1) \geq 0.$

PROOF. Let $f, g \in L^2(G)$ and $(x, \omega) \in G \times \widehat{G}$. To prove (i), we only have to use Hölder's inequality and the translation-invariance of the Haar integral. Then we obtain

$$|A_{f,g}(x,\omega)| = \left| \int_G \overline{f(t\Phi(x^{-1}))} g(t\Phi(x))\omega(t)\,dt \right| \leq \|f\|_2 \|g\|_2.$$

(ii) can also be shown by direct calculation:

$$\begin{aligned}
\overline{A_{f,g}(x,\omega)} &= \int_G f(t\,\Phi(x^{-1}))\overline{g(t\,\Phi(x))}\,\overline{\omega(t)}\,dt \\
&= \int_G \overline{g(t\,\Phi((x^{-1})^{-1}))} f(t\,\Phi(x^{-1}))\overline{\omega}(t)\,dt \\
&= A_{g,f}(x^{-1}, \overline{\omega}).
\end{aligned}$$

Finally,

$$A_f(e,1) = \int_G |f(t)|^2\,dt = \|f\|_2^2 \geq 0.$$

Hence (iii) has been proven. □

To conclude this subsection we want to examine the ambiguity function with respect to continuity. It will turn out that we obtain analogous results as for the classical ambiguity function on $\mathbb{R}$.

PROPOSITION 2.2.6. *Let G be a locally compact abelian group. Then, for all $f, g \in L^2(G)$,*

$$A_{f,g} \in C(G \times \widehat{G}).$$

PROOF. This is simply the fact that $A_{f,g}$ is a matrix coefficient of the representation ρ_G (Remark 2.2.4 (iii)) and that Φ is continuous. □

PROPOSITION 2.2.7. *Let G be a locally compact abelian group. Then the mapping*

$$(f, g) \mapsto A_{f,g}, \quad L^2(G) \times L^2(G) \to (C(G \times \widehat{G}), \|\cdot\|_\infty),$$

is continuous.

PROOF. This is also simply the fact that $A_{f,g}$ is a matrix coefficient (Remark 2.2.4 (iii)). □

2.2.2. Behaviour at infinity. In what follows, we are interested in the behaviour at infinity of the ambiguity function. It will turn out, that the ambiguity function vanishes at infinity for a large class of groups. First, we define this class of groups. It should be mentioned that the following definition is equivalent to [**Hey77**, Definition 3.1.1] by [**Hey77**, Theorem 3.1.4].

DEFINITION 2.2.8. Let $n \in \mathbb{N}$. A locally compact abelian group is called *n-root compact* if, for each compact subset C of G, the set

$$\{x \in G : x^n \in C\}$$

is compact.

Obviously, this definition is equivalent to requiring that the map $x \mapsto x^n$, $G \to G$, is proper.

REMARK 2.2.9. ([**Hey77**, Example 3.1.3]) Let $n \in \mathbb{N}$. Each compactly generated locally compact abelian group is n-root compact .

The next theorem classifies those locally compact abelian groups, for which the ambiguity function is continuous and vanishes at infinity.

THEOREM 2.2.10. *Let G be a locally compact abelian group. Then the following conditions are equivalent.*

(i) *$A_{f,g} \in C_0(G \times \widehat{G})$ for all $f, g \in L^2(G)$.*
(ii) *G is 2-root compact.*

PROOF. First, suppose that (i) holds. This implies

$$|A_{f,g}(\cdot, 1)| = \left|\int_G \overline{f(t)} g(t\,\Phi(\cdot^2))\,dt\right| \in C_0(G) \quad \text{for all } f, g \in L^2(G).$$

Towards a contradiction, assume that G is not 2-root compact. Then there exists a compact subset K of G such that the set $X := \{x \in G : x^2 \in K\}$ is non-compact. Let V be any compact neighbourhood of e and define $f, g \in L^2(G)$

by $f := \chi_V$ and $g := \chi_{V\Phi(K)}$. Then, for all $x \in X$, we obtain

$$\begin{aligned} |A_{f,g}(x,1)| &= \left|\int_G \overline{f(t)} g(t\,\Phi(x^2))\,dt\right| \\ &= \left|\int_G \chi_V(t)\chi_{V\Phi(K)}(t\,\Phi(x^2))\,dt\right| \\ &= \left|\int_G \chi_{V\cap V\Phi(x^{-2}K)}(t)\,dt\right| \\ &= |V|, \end{aligned}$$

a contradiction.

Now suppose that G is 2-root compact. Let $f, g \in L^2(G)$. By Proposition 2.2.6, $A_{f,g} \in C(G \times \widehat{G})$. Now let $\epsilon > 0$. For all $(x,\omega) \in G \times \widehat{G}$, we obtain

$$|A_{f,g}(x,\omega)| \leq \int_G |f(t)||g(t\,\Phi(x^2))|\,dt = (|f^*| * |g|)(\Phi(x^2)).$$

Since $f, g \in L^2(G)$, we obtain $|f^*| * |g| \in C_0(G)$. Hence there exists a compact set $\tilde{K} \subseteq G$ such that

$$(|f^*| * |g|)(y) < \frac{\epsilon}{2} \quad \text{for all } y \in G\backslash\tilde{K}.$$

Define $K \subseteq G$ by $K := \{x \in G : x^2 \in \Phi^{-1}(\tilde{K})\}$. Then

$$|A_{f,g}(x,\omega)| \leq (|f^*| * |g|)(\Phi(x^2)) < \frac{\epsilon}{2} \quad \text{for all } x \in G\backslash K,\ \omega \in \widehat{G}.$$

Since G is 2-root compact by hypothesis and since Φ is a topological isomorphism, K is compact.

On the other hand, $A_{f,g}$ may be rewritten as

$$A_{f,g}(x,\omega) = ((L_{\Phi(x)}\overline{f}) \cdot (L_{\Phi(x^{-1})}g))\widehat{\ }(\overline{\omega}) \quad ((x,\omega) \in G \times \widehat{G}).$$

Since both functions $L_{\Phi(x)}\overline{f}$ and $L_{\Phi(x^{-1})}g$ are square-integrable, we obtain $(L_{\Phi(x)}\overline{f}) \cdot (L_{\Phi(x^{-1})}g) \in L^1(G)$. Hence $A_{f,g}(x,\cdot) \in C_0(\widehat{G})$ for each $x \in G$. This implies that, for each $x \in G$, there exists a compact subset $\Gamma(x) \subseteq \widehat{G}$ such that

$$|A_{f,g}(x,\omega)| < \frac{\epsilon}{2} \quad \text{for all } \omega \in \widehat{G}\backslash\Gamma(x).$$

Now we use the sets $\Gamma(x)$, $x \in G$, to construct a compact subset Γ of $\widehat{G}$ such that $|A_{f,g}(x,\omega)| < \epsilon$ for all $(x,\omega) \in (G \times \widehat{G})\backslash(K \times \Gamma)$.

For this, let $(x,\omega) \in G \times \widehat{G}$ and let x_0 be an arbitrarily fixed element of G. By Remark 2.2.4 (iii),

$$\begin{aligned} |A_{f,g}(x,\omega)| &= |\langle \rho_G(\Phi(x^2 x_0^{-2})\Phi(x_0^2),\omega,1)g, f\rangle| \\ &= |\langle \rho_G(\Phi(x^2 x_0^{-2}),1,1)\rho_G(\Phi(x_0^2),\omega,1)g, f\rangle| \\ &= |A_{f,g}(x_0,\omega) + \langle \rho_G(\Phi(x_0^2),\omega,1)g, \rho_G(\Phi(x^{-2}x_0^2),1,1)f - f\rangle| \\ &\leq |A_{f,g}(x_0,\omega)| + \|g\|_2 \|\rho_G(\Phi(x^{-2}x_0^2),1,1)f - f\|_2. \end{aligned}$$

There exists a neighbourhood $V(x_0) \subseteq G$ of x_0 such that

$$\|\rho_G(\Phi(x^{-2}x_0^2), 1, 1)f - f\|_2 < \frac{\epsilon}{2\|g\|_2} \quad \text{for all } x \in V(x_0).$$

Since K is compact, it follows that we can choose finitely many elements $x_1, \ldots, x_N \in G$, $N \in \mathbb{N}$, in such a way that the union of the sets $V(x_i)$, $i = 1, \ldots, N$, covers K. Then we define $\Gamma \subset \widehat{G}$ by

$$\Gamma := \bigcup_{i=1}^{N} \Gamma(x_i).$$

Clearly, Γ is compact.

Now let $x \in K$. It remains to show that Γ satisfies the property mentioned above. There exists $i_0 \in \{1, \ldots, N\}$ such that $x \in V(x_{i_0})$. Hence, for all $\omega \in \widehat{G}\backslash\Gamma(x_{i_0})$,

$$\begin{aligned} |A_{f,g}(x,\omega)| &= |A_{f,g}(x_{i_0},\omega)| + \|g\|_2\|\rho_G(\Phi(x^{-2}x_{i_0}^2), 1, 1)f - f\|_2 \\ &< \frac{\epsilon}{2} + \|g\|_2 \frac{\epsilon}{2\|g\|_2} \\ &= \epsilon. \end{aligned}$$

In particular, this is true for all $\omega \in \widehat{G}\backslash\Gamma$. Therefore, we obtain

$$|A_{f,g}(x,\omega)| < \epsilon \quad \text{for all } (x,\omega) \in (G \times \widehat{G})\backslash(K \times \Gamma).$$

Thus (i) holds. □

2.2.3. Square-integrability. In this subsection we establish necessary and sufficient conditions for a locally compact abelian group G to force any ambiguity function to be square-integrable. This problem can be simplified in an easy way. For $G = \mathbb{R}$, a similar approach is contained in [**AT85**, Section 2].

For this, let $f, g \in L^2(G)$ and define the function $h_{f,g} : G \times G \to \mathbb{C}$ by

$$h_{f,g}(x,t) = \overline{f(t)}g(t\,\Phi(x^2)).$$

By Hölder's inequality, $h_{f,g}(x,\cdot) \in L^1(G)$ for each $x \in G$. Hence the Fourier transform of the function $t \mapsto h_{f,g}(x,t)$ exists and, by Remark 2.2.4 (i), we have the following equation.

$$\omega(\Phi(x))(h_{f,g}(x,\cdot))^\wedge(\overline{\omega}) = A_{f,g}(x,\omega) \quad \text{for all } (x,\omega) \in G \times \widehat{G}.$$

Now suppose that $h_{f,g} \in L^2(G \times G)$. Then the Fourier transform and the Plancherel transform of $h_{f,g}$ coincide and we obtain

$$\|A_{f,g}\|_2^2 = \|(x,\omega) \mapsto (h_{f,g}(x,\cdot))^\wedge(\omega)\|_2^2 = \|h_{f,g}\|_2^2. \tag{1}$$

Conversely, suppose that $A_{f,g}$ is square-integrable. Then, for almost all $x \in G$, we have $(h_{f,g}(x,\cdot))^\wedge \in L^2(\widehat{G})$. In particular, this is the Plancherel transform. This also implies (1). So we have proven the following lemma.

LEMMA 2.2.11. *Let G be a locally compact abelian group and let $f, g \in L^2(G)$. Then the following conditions are equivalent.*

(i) $A_{f,g} \in L^2(G \times \widehat{G})$.
(ii) $h_{f,g} \in L^2(G \times G)$.

Thus, in order to prove the square-integrability of an ambiguity function $A_{f,g}$, it suffices to show that $h_{f,g} \in L^2(G \times \widehat{G})$. Notice that, for $G = \mathbb{R}$, it is not very difficult to conclude from Lemma 2.2.11 that the ambiguity function is square-integrable for any two functions $f, g \in L^2(\mathbb{R})$ ([**AT85**, Theorem 2.1]). The case of an arbitrary locally compact abelian group is not so easy to deal with, since in this situation $A_{f,g}$ is not always square-integrable. Here our task will be to characterize those locally compact abelian groups G which satisfies $A_{f,g} \in L^2(G \times \widehat{G})$ for all $f, g \in L^2(G)$.

For the proofs of the following theorems, we need some preparation. Let G be a locally compact abelian group. For the remainder of this subsection, we define $\psi : G/\ker\varphi \to G^{(2)}$ by

$$\psi([x]) = x^2.$$

The function ψ has the following property.

LEMMA 2.2.12. *Let G be a σ-compact locally compact abelian group such that $G^{(2)}$ is closed. Then ψ is a topological isomorphism.*

PROOF. Obviously, the mapping $x \mapsto x^2$, $G \to G^{(2)}$, is a surjective and continuous homomorphism. Hence, by [**HR63**, Theorem 5.29], it is also open, since $G^{(2)}$ is locally compact. Now using [**HR63**, Theorem 5.27], the claim follows. $\square$

Recall that, if $G^{(2)}$ is open, $G^{(2)}$ is automatically closed.

The next remark shows that there exist locally compact abelian groups such that ψ is not a topological isomorphism.

REMARK 2.2.13. Let $H := \prod_{i=1}^{\infty} \mathbb{Z}_2$ (compare Remark 2.2.3) be endowed with the product topology. Then consider $G := \prod_{i=1}^{\infty} \mathbb{Z}_4$ and regard H as a subgroup of G, where we identify the elements of $\mathbb{Z}_2$ with the elements of $\mathbb{Z}_4$ of order ≤ 2. We endow G with the topology such that H is an open and compact subgroup of G. Obviously, $\varphi(H) = \{e\}$ and $G^{(2)} = H$. Hence the map $x \mapsto x^2$, $G \to G^{(2)}$, is not open. Note that this map is open if and only if ψ is open. Thus ψ is not a topological isomorphism.

Now suppose that $G^{(2)}$ is closed. In this case we give a condition equivalent to the square-integrability of the ambiguity function. This condition is much easier to check.

THEOREM 2.2.14. *Let G be a locally compact abelian group. Suppose that $G^{(2)}$ is closed. Then $\ker\varphi$ is compact, if $A_{f,g} \in L^2(G \times \widehat{G})$ for all $f, g \in L^2(G)$.*

If, in addition, G is σ-compact, then the following conditions are equivalent.

(i) *$A_{f,g} \in L^2(G \times \widehat{G})$ for all $f, g \in L^2(G)$.*
(ii) *$\ker\varphi$ is compact and $G^{(2)}$ is open.*

PROOF. In the following we will denote $\ker\varphi$ by K. Since K and $G^{(2)}$ are closed subgroups of G, there exist Haar measures on K, G/K, $G^{(2)}$ and $G/G^{(2)}$ such that for K and $G^{(2)}$ Weil's formula holds, respectively.

First, suppose that (i) holds. We claim that K is compact. Let $f, g \in L^2(G)$. By Weil's formula, we obtain

$$\begin{aligned}\infty &> \int_{\widehat{G}}\int_G |A_{f,g}(x,\omega)|^2\,dx\,d\omega \\ &= \int_{\widehat{G}}\int_{G/K}\left(\int_K |A_{f,g}(xk,\omega)|^2\,dk\right)\,d(xK)\,d\omega.\end{aligned}$$

This implies

$$\int_K |A_{f,g}(xk,\omega)|^2\,dk < \infty \quad \text{for almost all } xK \in G/K, \omega \in \widehat{G}.$$

But for all $k \in K$, we have $|A_{f,g}(xk,\omega)| = |A_{f,g}(x,\omega)|$. If K is non-compact, this implies $A_{f,g} = 0$ for all $f, g \in L^2(G)$, a contradiction.

Now it remains to prove, that $G^{(2)}$ has to be open. Assume, towards a contradiction, that $G^{(2)}$ is not open. Using Lemma 2.1.1 (i), for $f, g \in L^2(G)$, we obtain

$$\begin{aligned}\|h_{f,g}\|_2^2 &= \int_G |f(t)|^2 \int_G |g(t\Phi(x^2))|^2\,dx\,dt \\ &\geq \frac{1}{c}\int_G |f(\Phi(t))|^2 \int_G |g(\Phi(tx^2))|^2\,dx\,dt.\end{aligned}$$

Note that, if the function $t \mapsto |f(t)|^2 \int_G |g(t\Phi(x^2))|^2\,dx$ is not integrable, we obtain a contradiction at once by Lemma 2.2.11. In the other case the requirements of Lemma 2.1.1 (i) are fulfilled. Now it remains to prove the existence of functions $\tilde{f}, \tilde{g} \in L^2(G)$ such that

$$\int_G |\tilde{f}(t)|^2 \int_G |\tilde{g}(tx^2)|^2\,dx\,dt = \infty.$$

Then we can choose functions $f, g : G \to \mathbb{C}$ in the following way:

$$f(t) := \begin{cases} 0 & : \ t \in G\backslash H, \\ \tilde{f}(\Phi^{-1}(t)) & : \ t \in H, \end{cases}$$

and

$$g(t) := \begin{cases} 0 & : \ t \in G\backslash H, \\ \tilde{g}(\Phi^{-1}(t)) & : \ t \in H. \end{cases}$$

By Lemma 2.1.1 (ii), we obtain

$$\int_G |f(t)|^2\,dt = \int_G |\chi_H \tilde{f}(\Phi^{-1}(t))|^2\,dt = \int_H |\tilde{f}(\Phi^{-1}(t))|^2\,dt = \frac{1}{d}\int_G |\tilde{f}(t)|^2\,dt,$$

where the Haar measure on H is induced by the Haar measure on G. Hence $f, g \in L^2(G)$. Thus we would obtain $\|h_{f,g}\|_2^2 = \infty$. By Lemma 2.2.11, this is a contradiction.

By Lemma 2.2.12, ψ is a topological isomorphism. So, in particular, there exists a positive constant μ such that the Haar measure on G/K is equal to μ times the Haar measure on $G^{(2)}$. Without loss of generality we can assume that the Haar measures on G, K, G/K, $G^{(2)}$ and $G/G^{(2)}$ are normalized so that for K and $G^{(2)}$ Weil's formula holds, respectively, when we take on K the normalized Haar measure. By assumption, $G^{(2)}$ is not open in G. Hence $G/G^{(2)}$ is not discrete.

We claim that there exists a function $h \in L^1(G/G^{(2)}) \backslash L^2(G/G^{(2)})$. By the structure theorem for locally compact abelian groups [**HR63**, Theorem 24.30], there exist a compact abelian group J and some $p \geq 0$ such that $M := \mathbb{R}^p \times J$ is an open subgroup of $G/G^{(2)}$. Let the Haar measure on M be induced by the Haar measure on $G/G^{(2)}$. Then there exists a positive constant λ such that the Haar measure on M is equal to λ times the Haar measure on $\mathbb{R}^p \times J$, where we take on $\mathbb{R}^p$ the Lebesgue measure and on J the normalized Haar measure.

Clearly, if $p > 0$, there exist compact, disjoint subsets C_n, $n \in \mathbb{N}$, of $\mathbb{R}^p$ such that $|C_n| = \frac{1}{n^3}$ for any $n \in \mathbb{N}$. Now define $h_M : M \to \mathbb{R}$ by

$$h_M(x_1, x_2) := \begin{cases} n & : \quad x_1 \in C_n, \\ 0 & : \quad x_1 \in \mathbb{R}^p \backslash C_n \ \text{ for all } n \in \mathbb{N}, \end{cases}$$

and define $h : G/G^{(2)} \to \mathbb{R}$ by

$$h(x) := \begin{cases} h_M(x) & : \quad x \in M, \\ 0 & : \quad x \in (G/G^{(2)}) \backslash M. \end{cases}$$

Then

$$\int_{G/G^{(2)}} |h(x)|\, dx = \int_M |h_M(x)|\, dx = \lambda \sum_{n=1}^{\infty} n |C_n| = \lambda \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

and

$$\int_{G/G^{(2)}} |h(x)|^2\, dx = \int_M |h_M(x)|^2\, dx = \lambda \sum_{n=1}^{\infty} n^2 |C_n| = \lambda \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Now consider the case $p = 0$. First, suppose that $J = \mathbb{T}^r$, $r > 0$. Then there obviously exist compact, disjoint subsets C_n, $n \in \mathbb{N}$, of $\mathbb{T}^r$ such that $|C_n| = \frac{1}{\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)} \cdot \frac{1}{n^3}$ for all $n \in \mathbb{N}$. Next define $h_J : J \to \mathbb{R}$ by

$$h_J(x) := \begin{cases} n & : \quad x \in C_n, \\ 0 & : \quad x \in J \backslash C_n \ \text{ for all } n \in \mathbb{N}, \end{cases}$$

and $h : G/G^{(2)} \to \mathbb{R}$ by

$$h(x) := \begin{cases} h_J(x) & : \quad x \in J, \\ 0 & : \quad x \in (G/G^{(2)}) \backslash J. \end{cases}$$

Then

$$\int_{G/G^{(2)}} |h(x)|\, dx = \int_J |h_J(x)|\, dx = \lambda \sum_{n=1}^{\infty} n |C_n| = \lambda \frac{1}{\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

and

$$\int_{G/G^{(2)}} |h(x)|^2 \, dx = \int_J |h_J(x)|^2 \, dx = \lambda \sum_{n=1}^{\infty} n^2 |C_n| = \lambda \frac{1}{\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Secondly, let J be an arbitrary non-finite, compact abelian group. Since locally compact abelian groups are projective limits of Lie groups, there exists a system $\mathcal{A}$ of compact subgroups A of J, $\mathcal{A}$ downwards directed and $\bigcap_{A \in \mathcal{A}} A = \{e\}$, such that J/A is a Lie group for every $A \in \mathcal{A}$. Hence $J/A = \mathbb{T}^{r_A} \times F_A$, where $r_A > 0$ and F_A is a finite group. Now we fix some $A_0 \in \mathcal{A}$. By the preceding paragraph, there exists a function $h_{A_0} : J/A_0 \to \mathbb{R}$ such that $h_{A_0} \in L^1(J/A_0) \backslash L^2(J/A_0)$. Let $h_J : J \to \mathbb{R}$ be defined by

$$h_J(x) := h_{A_0}(xA_0)$$

and let $h : G/G^{(2)} \to \mathbb{R}$ be defined by

$$h(x) := \begin{cases} h_J(x) & : \quad x \in J, \\ 0 & : \quad x \in (G/G^{(2)}) \backslash J. \end{cases}$$

Without loss of generality, we can assume that the Haar measures on A_0 and J/A_0 are normalized so that Weil's formula holds, if we consider on A_0 the normalized Haar measure. Then we obtain by Weil's formula

$$\begin{aligned} \int_{G/G^{(2)}} |h(x)| \, dx &= \int_J |h_J(x)| \, dx \\ &= \int_J |h_{A_0}(xA_0)| \, dx \\ &= \int_{J/A_0} \int_{A_0} |h_{A_0}((xa)A_0)| \, da \, d(xA_0) \\ &= \int_{J/A_0} |h_{A_0}(xA_0)| \, d(xA_0) \\ &< \infty. \end{aligned}$$

A similar calculation shows

$$\int_{G/G^{(2)}} |h(x)|^2 \, dx = \infty.$$

Summarizing, we constructed a function $h : G/G^{(2)} \to \mathbb{R}$ such that $h \in L^1(G/G^{(2)}) \backslash L^2(G/G^{(2)})$.

By [**HR70**, Theorem 28.54 (iii)], there exists a function $\tilde{h} \in L^1(G)$ such that

$$h(tG^{(2)}) = \int_{G^{(2)}} \tilde{h}(tx) \, dx.$$

Now define $\tilde{f} : G \to \mathbb{R}$ by

$$\tilde{f}(y) := \sqrt{|\tilde{h}(y)|}.$$

Further, let $\tilde{g} := \tilde{f}$. Then, by using Weil's formula,

$$\begin{aligned}
&\int_G |\tilde{f}(t)|^2 \int_G |\tilde{g}(tx^2)|^2 \, dx \, dt \\
&= \int_G |\tilde{f}(t)|^2 \int_{G/K} \int_K |\tilde{f}(t(yh)^2)|^2 \, dh \, d(yK) \, dt \\
&= |K| \int_G |\tilde{f}(t)|^2 \int_{G/K} |\tilde{f}(ty^2)|^2 \, d(yK) \, dt \\
&= \mu \int_G |\tilde{f}(t)|^2 \int_{G^{(2)}} |\tilde{f}(ty)|^2 \, dy \, dt.
\end{aligned}$$

Note that, if the map $t \mapsto |\tilde{f}(t)|^2 \int_{G^{(2)}} |\tilde{f}(ty)|^2 \, dy$ does not belong to $L^1(G)$, we are done. Otherwise, using Weil's formula again, we obtain

$$\begin{aligned}
&\int_G |\tilde{f}(t)|^2 \int_G |\tilde{g}(tx^2)|^2 \, dx \, dt \\
&= \mu \int_G |\tilde{f}(t)|^2 \int_{G^{(2)}} |\tilde{f}(ty)|^2 \, dy \, dt \\
&= \mu \int_{G/G^{(2)}} \int_{G^{(2)}} |\tilde{f}(th)|^2 \int_{G^{(2)}} |\tilde{f}(thy)|^2 \, dy \, dh \, d(tG^{(2)}) \\
&= \mu \int_{G/G^{(2)}} \left[\int_{G^{(2)}} |\tilde{f}(th)|^2 \, dh \right]^2 d(tG^{(2)}) \\
&= \mu \int_{G/G^{(2)}} \left[\int_{G^{(2)}} |\tilde{h}(th)| \, dh \right]^2 d(tG^{(2)}) \\
&\geq \mu \int_{G/G^{(2)}} |h(tG^{(2)})|^2 \, d(tG^{(2)}) \\
&= \infty.
\end{aligned}$$

As mentioned above, this is a contradiction. Thus we proved (i) $\Rightarrow$ (ii).

Now suppose that (ii) holds. Let $f, g \in L^2(G)$. Using Weil's formula, we obtain

$$\begin{aligned}
&\|h_{f,g}\|_2^2 \\
&= \int_G |f(t)|^2 \int_G |g(t\,\Phi(x^2))|^2 \, dx \, dt \\
&= \int_G |f(t)|^2 \int_{G/K} \int_K |g(t\,\Phi((yh)^2))|^2 \, dh \, d(yK) \, dt \\
&= |K| \int_G |f(t)|^2 \int_{G/K} |g(t\,\Phi(y^2))|^2 \, d(yK) \, dt.
\end{aligned}$$

Now ψ is a topological isomorphism by Lemma 2.2.12. Hence, in particular, there exists a positive constant μ such that the Haar measure on G/K is equal to μ times the Haar measure on $G^{(2)}$. Since $G^{(2)}$ is open, there exists a positive constant ν such that the Haar measure on $G^{(2)}$ is ν times the measure on $G^{(2)}$

induced by the Haar measure on G. Thus, also using Lemma 2.1.1 (i), we obtain

$$\begin{aligned}
&|K| \int_G |f(t)|^2 \int_{G/K} |g(t\,\Phi(y^2))|^2 \, d(yK)\, dt \\
&= \mu|K| \int_G |f(t)|^2 \int_{G^{(2)}} |g(t\,\Phi(x))|^2 \, dx\, dt \\
&= \mu\nu|K| \int_G |f(t)|^2 \int_G 1_{G^{(2)}}(x) |g(t\,\Phi(x))|^2 \, dx\, dt \\
&\leq \mu\nu|K| \int_G |f(t)|^2 \int_G |g(t\,\Phi(x))|^2 \, dx\, dt \\
&\leq \mu\nu c|K| \int_G |f(t)|^2 \int_G |g(tx)|^2 \, dx\, dt \\
&= \mu\nu c|K| \|f\|_2^2 \|g\|_2^2.
\end{aligned}$$

Since, by hypothesis, K is compact, we have shown $h_{f,g} \in L^2(G \times \widehat{G})$. Now Lemma 2.2.11 yields (i). □

The condition that $G^{(2)}$ has to be closed is not very restrictive as it is shown in the next lemma.

PROPOSITION 2.2.15. *Let G be a 2-root compact, locally compact abelian group. Then $G^{(2)}$ is closed.*

PROOF. Towards a contradiction, assume that $G^{(2)}$ is not closed. Let $y \in G \backslash G^{(2)}$. Further, let $(x_\iota)_{\iota \in I}$ be a net in G such that $x_\iota^2 \to y$. Let K be a compact neighbourhood of y. Then there exists $\iota_0 \in I$ such that $x_\iota^2 \in K$ for all $\iota \geq \iota_0$. Notice that the set $\{x_\iota^2 : \iota \geq \iota_0\} \cup \{y\}$ is compact. Since G is 2-root compact, $\varphi^{-1}(\{x_\iota^2 : \iota \geq \iota_0\} \cup \{y\})$ is also compact. Hence there exists a convergent subnet $(x_{\iota_\lambda})_{\lambda \in I}$ of $(x_\iota)_{\iota \in I}$. Let $x \in G$ be defined by $x_{\iota_\lambda} \to x$. By continuity of φ, we obtain $y = x^2 \in G^{(2)}$, a contradiction. □

The preceding proposition and Remark 2.2.9 show that Theorem 2.2.14 may, for instance, be applied to all compactly generated locally compact abelian groups.

Under slightly stronger conditions than to suppose that $G^{(2)}$ is closed, we can find necessary and sufficient conditions for the ambiguity functions to be square-integrable, which are easier to check. However, by Theorem 2.2.14, the square-integrability of all ambiguity functions implies that $G^{(2)}$ is open. Thus the requirement that $G^{(2)}$ has to be open is not really restrictive.

THEOREM 2.2.16. *Let G be a locally compact abelian group. Suppose that $G^{(2)}$ is open. Consider the following statements.*

(i) $A_{f,g} \in L^2(G \times \widehat{G})$ *for all* $f, g \in L^2(G)$.
(ii) $\ker \varphi$ *is compact.*
(iii) G *is 2-root compact.*

Then one has the implications (i) $\Rightarrow$ (ii) *and* (iii) $\Rightarrow$ (ii).

If, in addition, G *is* σ*-compact, then* (i), (ii) *and* (iii) *are equivalent.*

PROOF. The connections between the statements (i) and (ii) are a direct conclusion from Theorem 2.2.14.

Now let G be a locally compact abelian group. Suppose that (iii) holds. The set $\{e\} \subseteq G$ is compact. Since G is supposed to be 2-root compact, also $\varphi^{-1}(\{e\}) = \ker \varphi$ is compact. Thus (ii) holds.

To finish the proof we show that (ii) implies (iii), if, in addition, G is σ-compact. Suppose that $G^{(2)}$ is open and $\ker \varphi$ is compact. Since ψ is a topological isomorphism, $\psi^{-1}(K) \subseteq G/\ker \varphi$ is compact for all compact subsets $K \subseteq G^{(2)}$. Now $\ker \varphi$ is compact. Thus (iii) follows immediately. $\square$

Thus, under a weak condition, the locally compact abelian groups, for which the ambiguity function is always square-integrable, are exactly those, for which the ambiguity function always vanishes at infinity.

COROLLARY 2.2.17. *Let* G *be a* σ*-compact locally compact abelian group. Suppose that* $G^{(2)}$ *is open. Then the following conditions are equivalent.*

(i) $A_{f,g} \in L^2(G \times \widehat{G})$ *for all* $f, g \in L^2(G)$.
(ii) $A_{f,g} \in C_0(G \times \widehat{G})$ *for all* $f, g \in L^2(G)$.

PROOF. This is an immediate consequence of Theorem 2.2.16 and Theorem 2.2.10. $\square$

It is natural to search for groups which satisfy the properties needed for the ambiguity function to be square-integrable. In fact, there exists a large class of such groups as the next corollary shows. However, first we want to exhibit a property for a locally compact abelian group G which forces $G^{(2)}$ to be open. This property is much easier to verify.

PROPOSITION 2.2.18. *Let* G *be a locally compact abelian group. Suppose that* G_0 *is open. Then* $G^{(2)}$ *is open.*

PROOF. Suppose that G_0 is open. By the structure theorem for locally compact abelian groups [**HR63**, Theorem 24.30], there exists a compact abelian group C and $p \geq 0$ such that $\mathbb{R}^p \times C$ is an open subgroup of G. We have $G_0 \subseteq \mathbb{R}^p \times C_0$. Since G_0 is open, also $\mathbb{R}^p \times C_0$ is open. Recall that compact, connected abelian groups are divisible ([**HR63**, Theorem 24.25]). Thus we have $C_0 \subseteq C^{(2)}$. This yields

$$\mathbb{R}^p \times C_0 \subseteq \mathbb{R}^p \times C^{(2)} \subseteq G^{(2)}.$$

Hence $G^{(2)}$ is open. $\square$

REMARK 2.2.19. It follows from Proposition 2.2.18 that, in particular, $G^{(2)}$ is open for all locally compact abelian Lie groups G.

COROLLARY 2.2.20. *Let* G *be a 2-root compact,* σ*-compact locally compact abelian group. Suppose that* G_0 *is open. Then, for all* $f, g \in L^2(G)$,

$$A_{f,g} \in L^2(G \times \widehat{G}).$$

PROOF. By Proposition 2.2.18, $G^{(2)}$ is open. Then the claim follows from Theorem 2.2.16. □

In particular, for all compactly generated locally compact abelian Lie groups und for all compactly generated locally compact abelian groups, for which G^c is connected, the ambiguity function is always square-integrable.

Now we want to compare Theorem 2.2.14 and Theorem 2.2.16. Let G be a locally compact abelian group. It might be an interesting question whether the closedness of $G^{(2)}$ suffices to guarantee the equivalence of (i) and (iii) in Theorem 2.2.16. The following remark shows that this is not the case in general.

REMARK 2.2.21. Let G be a 2-root compact, σ-compact locally compact abelian group. Suppose that $G^{(2)}$ is closed. It does not follow that $A_{f,g} \in L^2(G \times \widehat{G})$ for all $f, g \in L^2(G)$.

In fact, let $G := \prod_{i=1}^{\infty} \mathbb{Z}_2$ (compare Remark 2.2.3). By Tychonoff's theorem, G is compact. In particular, G is 2-root compact. It is easy to check that $G^{(2)} = \{0\}$, hence $G^{(2)}$ is closed. Since G is not finite, hence not discrete, $G^{(2)}$ is not open in G. By Theorem 2.2.14, $A_{f,g} \notin L^2(G \times \widehat{G})$ for suitable functions $f, g \in L^2(G)$.

The next proposition gives an estimate for the norm of the ambiguity function, which will be needed in Section 2.4. The idea of the proof of (i) is again using the function $h_{f,g}$. This occurs already in [**AT85**, Section 2] for $G = \mathbb{R}$.

PROPOSITION 2.2.22. *Let G be a σ-compact locally compact abelian group. Suppose that $G^{(2)}$ is closed and that $A_{f,g} \in L^2(G \times \widehat{G})$ for all $f, g \in L^2(G)$.*

(i) *There exists a positive constant C such that, for all $f, g \in L^2(G)$,*

$$\|A_{f,g}\|_2^2 \leq C\|f\|_2^2\|g\|_2^2.$$

(ii) *The mapping*

$$(f, g) \mapsto A_{f,g}, \quad L^2(G) \times L^2(G) \to L^2(G \times \widehat{G}),$$

is continuous.

PROOF. Let $f, g \in L^2(G)$. Using the proof of Lemma 2.2.11, we obtain $\|A_{f,g}\|_2^2 = \|h_{f,g}\|_2^2$. Furthermore, there exist suitable positive constants μ and ν such that

$$\|h_{f,g}\|_2^2 \leq \mu\nu c|\ker\varphi|\|f\|_2^2\|g\|_2^2$$

as it was shown in the proof of Theorem 2.2.14 ((ii) $\Rightarrow$ (i)). This can be applied, since, by Theorem 2.2.14 ((i) $\Rightarrow$ (ii)), $\ker\varphi$ is compact and $G^{(2)}$ is open. In addition, we obtain $0 < |\ker\varphi| < \infty$. This proves (i).

The claim in (ii) is a direct conclusion from (i). □

2.2.4. Injectivity. The injectivity of the classical ambiguity function on $L^2(\mathbb{R})$ has been exhibited in [**AT85**]. There Auslander and Tolimieri proved that the classical ambiguity function is essentially injective in the following sense. For all $f, g \in L^2(G)$,

$$A_f = A_g \text{ if and only if there exists some } \lambda \in \mathbb{T} \text{ such that } f = \lambda g.$$

Now it seems natural to ask to which extent this result can be generalized to locally compact abelian groups.

Let G be a locally compact abelian group. Suppose that $H^{(2)}$ is closed. Then we define $H(G)_H$ by

$$H(G)_H := H^{(2)} \times \widehat{G} \times \mathbb{T} < G \times \widehat{G} \times \mathbb{T}$$

endowed with the group multiplication of $H(G)$, the Heisenberg group associated with G. In addition, we will consider the following unitary representation:

$$\pi_G : H(G)_H \to \mathcal{U}(L^2(G)), \quad \pi_G := \rho_G|_{H(G)_H}.$$

In the next theorem we establish a necessary and sufficient condition for the injectivity of the ambiguity function. The idea of using Schur's Lemma and the definition of the sets A and B and of the function T in the proof of (i) $\Rightarrow$ (ii) already appears in [**AT85**, Theorem 3.6] for $G = \mathbb{R}$.

THEOREM 2.2.23. *Let G be a locally compact abelian group. Suppose that $H^{(2)}$ is closed. Let $\mathcal{H}$ be a closed, π_G-invariant subspace of $L^2(G)$. Then the following conditions are equivalent.*

(i) *$\pi_G|_{\mathcal{H}}$ is irreducible.*
(ii) *For all $f, g \in \mathcal{H}$, the following conditions are equivalent.*
  (a) *$A_f = A_g$.*
  (b) *There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.*

PROOF. First, suppose that (i) holds. Let $f, g \in L^2(G)$. By Remark 2.2.4 (iii), $A_f = A_g$ holds if and only if, for all $(x, \omega) \in G \times \widehat{G}$,

$$\langle \rho_G(\Phi(x^2), \omega, 1) f, f\rangle = \langle \rho_G(\Phi(x^2), \omega, 1) g, g\rangle. \tag{2}$$

Clearly, (2) is fulfilled, if there exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$. Now suppose that (2) holds. Using the definition of π_G, we obtain

$$\langle \pi_G(x, \omega, z) f, f\rangle = \langle \pi_G(x, \omega, z) g, g\rangle \quad \text{for all } (x, \omega, z) \in H(G)_H. \tag{3}$$

Let $A, B \subseteq L^2(G)$ be defined by

$$A := \{\pi_G(x, \omega, z) f : (x, \omega, z) \in H(G)_H\} \text{ and}$$

$$B := \{\pi_G(x, \omega, z) g : (x, \omega, z) \in H(G)_H\},$$

and let the mapping $T : A \to B$ be defined by

$$T(\pi_G(x, \omega, z) f) = \pi_G(x, \omega, z) g.$$

First, we claim that T is well-defined. For this, let (x_1, ω_1, z_1), $(x_2, \omega_2, z_2) \in H(G)_H$ be such that $\pi_G(x_1, \omega_1, z_1)f = \pi_G(x_2, \omega_2, z_2)f$. By (3), it follows that, for all $(x, \omega, z) \in H(G)_H$ and $i = 1, 2$,

$$\langle \pi_G(x_i, \omega_i, z_i)f, \pi_G(x, \omega, z)f \rangle = \langle \pi_G(x_i, \omega_i, z_i)g, \pi_G(x, \omega, z)g \rangle. \tag{4}$$

This implies that

$$\begin{aligned} \langle \pi_G(x_1, \omega_1, z_1)g, \pi_G(x, \omega, z)g \rangle &= \langle \pi_G(x_1, \omega_1, z_1)f, \pi_G(x, \omega, z)f \rangle \\ &= \langle \pi_G(x_2, \omega_2, z_2)f, \pi_G(x, \omega, z)f \rangle \\ &= \langle \pi_G(x_2, \omega_2, z_2)g, \pi_G(x, \omega, z)g \rangle. \end{aligned}$$

By hypothesis, $\pi_G|_{\mathcal{H}}$ is irreducible, so B spans a dense subset. Thus we have

$$\pi_G(x_1, \omega_1, z_1)g = \pi_G(x_2, \omega_2, z_2)g.$$

This proves our claim.

Now (4) implies that T extends to a unitary operator on $\mathcal{H}$. It is easy to check that $T \circ \pi_G \circ T^{-1} = \pi_G$. By Schur's Lemma, there exists some $\lambda \in \mathbb{T}$ such that $T = \lambda \, \mathrm{Id}_{\mathcal{H}}$. Hence

$$\lambda f = T(f) = T(\pi_G(e, 1, 1)f) = \pi_G(e, 1, 1)g = g.$$

Conversely, suppose that $\mathcal{H}$ is a closed, π_G-invariant subspace of $L^2(G)$ which satisfies (ii). Assume, towards a contradiction, that $\pi_G|_{\mathcal{H}}$ is not irreducible. Hence there exists a closed, π_G-invariant, proper subspace $E \neq \{0\}$ of $\mathcal{H}$. Note that the complement of E in $\mathcal{H}$, $E^\perp$, is also a closed, π_G-invariant, proper subspace $\neq \{0\}$ of $\mathcal{H}$. Now we choose $f, g \in L^2(G)$ such that $f \in E$, $f \neq 0$ and $g \in E^\perp$, $g \neq 0$. Then we define $h_1, h_2 \in \mathcal{H}$ by

$$h_1 := f + g \quad \text{and} \quad h_2 := f - g.$$

Towards a contradiction, assume that there exists some $\lambda \in \mathbb{T}$ such that $h_1 = \lambda h_2$. Then $(\lambda+1)g = (\lambda-1)f$. This implies that either $\lambda = -1$ and hence $\lambda = 1$, since $f \neq 0$, or $g = \frac{\lambda-1}{\lambda+1}f$ and hence $g \in E$. This is a contradiction.

Thus, by the construction of h_1 and h_2, there exists no $\lambda \in \mathbb{T}$ such that $h_1 = \lambda h_2$. On the other hand, $A_{h_1} = A_{h_2}$ holds if and only if, for all $(x, \omega, z) \in H(G)_H$,

$$\langle \pi_G(x, \omega, z)(f + g), f + g \rangle = \langle \pi_G(x, \omega, z)(f - g), f - g \rangle.$$

Since $f \in E$ and $g \in E^\perp$ and since E and $E^\perp$ are π_G-invariant subspaces, both sides of this equation coincide. Thus $A_{h_1} = A_{h_2}$. This is a contradiction to (ii). ☐

The next lemma treats an important special case of Theorem 2.2.23. For this, suppose that $H^{(2)}$ is closed. Then define $\mathcal{H}_y \subseteq L^2(G)$ by

$$\mathcal{H}_y := \{f \in L^2(G) : \mathrm{supp} f \subseteq yH^{(2)}\} \quad (y \in G).$$

Note that $\mathcal{H}_y$ is a closed subspace of $L^2(G)$. If $H^{(2)}$ is open, we obtain $\mathcal{H}_y \cong L^2(H^{(2)})$. On the other hand, if $H^{(2)}$ is not open, but closed, $\mathcal{H}_y = \{0\}$. Hence this case is of no interest. We intend to apply Theorem 2.2.23 to $\mathcal{H}_y$. For this, we need the following lemma. In the following we will restrict to open subgroups $H^{(2)}$, since otherwise we have $\mathcal{H}_y = \{0\}$.

LEMMA 2.2.24. *Let G be a locally compact abelian group. Suppose that $H^{(2)}$ is open and let $y \in G$. Then $\mathcal{H}_y$ is a closed, π_G-invariant subspace of G and $\pi_G|_{\mathcal{H}_y}$ is irreducible.*

PROOF. Let $y \in G$. For $f \in \mathcal{H}_y$ and $(x, \omega, z) \in H(G)_H$, we have

$$\begin{aligned} \operatorname{supp}(\pi_G(x,\omega,z)f) &= \{t \in G : z\omega(t)f(tx) \neq 0 \quad \text{almost everywhere}\} \\ &\subseteq \{t \in G : tx \in yH^{(2)}\} \\ &= yH^{(2)}. \end{aligned}$$

This implies that $\mathcal{H}_y$ is π_G-invariant. Since $\mathcal{H}_y$ is closed, it remains to show that $\pi_G|_{\mathcal{H}_y}$ is irreducible. For this, notice that ρ_G is irreducible and we have

$$\rho_G = \operatorname{ind}_{\widehat{G}\times\mathbb{T}}^{H(G)} \sigma = \operatorname{ind}_{H(G)_H}^{H(G)} \left(\operatorname{ind}_{\widehat{G}\times\mathbb{T}}^{H(G)_H} \sigma \right),$$

where $\sigma : \widehat{G} \times \mathbb{T} \to \mathcal{U}(\mathbb{T})$ is defined by

$$\sigma(\omega, z) = z.$$

This implies that $\pi_G|_{L^2(H^{(2)})} = \operatorname{ind}_{\widehat{G}\times\mathbb{T}}^{H(G)_H} \sigma$ is irreducible on $L^2(H^{(2)})$. Thus also $\pi_G|_{\mathcal{H}_y}$ is irreducible. □

COROLLARY 2.2.25. *Let G be a locally compact abelian group. Suppose that $H^{(2)}$ is open and let $y \in G$. For $f, g \in \mathcal{H}_y$, the following conditions are equivalent.*

(i) $A_f = A_g$.
(ii) *There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.*

PROOF. The corollary follows immediately from Lemma 2.2.24 and Theorem 2.2.23. □

Note that, if $H^{(2)} = G$, as, for example, in the cases $G = \mathbb{R}^p, \mathbb{T}^r$, $p, r \geq 0$, the injectivity condition holds for all functions $f, g \in L^2(G)$.

Next we exhibit properties of the group G that force $H^{(2)}$ to be closed or even open.

PROPOSITION 2.2.26. *Let G be a σ-compact locally compact abelian group.*

(i) *Suppose that G is 2-root compact. Then $H^{(2)}$ is closed.*
(ii) *Suppose that G is a Lie group. Then $H^{(2)}$ is open.*

PROOF. First, we prove (i). Since G is 2-root compact, $G^{(2)}$ is closed (Proposition 2.2.15). Hence $G^{(2)}$ is locally compact. Now the mapping $x \mapsto x^2, G \to G^{(2)}$, is open ([**HR63**, Theorem 5.29]). Thus $\varphi(H) = H^{(2)} < G^{(2)}$ is open and the claim follows.

Now suppose that G is a Lie group. Then H is always open (Proposition 2.2.2) and, by Remark 2.2.19, $G^{(2)}$ is open in G. With the same argument as before we obtain that $H^{(2)}$ is open in $G^{(2)}$. Hence $H^{(2)}$ is open in G. □

Note that for the proof of (ii) we only need that H is closed.

Let $\mathcal{H}$ be a closed subspace of $L^2(G)$ which contains $\mathcal{H}_y$ for some element $y \in G$. It seems natural to ask under which conditions the ambiguity function on $\mathcal{H}$ satisfies this special injectivity condition. It will turn out that the subspaces $\mathcal{H}_y$, $y \in G$, are maximal in a certain sense.

PROPOSITION 2.2.27. *Let G be a locally compact abelian group. Suppose that $H^{(2)}$ is open. Furthermore, suppose that $\mathcal{H}$ is a closed subspace of $L^2(G)$ with the properties that there exists some $y_0 \in G$ such that $\mathcal{H}_{y_0} \subseteq \mathcal{H}$ and that, for all $f, g \in \mathcal{H}$, the following conditions are equivalent.*

(i) $A_f = A_g$.
(ii) *There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.*

Then $\mathcal{H} = \mathcal{H}_{y_0}$.

PROOF. Towards a contradiction, assume that $\mathcal{H}_{y_0} \neq \mathcal{H}$. Then there exist a function $h \in \mathcal{H}$ and a subset $W \subseteq G \backslash y_0 H^{(2)}$ of positive measure such that $h(x) \neq 0$ for almost all $x \in W$. Note that $h\chi_{y_0 H^{(2)}} \in L^2(G)$. Hence we have $h\chi_{y_0 H^{(2)}} \in \mathcal{H}_{y_0} \subseteq \mathcal{H}$. This yields

$$g := h - h\chi_{y_0 H^{(2)}} \in \mathcal{H}.$$

Now let K be a relatively compact subset of $y_0 H^{(2)}$ such that $K^\circ \neq \emptyset$. Then $f := \chi_K \in \mathcal{H}_{y_0} \subseteq \mathcal{H}$. Next we define functions h_1 and h_2 by

$$h_1 := f + g \quad \text{and} \quad h_2 := f - g.$$

Note that $h_1, h_2 \in \mathcal{H}$. Now we are in an almost similar situation as in the second part of the proof of Theorem 2.2.23 (with $E := \mathcal{H}_{y_0}$). The calculation there shows that there exists no $\lambda \in \mathbb{T}$ such that $h_1 = \lambda h_2$. Therefore, it suffices to prove $A_{h_1} = A_{h_2}$ to obtain a contradiction.

By the second part of the proof of Theorem 2.2.23, it suffices to show that, for all $(x, \omega, z) \in H(G)_H$,

$$\langle \pi_G(x, \omega, z) f, g\rangle = 0 \text{ and } \langle \pi_G(x, \omega, z) g, f\rangle = 0.$$

The first equation follows immediately from the fact that $\mathcal{H}_{y_0}$ is π_G-invariant. To prove the second equation notice that

$$\operatorname{supp}(\pi_G(x, \omega, z) g) = x^{-1} \operatorname{supp} g.$$

Since $\operatorname{supp} g \subseteq G \backslash y_0 H^{(2)}$, we obtain

$$\operatorname{supp}(\pi_G(x, \omega, z) g) \cap \operatorname{supp} f = \emptyset.$$

Summarizing, we chose h_1 and h_2 in such a way that $A_{h_1} = A_{h_2}$, but that there exists no $\lambda \in \mathbb{T}$ such that $h_1 = \lambda h_2$, a contradiction. ☐

Now it remains to study non-π_G-invariant subspaces $\mathcal{H}$. Here we may also try to establish results similar to Theorem 2.2.23. Unfortunately, we can give examples, where $\mathcal{H}$ is a closed, non-π_G-invariant subspace of $L^2(G)$ and one time the injectivity condition is fulfilled and the other time not.

EXAMPLE 2.2.28. Let G be a locally compact abelian group such that $H^{(2)} \neq \{e\}$ and let $h \in L^2(G)$, $h \neq$ const. Then, for $f, g \in \mathcal{H} := \operatorname{span}\{h\}$, the following conditions are equivalent.

(i) $A_f = A_g$.
(ii) There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.

To check this claim, let $\eta_1, \eta_2 \in \mathbb{C}$. By the injectivity of the Fourier transform and Remark 2.2.4 (i), $A_{\eta_1 h} = A_{\eta_2 h}$ holds if and only if, for each $x \in G$, there exists a zero set $N_x \subseteq G$ such that

$$\overline{\eta_1}\overline{h(y)}\eta_1 h(y\Phi(x^2)) = \overline{\eta_2}\overline{h(y)}\eta_2 h(y\Phi(x^2)) \quad \text{for all } y \in G \backslash N_x.$$

This holds if and only if there exists some $\lambda \in \mathbb{T}$ such that $\eta_1 = \lambda \eta_2$. This proves the claim.

Note that there exist $h \in L^2(G)$, $h \neq$ const, $x_0 \in H^{(2)}$ and $\omega_0 \in \widehat{G}$ such that $(L_{x_0} h) \cdot \omega_0 \notin \mathcal{H}$. Then $\mathcal{H}$ is not π_G-invariant.

EXAMPLE 2.2.29. Let G be a locally compact abelian group such that $G^{(2)}$ is open and $G^{(2)} \neq G$, and let $\Phi : G \to G$ be defined by $\Phi := \mathrm{Id}_G$. Furthermore, let $K_1, K_2 \subseteq G$, $K_1^\circ, K_2^\circ \neq \emptyset$, be compact sets such that $K_1 \subseteq G^{(2)}$ and $K_2 \subseteq G \backslash G^{(2)}$. Now define $\mathcal{H} \subseteq L^2(G)$ by $\mathcal{H} := \operatorname{span}\{\chi_{K_1}, \chi_{K_2}\}$. Then defining f and g by $f := \chi_{K_1} + \chi_{K_2} \in \mathcal{H}$ and $g := \chi_{K_1} - \chi_{K_2} \in \mathcal{H}$, we have $A_f = A_g$, but there exists no $\lambda \in \mathbb{T}$ such that $f = \lambda g$.

For instance, a special choice would be $G = \mathbb{Z}$. Then $G^{(2)} = 2\mathbb{Z}$. The compact sets K_1 and K_2 may be chosen by $K_1 := \{0\}$ and $K_2 := \{1\}$.

It remains to prove the claim. For all $(x, \omega) \in G \times \widehat{G}$,

$$\begin{aligned} A_f(x,\omega) &= \omega(x) \int_G \overline{f(t)} f(tx^2)\omega(t)\,dt \\ &= \omega(x) \int_G (\chi_{K_1} + \chi_{K_2})(t)(\chi_{K_1} + \chi_{K_2})(tx^2)\omega(t)\,dt \\ &= \omega(x) \int_G \chi_{K_x}(t)\omega(t)\,dt, \end{aligned}$$

where $K_x := (K_1 \cup K_2) \cap (x^{-2}K_1 \cup x^{-2}K_2)$. On the other hand, for all $(x, \omega) \in G \times \widehat{G}$, we have

$$\begin{aligned} A_g(x,\omega) &= \omega(x) \int_G \overline{g(t)} g(tx^2)\omega(t)\,dt \\ &= \omega(x) \int_G (\chi_{K_1} - \chi_{K_2})(t)(\chi_{K_1} - \chi_{K_2})(tx^2)\omega(t)\,dt. \end{aligned}$$

For simplicity, let the function $h_x : G \to \mathbb{C}$ be defined by

$$\begin{aligned} h_x(t) &:= (\chi_{K_1} - \chi_{K_2})(t)(\chi_{K_1} - \chi_{K_2})(tx^2) \\ &= (\chi_{K_1} - \chi_{K_2})(t)(\chi_{x^{-2}K_1} - \chi_{x^{-2}K_2})(t). \end{aligned}$$

Note that $h_x(t) \neq -1$ for all $x, t \in G$, since $K_1 \cap x^{-2}K_2 = \emptyset$ and $K_2 \cap x^{-2}K_1 = \emptyset$ by definition of K_1 and K_2. Furthermore, $h_x(t) = 1$ holds if and only if

$t \in (K_1 \cap x^{-2}K_1) \cup (K_2 \cap x^{-2}K_2) = K_x$. Hence we proved that $A_f = A_g$. Obviously, there exists no $\lambda \in \mathbb{T}$ such that $f = \lambda g$. Thus the claim holds.

By the choice of K_1 and K_2, there exist elements $x_0 \in H^{(2)} = G^{(2)}$, $\omega_0 \in \widehat{G}$ and $h \in \mathcal{H}$ such that $(L_{x_0}h) \cdot \omega_0 \notin \mathcal{H}$. This implies that $\mathcal{H}$ is not π_G-invariant.

2.3. The Wigner distribution on locally compact abelian groups

In this section we give a definition of a generalized Wigner distribution on locally compact abelian groups. Investigating this function, it will turn out that nearly all properties of the generalized ambiguity function carry over to the generalized Wigner distribution.

2.3.1. Definition and some basic facts. Recall the discussion in the beginning of Subsection 2.2.1. There we mentioned that we intend to define a general Wigner distribution in a manner analogous to the definition of the general ambiguity function (compare Definition 2.2.1). Hence let H again be an open subgroup of a locally compact abelian group G, and let $\Phi : G \to H$ be a topological isomorphism.

DEFINITION 2.3.1. Let G be a locally compact abelian group. For $f, g \in L^2(G)$, the *Wigner distribution of f and g* on $G \times \widehat{G}$ (which depends on H and Φ) is defined by

$$W_{f,g}(\omega, x) := \int_G \overline{f(x\Phi(t^{-1}))} g(x\Phi(t)) \overline{\omega(t)}\, dt \quad ((\omega, x) \in \widehat{G} \times G).$$

Further, we denote $W_{f,f}$ by W_f.

Note that this definition generalizes, for example, the definition of the Wigner distribution of discrete-time signals defined in [**CM80b**].

It it easily checked that $W_{f,g}(\omega, x)$ is defined for each $(\omega, x) \in \widehat{G} \times G$. For this, let $f, g \in L^2(G)$ and let $(\omega, x) \in \widehat{G} \times G$. To prove this for $G = \mathbb{R}$, Lieb ([**Lie90**, Section I and II]) used Hölder's inequality. In addition, here we have to use Lemma 2.1.1 (i) to obtain

$$\begin{aligned} |W_{f,g}(\omega, x)| &\leq \int_G |f(x\Phi(t^{-1}))||g(x\Phi(t))|\, dt \\ &\leq c \int_G |f(xt^{-1})||g(xt)|\, dt \\ &\leq c \left(\int_G |f(xt^{-1})|^2\, dt \right)^{\frac{1}{2}} \left(\int_G |g(xt)|^2\, dt \right)^{\frac{1}{2}} \\ &= c\|f\|_2 \|g\|_2 < \infty. \end{aligned}$$

Notice that we need essentially the openness of H. Indeed, suppose that H is not open. Then the set $H = \{\Phi(t) : t \in G\}$ is a zero set in G. Let $f, g \in L^2(G)$ be such that $f(t) = g(t^{-1})$ for all $t \in H$. Then the Wigner distribution of f and g may not be defined in $(1,e)$, since

$$W_{f,g}(1,e) = \int_G \overline{f(\Phi(t^{-1}))} g(\Phi(t))\, dt = \int_G |f(\Phi(t))|^2\, dt.$$

However, H is automatically open for a large class of locally compact abelian groups (compare Proposition 2.2.2).

It remains to ensure that the term Wigner distribution is justified here. For this, let $G = \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ be defined by $\Phi(x) = \frac{x}{2}$ as for the ambiguity function. Then we obtain

$$W_{f,g}(y,x) = \int_{\mathbb{R}} \overline{f\left(x - \frac{t}{2}\right)} g\left(x + \frac{t}{2}\right) e^{-2\pi i y t}\, dt \quad ((y,x) \in \mathbb{R} \times \mathbb{R}).$$

This is the definition of the classical Wigner distribution on $L^2(\mathbb{R})$. So indeed, our definition is a suitable generalization.

REMARK 2.3.2. Recall that for $G = \mathbb{R}$ we have $W_{f,g}(y,x) = 2A_{f^{\check{}},g}(2x,-2y)$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$, $f,g \in L^2(\mathbb{R})$ and $f^{\check{}} \in L^2(\mathbb{R})$ defined by $f^{\check{}}(t) = f(-t)$ ([**Lie90**, Section I]). In the general setting we can find such a relation between the Wigner distribution and the ambiguity function only if $H = G$. Then, for all $(x,\omega) \in G \times \widehat{G}$, $f,g \in L^2(G)$ and $f^{\check{}} \in L^2(G)$ defined by $f^{\check{}}(t) = f(t^{-1})$, we obtain, using Lemma 2.1.1 (ii),

$$\begin{aligned}
W_{f,g}(\omega,x) &= \int_G \overline{f(x\Phi(t^{-1}))} g(x\Phi(t)) \overline{\omega(t)}\, dt \\
&= d \int_G \overline{f(t^{-1}x)} g(tx) \overline{\omega(\Phi^{-1}(t))}\, dt \\
&= d \int_G \overline{f^{\check{}}(t\Phi((\Phi^{-1}(x))^{-1}))} g(t\Phi(\Phi^{-1}(x))) (\overline{\omega \circ \Phi^{-1}})(t)\, dt \\
&= d\, A_{f^{\check{}},g}(\Phi^{-1}(x), \overline{\omega \circ \Phi^{-1}}).
\end{aligned}$$

Hence this case is easy to deal with. But if H is a proper subgroup of G, we have to use different arguments.

First, we will check some basic properties, which are well-known for $G = \mathbb{R}$ (compare [**Lie90**, Section I and II] and [**CM80a**, Subsection 2.3]). The proofs carry over in a straightforward manner.

PROPOSITION 2.3.3. *Let G be a locally compact abelian group. For all $f,g \in L^2(G)$, the following hold.*

(i) *There exists a positive constant c such that, for all $(\omega,x) \in \widehat{G} \times G$,*

$$|W_{f,g}(\omega,x)| \le c\|f\|_2\|g\|_2.$$

(ii) *For all $(\omega,x) \in \widehat{G} \times G$, we have*

$$\overline{W_{f,g}(\omega,x)} = W_{g,f}(\omega,x).$$

In particular, this means that W_f is real-valued.

PROOF. (i) was already shown above. For all $(\omega, x) \in \widehat{G} \times G$, we have

$$\begin{aligned}\overline{W_{f,g}(\omega,x)} &= \int_G f(x\Phi(t^{-1}))\overline{g(x\Phi(t))}\omega(t)\,dt \\ &\overset{t\mapsto t^{-1}}{=} \int_G \overline{g(x\Phi(t^{-1}))}f(x\Phi(t))\omega(t^{-1})\,dt \\ &= W_{g,f}(\omega,x).\end{aligned}$$

This proves (ii). □

Next we intend to study the generalized Wigner distribution with respect to continuity. The following results are the same as for the ambiguity function. Notice that, for $G = \mathbb{R}$, they are obvious because of Remark 2.3.2 and the analogous well-known result for the ambiguity function. However, in the general setting some arguments are required.

PROPOSITION 2.3.4. *Let G be a locally compact abelian group. Then, for all $f, g \in L^2(G)$,*

$$W_{f,g} \in C(\widehat{G} \times G).$$

PROOF. Let $(\omega_0, x_0) \in \widehat{G} \times G$ and $\epsilon > 0$. Then

$$\begin{aligned}&|W_{f,g}(\omega,x) - W_{f,g}(\omega_0,x_0)| \\ &\leq |W_{f,g}(\omega,x) - W_{f,g}(\omega,x_0)| + |W_{f,g}(\omega,x_0) - W_{f,g}(\omega_0,x_0)|\end{aligned}$$

for all $(\omega, x) \in \widehat{G} \times G$. The next step is to estimate the two summands.

We start with the first summand. Using Hölder's inequality and Lemma 2.1.1 (i),

$$\begin{aligned}&|W_{f,g}(\omega,x) - W_{f,g}(\omega,x_0)| \\ &= \Bigg|\int_G \Big[\overline{f(x\Phi(t^{-1}))}g(x\Phi(t)) - \overline{f(x_0\Phi(t^{-1}))}g(x\Phi(t)) \\ &\qquad + \overline{f(x_0\Phi(t^{-1}))}g(x\Phi(t)) - \overline{f(x_0\Phi(t^{-1}))}g(x_0\Phi(t))\Big]\,\omega(t)\,dt\Bigg| \\ &\leq \int_G |g(x\Phi(t))||f(x\Phi(t^{-1})) - f(x_0\Phi(t^{-1}))|\,dt \\ &\qquad + \int_G |f(x_0\Phi(t^{-1}))||g(x\Phi(t)) - g(x_0\Phi(t))|\,dt \\ &\leq \left(\int_G |g(x\Phi(t))|^2\,dt\right)^{\frac{1}{2}} \left(\int_G |f(x\Phi(t^{-1})) - f(x_0\Phi(t^{-1}))|^2\,dt\right)^{\frac{1}{2}} \\ &+ \left(\int_G |f(x_0\Phi(t^{-1}))|^2\,dt\right)^{\frac{1}{2}} \left(\int_G |g(x\Phi(t)) - g(x_0\Phi(t))|^2\,dt\right)^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned} &\leq \sqrt{c}\|g\|_2 \left(c\int_G |f(xt^{-1}) - f(x_0t^{-1})|^2\,dt\right)^{\frac{1}{2}} \\ &\quad + \sqrt{c}\|f\|_2 \left(c\int_G |g(xt) - g(x_0t)|^2\,dt\right)^{\frac{1}{2}}. \end{aligned}$$

Note that the mapping $x \mapsto L_x h$, $G \to L^2(G)$, is uniformly continuous for all $h \in L^2(G)$ ([**HR63**, Theorem 20.4]). Hence, for each $h \in L^2(G)$, there exists a neighbourhood U_h of x_0 in G such that

$$\|L_{x^{-1}}h - L_{x_0^{-1}}h\|_2 < \frac{\epsilon}{4c(\|g\|_2 + \|f\|_2)}$$

for all $x \in U_h$. So, for all $x \in U_f \cap U_g$,

$$\begin{aligned} &|W_{f,g}(\omega, x) - W_{f,g}(\omega, x_0)| \\ &\leq c(\|g\|_2\|L_{x^{-1}}f - L_{x_0^{-1}}f\|_2 + \|f\|_2\|L_{x^{-1}}g - L_{x_0^{-1}}g\|_2) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Now we turn to the second summand. There we obtain

$$\begin{aligned} &|W_{f,g}(\omega, x_0) - W_{f,g}(\omega_0, x_0)| \\ &= \left|(\overline{f(x_0\Phi(\cdot^{-1}))}g(x_0\Phi(\cdot))^\wedge(\omega) - (\overline{f(x_0\Phi(\cdot^{-1}))}g(x_0\Phi(\cdot))^\wedge(\omega_0)\right|. \end{aligned}$$

The function $t \mapsto \overline{f(x_0\Phi(t^{-1}))}g(x_0\Phi(t))$ is integrable, since $f, g \in L^2(G)$ and by Lemma 2.1.1 (i). Hence the Fourier transform belongs to $C_0(\widehat{G})$. This implies that there exists a neighbourhood V of ω_0 in $\widehat{G}$ such that, for all $\omega \in V$,

$$|W_{f,g}(\omega, x_0) - W_{f,g}(\omega_0, x_0)| < \frac{\epsilon}{2}.$$

Summarizing, we have

$$|W_{f,g}(\omega, x) - W_{f,g}(\omega_0, x_0)| < \epsilon \quad \text{for all } (\omega, x) \in V \times (U_f \cap U_g).$$

□

PROPOSITION 2.3.5. *Let G be a locally compact abelian group. Then the map*

$$(f, g) \mapsto W_{f,g}, \quad L^2(G) \times L^2(G) \to (C(\widehat{G} \times G), \|\cdot\|_\infty),$$

is continuous.

PROOF. Let $f, g, f', g' \in L^2(G)$ and $(\omega, x) \in \widehat{G} \times G$. It is easy to check that

$$|W_{f,g}(\omega, x) - W_{f',g'}(\omega, x)| \leq |W_{f-f',g}(\omega, x)| + |W_{f',g-g'}(\omega, x)|.$$

Using Proposition 2.3.3 (i), this implies

$$|W_{f,g}(\omega, x) - W_{f',g'}(\omega, x)| \leq c(\|f - f'\|_2\|g\|_2 + \|f'\|_2\|g - g'\|_2).$$

This finishes the proof. □

2.3.2. Behaviour at infinity. A natural question to ask is whether the Wigner distribution vanishes at infinity. It will turn out that indeed the analogue of Theorem 2.2.10 holds. The main steps of the proof are the same but treating this case often requires different arguments.

THEOREM 2.3.6. *Let G be a locally compact abelian group. Then the following conditions are equivalent.*

(i) $W_{f,g} \in C_0(\widehat{G} \times G)$ *for all* $f, g \in L^2(G)$.
(ii) G *is 2-root compact.*

PROOF. First, suppose that (i) holds. By Lemma 2.1.1 (ii), this implies

$$\begin{aligned} |W_{f,g}(1, \cdot)| &= d\,\Big|\int_H \overline{f(\cdot\, t^{-1})} g(\cdot\, t)\, dt\Big| \\ &= d\,\Big|\int_H \overline{f(t^{-1})} g(\cdot^2\, t)\, dt\Big| \\ &\in C_0(G) \quad \text{for all } f, g \in L^2(G). \end{aligned}$$

Towards a contradiction, assume that G is not 2-root compact. Then there exists a compact subset K of G such that the set $X := \{x \in G : x^2 \in K\}$ is not compact. Let V be any compact, symmetric neighbourhood of e in H and define $f, g \in L^2(G)$ by $f := \chi_V$ and $g := \chi_{VK}$. Then, for all $x \in X$, we obtain

$$\begin{aligned} |W_{f,g}(1, x)| &= d\,\Big|\int_H \overline{f(t^{-1})} g(x^2 t)\, dt\Big| \\ &= d\,\Big|\int_H \chi_V(t^{-1}) \chi_{VK}(x^2 t)\, dt\Big| \\ &= d\,\Big|\int_H \chi_{V^{-1} \cap V x^{-2} K}(t)\, dt\Big| \\ &= d\,|V|, \end{aligned}$$

a contradiction. Thus we proved (i) $\Longrightarrow$ (ii).

Secondly, suppose that G is 2-root compact. Let $f, g \in L^2(G)$. By Proposition 2.3.4, $W_{f,g}$ is continuous. Now let $\epsilon > 0$. On the one hand, by Lemma 2.1.1 (i), we have, for $(\omega, x) \in \widehat{G} \times G$,

$$\begin{aligned} |W_{f,g}(\omega, x)| &\leq \int_G |f(x\Phi(t^{-1}))||g(x\Phi(t))|\, dt \\ &\leq c \int_G |f(xt^{-1})||g(xt)|\, dt \\ &\overset{t \mapsto xt}{=} c \int_G |f(t^{-1})||g(x^2 t)|\, dt \\ &= c\,(|f| * |g|)(x^2). \end{aligned}$$

Since $f, g \in L^2(G)$ implies $|f| * |g| \in C_0(G)$, there exists a compact set $\tilde{K} \subseteq G$ such that

$$(|f| * |g|)(x) < \frac{\epsilon}{2c} \quad \text{for all } x \in G \backslash \tilde{K}.$$

The set $K := \{x \in G : x^2 \in \tilde{K}\}$ is compact, since G is supposed to be 2-root compact. We obtain

$$|W_{f,g}(\omega, x)| \le c(|f| * |g|)(x^2) < \frac{\epsilon}{2} \quad \text{for all } x \in G \backslash K.$$

On the other hand, we may apply Lemma 2.1.1 (ii) to $W_{f,g}$. Then, for all $(\omega, x) \in \widehat{G} \times G$, we obtain

$$\begin{aligned} W_{f,g}(\omega, x) &= \int_G \overline{f(x\Phi(t^{-1}))} g(x\Phi(t)) \overline{\omega(\Phi^{-1} \circ \Phi(t))}\, dt \\ &= d \int_H \overline{f(xt^{-1})} g(xt) \overline{\omega(\Phi^{-1}(t))}\, dt \\ &= d\, (((L_x f^*) \cdot (L_{x^{-1}} g))|_H)\widehat{\ }(\omega \circ \Phi^{-1}). \end{aligned}$$

Now $f, g \in L^2(G)$ implies $((L_x f^*) \cdot (L_{x^{-1}} g)) \in L^1(G)$. Since H is open, we have $((L_x f^*) \cdot (L_{x^{-1}} g))|_H \in L^1(H)$. Hence $(((L_x f^*) \cdot (L_{x^{-1}} g))|_H)\widehat{\ } \in C_0(\widehat{H})$. This implies the existence of a compact set $\tilde{\Gamma}(x) \subseteq \widehat{H}$ for each $x \in G$ such that

$$|W_{f,g}(\omega, x)| < \frac{\epsilon}{2} \quad \text{for all } \omega \circ \Phi^{-1} \in \widehat{H} \backslash \tilde{\Gamma}(x).$$

Now define $\Gamma(x) \subseteq \widehat{G}$ by $\Gamma(x) := \{\omega \in \widehat{G} : \omega \circ \Phi^{-1} \in \tilde{\Gamma}(x)\}$ for each $x \in G$. Obviously, the map $\phi : \widehat{H} \to \widehat{G}$ defined by $\phi(\omega) = \omega \circ \Phi$ is continuous. Note that $\Gamma(x) = \phi(\tilde{\Gamma}(x))$ for all $x \in G$. Hence $\Gamma(x)$ is always compact. We finally obtain

$$|W_{f,g}(\omega, x)| < \frac{\epsilon}{2} \quad \text{for all } \omega \in \widehat{G} \backslash \Gamma(x).$$

Next we use the sets $\Gamma(x)$, $x \in G$, just defined to construct a compact set $\Gamma \in \widehat{G}$, such that $|W_{f,g}(\omega, x)| < \epsilon$ for all $(\omega, x) \in (\widehat{G} \times G) \backslash (\Gamma \times K)$. This yields the claim.

For this, let $(\omega, x) \in \widehat{G} \times G$ and let x_0 belong to G. By Lemma 2.1.1 (ii), we obtain

$$\begin{aligned} |W_{f,g}(\omega, x)| &= d \left| \int_H \overline{f(xt^{-1})} g(xt) \overline{\omega(\Phi^{-1}(t))}\, dt \right| \\ &= d \left| \int_H \overline{f(x_0^{-1} x_0 x t^{-1})} g(x_0^{-1} x_0 x t) \overline{\omega(\Phi^{-1}(t))}\, dt \right|. \end{aligned}$$

Since H is open, there exists a neighbourhood $U(x_0)$ of x_0 in G such that $x_0 x^{-1} \in H$ for all $x \in U(x_0)$. Hence we may substitute t by $t x_0 x^{-1}$. Now, using the openness of H, Lemma 2.1.1 (ii) and Hölder's inequality, we obtain

$$\begin{aligned}
&|W_{f,g}(\omega, x)| \\
&\quad = \quad d\left|\int_H \overline{f(x_0^{-1}x_0xt^{-1})}g(x_0^{-1}x_0xt)\overline{\omega(\Phi^{-1}(t))}\,dt\right| \\
&\quad \overset{t\mapsto tx_0x^{-1}}{=} \quad d\left|\int_H \overline{f(x_0^{-1}x_0xx_0^{-1}xt^{-1})}g(x_0t)\overline{\omega(\Phi^{-1}(tx_0x^{-1}))}\,dt\right| \\
&\quad = \quad d\left|\int_H \overline{f(x_0^{-1}x^2t^{-1})}g(x_0t)\overline{\omega(\Phi^{-1}(t))}\,dt\right| \\
&\quad \leq \quad d\left[\left|\int_H \overline{f(x_0t^{-1})}g(x_0t)\overline{\omega(\Phi^{-1}(t))}\,dt\right| \right. \\
&\qquad\qquad \left. + \int_G |g(x_0t)||f(x_0^{-1}x^2t^{-1}) - f(x_0t^{-1})|\,dt\right] \\
&\quad = \quad |W_{f,g}(\omega, x_0)| + d\int_G |g(x_0t)||f(x_0^{-2}x^2(x_0t^{-1})) - f(x_0t^{-1})|\,dt \\
&\quad \leq \quad |W_{f,g}(\omega, x_0)| + d\|g\|_2\|L_{x_0^2x^{-2}}f - f\|_2.
\end{aligned}$$

Recall that the map $x \mapsto L_xf$, $G \to L^2(G)$, is uniformly continuous ([**HR63**, Theorem 20.4]). Hence there exists a neighbourhood $V(x_0)$ of x_0 in G such that

$$\|L_{x_0^2x^{-2}}f - f\|_2 \leq \frac{\epsilon}{2d\|g\|_2} \quad \text{for all } x \in V(x_0).$$

Since K is compact, there exist finitely many elements $x_1, \dots, x_N \in G$, $N \in \mathbb{N}$, such that union of the open sets $V(x_i) \cap U(x_i)$, $i \in \{1, \dots, N\}$, covers K. Now define $\Gamma \subseteq \widehat{G}$ by

$$\Gamma := \bigcup_{i=1}^{N} \Gamma(x_i).$$

Clearly, Γ is compact.

Let $x \in K$. It remains to prove that Γ satisfies the property mentioned above. There exists $i_0 \in \{1, \dots, N\}$ such that $x \in V(x_{i_0}) \cap U(x_{i_0})$. Hence, for all $\omega \in \widehat{G} \backslash \Gamma(x_{i_0})$,

$$\begin{aligned}
|W_{f,g}(\omega, x)| &\leq |W_{f,g}(\omega, x_{i_0})| + d\|g\|_2\|L_{x_{i_0}^2x^{-2}}f - f\|_2 \\
&< \frac{\epsilon}{2} + d\|g\|_2\frac{\epsilon}{2d\|g\|_2} \\
&= \epsilon.
\end{aligned}$$

This is especially true for all $\omega \in \widehat{G} \backslash \Gamma$. This implies that

$$|W_{f,g}(\omega, x)| < \epsilon \quad \text{for all } (\omega, x) \in (\widehat{G} \times G) \backslash (\Gamma \times K).$$

Thus we have shown (i). □

2.3.3. Square-integrability. In this subsection we shall give a characterization of those locally compact abelian groups, for which the associated Wigner distribution is always square-integrable. Because of the following theorem, we may carry over most results obtained in Subsection 2.2.3 for the ambiguity function.

THEOREM 2.3.7. *Let G be a locally compact abelian group and let $f, g \in L^2(G)$. Then*

$$\|A_{f,g}\|_2 = \|W_{f,g}\|_2.$$

In particular, the following conditions are equivalent.

(i) $A_{f,g} \in L^2(G \times \widehat{G})$.
(ii) $W_{f,g} \in L^2(\widehat{G} \times G)$.

PROOF. Consider the function $\tilde{h}_{f,g} : G \times G \to \mathbb{C}$ defined by

$$\tilde{h}_{f,g}(x,t) = \overline{f(x\Phi(t^{-1}))}g(x\Phi(t)).$$

By Hölder's inequality and Lemma 2.1.1 (i), $\tilde{h}_{f,g}(x,\cdot) \in L^1(G)$. Hence the Fourier transform of $t \mapsto \tilde{h}_{f,g}(x,t)$ exists and we obtain

$$(\tilde{h}_{f,g}(x,\cdot))^\frown(\omega) = W_{f,g}(\omega, x) \quad \text{for all } (\omega, x) \in \widehat{G} \times G.$$

Assume that $\tilde{h}_{f,g} \in L^2(G \times G)$. Then the Fourier transform coincides with the Plancherel transform and we obtain

$$\|W_{f,g}\|_2^2 = \|(\omega, x) \mapsto (\tilde{h}_{f,g}(x,\cdot))^\frown(\omega)\|_2^2 = \|\tilde{h}_{f,g}\|_2^2. \tag{5}$$

In particular, $W_{f,g} \in L^2(\widehat{G} \times G)$.

Conversely, assume that $W_{f,g} \in L^2(\widehat{G} \times G)$. Hence $(\tilde{h}_{f,g}(x,\cdot))^\frown \in L^2(\widehat{G})$ for almost all $x \in G$. Thus, in particular, this is the Plancherel transform. So, equation (5) follows.

This proves that

$$\|W_{f,g}\|_2 = \|\tilde{h}_{f,g}\|_2.$$

Let $h_{f,g}$ be defined as in the preliminaries of Subsection 2.2.3. In the proof of Lemma 2.2.11 we showed that

$$\|A_{f,g}\|_2 = \|h_{f,g}\|_2.$$

Since

$$\begin{aligned} \|h_{f,g}\|_2^2 &= \int_G \int_G |f(t)|^2 |g(t\Phi(x^2))|^2 \, dt \, dx \\ &\overset{t \mapsto t\Phi(x^{-1})}{=} \int_G \int_G |f(t\Phi(x^{-1}))|^2 |g(t\Phi(x))|^2 \, dt \, dx \\ &= \|\tilde{h}_{f,g}\|_2^2, \end{aligned}$$

this finishes the proof. ∎

Theorem 2.3.7 shows that all results proven for the ambiguity function in Subsection 2.2.3 remain true for the Wigner distribution. In [**Lie90**, Section I and II] the properties $\|W_{f,g}\|_2 = \|A_{f,g}\|_2$ and $W_{f,g} \in L^2(\mathbb{R} \times \mathbb{R})$ for all $f, g \in L^2(\mathbb{R})$ are proven by using the relation between the Wigner distribution and the ambiguity function (compare Remark 2.3.2). Here this relation is not true except in the case $H = G$, but we still have Theorem 2.3.7. In the following we list the results concerning square-integrability of the Wigner distribution without proofs, since they are obvious by using Theorem 2.3.7.

COROLLARY 2.3.8. *Let G be a locally compact abelian group. Suppose that $G^{(2)}$ is closed. Then* $\ker\varphi$ *is compact, if $W_{f,g} \in L^2(\widehat{G} \times G)$ for all $f, g \in L^2(G)$.*

If, in addition, G is σ-compact, then the following conditions are equivalent.

(i) $W_{f,g} \in L^2(\widehat{G} \times G)$ *for all* $f, g \in L^2(G)$.
(ii) $\ker\varphi$ *is compact and* $G^{(2)}$ *is open.*

COROLLARY 2.3.9. *Let G be a locally compact abelian group. Suppose that $G^{(2)}$ is open. Consider the following statements.*

(i) $W_{f,g} \in L^2(\widehat{G} \times G)$ *for all* $f, g \in L^2(G)$.
(ii) $\ker\varphi$ *is compact.*
(iii) G *is 2-root compact.*

Then one has the implications (i) $\Rightarrow$ (ii) *and* (iii) $\Rightarrow$ (ii).

If, in addition, G is σ-compact, then (i), (ii) *and* (iii) *are equivalent.*

COROLLARY 2.3.10. *Let G be a σ-compact locally compact abelian group. Suppose that $G^{(2)}$ is open. Then the following conditions are equivalent.*

(i) $W_{f,g} \in L^2(\widehat{G} \times G)$ *for all* $f, g \in L^2(G)$.
(ii) $W_{f,g} \in C_0(\widehat{G} \times G)$ *for all* $f, g \in L^2(G)$.

COROLLARY 2.3.11. *Let G be a 2-root compact, σ-compact locally compact abelian group. Suppose that G_0 is open. Then, for all $f, g \in L^2(G)$,*

$$W_{f,g} \in L^2(\widehat{G} \times G).$$

COROLLARY 2.3.12. *Let G be a σ-compact locally compact abelian group. Suppose that $G^{(2)}$ is closed and that $W_{f,g} \in L^2(\widehat{G} \times G)$ for all $f, g \in L^2(G)$.*

(i) *There exists a positive constant C such that, for all $f, g \in L^2(G)$,*

$$\|W_{f,g}\|_2^2 \leq C\|f\|_2^2\|g\|_2^2.$$

(ii) *The mapping*

$$(f, g) \mapsto W_{f,g}, \quad L^2(G) \times L^2(G) \to L^2(\widehat{G} \times G),$$

is continuous.

2.3.4. Injectivity. As in the previous subsection also the injectivity properties of the ambiguity function carry over to the Wigner distribution for nearly all locally compact abelian groups and for all locally compact abelian groups in the case $H = G$. In fact, Remark 2.3.2 yields the following results by using Theorem 2.2.23, Corollary 2.2.25 and Proposition 2.2.27. In the following let π_G be defined as in Subsection 2.2.4.

COROLLARY 2.3.13. *Let G be a locally compact abelian group. Suppose that $H = G$ and that $H^{(2)}$ is closed. Let $\mathcal{H}$ be a closed, π_G-invariant subspace of $L^2(G)$. Then the following conditions are equivalent.*

(i) *$\pi_G|_{\mathcal{H}}$ is irreducible.*
(ii) *For all $f, g \in \mathcal{H}$, the following conditions are equivalent.*
 (a) *$W_f = W_g$.*
 (b) *There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.*

COROLLARY 2.3.14. *Let G be a locally compact abelian group. Suppose that $H = G$ and that $H^{(2)}$ is open. Further, let $y \in G$. For $f, g \in \mathcal{H}_y$, the following conditions are equivalent.*

(i) *$W_f = W_g$.*
(ii) *There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.*

PROPOSITION 2.3.15. *Let G be a locally compact abelian group. Suppose that $H = G$ and that $H^{(2)}$ is open. Furthermore, suppose that $\mathcal{H}$ is a closed subspace of $L^2(G)$ with the properties that there exists some $y_0 \in G$ such that $\mathcal{H}_{y_0} \subseteq \mathcal{H}$ and that, for all $f, g \in \mathcal{H}$, the following conditions are equivalent.*

(i) *$W_f = W_g$.*
(ii) *There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.*

Then $\mathcal{H} = \mathcal{H}_{y_0}$.

To show that there exists a closed, non-π_G-invariant subspace of $L^2(G)$ such that one time the injectivity condition is satisfied and the other time not, we may use the same examples as in Subsection 2.2.4.

EXAMPLE 2.3.16. Let G be a locally compact abelian group such that $H^{(2)} \neq \{e\}$ and let $h \in L^2(G)$ be such that $L_{x_0}h \notin \operatorname{span}\{h\}$ for some $x_0 \in H^{(2)}$. Then, for $f, g \in \mathcal{H} := \operatorname{span}\{h\}$, the following conditions are equivalent.

(i) $W_f = W_g$.
(ii) There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.

EXAMPLE 2.3.17. Let G be a locally compact abelian group such that $G^{(2)}$ is open and $G^{(2)} \neq G$, and let $\Phi : G \to G$ be defined by $\Phi := \mathrm{Id}_G$. Furthermore, let $K_1, K_2 \subseteq G$, $K_1^\circ, K_2^\circ \neq \emptyset$, be compact sets such that $K_1 \subseteq G^{(2)}$ and $K_2 \subseteq G \backslash G^{(2)}$. Now define $\mathcal{H} \subseteq L^2(G)$ by $\mathcal{H} := \operatorname{span}\{\chi_{K_1}, \chi_{K_2}\}$. Then defining f and g by $f := \chi_{K_1} + \chi_{K_2} \in \mathcal{H}$ and $g := \chi_{K_1} - \chi_{K_2} \in \mathcal{H}$, we have $W_f = W_g$, but there exists no $\lambda \in \mathbb{T}$ such that $f = \lambda g$.

For instance, a special choice would be $G = \mathbb{Z}$. Then $G^{(2)} = 2\mathbb{Z}$. The compact sets K_1 and K_2 may be chosen by $K_1 := \{0\}$ and $K_2 := \{1\}$.

Now we deal with the case that H is a proper subgroup of G. For this, we need a result from the next section. It can be used here, since Section 2.4 is independent of the results appearing in this subsection. The following theorem holds.

THEOREM 2.3.18. *Let G be a 2-root compact, second countable locally compact abelian group. Suppose that $G^{(2)}$ is open. Then, for all $f, g \in L^2(G)$, the following conditions are equivalent.*

(i) $A_f = A_g$.
(ii) $W_f = W_g$.

PROOF. This follows immediately from Theorem 2.4.4 and the injectivity of the Plancherel transform. □

Using this theorem, we can start carrying over the results of the ambiguity function. Note that, by using Theorem 2.3.18, we have to restrict to 2-root compact, second countable locally compact abelian groups G, where $G^{(2)}$ is open.

COROLLARY 2.3.19. *Let G be a 2-root compact, second countable locally compact abelian group. Suppose that $G^{(2)}$ is open. Further, let $\mathcal{H}$ be a closed, π_G-invariant subspace of $L^2(G)$. Then the following conditions are equivalent.*

(i) *$\pi_G|_{\mathcal{H}}$ is irreducible.*
(ii) *For all $f, g \in L^2(G)$, the following are equivalent.*
  (a) $W_f = W_g$.
  (b) *There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.*

PROOF. By Proposition 2.2.26 (i), $H^{(2)}$ is closed. Hence the claim follows from Theorem 2.3.18 and Theorem 2.2.23. □

COROLLARY 2.3.20. *Let G be a 2-root compact, second countable locally compact abelian group. Suppose that $H^{(2)}$ is open and let $y \in G$. For $f, g \in \mathcal{H}_y$, the following conditions are equivalent.*

(i) $W_f = W_g$.
(ii) *There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.*

PROOF. The corollary follows immediately from Corollary 2.2.25 and Theorem 2.3.18, since $G^{(2)}$ is also open. □

COROLLARY 2.3.21. *Let G be a 2-root compact, second countable locally compact abelian group. Suppose that $H^{(2)}$ is open. Furthermore, suppose that $\mathcal{H}$ is a closed subspace of $L^2(G)$ with the properties that there exists some $y_0 \in G$ such that $\mathcal{H}_{y_0} \subseteq \mathcal{H}$ and that, for all $f, g \in \mathcal{H}$, the following conditions are equivalent.*

(i) $W_f = W_g$.
(ii) *There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.*

Then $\mathcal{H} = \mathcal{H}_{y_0}$.

PROOF. Note that, since $H^{(2)}$ is open, $G^{(2)}$ is open. Thus the result is a conclusion from Proposition 2.2.27 and Theorem 2.3.18. □

Also the two example of Subsection 2.2.4 carry over at once.

EXAMPLE 2.3.22. Let G be a 2-root compact, second countable locally compact abelian group such that $H^{(2)} \neq \{e\}$. Suppose that $G^{(2)}$ is open. Furthermore, let $h \in L^2(G)$ be such that $L_{x_0}h \notin \text{span}\{h\}$ for some $x_0 \in H^{(2)}$. Then, for $f, g \in \mathcal{H} := \text{span}\{h\}$, the following conditions are equivalent.

(i) $W_f = W_g$.
(ii) There exists some $\lambda \in \mathbb{T}$ such that $f = \lambda g$.

This is an example of a closed subspace $\mathcal{H}$ which is not π_G-invariant, but the corresponding Wigner distribution satisfies the injectivity condition.

EXAMPLE 2.3.23. Let G be a 2-root compact, second countable locally compact abelian group such that $G^{(2)}$ is open and $G^{(2)} \neq G$. Let $\Phi : G \to G$ be defined by $\Phi := \text{Id}_G$. Let $K_1, K_2 \subseteq G$, $K_1^\circ, K_2^\circ \neq \emptyset$, be compact sets such that $K_1 \subseteq G^{(2)}$ and $K_2 \subseteq G \backslash G^{(2)}$. Define $\mathcal{H} \subseteq L^2(G)$ by $\mathcal{H} := \text{span}\{\chi_{K_1}, \chi_{K_2}\}$. Then defining f and g by $f := \chi_{K_1} + \chi_{K_2} \in \mathcal{H}$ and $g := \chi_{K_1} - \chi_{K_2} \in \mathcal{H}$, we obtain $W_f = W_g$ but there exists no $\lambda \in \mathbb{T}$ such that $f = \lambda g$.

For instance, a special choice in this case would be $G = \mathbb{Z}$. Then $G^{(2)} = 2\mathbb{Z}$. The compact sets K_1 and K_2 may be chosen by $K_1 := \{0\}$ and $K_2 := \{1\}$.

This is an example of a closed subspace $\mathcal{H}$ which is neither π_G-invariant nor the corresponding Wigner distribution satisfies the injectivity condition.

2.4. $\widehat{A_{f,g}} = W_{f,g}$

This section contains the main result of this chapter. We will show that the Plancherel transform of the ambiguity function coincides with the Wigner distribution for a large class of locally compact abelian groups. We already mentioned that this result cannot be achieved using the generalization of the ambiguity function used in other publications, for example [**FS98**, Subsection 7.6.1], together with a similar generalization of the Wigner distribution. For $G = \mathbb{R}$, this fact has various applications, for example, it is used for the description of Cohen's class by characteristic functions (see [**Coh95**, Chapter 9]).

We need some preparations before we give the theorem and its proof. The following theorem is [**HR70**, Theorem 31.13]. We give a version which is slightly weaker, since we don't need the whole result.

THEOREM 2.4.1. *Let G be a locally compact abelian group such that $\widehat{G}$ is σ-compact. Let $C_0^+(G)$ and $C_0^+(\widehat{G})$ denote the space of functions which have only positive values and which belong to $C_0(G)$ and $C_0(\widehat{G})$, respectively. Then there exist sequences $(k_n)_{n\in\mathbb{N}} \subseteq C_0^+(\widehat{G}) \cap L^1(\widehat{G})$ and $(\psi_n)_{n\in\mathbb{N}} \subseteq C_0^+(G) \cap L^1(G)$ such that for all $n \in \mathbb{N}$, $\omega \in \widehat{G}$, $x \in G$ and $f \in C_c(G)$ the following is true.*

(i) $k_n(\widehat{G}) \subseteq [0, 1]$ *and* $k_n = k_n^*$.

(ii) $\lim_{n\to\infty} k_n(\omega) = 1$.

(iii) $\check{k_n} = \psi_n$, *where* ˇ *denotes the inverse Fourier transform.*

(iv) $\int_G \psi_n(x)\,dx = 1$ *and* $\psi_n = \psi_n^*$.

(v) $\lim_{n\to\infty} (\omega * \psi_n)(x) = \omega(x)$.

(vi) $\lim_{n\to\infty} (f * \psi_n)(x) = f(x)$.

The next two lemmas will be used in the proof of the theorem.

LEMMA 2.4.2. *Let G be a 2-root compact, locally compact abelian group such that $\widehat{G}$ is σ-compact. Let $f, g \in C_c(G)$ and define the function $\tilde{h}_{f,g} : G \times G \to \mathbb{C}$ (as in the proof of Theorem 2.3.7) by*

$$\tilde{h}_{f,g}(x,t) = \overline{f(x\Phi(t^{-1}))}g(x\Phi(t)).$$

Let $(\psi_n)_{n\in\mathbb{N}}$ be as in Theorem 2.4.1. Then

(i) $\tilde{h}_{f,g} \in C_c(G \times G)$.

(ii) *For each $x \in G$, $(\tilde{h}_{f,g}(\cdot,t) * \overline{\psi}_n)(x)$ converges uniformly in $t \in G$ to $\tilde{h}_{f,g}(x,t)$.*

PROOF. Clearly, $\tilde{h}_{f,g}$ is continuous. Define $T \subseteq G$ by $T := \mathrm{supp} f \cup \mathrm{supp} g$. Clearly, T is compact. Furthermore, define $C_1, C_2 \subseteq G$ by

$$C_1 := \{t \in G : t^2 \in \Phi^{-1}(T^{-1}T)\} \quad \text{and} \quad C_2 := \Phi(C_1)T \cap \Phi(C_1)^{-1}T.$$

Obviously, C_1 and C_2 are also compact, especially since G is supposed to be 2-root compact. Now let $t \in G\backslash C_1$. Then $\Phi(t^2) \notin T^{-1}T$, hence $\Phi(t)T \cap \Phi(t)^{-1}T = \emptyset$. For each $x \in G$, this implies that either $f(x\Phi(t^{-1})) = 0$ or $g(x\Phi(t)) = 0$. Hence $\tilde{h}_{f,g}(x,t) = 0$ for all $x \in G$. Now let $x \in G\backslash C_2$. For each $t \in C_1$, we have either $f(x\Phi(t^{-1})) = 0$ or $g(x\Phi(t)) = 0$. Hence $\tilde{h}_{f,g}(x,t) = 0$ for all $t \in C_1$. Thus we have shown that $\mathrm{supp}\tilde{h}_{f,g} \subseteq C_2 \times C_1$. This implies (i).

Now let $x, t \in G$. By Theorem 2.4.1 (vi), the convergence of $(\tilde{h}_{f,g}(\cdot,t) * \overline{\psi}_n)(x)$ to $\tilde{h}_{f,g}(x,t)$ is pointwise. By (i), we obtain $(t \mapsto (\tilde{h}_{f,g}(\cdot,t) * \overline{\psi}_n)(x)) \in C_c(G)$ and the support is contained in C_1. Again using (i), we obtain $\tilde{h}_{f,g}(x,\cdot) \in C_c(G)$ and $\mathrm{supp}(\tilde{h}_{f,g}(x,\cdot)) \subseteq C_1$. This implies that the convergence is uniform. □

LEMMA 2.4.3. *Let G be a 2-root compact, locally compact abelian group such that $\widehat{G}$ is σ-compact. Further, let $f, g \in C_c(G)$ and let $(k_n)_{n\in\mathbb{N}}$ be as in Theorem 2.4.1. Then, for all $n \in \mathbb{N}$,*

$$A_{f,g} \cdot k_n \in L^1(G \times \widehat{G}).$$

PROOF. Let $n \in \mathbb{N}$. By Theorem 2.4.1 (iii) and using the fact that $(k_n)_{n\in\mathbb{N}} \subseteq C_0^+(\widehat{G}) \cap L^1(\widehat{G})$, we obtain

$$\begin{aligned}
&\int_{G\times\widehat{G}} |A_{f,g}(x,\omega)k_n(\omega)|\, d(x,\omega) \\
&\leq \int_{\widehat{G}}\int_G\int_G |f(t\Phi(x^{-1}))||g(t\Phi(x))|\, dt\, k_n(\omega)\, dx\, d\omega \\
&= \int_G\int_G\int_{\widehat{G}} k_n(\omega)\, d\omega |f(t\Phi(x^{-1}))||g(t\Phi(x))|\, dx\, dt \\
&= \psi_n(e)\int_G\int_G |\tilde{h}_{f,g}(t,x)|\, dx\, dt \\
&< \infty.
\end{aligned}$$

This proves the lemma. □

We now present the main result of this chapter. For $G = \mathbb{R}$, this result is contained in [**CM80c**, Section 2] but without a detailed proof.

THEOREM 2.4.4. *Let G be a 2-root compact, second countable locally compact abelian group. Suppose that $G^{(2)}$ is open. Then, for all $f, g \in L^2(G)$,*

$$\widehat{A_{f,g}} = W_{f,g} \quad \text{in } L^2(\widehat{G}\times G).$$

PROOF. Note that both G and $\widehat{G}$ are σ-compact. Further, by Theorem 2.2.16, $A_{f,g} \in L^2(G\times\widehat{G})$ for all $f, g \in L^2(G)$.

First, we restrict to the case $f, g \in C_c(G)$. Let $(k_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be as in Theorem 2.4.1. Since $k_n \in C_0(\widehat{G})$, the function

$$(x,\omega) \mapsto A_{f,g}(x,\omega)k_n(\omega), \quad G\times\widehat{G} \to \mathbb{C},$$

is square-integrable. Hence the Plancherel transforms of both $A_{f,g}$ and $A_{f,g}\cdot k_n$ exist. By Theorem 2.4.1 (ii), for all $(x,\omega) \in G\times\widehat{G}$,

$$\lim_{n\to\infty} A_{f,g}(x,\omega)k_n(\omega) = A_{f,g}(x,\omega).$$

In addition, by Theorem 2.4.1 (i), we obtain

$$|A_{f,g}(x,\omega)k_n(\omega) - A_{f,g}(x,\omega)|^2 = |k_n(\omega)-1|^2|A_{f,g}(x,\omega)|^2 \leq |A_{f,g}(x,\omega)|^2$$

for all $(x,\omega) \in G\times\widehat{G}$. Then, by the theorem of dominated convergence,

$$\lim_{n\to\infty} \|A_{f,g}\cdot k_n - A_{f,g}\|_2 = 0,$$

hence, by the Plancherel theorem,

$$\lim_{n\to\infty} \|\widehat{A_{f,g}\cdot k_n} - \widehat{A_{f,g}}\|_2 = 0.$$

This implies that it suffices to prove

$$\lim_{n\to\infty} \widehat{A_{f,g}\cdot k_n}(\omega,x) = W_{f,g}(\omega,x)$$

for almost all $(\omega,x) \in \widehat{G}\times G$.

For this, notice that the Plancherel transform of $A_{f,g} \cdot k_n$ coincides with the Fourier transform by Lemma 2.4.3. Then, for almost all $(\omega, x) \in \widehat{G} \times G$,

$$\begin{aligned}
&\widehat{A_{f,g} \cdot k_n}(\omega, x) \\
&= \int_G \int_{\widehat{G}} A_{f,g}(t,\chi) k_n(\chi) \overline{\omega(t)}\, \overline{\chi(x)}\, d\chi\, dt \\
&= \int_G \int_{\widehat{G}} \int_G \overline{f(y\Phi(t^{-1}))} g(y\Phi(t)) \chi(y)\, dy k_n(\chi) \overline{\omega(t)}\, \overline{\chi(x)}\, d\chi\, dt \\
&= \int_G \int_G \int_{\widehat{G}} k_n(\chi) \chi(yx^{-1})\, d\chi \overline{f(y\Phi(t^{-1}))} g(y\Phi(t)) \overline{\omega(t)}\, dy\, dt.
\end{aligned}$$

Since $k_n \in L^1(\widehat{G})$ as well as $fg \in L^1(G)$, we are able to use Fubini's theorem in the last step. Then, by Theorem 2.4.1 (iii), (iv),

$$\begin{aligned}
&\widehat{A_{f,g} \cdot k_n}(\omega, x) \\
&= \int_G \int_G \int_{\widehat{G}} k_n(\chi) \chi(yx^{-1})\, d\chi \overline{f(y\Phi(t^{-1}))} g(y\Phi(t)) \overline{\omega(t)}\, dy\, dt \\
&= \int_G \int_G \psi_n(yx^{-1}) \overline{f(y\Phi(t^{-1}))} g(y\Phi(t)) \overline{\omega(t)}\, dy\, dt \\
&= \int_G ((\overline{f(\cdot\Phi(t^{-1}))} g(\cdot\Phi(t))) * \overline{\psi}_n)(x) \overline{\omega(t)}\, dt.
\end{aligned}$$

Hence, using Theorem 2.4.1 (vi) and Lemma 2.4.2 (ii), we obtain, for all $(\omega, x) \in \widehat{G} \times G$,

$$\begin{aligned}
W_{f,g}(\omega, x) &= \int_G \overline{f(x\Phi(t^{-1}))} g(x\Phi(t)) \overline{\omega(t)}\, dt \\
&= \int_G \lim_{n\to\infty} ((\overline{f(\cdot\Phi(t^{-1}))} g(\cdot\Phi(t))) * \overline{\psi}_n)(x) \overline{\omega(t)}\, dt \\
&= \lim_{n\to\infty} \int_G ((\overline{f(\cdot\Phi(t^{-1}))} g(\cdot\Phi(t))) * \overline{\psi}_n)(x) \overline{\omega(t)}\, dt \\
&= \lim_{n\to\infty} \widehat{A_{f,g} \cdot k_n}(\omega, x).
\end{aligned}$$

This shows

$$\widehat{A_{f,g}} = W_{f,g} \quad \text{for all } f, g \in C_c(G).$$

By Proposition 2.2.22 (i), Corollary 2.3.12 (i) and the Plancherel theorem, the claim follows. □

Note that, by Theorem 2.2.14, the condition that $G^{(2)}$ is supposed to be open is necessary for $A_{f,g}$ to be square-integrable.

2.5. Some examples

In this section we investigate an abundance of examples of locally compact abelian groups with respect to the existence of ambiguity functions and Wigner distributions and properties of them. In particular, for each group G dealt with we give examples of automorphisms $\Phi : G \to H < G$, where H is open. Clearly, H might always be chosen by $H = G$ and Φ by $\Phi = \mathrm{Id}_G$, but these are not the interesting cases. Furthermore, we check whether G is 2-root compact, since this implies that the ambiguity function and Wigner distribution vanish at infinity (compare Theorem 2.2.10 and Theorem 2.3.6). Moreover, we examine whether $G^{(2)}$ is open and $\ker \varphi$ is compact (Clearly, this second condition is always satisfied, if G is 2-root compact). If, in addition, G is σ-compact, then both the ambiguity function and Wigner distribution are square-integrable (compare Theorem 2.2.14 and Corollary 2.3.8). Finally, if all just mentioned properties of G are satisfied and G is second countable, the Wigner distribution coincides with the Plancherel transform of the ambiguity function (compare Theorem 2.4.4).

EXAMPLE 2.5.1. *Compactly generated locally compact abelian Lie groups.*
For applications the most important class of locally compact abelian groups is the class of compactly generated locally compact abelian Lie groups. Let G be a compactly generated locally compact abelian Lie group. Then G is 2-root compact (Remark 2.2.9) and $G^{(2)}$ is open (Remark 2.2.19). Also notice that H is open (Proposition 2.2.2), if it is supposed to be closed.

By the structure theorem for compactly generated locally compact abelian Lie groups, G is of the form $G = \mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^r \times F$, where $p, q, r \in \mathbb{N}_0$, and F is a finite abelian group. An automorphism Φ might, for example, be constructed from automorphism of the groups $\mathbb{R}^p, \mathbb{Z}^q, \mathbb{T}^r$ and F.

For $G = \mathbb{R}^p$ or $\mathbb{T}^r$, an open subgroup H of G always equals G, since G is connected. So all possible topological isomorphisms Φ may be found in [**HR63**, Example 26.18 (h) and (i)].

For $G = \mathbb{Z}^q$, H is an arbitrary subgroup of G. It is easy to check (see also [**HR63**, Example 26.18 (g)]) that each topological isomorphism $\Phi : G \to H$ is given by an element of the discrete group of $q \times q$ matrices A having integer entries and for which $\det A \neq 0$.

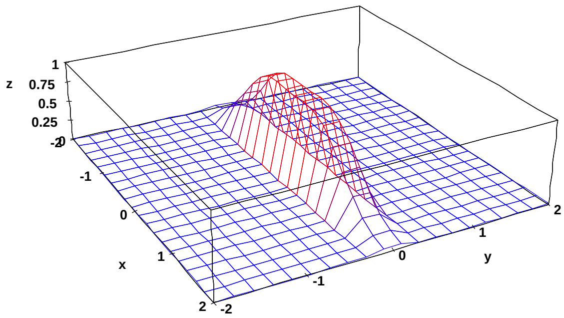
For $G = F$, H always equals G. By [**HR63**, Example 23.27 (d)], G is isomorphic to $\mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_s}$ for integers $m_1, \ldots, m_s$ greater than 1, each of which is a power of a prime. Now let $G = \mathbb{Z}_m$, $m > 1$ and a power of a prime, and let $a, b \in G$ be generators of G. Then $\Phi : G \to G$, $\Phi(a^n) = b^n$, $n \in \mathbb{Z}$, is a topological isomorphism and each of the topological automorphisms of G can be constructed in such a way.

To give a clear picture of how the Wigner distribution works as a time-frequency representation on different groups and with different H and Φ, we consider the cases $G = \mathbb{R}, \mathbb{Z}, \mathbb{T}$ and $\mathbb{Z}_7$.

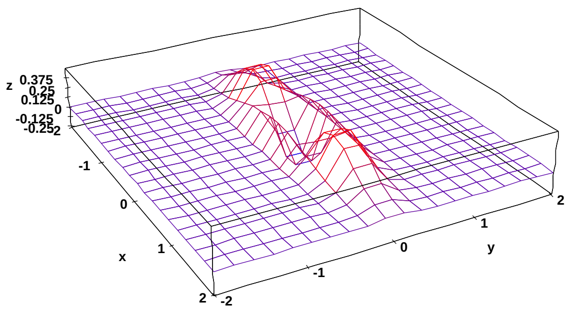
Case $G = \mathbb{R}$.

As mentioned above, all topological isomorphisms from $\mathbb{R}$ onto an open subgroup of $\mathbb{R}$ are of the form $x \mapsto \alpha x$, $\mathbb{R} \to \mathbb{R}$, $\alpha \in \mathbb{R}^*$. To exhibit the influence of α, for each example, we choose $\alpha = \frac{1}{4}, \frac{1}{2}, 1$. Recall that for the classical Wigner distribution α is chosen by $\alpha = \frac{1}{2}$. First, we consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=e^{-\frac{x^2}{2}}\sin(x)$ and calculate W_f (Figure 2.1).

$\alpha=\frac{1}{4}$:



$\alpha=\frac{1}{2}$:



$\alpha=1$:

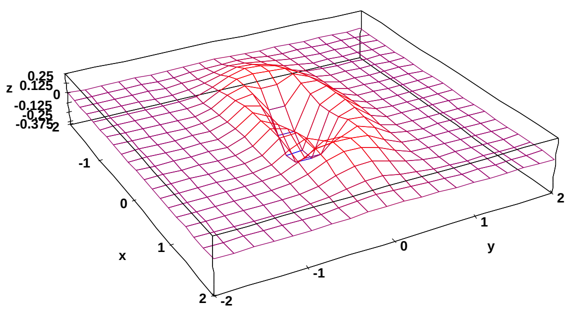
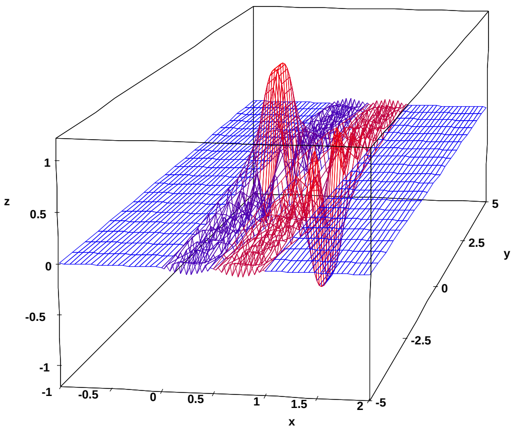


FIGURE 2.1. W_f, where $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^{-\frac{x^2}{2}}\sin(x)$, and $\alpha=\frac{1}{4},\frac{1}{2},1$.

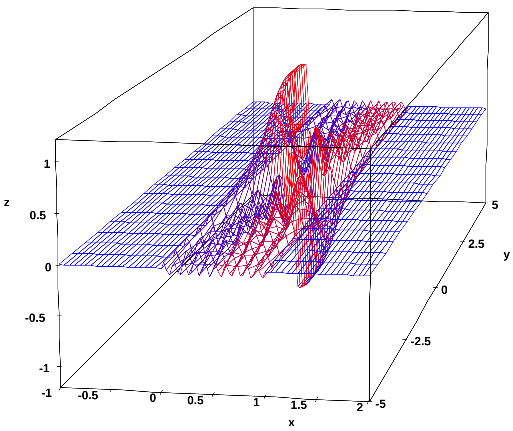
We see that the Wigner distribution is concentrated around the single frequency of the sinusoid for $\alpha=\frac{1}{4},\frac{1}{2},1$, but for $\alpha=1$ it is more spreading.

Secondly, we apply the Wigner distribution to the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \chi_{[0,1)}(x)$ (Figure 2.2).

$\alpha = \frac{1}{4}$:



$\alpha = \frac{1}{2}$:



$\alpha = 1$:

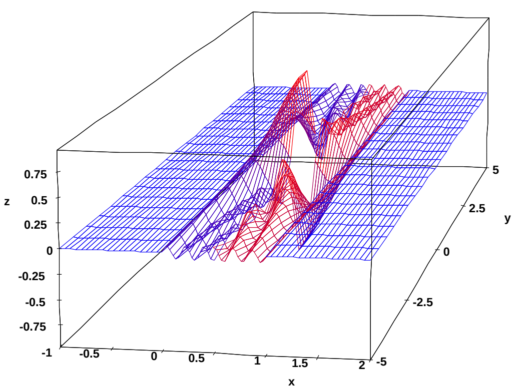


FIGURE 2.2. W_g, where $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \chi_{[0,1)}(x)$, and $\alpha = \frac{1}{4}, \frac{1}{2}, 1$.

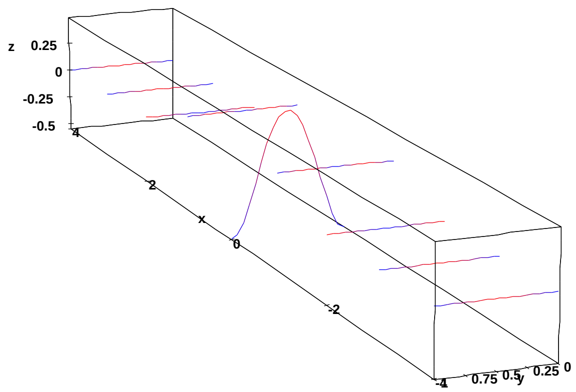
These pictures indicate that certainly the Wigner distribution becomes more smooth on $\mathbb{R}^2 \setminus \{(0,0)\}$ the greater α is, but already for $\alpha = 1$, we almost obtain a singularity at $(0,0)$.

Even these two examples show that depending on the function to analyze the parameter α has to be chosen differently to obtain better results.

Case $G = \mathbb{Z}$.

Now $H = G$ and each topological automorphism of $\mathbb{Z}$ is of the form $n \mapsto mn$, $\mathbb{Z} \to \mathbb{Z}$, $m \in \mathbb{Z}$. To analyze the influence of m, for each example, we choose $m = 1$ and 2. Here $\widehat{\mathbb{Z}} = \mathbb{T}$ is identified with $[0,1)$. For better comparison of the different groups, here we consider again the same function f as before, now defined on $\mathbb{Z}$ (Figure 2.3).

$m = 1$:



$m = 2$:

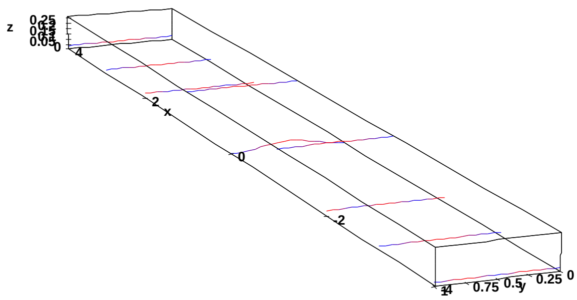


FIGURE 2.3. W_f, where $f : \mathbb{Z} \to \mathbb{R}$, $f(n) = e^{-\frac{n^2}{2}} \sin(n)$, and $m = 1$ and 2.

We see that also the Wigner distribution on $\mathbb{Z}$ detects the single frequency very good. Note that the greater m is the less is the detection of the single frequency. Already for $m = 2$ details at $x = 0$ are difficult to recognize.

The next function for consideration is the function g of the previous example restricted to $\mathbb{Z}$ (Figure 2.4).

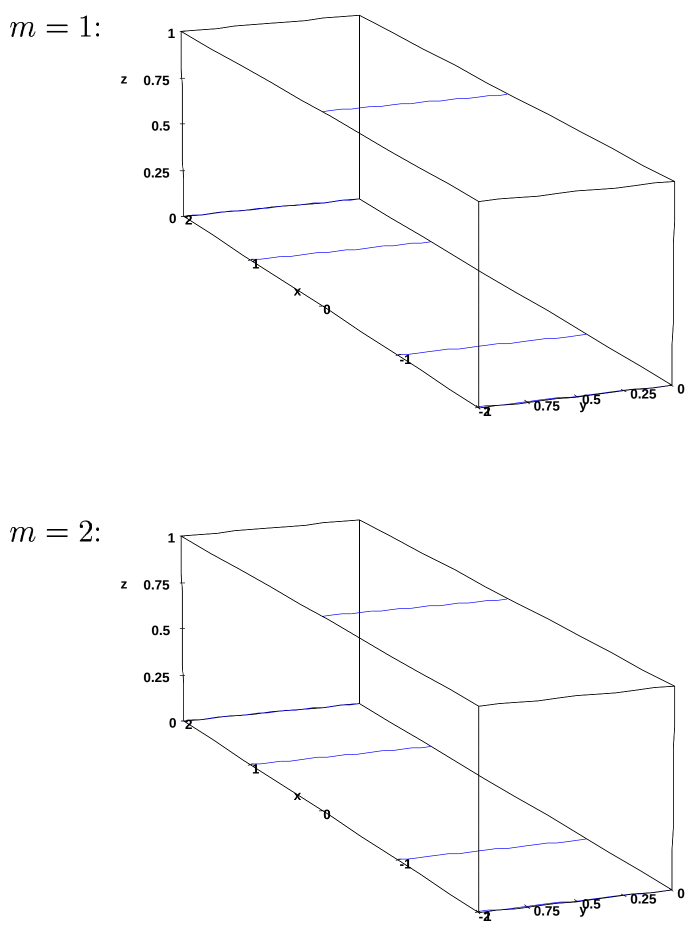


FIGURE 2.4. W_g, where $g : \mathbb{Z} \to \mathbb{R}$, $g(n) = \chi_{\{0\}}(n)$, and $m = 1$ and 2.

The result is as we might expected, since $\mathbb{Z}$ as a discretization of $\mathbb{R}$ is not fine enough to give information about frequencies contained in the signal $x \mapsto \chi_{[0,1)}(x)$, $\mathbb{R} \to \mathbb{R}$. Furthermore, it turns out that W_g does not depend on m.

We see that also for $G = \mathbb{Z}$ the Wigner distribution turns out to be a helpful tool for analyzing functions but not for all functions.

Case $G = \mathbb{T}$.

Here $H = G$ and $\Phi(z) = z$ or $\Phi(z) = z^{-1}$, $\Phi : \mathbb{T} \to \mathbb{T}$. Notice that the Wigner distribution W_h, $h \in L^2(\mathbb{T})$, with respect to both topological automorphisms of $\mathbb{T}$ coincide, if h is real-valued. Here $\mathbb{T}$ is identified with $[0, 1)$. First, we consider the function $f : \mathbb{T} \to \mathbb{R}$, $f(z) = \sin(2\pi z)$ (Figure 2.5).

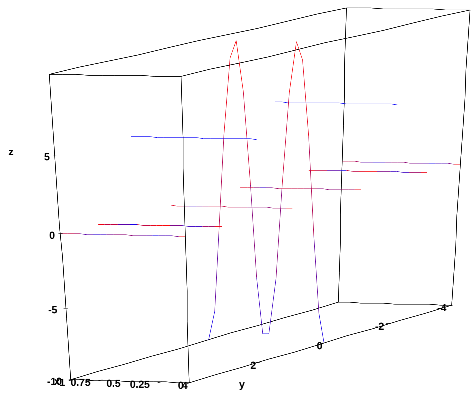


FIGURE 2.5. W_f, where $f : \mathbb{T} \to \mathbb{R}$, $f(z) = \sin(2\pi z)$.

As in the previous two cases the Wigner distribution is concentrated around the single frequency. If we compare this picture with Figure 2.3, we immediately see that they look indeed quite similar.

The Wigner distribution of $g : \mathbb{T} \to \mathbb{R}$, $g(z) = \chi_{[0,\frac{1}{2})}(z)$, yields a totally different picture than Figure 2.4 (Figure 2.6).

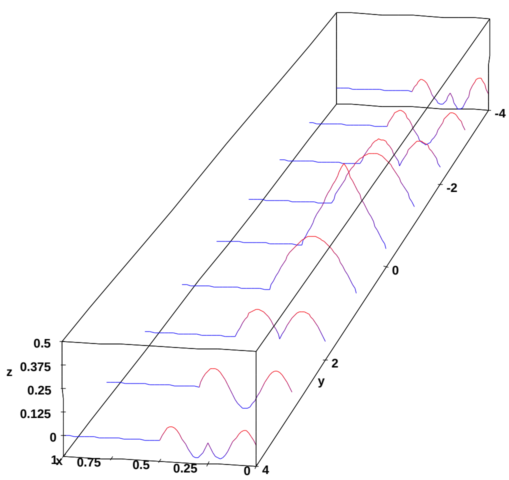


FIGURE 2.6. W_g, where $g : \mathbb{T} \to \mathbb{R}$, $g(z) = \chi_{[0,\frac{1}{2})}(z)$.

Here the different frequencies contained in the function are good indicated.

Hence also in this case the Wigner distribution seems to be a useful time-frequency representation.

Case $G = \mathbb{Z}_7$. Here we choose $\Phi : \mathbb{Z}_7 \to \mathbb{Z}_7$ by $\Phi(n) = \frac{n}{2}$. As a first function to analyze we choose in analogy to $G = \mathbb{T}$, the function $f : \mathbb{Z}_7 \to \mathbb{R}$, $f(n) = \sin(2\pi\frac{n}{7})$. We obtain Figure 2.7.

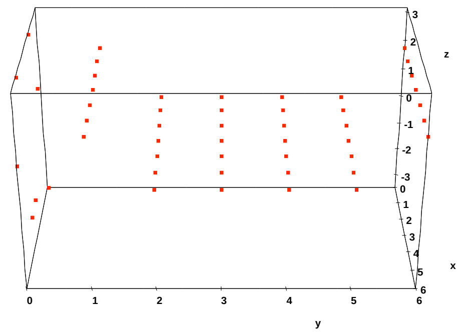


FIGURE 2.7. W_f, where $f : \mathbb{Z}_7 \to \mathbb{R}$, $f(n) = \sin(2\pi\frac{n}{7})$.

This seems to be a discrete version of Figure 2.5. Here the Wigner distribution also detects the single frequency of the sinusoid in a very good way.

Secondly, we consider the function $g : \mathbb{Z}_7 \to \mathbb{R}$, $g(n) = \chi_{\{3\}}(n)$ (Figure 2.8).

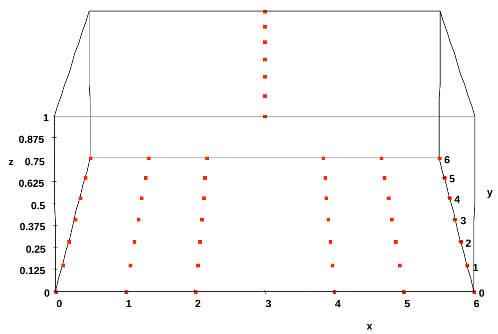


FIGURE 2.8. W_g, where $g : \mathbb{Z}_7 \to \mathbb{R}$, $g(n) = \chi_{\{3\}}(n)$.

This is a discrete version of Figure 2.4. Here we have the same problem. Concluding the Wigner distribution for this finite group can help analyzing functions but regarded as a discretization it is often not fine enough.

EXAMPLE 2.5.2. *Discrete abelian groups.*

Let G be a discrete abelian group. Clearly, H is always open. It is easy to check that G is 2-root compact if and only if $|\varphi^{-1}(e)| < \infty$. This is equivalent to the property that $\ker \varphi$ is compact. Note that $G^{(2)}$ is always open.

In this context we mention two special examples. First, we consider $G = \mathbb{Z}$. Here each mentioned property is satisfied (compare Example 2.5.1).

Secondly, let G be an infinite group where the order of each element except the neutral element equals 2. Let G be endowed with the discrete topology. There exists an automorphism Φ, because Φ may always be chosen by $\Phi = \mathrm{Id}_G$. Note that this discrete group is not 2-root compact.

EXAMPLE 2.5.3. *Compact abelian groups.*

Let G be a compact abelian group. In this case G is always 2-root compact and $G^{(2)}$ is closed.

Recall that $\mathbb{T}^{(2)}$ is open (Example 2.5.1). But if we choose $G = \prod_{i=1}^{\infty} \mathbb{Z}_2$, the subgroup $G^{(2)}$ is not open but only closed (Remark 2.2.21). Note that for this group we could choose a subgroup H of G which is not open but topological isomorphic to G (compare Remark 2.2.3). A more suitable choice of H and Φ might be the following. Let $H := G$ and define $\Phi : G \to H$ by $\Phi((x_n)_{n\in\mathbb{N}}) = (y_n)_{n\in\mathbb{N}}$, where $y_n := x_{n-1}$ for $n \geq 2$ and $y_1 := 0$.

EXAMPLE 2.5.4. *p-adic group, p prime.*

Let p be a prime and let G be the p-adic group. For each integer k, let Λ_k be the set of all $(x_n)_{n\in\mathbb{Z}} \in G$ such that $x_n = 0$ for all $n < k$. By [**HR63**, 10.16 (a)], the only proper closed subgroups of G are the subgroups Λ_k. All subgroups Λ_k are even open by the definition of the topology of G. Hence the only possible choice for H is $H := G$ or $H := \Lambda_k$, $k \in \mathbb{Z}$. But since the subgroups Λ_k are compact and G is non-compact, Λ_k is not topological isomorphic to G for each $k \in \mathbb{Z}$. Thus H has to be chosen to equal G. Then all possible automorphisms Φ may be found in [**HR63**, Example 26.18 (d)].

Next we are going to prove that $G^{(2)} = G$. Then we have shown that $G^{(2)}$ is open. For this, let $(y_n)_{n\in\mathbb{Z}} \in G$. We will construct some element $(x_n)_{n\in\mathbb{Z}} \in G$ such that $(x_n)_{n\in\mathbb{Z}} + (x_n)_{n\in\mathbb{Z}} = (y_n)_{n\in\mathbb{Z}}$. Let $n_0 \in \mathbb{Z}$ be such that $y_n = 0$ for all $n < n_0$ and $y_{n_0} \neq 0$. We set $x_n := 0$ for all $n < n_0$ and define x_{n_0} as follows.

Case y_{n_0} is even.

Put $x_{n_0} = \frac{1}{2} y_{n_0}$. Then $2x_{n_0} = y_{n_0} + t_{n_0} p$ with $t_{n_0} = 0$.

Case y_{n_0} is odd.

Put $x_{n_0} = \frac{1}{2}(y_{n_0} + p)$. Then $2x_{n_0} = y_{n_0} + t_{n_0} p$ with $t_{n_0} = 1$.

Suppose that $x_{n_0}, x_{n_0+1}, \dots, x_k$ and $t_{n_0}, t_{n_0+1}, \dots, t_k$, $k \geq n_0$ have been defined such that

$$2x_n + t_{n-1} = y_n + t_n p \quad \text{for all } n_0 \leq n \leq k \text{ and with } t_{n_0-1} := 0.$$

Now we construct x_{k+1} and t_{k+1} such that $2x_{k+1} + t_k = y_{k+1} + t_{k+1} p$.

Case y_{k+1} is even and $t_k = 0$.
Put $x_{k+1} = \frac{1}{2} y_{k+1}$ and $t_{k+1} = 0$.

Case y_{k+1} is even and $t_k = 1$.
Put $x_{k+1} = \frac{1}{2}(y_{k+1} + p - 1)$ and $t_{k+1} = 1$.

Case y_{k+1} is odd and $t_k = 0$.
Put $x_{k+1} = \frac{1}{2}(y_{k+1} + p)$ and $t_{k+1} = 1$.

Case y_{k+1} is odd and $t_k = 1$.
Put $x_{k+1} = \frac{1}{2}(y_{k+1} - 1)$ and $t_{k+1} = 0$.

By induction, $(x_n)_{n\in\mathbb{Z}}$ is defined and satisfies $2(x_n)_{n\in\mathbb{Z}} = (y_n)_{n\in\mathbb{Z}}$. Thus the claim is proven.

G is also 2-root compact. We prove this by showing that in this case the continuous homomorphism φ is a topological isomorphism. It is easily seen that the element $(x_n)_{n\in\mathbb{Z}}$ constructed above is uniquely determined. Hence φ is bijective. It remains to prove that φ is open. The proof of $G^{(2)} = G$ implies that $\varphi(\Lambda_k) = \Lambda_k$ for all $k \in \mathbb{Z}$. Hence, by the definition of the topology of G, φ is open.

CHAPTER 3

The Zak transform

This chapter focuses on the generalization of the Zak transform to locally compact groups and important properties of it.

The first section deals with the abelian case. We prove that the Zak transform is a Hilbert space isomorphism. Further, we investigate in which cases a Zak transform has a zero. Finally, several applications are exhibited.

A definition of the Zak transform for a large class of locally compact groups, which includes, for example, all connected and simply connected 2-step nilpotent Lie groups, is given in the second section. Moreover, several conditions are established under which the Zak transform is a Hilbert space isomorphism. In the last part of this section we present some classes of groups which satisfy these conditions.

3.1. The abelian case

In this section we focus on locally compact abelian groups and their Zak transforms. Partly, this section, especially Theorem 3.1.9, is joint work with Kaniuth [**KK98**].

3.1.1. Definition and some properties. For $f \in L^2(\mathbb{R})$, the Zak transform Zf is the function on $\mathbb{R} \times \mathbb{R}$ defined by

$$Zf(x,y) = \sum_{k \in \mathbb{Z}} f(x+k) e^{2\pi i y k}.$$

A striking property of the Zak transform is that the so-called quasi-periodicity relation is satisfied, that means, for all $(x,y) \in \mathbb{R} \times \mathbb{R}$ and $(l,m) \in \mathbb{Z} \times \mathbb{Z}$, we have

$$Zf(x+l, y+m) = e^{-2\pi i y l} Zf(x,y).$$

Thus the function Zf is uniquely determined by its values on $[0,1) \times [0,1)$. Now it is well-known that $Z(L^2(\mathbb{R}))|_{[0,1)^2} \subseteq L^2([0,1)^2)$. In particular, this implies that $Zf(x,y)$ is defined for almost all $(x,y) \in \mathbb{R} \times \mathbb{R}$. Furthermore, the mapping $Z : L^2(\mathbb{R}) \to L^2([0,1)^2)$ is a Hilbert space isomorphism. As a general reference for properties of the Zak transform on $\mathbb{R}$ we mention [**Jan88**].

Now the question arises of how to generalize the definition of the classical Zak transform to functions in $L^2(G)$, where G is a locally compact abelian group. At first, note that the subgroup $\mathbb{Z}$ is a uniform lattice in $\mathbb{R}$. Thus for a locally compact abelian group G it seems natural to replace $\mathbb{Z}$ by a uniform lattice K in G. Hence the notion of Zak transform admits a natural generalization to locally compact abelian groups having uniform lattices (Definition 3.1.4). The map

$$(x,\omega) \mapsto \sum_{k\in K} f(xk)\omega(k), \quad G \times \widehat{G} \to \mathbb{C}, \tag{6}$$

is a natural candidate for the Zak transform of a function $f \in L^2(G)$. But as in the case $G = \mathbb{R}$ we have to check the convergence of the series.

For this, we need an analogue of the set $[0,1) \times [0,1) \subseteq \mathbb{R} \times \mathbb{R}$. Note that $[0,1)$ is a relatively compact Borel set such that every $x \in \mathbb{R}$ can be uniquely written in the form $x = sk$ where $s \in [0,1)$ and $k \in \mathbb{Z}$. A canonical generalization is the following.

Definition 3.1.1. Let G be a locally compact abelian group and let K be a uniform lattice in G. Then a *fundamental domain* for K is a Borel subset S_K of G such that every $x \in G$ can be uniquely written in the form $x = sk$ where $s \in S_K$ and $k \in K$.

The first step is to guarantee the existence of a fundamental domain for K.

Lemma 3.1.2. *Let G be a locally compact abelian group and let K be a uniform lattice in G. Then there exists a relatively compact fundamental domain for K.*

Proof. We assume first that G is compactly generated. Since G is a projective limit of second countable groups [**MZ55**, p.175] and K is discrete, there exists a compact subgroup C of G such that $C \cap K = \{e\}$ and G/C is second countable. By [**Mac52**, Lemma 1.1], there exists a relatively compact fundamental domain Q for KC/C in G/C. Let $q : G \to G/C$ denote the quotient homomorphism, and set $S = q^{-1}(Q)$. Clearly, S is a relatively compact Borel set, and, using the fact that $K \cap C = \{e\}$, it is easy to check that S is indeed a fundamental domain for K.

Now drop the assumption that G is compactly generated and choose an open compactly generated subgroup H of G. Since $K \cap H$ is a uniform lattice in H, by the preceding paragraph, there exists a relatively compact fundamental domain S for $K \cap H$ in H. As H is open and G/K is compact, KH has finite index in G. Let F be a coset representative system for KH in G, and let $T = FS$. Then T is a relatively compact Borel set, and as above it is straightforward to verify that T is a fundamental domain for K in G. □

Furthermore, note that $\mathbb{Z} = \{y \in \mathbb{R} : e^{2\pi ixy} = 1 \text{ for all } x \in \mathbb{Z}\} = A(\mathbb{Z}, \widehat{\mathbb{R}}) < \widehat{\mathbb{R}} = \mathbb{R}$ (compare Remark 1.0.1). Hence concerning the second component $[0,1)$ is also a relatively compact fundamental domain for $A(\mathbb{Z}, \widehat{\mathbb{R}})$.

Now let G be a locally compact abelian group and let K be a uniform lattice in G. By Remark 1.0.2, $A(K, \widehat{G})$ is also a uniform lattice in $\widehat{G}$ and, by Lemma 3.1.2, there exist relatively compact fundamental domains S_K for K

in G and Ω_K for $A(K,\widehat{G})$ in $\widehat{G}$. Let the Haar measure on G be normalized so that Weil's formula holds, when we take on G/K the normalized Haar measure and the counting measure on K. Clearly, if G is σ-compact (equivalently, K is countable) then S_K has positive measure, ($|S_K| > 0$). However, this is also true in the general case. To see this, choose a compactly generated open subgroup H of G containing S_K and observe that $S_K k \cap H \neq \emptyset$ if and only if $k \in H$. Since H is σ-compact and K is discrete, there are only countably many $k \in K \cap H$. Thus H is contained in a countable union of sets $S_K k$, $k \in K$, whence $|S_K| > 0$.

The map $\Phi : S_K \to G/K$, $x \mapsto xK$, is a continuous bijection. For each measurable subset M of S_K, Weil's formula gives

$$|M| = \int_G \chi_M(x)\,dx = \int_{G/K} \left(\sum_{k\in K} \chi_M(xk)\right) d(xK) = |\Phi(M)|.$$

Hence Φ maps the measure on S_K induced by the Haar measure on G to the normalized Haar measure on G/K.

Similarly, normalizing the Haar measures on $\widehat{G}$ and $\widehat{G}/A(K,\widehat{G})$ appropriately, the mapping $\Omega_K \to \widehat{G}/A(K,\widehat{G})$, $\omega \mapsto \omega A(K,\widehat{G})$, transforms the induced measure on Ω_K into the Haar measure on $\widehat{G}/A(K,\widehat{G})$, and $|\Omega_K| = 1$.

The next lemma will show that, for an arbitrary locally compact abelian group, the Zak transform can be defined as in (6). Furthermore, it proves that then the Zak transform $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ is an isometry. The proof follows mostly the steps of the proof for $G = \mathbb{R}$ ([**BF94**, Theorem 3.15]).

LEMMA 3.1.3. *Let G be a locally compact abelian group, let K be a uniform lattice in G and let $f \in L^2(G)$. Furthermore, let S_K and Ω_K be relatively compact fundamental domains for K in G and for $A(K,\widehat{G})$ in $\widehat{G}$, respectively. Then, for almost all $(x,\omega) \in S_K \times \Omega_K$,*

$$Zf(x,\omega) = \sum_{k\in K} f(xk)\omega(k)$$

converges, and the function Zf belongs to $L^2(S_K \times \Omega_K)$ and satisfies $\|Zf\|_2 = \|f\|_2$.

PROOF. For $k \in K$, define $f_k \in L^2(S_K \times \Omega_K)$ by $f_k(x,\omega) := f(xk)\omega(k)$. Then

$$\sum_{k\in K} \|f_k\|_2^2 = \sum_{k\in K} \int_{S_K}\int_{\Omega_K} |f_k(x,\omega)|^2\,d\omega\,dx$$

$$= \sum_{k\in K} \int_{S_K} |f(xk)|^2\,dx = \|f\|_2^2.$$

Now let $k,l \in K$ such that $k \neq l$. We claim that $\langle f_k, f_l\rangle = 0$. To show this recall that if C is a compact abelian group and φ a non-trivial character of C, then $\int_C \varphi(y)\, dy = 0$ [**HR63**, Lemma 23.19]. Applying this to $C = \widehat{G}/A(K, \widehat{G})$ and the character φ defined by

$$\varphi(\omega A(K, \widehat{G})) = \omega(kl^{-1}), \quad \omega \in \widehat{G},$$

we obtain

$$\int\limits_{\Omega_K} \omega(kl^{-1})\, d\omega = \int\limits_{\widehat{G}/A(K,\widehat{G})} \varphi(\omega A(K, \widehat{G}))\, d(\omega A(K, \widehat{G})) = 0,$$

and this in turn implies

$$\langle f_k, f_l \rangle = \int\limits_{S_K} \int\limits_{\Omega_K} f(xk)\overline{f(xl)}\omega(kl^{-1})\, d\omega\, dx = 0.$$

It follows that the series $\sum_{k\in K} f_k$ converges in $L^2(S_K \times \Omega_K)$ and satisfies

$$\|\sum_{k\in K} f_k\|_2^2 = \sum_{k\in K} \|f_k\|_2^2 = \|f\|_2^2.$$

In particular, $Zf(x, \omega)$ exists for almost all $(x, \omega) \in S_K \times \Omega_K$. □

Now we can define the Zak transform Zf for $f \in L^2(G)$. Notice first that, for every $(k, \gamma) \in K \times A(K, \widehat{G})$ and any finite subset H of K,

$$\sum_{h\in H} f(xkh)(\omega\gamma)(h) = \overline{\omega(k)} \sum_{l\in kH} f(xl)\omega(l).$$

Thus $Zf(xk, \omega\gamma)$ converges if and only if $Zf(x, \omega)$ does. It follows from Lemma 3.1.3 that

$$Zf(x, \omega) = \sum_{k\in K} f(xk)\omega(k)$$

is defined for locally almost all $(x, \omega) \in G \times \widehat{G}$ (and, in fact, for almost all (x, ω), if G is σ-compact).

DEFINITION 3.1.4. Let G be a locally compact abelian group and let K be a uniform lattice in G. The *Zak transform associated with* K of $f \in L^2(G)$ is defined on $G \times \widehat{G}$ by

$$Zf(x, \omega) = \sum_{k\in K} f(xk)\omega(k).$$

Now the question arises whether each locally compact abelian group contains a uniform lattice.

REMARK 3.1.5. In general, a locally compact abelian group G need not contain a uniform lattice. The following example was kindly communicated by the referee of [**KK98**].

Suppose G is the group $\prod_{i=1}^{\infty} \mathbb{Z}_4$ with the topology obtained when the subgroup C generated by all elements of order 2 is declared to be open and compact. Then every discrete subgroup K of G has to be finite. Indeed, $K \cap C$ is finite and $x \mapsto x^2$ is a homomorphism from K into $K \cap C$ with kernel $K \cap C$.

However, if G is of the form $G = \mathbb{R}^p \times D \times C$, where D is discrete and C is compact, then we can take $K = \mathbb{Z}^p \times D$. More specifically, if G is compactly generated, say $G = \mathbb{R}^p \times \mathbb{Z}^q \times C$, then an abundance of uniform lattices can be constructed as follows. Let h_1 be a homomorphism of $\mathbb{Z}^p \subseteq \mathbb{R}^p$ into C and let h_2 and h_3 be homomorphisms of $\mathbb{Z}^q$ into $\mathbb{R}^p$ and C, respectively. Then

$$K = \{(x_1 + h_2(x_2), x_2, h_1(x_1) + h_3(x_2)) : x_1 \in \mathbb{Z}^p, x_2 \in \mathbb{Z}^q\}$$

is a uniform lattice in G.

Also in the general case the Zak transform satisfies a quasi-periodicity relation. The proof is a straightforward calculation.

LEMMA 3.1.6. *Let G be a locally compact abelian group, let K be a uniform lattice in G and let $f \in L^2(G)$. Then, for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times A(K, \widehat{G})$,*

$$Zf(xk, \omega\gamma) = \overline{\omega(k)} Zf(x, \omega).$$

PROOF. For $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times A(K, \widehat{G})$, we have

$$Zf(xk, \omega\gamma) = \sum_{h \in K} f(xkh)(\omega\gamma)(h) = \sum_{h \in K} f(xkh)\omega(h) = \overline{\omega(k)} Zf(x, \omega).$$

□

In particular, Lemma 3.1.6 shows, that the function Zf is uniquely determined by its values on the fundamental domain $S_K \times \Omega_K$. Finally, we show that, for all second countable locally compact abelian groups G having uniform lattices, the map $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ is even a Hilbert space isomorphism as in the case $G = \mathbb{R}$. The proof is a straightforward generalization of the proof for $G = \mathbb{R}$ ([**BF94**, Theorem 3.15]). In order to complete the picture, we give the proof here.

THEOREM 3.1.7. *Let G be a second countable locally compact abelian group, let K be a uniform lattice in G and let Z denote the associated Zak transform. Furthermore, let S_K and Ω_K be relatively compact fundamental domains for K in G and for $A(K, \widehat{G})$ in $\widehat{G}$, respectively. Then*

$$Z : L^2(G) \to L^2(S_K \times \Omega_K)$$

is a Hilbert space isomorphism.

PROOF. By Lemma 3.1.3, $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ is an isometry. Obviously, Z is also linear. Hence it remains to show that $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ is also surjective.

Consider the sets $M_1 \subseteq L^2(G)$ and $M_2 \subseteq L^2(S_K \times \Omega_K)$ defined by

$$M_1 := \{\gamma \cdot L_k \chi_{S_K} : (k, \gamma) \in K \times A(K, \widehat{G})\}$$

and

$$M_2 := \{\gamma \cdot \widehat{k} : (k,\gamma) \in K \times A(K,\widehat{G})\}.$$

Note that, for all $f \in L^2(G)$ and $(x,\omega) \in S_K \times \Omega_K$, $(k,\gamma) \in K \times A(K,\widehat{G})$, we obtain

$$\begin{aligned} Z(\gamma \cdot L_k \chi_{S_K})(x,\omega) &= \sum_{l \in K} \omega(l)\gamma(xl)\chi_{S_K}(xlk^{-1}) \\ &= \gamma(x) \sum_{l \in K} \omega(l)\chi_{S_K}(xlk^{-1}) \\ &= \gamma(x)\omega(k) \sum_{l \in K} \omega(l)\chi_{S_K}(xl) \\ &= \gamma \cdot \widehat{k}(x,\omega). \end{aligned}$$

It remains to prove that M_2 is an orthonormal base of $L^2(S_K \times \Omega_K)$.

Using [**HR63**, Lemma 23.19], for all $(k_1,\gamma_1), (k_2,\gamma_2) \in K \times A(K,\widehat{G})$,

$$\begin{aligned} \langle \gamma_1 \cdot \widehat{k_1}, \gamma_2 \cdot \widehat{k_2} \rangle &= \int_{S_K} (\gamma_1 \overline{\gamma_2})(x)\, dx \int_{\Omega_K} \omega(k_1 k_2^{-1})\, d\omega \\ &= \delta_{\gamma_1,\gamma_2}\delta_{k_1,k_2}. \end{aligned}$$

By [**Mac52**, Theorem 1.1] and [**Mac57**, Theorem 4.2], the canonical mappings $S_K \to G/K$ and $\Omega_K \to \widehat{G}/A(K,\widehat{G})$ are Borel isomorphisms (that is, S_K and Ω_K arise from Borel cross-sections $G/K \to G$ and $\widehat{G}/A(K,\widehat{G}) \to \widehat{G}$), since the existence of such cross-sections is guaranteed when G (and hence $\widehat{G}$) is second countable. Hence they induce Hilbert space isomorphisms $L^2(S_K) \to L^2(G/K)$ and $L^2(\Omega_K) \to L^2(\widehat{G}/A(K,\widehat{G}))$. Now $\{\gamma : \gamma \in A(K,\widehat{G})\}$ is a complete subset of $L^2(G/K) = L^2(S_K)$ and $\{\widehat{k} : k \in K\}$ is a complete subset of $L^2(\widehat{G}/A(K,\widehat{G})) = L^2(\Omega_K)$. Moreover, M_2 is an orthonormal system in $L^2(S_K \times \Omega_K)$. Thus M_2 is an orthonormal basis of $L^2(S_K \times \Omega_K)$. This finishes the proof. □

3.1.2. Zeros of the Zak transform. A striking property of the Zak transform, independently shown by Zak [**BZ81**] and Janssen [**Jan82**], is that Zf has a zero whenever Zf is continuous on $\mathbb{R} \times \mathbb{R}$. Actually, in certain special cases like f the Gaussian, this follows from elementary properties of theta series.

In the first part of this subsection we investigate to what extent this result can be generalized to locally compact abelian groups. It will turn out, that this is indeed the case for all compactly generated locally compact abelian groups with non-compact connected component of the identity (Theorem 3.1.9). The proof of the theorem only uses the quasi-periodicity relation of the Zak transform. Therefore, this theorem will be first formulated for functions satisfying the quasi-periodicity relation. Then the generalization of Zak's and Janssen's result follows as a corollary.

The second part contains some remarks about the corollary. Moreover, an application of Corollary 3.1.11 to generalized Gabor systems is shown. Using

Corollary 3.1.11 in the general situation, one obtains more information about a Gabor system by means of the Zak transform of the associated function.

3.1.2.1. *The main theorem.* Let G be a locally compact abelian group. As mentioned above, we first focus on functions $f \in L^2(G)$ satisfying the quasi-periodicity relation. The following lemma is required in the proof of the theorem.

LEMMA 3.1.8. *Let $\mathcal{H}$ be a downwards directed system of compact subgroups of G (with normalized Haar measures) such that $\bigcap_{H\in\mathcal{H}} H = \{e\}$. Let g be a continuous function on $G \times \widehat{G}$ such that*

$$g(xk, \omega\gamma) = \overline{\omega(k)}g(x, \omega)$$

for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times A(K, \widehat{G})$. For each $H \in \mathcal{H}$, define g_H on $G \times \widehat{G}$ by

$$g_H(x, \omega) = \int\limits_H g(xh, \omega)\, dh.$$

Then g_H is continuous and satisfies $g_H(xk, \omega\gamma) = \overline{\omega(k)}g_H(x, \omega)$. If every g_H has a zero, then g has a zero.

PROOF. That g_H is continuous follows immediately from the uniform continuity of g on compact subsets of $G \times \widehat{G}$. Moreover, for $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times A(K, \widehat{G})$,

$$g_H(xk, \omega\gamma) = \int\limits_H g(xkh, \omega)\, dh = \overline{\omega(k)} \int\limits_H g(xh, \omega)\, dh = \overline{\omega(k)}g_H(x, \omega).$$

Now suppose that every g_H has a zero. Since G/K and $\widehat{G}/A(K, \widehat{G})$ are compact, there exist compact subsets C of G and Δ of $\widehat{G}$ such that $G = CK$ and $\widehat{G} = \Delta A(K, \widehat{G})$. Due to the quasi-periodicity, for each $H \in \mathcal{H}$, there exist $x_H \in C$ and $\omega_H \in \Delta$ such that $g_H(x_H, \omega_H) = 0$. C and Δ being compact, by passing to a subnet if necessary, we can assume that $x_H \to x$ and $\omega_H \to \omega$ for some $x \in C$ and $\omega \in \Delta$. Finally, employing the uniform continuity of g on compact sets once more, we obtain that

$$|g(x, \omega)| = |g(x, \omega) - g_H(x_H, \omega_H)|$$

$$= \Big| \int\limits_H \Big(g(x, \omega) - g(x_H h, \omega_H)\Big)\, dh\Big| \leq \int\limits_H |g(x, \omega) - g(x_H h, \omega_H)|\, dh,$$

which converges to zero as $H \to \{e\}$. □

Now the following theorem holds.

THEOREM 3.1.9. *Let G be a compactly generated locally compact abelian group with non-compact connected component of the identity and let K be a uniform lattice in G. Suppose that $g : G \times \widehat{G} \to \mathbb{C}$ is a continuous function satisfying the quasi-periodicity relation*

$$g(xk, \omega\gamma) = \overline{\omega(k)}g(x, \omega)$$

for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times A(K, \widehat{G})$. Then g has a zero.

PROOF. Notice first that, by the structure theorem for compactly generated locally compact abelian groups [**HR63**, Theorem 9.8], G is of the form $G = \mathbb{R}^p \times \mathbb{Z}^q \times C$, where C is a compact group and $p \geq 1$, since, by hypothesis, G has a non-compact connected component of the identity. Now compact groups are projective limits of Lie groups [**MZ55**, p.175]. Therefore, there exists a system $\mathcal{H}$ of closed subgroups H of C as in Lemma 3.1.8 such that C/H is a Lie group for every $H \in \mathcal{H}$. Thus, for each $H \in \mathcal{H}$, there is a closed subgroup L_H of C such that $H \subseteq L_H, L_H$ is of finite index in C and $L_H/H = \mathbb{T}^{r_H}$ for some $r_H \in \mathbb{N}_0$.

By Lemma 3.1.8, for any such H, g_H is continuous, and once we have established that g_H has a zero on $G \times \widehat{G}$, it follows that g has a zero as well. To that end, fix H and set $L = L_H$ and $r = r_H$. Replacing g_H by g, we can therefore assume that g is constant on cosets of H. Let $\pi : G \to G/H$ denote the quotient homomorphism. Then $\pi(K) = KH/H$ is a uniform lattice in G/H, and

$$A(\pi(K), \widehat{G/H}) = \{\chi \in \widehat{G/H} : \chi \circ \pi \in A(K, \widehat{G})\}.$$

Now the function $\tilde{g} : G/H \times \widehat{G/H} \to \mathbb{C}$ defined by

$$\tilde{g}(\pi(x), \chi) = g(x, \chi \circ \pi),$$

$x \in G$, $\chi \in \widehat{G/H}$, is continuous and satisfies the equation

$$\tilde{g}(\pi(x)\pi(k), \chi\,\delta) = \tilde{g}(\pi(x), \chi)\overline{(\chi \circ \pi)(k)}$$

for all $x \in G$, $k \in K$, $\chi \in \widehat{G/H}$ and $\delta \in A(\pi(K), \widehat{G/H})$. It suffices to show that $\tilde{g}$ has a zero. Thus, after moving to G/H, we can assume that $L = \mathbb{T}^r$. Towards a contradiction, suppose that $g(x, \omega) \neq 0$ for all $(x, \omega) \in G \times \widehat{G}$.

When convenient, we shall identify $\mathbb{R}^p$ with $\widehat{\mathbb{R}^p}$ (compare Remark 1.0.1). Let $e_r : \mathbb{R}^r \to \mathbb{T}^r$ be the covering homomorphism given by

$$e_r(u) = (e^{2\pi i u_1}, \ldots, e^{2\pi i u_r})$$

for $u \in \mathbb{R}^r$. We define homomorphisms

$$\varphi_1 : \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^p \times \{0\} \times \mathbb{T}^r \subseteq G, \quad (x_1, u) \mapsto (x_1, 0, e_r(u))$$

and

$$\varphi_2 : \widehat{\mathbb{R}^p} \times \widehat{\mathbb{R}^q} \to \widehat{\mathbb{R}^p} \times \widehat{\mathbb{Z}^q} \times \{1_C\} \subseteq \widehat{G}, \quad (\omega_1, \chi) \mapsto (\omega_1, \chi|_{\mathbb{Z}^q}, 1_C).$$

Since g is continuous and has no zero on $G\times\widehat{G}$, we can consider the continuous function

$$(x_1,u,\omega_1,\chi)\mapsto \frac{g(\varphi_1(x_1,u),\varphi_2(\omega_1,\chi))}{|g(\varphi_1(x_1,u),\varphi_2(\omega_1,\chi))|}$$

on $S=\mathbb{R}^p\times\mathbb{R}^r\times\widehat{\mathbb{R}^p}\times\widehat{\mathbb{R}^q}$. S being simply connected, there exists a continuous function $\varphi: S\to\mathbb{R}$ such that

$$\exp 2\pi i\varphi(x,\omega)=\frac{g(\varphi_1(x),\varphi_2(\omega))}{|g(\varphi_1(x),\varphi_2(\omega))|}$$

for all $x\in\mathbb{R}^p\times\mathbb{R}^r$ and $\omega\in\widehat{\mathbb{R}^p}\times\widehat{\mathbb{R}^q}$. Since φ_1 and φ_2 are homomorphisms, the quasi-periodicity relation for g implies that

$$\exp 2\pi i[\varphi(xk,\omega)-\varphi(x,\omega)]=\overline{\varphi_2(\omega)(\varphi_1(k))}$$

$$=\overline{\omega_1(k_1)}=\exp(-2\pi i\langle\omega_1,k_1\rangle)$$

for all $(x,\omega)\in S$ and $k\in\varphi_1^{-1}(K)$, and

$$\exp 2\pi i[\varphi(x,\omega\gamma)-\varphi(x,\omega)]=1$$

for all $(x,\omega)\in S$ and $\gamma\in\varphi_2^{-1}(A(K,\widehat{G}))$.

Since S is connected and φ is continuous, it follows that given k and γ, there are integers $m_1(k)$ and $m_2(\gamma)$ such that

(1) $\varphi(xk,\omega)-\varphi(x,\omega)+\langle k_1,\omega_1\rangle\ =m_1(k)$

and

(2) $\varphi(x,\omega\gamma)-\varphi(x,\omega)=m_2(\gamma)$

for all $x\in\mathbb{R}^p\times\mathbb{R}^r$ and $\omega\in\widehat{\mathbb{R}^p}\times\widehat{\mathbb{R}^q}$. Applying (1) first and then (2) yields

$$\begin{aligned}\varphi(xk,\omega\gamma) &= \varphi(x,\omega\gamma)-\langle\, k_1,\omega_1\rangle\ -\ \langle k_1,\gamma_1\rangle\ +\ m_1(k)\\ &= \varphi(x,\omega)+m_2(\gamma)-\ \langle k_1,\omega_1\rangle\ -\ \langle k_1,\gamma_1\rangle\ +m_1(k).\end{aligned}$$

On the other hand, applying (1) and (2) in the reverse order gives

$$\begin{aligned}\varphi(xk,\omega\gamma) &= \varphi(xk,\omega)+m_2(\gamma)\\ &= \varphi(x,\omega)-\ \langle k_1,\omega_1\rangle\ +m_1(k)+m_2(\gamma).\end{aligned}$$

Subtracting these two equations shows that

$$\langle k_1,\gamma_1\rangle\ =0$$

for all pairs (k_1,γ_1) such that $(k_1,0,k_3)\in K$ for some $k_3\in\mathbb{T}^r$ and $(\gamma_1,\gamma_2,1_C)\in A(K,\widehat{G})$ for some $\gamma_3\in\widehat{\mathbb{Z}^q}$.

We are now going to show that this is impossible. For that, notice first that, since $G'=\mathbb{R}^p\times L$ is open in G, $G'/(G'\cap K)$ is topologically isomorphic to $G'K/K\subseteq G/K$, which is compact. Hence $K\cap G'$ is cocompact in G'. Let K_1 denote the set of first components of elements in $K\cap G'$. Then K_1 contains a vector space basis for $\mathbb{R}^p$. Indeed, otherwise

$$K\cap G'\subseteq K_1\times L\subseteq V\times L\subseteq\mathbb{R}^p\times L=G'$$

for some proper subspace V of $\mathbb{R}^p$, which contradicts the fact that $G'/(K\cap G')$ is compact.

Thus it only remains to verify that there exist $\gamma_1 \in \widehat{\mathbb{R}^p}$ and $\gamma_2 \in \widehat{\mathbb{Z}^q}$ such that $\gamma_1 \neq 0$ and $(\gamma_1, \gamma_2, 1_C) \in A(K, \widehat{G})$. Assume that $(\gamma_1, \gamma_2, 1_C) \in A(K, \widehat{G})$ only if $\gamma_1 = 0$. Then

$$A(KC, \widehat{G}) \subseteq A(\mathbb{R}^p \times C, \widehat{G}),$$

and hence $KC \supseteq \mathbb{R}^p \times C$, whence

$$K/(K \cap C) = KC/C \supseteq \mathbb{R}^p,$$

which is impossible, since K is discrete. This finishes the proof of the theorem. □

The idea of writing $g/|g|$, when possible, as the exponential of some continuous function occurs already in the proofs that Zak [**BZ81**] and Janssen [**Jan82**] gave for the existence of a zero in the case $G = \mathbb{R}$. For a different proof, compare [**AT75**, p.18].

DEFINITION 3.1.10. We say that Zf *is continuous on* $G\times\widehat{G}$ if there exists a continuous function g on $G\times\widehat{G}$ which agrees with Zf locally almost everywhere on $G \times \widehat{G}$.

Of course, such a function g then satisfies the quasi-periodicity relation $g(xk, \omega\gamma) = \overline{\omega(k)}g(x, \omega)$ for all $(x, \omega) \in G\times\widehat{G}$ and $(k, \gamma) \in K\times A(K, \widehat{G})$. Hence an application of the theorem yields the following corollary, which generalizes Zak's [**BZ81**] and Janssen's [**Jan82**] result for $G = \mathbb{R}$.

COROLLARY 3.1.11. *Let G be a compactly generated locally compact abelian group with non-compact connected component of the identity, let K be a uniform lattice in G and let Z denote the associated Zak transform. Let $f \in L^2(G)$ and suppose that Zf is continuous on $G \times \widehat{G}$. Then Zf has a zero.*

PROOF. By Lemma 3.1.6, we may apply Theorem 3.1.9 to the Zak transform. □

3.1.2.2. *Some remarks and an application.* Here we want to present some remarks concerning Corollary 3.1.11 and an application of the Zak transform to Gabor systems.

At first, it seems natural to ask, whether Corollary 3.1.11 still holds, if G_0 is supposed to be compact.

REMARK 3.1.12. Let G be a compactly generated locally compact abelian group. Then G_0 must be non-compact provided that G has the following property: For every uniform lattice K in G and $f \in L^2(G)$, Zf has a zero whenever Zf is continuous.

In fact, suppose that G_0 is compact so that $G = D \times C$ where D is discrete and C is compact. Choosing $K = D$ and $f = \chi_C$, one obtains, for $x = dc$, $d \in D$, $c \in C$, and $\omega \in \widehat{G}$,

$$Zf(x,\omega) = \sum_{k \in D} f(xk)\omega(k) = \overline{\omega(d)}.$$

This formula shows that Zf is continuous and of modulus 1.

The next remark deals with the question whether Theorem 3.1.9 and Corollary 3.1.11 are actually equivalent.

REMARK 3.1.13. Let G be a first countable compactly generated locally compact abelian group, and let K be a uniform lattice in G. Choose relatively compact Borel sets S_K in G and Ω_K in $\widehat{G}$ such that the quotient mappings are Borel isomorphisms (see the proof of Theorem 3.1.7). Suppose that g is a continuous function on $G \times \widehat{G}$ satisfying the quasi-periodicity relation $g(xk, \omega\gamma) = \overline{\omega(k)}\, g(x,\omega)$ for all $(x,\omega) \in G \times \widehat{G}$ and $(k,\gamma) \in K \times A(K,\widehat{G})$. Then, since $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ is surjective, there exists $f \in L^2(G)$ such that $Zf = g$ almost everywhere on $S_K \times \Omega_K$, hence almost everywhere on $G \times \widehat{G}$. Thus in this situation the theorem and the corollary are equivalent.

Let $f \in L^2(G)$. If Zf is continuous and if the hypotheses of Corollary 3.1.11 are fulfilled, Zf has a zero. Hence it is of great importance to give sufficient conditions that force Zf to be continuous.

REMARK 3.1.14. The condition that Zf be continuous is satisfied whenever f is continuous and rapidly decreasing outside of compact subsets of G. More precisely, it is well-known that if f is a continuous function on $\mathbb{R}^d$ such that $|f(x)| \leq c(1 + \|x\|_2)^{-\alpha}$ for some $\alpha > 1$ and $c > 0$, then Zf is continuous. Slightly more general, it is not difficult to see that, for $G = \mathbb{R}^p \times \mathbb{Z}^q \times C \subseteq \mathbb{R}^p \times \mathbb{R}^q \times C$, a similar hypothesis with respect to the $\mathbb{R}^p$ and $\mathbb{R}^q$ variables is sufficient.

In the case $G = \mathbb{R}$ the Zak transform and the existence of zeros play an important role concerning so-called Gabor systems. We want to investigate whether a similar result holds in the situation discussed here.

The notion of Gabor systems admits a natural generalization to locally compact abelian groups.

DEFINITION 3.1.15. Let G be a locally compact abelian group and let K be a uniform lattice in G. Further, let $f \in L^2(G)$. The system

$$\{\varphi_{k,\gamma}(f) := \overline{\gamma} \cdot L_k f : (k,\gamma) \in K \times A(K,\widehat{G})\}$$

is called the *Gabor system associated with* f.

Note that we have

$$\varphi_{k,\gamma}(f) = U^{-1}(\rho_G(k,\gamma,1)(Uf)),$$

where $U \in \mathcal{U}(L^2(G))$ is the operator defined by $Uf(t) = f(t^{-1})$.

The connection between the Zak transform and the Gabor system both corresponding to some function $f \in L^2(G)$ will be investigated in the next lemma. It is a direct generalization of the same result for $G = \mathbb{R}$ ([**BF94**, Definition 3.10 (d)]).

LEMMA 3.1.16. *Let G be a locally compact abelian group, let K be a uniform lattice in G and let $f \in L^2(G)$. Further, let Z denote the Zak transform associated with K. Then, for all $(x,\omega) \in G \times \widehat{G}$ and $(k,\gamma) \in K \times A(K,\widehat{G})$,*

$$Z(\varphi_{k,\gamma}(f))(x,\omega) = \overline{\gamma(x)}\omega(k)Zf(x,\omega).$$

PROOF. For all $(x,\omega) \in G \times \widehat{G}$ and $(k,\gamma) \in K \times A(K,\widehat{G})$, we have

$$\begin{aligned} Z(\varphi_{k,\gamma}(f))(x,\omega) &= \sum_{l\in K} \omega(l)\overline{\gamma(xl)}f(xlk^{-1}) \\ &= \overline{\gamma(x)}\sum_{l\in K}\omega(l)f(xlk^{-1}) \\ &= \overline{\gamma(x)}\omega(k)Zf(x,\omega). \end{aligned}$$

□

For applications, it has been a matter of great importance whether a Gabor system is complete, minimal, a frame or even an orthonormal basis. In the classical situation $G = \mathbb{R}$ the Zak transform proved to be very useful (compare [**BF94**, Theorem 3.16]). This result can be generalized in a straightforward manner. The proof is mentioned here to complete the picture. For the definition and properties of frames in Hilbert spaces, we mention [**HW89**, Section 2].

THEOREM 3.1.17. *Let G be a second countable locally compact abelian group, let K be a uniform lattice in G and let $f \in L^2(G)$. Further, let Z denote the Zak transform associated with K and let S_K and Ω_K be relatively compact fundamental domains for K in G and for $A(K,\widehat{G})$ in $\widehat{G}$, respectively.*

(i) $\{\varphi_{k,\gamma}(f) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ *is complete in* $L^2(G)$
$\iff Zf \neq 0$ *almost everywhere.*
(ii) $\{\varphi_{k,\gamma}(f) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ *is minimal and complete in* $L^2(G)$
$\iff \frac{1}{Zf} \in L^2(S_K \times \Omega_K)$.
(iii) $\{\varphi_{k,\gamma}(f) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ *is a frame for* $L^2(G)$ *with bounds* A *and* B
$\iff A \leq |Zf|^2 \leq B$ *almost everywhere.*
In this case $\{\varphi_{k,\gamma}(f) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ *is an exact frame.*
(iv) $\{\varphi_{k,\gamma}(f) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ *is an orthonormal basis of* $L^2(G)$
$\iff |Zf| = 1$ *almost everywhere.*

We need some preparation before we can give the proof of this theorem.

LEMMA 3.1.18. *Let G, K, S_K and Ω_K be as before. Further, let $f \in L^2(G)$.*

(i) $\{\varphi_{k,\gamma}(f) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ *is complete (minimal) in* $L^2(G)$
$\iff \{Z(\varphi_{k,\gamma}(f)) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ *is complete (minimal) in* $L^2(S_K \times \Omega_K)$.

(ii) $\{\varphi_{k,\gamma}(f) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ *is a frame for* $L^2(G)$ *with bounds A and B*
$\iff \{Z(\varphi_{k,\gamma}(f)) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ *is a frame for* $L^2(S_K \times \Omega_K)$ *with bounds A and B.*

PROOF. This is an immediate consequence of the fact that $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ is a Hilbert space isomorphism (Theorem 3.1.7). □

PROOF OF THEOREM 3.1.17. We first prove (i). For this, suppose that $\{\varphi_{k,\gamma}(f) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ is complete in $L^2(G)$. Then Lemma 3.1.18 implies that $\{Z(\varphi_{k,\gamma}(f)) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ is complete in $L^2(S_K \times \Omega_K)$. Now define $W \subseteq S_K \times \Omega_K$ by

$$W := \{(x,\omega) \in S_K \times \Omega_K : Zf(x,\omega) = 0\}.$$

By Lemma 3.1.16, we obtain

$$\langle \chi_W, Z(\varphi_{k,\gamma}(f))\rangle = \langle \chi_W, \overline{\gamma} \cdot \widehat{k} \cdot Zf\rangle = 0$$

for all $(k,\gamma) \in K \times A(K,\widehat{G})$. Hence $\|\chi_W\|_2 = 0$. Thus $Zf \neq 0$ almost everywhere.

Conversely, suppose that $Zf \neq 0$ almost everywhere. Let $h \in L^2(S_K \times \Omega_K)$ be such that

$$\langle h, Z(\varphi_{k,\gamma}(f))\rangle = 0$$

for all $(k,\gamma) \in K \times A(K,\widehat{G})$. Note that it suffices to show that $\|h\|_2 = 0$. It is easily checked that, for all $(k,\gamma) \in K \times A(K,\widehat{G})$,

$$0 = \langle h, Z(\varphi_{k,\gamma}(f))\rangle = \langle h, \overline{\gamma} \cdot \widehat{k} \cdot Zf\rangle = \langle \overline{Zf} \cdot h, \overline{\gamma} \cdot \widehat{k}\rangle = \widehat{\overline{Zf} \cdot h}(\overline{\gamma}, k),$$

since $\overline{Zf}, h \in L^2(S_K \times \Omega_K)$ and hence, by Hölder's inequality, $\overline{Zf} \cdot h \in L^1(S_K \times \Omega_K)$. Recall that the Fourier transform is injective. This implies that $\overline{Zf} \cdot h = 0$ almost everywhere. Since, by hypothesis, $Zf \neq 0$ almost everywhere, we obtain $h = 0$ almost everywhere. This shows $\|h\|_2 = 0$.

To prove (ii), suppose that $\{\varphi_{k,\gamma}(f) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ is minimal and complete in $L^2(G)$. By (i), $Zf \neq 0$ almost everywhere, and, by Lemma 3.1.18 (i), also $\{Z(\varphi_{k,\gamma}(f)) : (k,\gamma) \in K \times A(K,\widehat{G})\}$ is minimal and complete in $L^2(S_K \times \Omega_K)$. Then, for all $(k,\gamma) \in K \times A(K,\widehat{G})$, [**HR63**, Lemma 23.19] implies

$$\langle \frac{1}{\overline{Zf}}, Z(\varphi_{k,\gamma}(f))\rangle = \langle \frac{1}{\overline{Zf}}, \overline{\gamma} \cdot \widehat{k} \cdot Zf\rangle = \langle 1, \overline{\gamma} \cdot \widehat{k}\rangle = \delta_{k,e}\delta_{\gamma,1}. \tag{7}$$

Let $X_{e,1}$ be defined by

$$X_{e,1} := \{Z(\varphi_{k,\gamma}(f)) : (k,\gamma) \in (K \times A(K,\widehat{G}))\backslash\{(e,1)\}\}.$$

The minimality implies that $Z(\varphi_{e,1}(f)) \notin \overline{\text{span} X_{e,1}}$. Hence, by the Hahn-Banach theorem, there exists some $H \in L^2(S_K \times \Omega_K)^*$, where $L^2(S_K \times \Omega_K)^*$ denotes the dual space of $L^2(S_K \times \Omega_K)$, such that $H|_{X_{e,1}} = 0$ and $H(Z(\varphi_{e,1}(f))) = 1$. By Riesz's theorem, there exists a uniquely determined $\tilde{H} \in L^2(S_K \times \Omega_K)$ such that

$$\langle \tilde{H}, Z(\varphi_{k,\gamma}(f)) \rangle = \delta_{k,e}\delta_{\gamma,1}$$

for all $(k, \gamma) \in K \times A(K, \widehat{G})$. Since $\overline{Zf}, \tilde{H} \in L^2(S_K \times \Omega_K)$, Hölder's inequality implies that $\overline{Zf} \cdot \tilde{H} \in L^1(S_K \times \Omega_K)$, hence $1 - \overline{Zf} \cdot \tilde{H} \in L^1(S_K \times \Omega_K)$. Using (7), we obtain

$$0 = \langle \frac{1}{\overline{Zf}} - \tilde{H}, Z(\varphi_{k,\gamma}(f)) \rangle = \langle \frac{1}{\overline{Zf}} - \tilde{H}, \overline{\gamma} \cdot \widehat{k} \cdot Zf \rangle$$

$$= \langle \overline{Zf} \cdot \left(\frac{1}{\overline{Zf}} - \tilde{H} \right), \overline{\gamma} \cdot \widehat{k} \rangle = \left(1 - \overline{Zf} \cdot \tilde{H}\right)^{\widehat{}} (\overline{\gamma}, k).$$

Recall again that the Fourier transform is injective. This implies that $\overline{Zf} \cdot (\frac{1}{\overline{Zf}} - \tilde{H}) = 1 - \overline{Zf} \cdot \tilde{H} = 0$ almost everywhere. Since, by hypothesis, $Zf \neq 0$ almost everywhere, we obtain $(\frac{1}{\overline{Zf}} - \tilde{H}) = 0$ almost everywhere. This shows $\frac{1}{\overline{Zf}} = \tilde{H} \in L^2(S_K \times \Omega_K)$.

Conversely, suppose that $\frac{1}{Zf} \in L^2(S_K \times \Omega_K)$. Then $Zf \neq 0$ almost everywhere. Hence, by (i), the set $\{\varphi_{k,\gamma}(f) : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is complete in $L^2(G)$. Let $(l, \tau) \in K \times A(K, \widehat{G})$ and let $X_{l,\tau}$ be defined by

$$X_{l,\tau} := \{Z(\varphi_{k,\gamma}(f)) : (k, \gamma) \in (K \times A(K, \widehat{G})) \backslash \{(l, \tau)\}\}.$$

Note that, by Lemma 3.1.18 (i), it suffices to show that $X_{l,\tau}$ is incomplete. Since $\{\overline{\gamma} \cdot \widehat{k} : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is an orthonormal basis of $L^2(S_K \times \Omega_K)$, it follows from Lemma 3.1.16 that

$$0 = \langle \overline{\tau} \cdot \widehat{l}, \overline{\gamma} \cdot \widehat{k} \rangle = \langle \overline{\tau} \cdot \widehat{l} \cdot \frac{1}{\overline{Zf}}, \overline{\gamma} \cdot \widehat{k} \cdot Zf \rangle = \langle \overline{\tau} \cdot \widehat{l} \cdot \frac{1}{\overline{Zf}}, Z(\varphi_{k,\gamma}(f)) \rangle$$

for all $(k, \gamma) \in K \times A(K, \widehat{G})$, $(k, \gamma) \neq (l, \tau)$. Hence $\overline{\tau} \cdot \widehat{l} \cdot \frac{1}{\overline{Zf}}$ $(\in L^2(S_K \times \Omega_K))$ is orthogonal to $X_{l,\tau}$. Thus $X_{l,\tau}$ is incomplete.

Now we have to prove part (iii). For this, suppose that $\{\varphi_{k,\gamma}(f) : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is a frame for $L^2(G)$ with bounds A and B. Then, using also Lemma 3.1.16, Lemma 3.1.18 (ii) implies that $\{\overline{\gamma} \cdot \widehat{k} \cdot Zf : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is a frame for $L^2(S_K \times \Omega_K)$ with bounds A and B. Let $g \in L^2(S_K \times \Omega_K)$. Since $\{\overline{\gamma} \cdot \widehat{k} : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is an orthonormal basis of $L^2(S_K \times \Omega_K)$, we have

$$\sum_{k \in K} \sum_{\gamma \in A(K, \widehat{G})} |\langle g, \overline{\gamma} \cdot \widehat{k} \cdot Zf \rangle|^2 = \|g \cdot \overline{Zf}\|^2. \tag{8}$$

This implies

$$A\|g\|_2^2 \leq \|g \cdot \overline{Zf}\|_2^2 \leq B\|g\|_2^2. \tag{9}$$

So we obtain

$$A \leq \frac{1}{\|g\|_2^2}\|g \cdot \overline{Zf}\|_2^2 \leq B$$

for all $g \in L^2(S_K \times \Omega_K)$. It it easily checked that then $A \leq |Zf|^2 \leq B$.

Conversely, suppose that $A \leq |Zf|^2 \leq B$. This implies (9). Then, using equation (8), $\{\overline{\gamma} \cdot \widehat{k} \cdot Zf : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is a frame for $L^2(S_K \times \Omega_K)$ with bounds A and B. Lemma 3.1.16 and Lemma 3.1.18 (ii) yields the claim.

To show that in this case $\{\varphi_{k,\gamma}(f) : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is an exact frame, let $(l, \tau) \in K \times A(K, \widehat{G})$ and let $X_{l,\tau}$ be defined as in the proof of part (ii). Since $A \leq |Zf|^2 \leq B$, we obtain $\frac{1}{\overline{Zf}} \in L^2(S_K \times \Omega_K)$. As in the proof of (ii) the incompleteness of $X_{l,\tau}$ follows. Hence $\{\varphi_{k,\gamma}(f) : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is an exact frame by [**HW89**, Theorem 2.1.6].

Finally, it remains to prove the equivalence of (iv). For this, suppose that $\{\varphi_{k,\gamma}(f) : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is an orthonormal basis of $L^2(G)$. Hence $\{\varphi_{k,\gamma}(f) : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is an exact frame with bounds $A = B = 1$. Part (iii) implies that $1 \leq |Zf|^2 \leq 1$ almost everywhere. Hence $|Zf| = 1$ almost everywhere.

Conversely, suppose that $|Zf| = 1$ almost everywhere. Again, by part (iii), $\{\varphi_{k,\gamma}(f) : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is a frame with bounds $A = B = 1$ and we obtain

$$\begin{aligned}
\|\varphi_{k,\gamma}(f)\|_2^2 &= \|Z(\varphi_{k,\gamma}(f))\|_2^2 \\
&= \int_{S_K}\int_{\Omega_K} |\widehat{k}(\omega)\overline{\gamma(x)}Zf(x,\omega)|^2 \, d\omega \, dx \\
&= \int_{S_K}\int_{\Omega_K} |Zf(x,\omega)|^2 \, d\omega \, dx \\
&= 1.
\end{aligned}$$

Hence $\{\varphi_{k,\gamma}(f) : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is an orthonormal basis of $L^2(G)$. □

Now, using Corollary 3.1.11 and Theorem 3.1.17 (iii), we obtain the following.

COROLLARY 3.1.19. *Let G be a second countable compactly generated locally compact abelian group, let K be a uniform lattice in G and let $f \in L^2(G)$. Further, let Z denote the Zak transform associated with K. If Zf is continuous, the Gabor system $\{\varphi_{k,\gamma}(f) : (k, \gamma) \in K \times A(K, \widehat{G})\}$ is not a frame for $L^2(G)$.*

3.2. The non-abelian case

The purpose of this section is to define a Zak transform for a class of locally compact groups and study its properties. Finally, we examine Zak transforms of several classes of groups.

3.2.1. Definition and some properties. Let J be a locally compact abelian group. Then the Zak transform associated with a uniform lattice D in J of $f \in L^2(J)$ is defined on $J \times \widehat{J}$ by

$$Zf(x,\omega) = \sum_{k \in D} f(xk)\omega(k)$$

(compare Definition 3.1.4). Let S_D and Ω_D be fundamental domains for D in J and for $A(D, \widehat{J})$ in $\widehat{J}$, respectively. Since the Zak transform satisfies some quasi-periodicity relation (Lemma 3.1.6), the function Zf is uniquely determined by its values on $S_D \times \Omega_D$. By Theorem 3.1.7, the mapping $Z : L^2(J) \to L^2(S_D \times \Omega_D)$ is an isometry. Moreover, if J is second countable, it is even a Hilbert space isomorphism.

Now we are interested in a generalization of this definition of the Zak transform to a locally compact group G. This should be done in such a way that the above mentioned properties remain true. Notice first that we may rewrite the Zak transform in the following way

$$Zf(x,\omega) = \sum_{k \in D} (\rho_J(x,\omega,1)f)(k),$$

where ρ_J denotes the Schrödinger representation (compare Chapter 1).

First, we define a subgroup of G, which plays the role of the uniform lattice in the abelian case. For this, consider a discrete subgroup K of G. The following definition generalizes the notion of fundamental domain to the non-abelian case.

DEFINITION 3.2.1. A *fundamental domain* for K is a measurable cross section S_K, that means a measurable set $S_K \subseteq G$ such that every $x \in G$ can be uniquely written in the form $x = ks$ where $k \in K$ and $s \in S_K$.

Now ρ_J is a representation of the Heisenberg group associated with J, $H(J)$, on $L^2(J)$ (compare Chapter 1). Note that we have $H(J) = J \ltimes_\sigma (\widehat{J} \times \mathbb{T})$, where $\sigma : J \to \mathrm{Aut}(\widehat{J} \times \mathbb{T})$ is given by $\sigma_x(\omega, z) = (\omega, z\omega(x))$. In addition, we have $Z(H(J)) = \mathbb{T}$. To generalize this let L and Z be locally compact abelian groups and let $\tau : G \to \mathrm{Aut}(L \times Z)$ be an action such that $Z(G \ltimes_\tau (L \times Z)) = Z$. Then in the non-abelian case the group $G \ltimes_\tau (L \times Z)$ will play a similar role as the group $H(J)$.

In the abelian situation consider again the group $H(J) = J \ltimes_\sigma (\widehat{J} \times \mathbb{T})$ and let $S_{(e,1)}$ denote the stabilizer of $(e,1) \in \widehat{\widehat{J}} \times \widehat{\mathbb{T}} = J \times \mathbb{Z}$. It is easily checked that $S_{(e,1)} = \{e\}$. It is well-known that the Schrödinger representation ρ_J is equivalent to the representation induced by the character $(e,1) \in (\widehat{J} \times \mathbb{T})\widehat{\ }$. A suitable generalization to G might be the following. According to the abelian case suppose that there exists some $\chi \in \widehat{Z}$ such that $S_{(1,\chi)} = \{e\}$. Then the induced representation

$$\rho := \operatorname{ind}_{\{e\} \ltimes_\tau (L \times Z)}^{G \ltimes_\tau (L \times Z)} (1_{\widehat{L}}, \chi) : G \ltimes_\tau (L \times Z) \to \mathcal{U}(L^2(G))$$

replaces the Schrödinger representation.

Thus, for the remainder of this subsection, let G be a locally compact group. In addition, suppose that

(I) there exists a discrete subgroup K of G and a relatively compact fundamental domain S_K for K,

(II) there exist locally compact abelian groups L and Z and some action $\tau : G \to \operatorname{Aut}(L \times Z)$ (recall that we write $\tau = (\tau^{(1)}, \tau^{(2)})$ as mentioned in Chapter 1) such that $Z(G \ltimes_\tau (L \times Z)) = Z$ and such that the map $y \mapsto \tau_k^{(1)}(y, e)$, $L \to L$, is an isomorphism for all $k \in K$,

(III) there exists some $\chi \in \widehat{Z}$ such that the map

$$x \mapsto \chi(\tau_x^{(2)}(\cdot, e)), \quad G \to \widehat{L},$$

is injective.

For the definition of the representation, which shall generalize the Schrödinger representation, we need the following lemma.

LEMMA 3.2.2. *We have*

$$S_{(1,\chi)} = \{e\}.$$

PROOF. Using the fact that $Z(G \ltimes_\tau (L \times Z)) = Z$, it is easily checked that

$$S_{(1,\chi)} = \{x \in G : \chi(\tau_x^{(2)}(y, e)) = 1 \quad \text{for all } y \in L\}.$$

Hence the claim follows from (III) of the preceding assumptions. □

From now on let $\rho : G \ltimes_\tau (L \times Z) \to \mathcal{U}(L^2(G))$ be the unitary representation defined by

$$\rho := \operatorname{ind}_{S_{(1,\chi)} \ltimes_\tau (L \times Z)}^{G \ltimes_\tau (L \times Z)} (1_{\widehat{L}}, \chi).$$

By [**Rie79**], we obtain

$$(\rho(x, y, z)f)(t) = \chi(z)\chi(\tau_t^{(2)}(y, e))f(tx)$$

for all $(x, y, z) \in G \ltimes_\tau (L \times Z), t \in G$ and $f \in L^2(G)$.

DEFINITION 3.2.3. Then the *Zak transform associated with K (and L, Z, τ and χ)* of $f \in L^2(G)$ is defined on $G \times L$ by

$$Zf(x, y) = \sum_{k \in K} (\rho(x, y, e)f)(k) = \sum_{k \in K} \chi(\tau_k^{(2)}(y, e))f(kx).$$

Concerning convergence of the series compare Proposition 3.2.9.

The next lemma is easily seen, but it is mentioned here, since the result is important for all the following calculations.

LEMMA 3.2.4. *Let $\alpha : G \to \mathrm{Aut}(L \times Z)$ be an action. Then the following is true.*

(i) *The following conditions are equivalent.*
  (a) $Z(G \ltimes_\alpha (L \times Z)) = Z$
  (b) *For all $x \in G$ and $(y, z) \in L \times Z$, we have $\alpha_x(y, z) = (e, z)\alpha_x(y, e)$.*

(ii) *Suppose that $Z(G \ltimes_\alpha (L \times Z)) = Z$. Then, for all $x, x' \in G$, $y \in L$,*

$$\alpha_x^{(2)}(\alpha_{x'}^{(1)}(y, e), e) = \alpha_{xx'}^{(2)}(y, e)(\alpha_{x'}^{(2)}(y, e))^{-1}.$$

PROOF. $Z(G \ltimes_\alpha (L \times Z)) = Z$ holds if and only if, for all $(x, y, z) \in G \ltimes_\alpha (L \times Z)$ and $z' \in Z$,

$$(x, (y, z)\alpha_x(e, z')) = (x, y, z)(e, e, z') = (e, e, z')(x, y, z) = (x, y, z'z).$$

Hence $Z(G \ltimes_\alpha (L \times Z)) = Z$ is equivalent as to demand that

$$\alpha_x(e, z') = (e, z') \quad \text{for all } x \in G, z' \in Z.$$

Since $\alpha_x \in \mathrm{Aut}(L \times Z)$, this yields (i).

By (i),

$$\begin{aligned} \alpha_{xx'}(y, e) &= \alpha_x(\alpha_{x'}(y, e)) \\ &= \alpha_x(\alpha_{x'}^{(1)}(y, e), \alpha_{x'}^{(2)}(y, e)) \\ &= (e, \alpha_{x'}^{(2)}(y, e))\alpha_x(\alpha_{x'}^{(1)}(y, e), e). \end{aligned}$$

This implies (ii). □

It is desirable that the Zak transform of Definition 3.2.3 coincides with the classical Zak transform for $G = \mathbb{R}$. This is the subject of the following example.

EXAMPLE 3.2.5. Consider the case $G = \mathbb{R}$. Each discrete subgroup of $\mathbb{R}$ is of the form $K = r\mathbb{Z}$ with $r \in \mathbb{R}^*$. Notice that $S_K := [0, r)$ is a relatively compact fundamental domain for K. Now let $L := Z := \mathbb{R}$. We intend to calculate possible Zak transforms in the sense of Definition 3.2.3.

Let $\tau : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ be an action. By Lemma 3.2.4 (i), there exist functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ such that τ is of the form

$$\tau_x = \begin{pmatrix} \varphi(x) & 0 \\ \psi(x) & 1 \end{pmatrix} \quad \text{for all } x \in \mathbb{R}.$$

Since τ is a homomorphism, we obtain, for all $x, y \in \mathbb{R}$,

$$\varphi(x + y) = \varphi(x)\varphi(y) \tag{10}$$

and

$$\psi(x + y) = \psi(x)\varphi(y) + \psi(y) = \psi(y)\varphi(x) + \psi(x). \tag{11}$$

Equation (10) and the fact that τ is an action imply that there either exists $a > 0$ such that, for all $x \in \mathbb{R}$,

$$\varphi(x) = a^x$$

or

$$\varphi \equiv 0.$$

But $\tau_x \in \mathrm{Aut}(\mathbb{R}^2)$ for all $x \in \mathbb{R}$ implies $\varphi \not\equiv 0$. Now we have to consider two cases.

Case $a \neq 1$.

Then (11) can be rewritten in the following way

$$\psi(x)(a^y - 1) = \psi(y)(a^x - 1) \quad \text{for all } x, y \in \mathbb{R}.$$

Choosing $y = 1$ yields the existence of $b (= \psi(1)) \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$\psi(x) = \frac{b}{a-1}(a^x - 1).$$

It is easily checked that (II) is satisfied and that (III) is fulfilled if and only if $b \neq 0$. Hence τ yields a Zak transform in the sense of Definition 3.2.3, which does not coincide with the classical Zak transform. But notice that the Zak transform associated with τ satisfies no quasi-periodicity relation, since the subgroup Γ_K defined below is trivial.

Case $a = 1$.

Then (11) implies that ψ is a homomorphism and hence there exists $s \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$\psi(x) = sx.$$

Note that (II) is always fulfilled and that (III) is satisfied if and only if $s \neq 0$. Then $\mathbb{R} \ltimes_\tau (\mathbb{R} \times \mathbb{R})$ is the classical Heisenberg group associated with $\mathbb{R}$. Hence ρ is the classical Schrödinger representation and therefore τ yields the classical Zak transform with the additional factor r. But this factor sometimes also appears in the definition of the Zak transform ([**Jan88**, Section 1]).

Hence we proved that, for $L = Z = \mathbb{R}$, Definition 3.2.3 does yield different Zak transforms, but the only ones fulfilling further important properties are the Zak transforms coming from the classical one with an additional factor. If L and Z are chosen in a different way, clearly this yields a different transform, for example, with $L = \mathbb{R}^2$ and $Z = \mathbb{R}$.

Furthermore, the Zak transform on a locally compact abelian group is also a Zak transform in the sense of Definition 3.2.3. For this, compare Subsection 3.2.2.2.

Next we have to check whether the properties of the Zak transform in the abelian case carry over to the non-abelian case. So the proofs of the following results often use ideas which already appear in the abelian case.

First, we will investigate whether the general Zak transform also satisfies a quasi-periodicity relation. For this, we need a replacement of the annihilator in the abelian case. Thus, for the remainder of this subsection, let $\Gamma_K < L$ be defined by

$$\Gamma_K := \{m \in L : \chi(\tau_k^{(2)}(m,e)) = 1 \quad \text{for all } k \in K\}.$$

In the classical situation of Section 3.1 the subgroup Γ_K is just the annihilator of K in $\widehat{G}$ (compare Subsection 3.2.2.2).

PROPOSITION 3.2.6. *For all $f \in L^2(G)$ and $(x,y) \in G \times L, (l,m) \in K \times \Gamma_K$,*

$$Zf(lx, m\tau_l^{(1)}(y,e)) = \overline{\chi(\tau_l^{(2)}(y,e))} Zf(x,y).$$

PROOF. Let $f \in L^2(G)$ and let $(x,y) \in G \times L, (l,m) \in K \times \Gamma_K$. The definition of Γ_K implies that

$$\sum_{k\in K} (\rho(l,m,e)f)(k) = \sum_{k \in K} \chi(\tau_k^{(2)}(m,e)) f(kl) \overset{k \mapsto kl^{-1}}{=} \sum_{k \in K} f(k).$$

This yields

$$\begin{aligned} Zf(lx, m\tau_l^{(1)}(y,e)) &= \sum_{k \in K} (\rho((l,m,(\tau_l^{(2)}(y,e))^{-1})(x,y,e))f)(k) \\ &= \sum_{k\in K} \rho(l,m,(\tau_l^{(2)}(y,e))^{-1})(\rho(x,y,e)f)(k) \\ &= \overline{\chi(\tau_l^{(2)}(y,e))} \sum_{k \in K} \rho(l,m,e)(\rho(x,y,e)f)(k). \end{aligned}$$

Now the claim follows from the first part of the proof. □

The next lemma establishes an important property of the relationship between Γ_K and τ.

LEMMA 3.2.7. *For all $k \in K$,*

$$\tau_k^{(1)}(\Gamma_K \times \{e\}) = \Gamma_K.$$

PROOF. Let $k \in K$. First, we prove that $\tau_k^{(1)}(\Gamma_K \times \{e\}) < \Gamma_K$. For this, let $m \in \Gamma_K$. Then, by Lemma 3.2.4 (ii) and the definition of Γ_K,

$$\chi(\tau_l^{(2)}(\tau_k^{(1)}(m,e),e)) = \overline{\chi(\tau_k^{(2)}(m,e))} \chi(\tau_{lk}^{(2)}(m,e)) = 1$$

for all $l \in K$. This implies $\tau_k^{(1)}(m,e) \in \Gamma_K$.

Now assume, towards a contradiction, that there exist $y \in L \backslash \Gamma_K$ and $k \in K$ such that $\tau_k^{(1)}(y,e) \in \Gamma_K$. This implies both the existence of some $k_0 \in K$ such that

$$\chi(\tau_{k_0}^{(2)}(y,e)) \neq 1$$

and

$$1 = \chi(\tau_l^{(2)}(\tau_k^{(1)}(y,e),e)) = \overline{\chi(\tau_k^{(2)}(y,e))} \chi(\tau_{lk}^{(2)}(y,e))$$

for all $l \in K$. Hence

$$\chi(\tau_{lk}^{(2)}(y,e)) = \chi(\tau_k^{(2)}(y,e))$$

for all $l \in K$. Choosing $l \in K$ as $l := k_0k^{-1}$, we obtain, by the choice of k_0,

$$\chi(\tau_k^{(2)}(y,e)) \neq 1$$

and hence

$$\chi(\tau_{lk}^{(2)}(y,e)) \neq 1$$

for all $l \in K$. With $l := k^{-1}$, we get a contradiction.

Summarizing, we proved $\tau_k^{(1)}(\Gamma_K \times \{e\}) < \Gamma_K$ and $\tau_k^{(1)}((L\backslash\Gamma_K) \times \{e\}) \subseteq L\backslash\Gamma_K$ for all $k \in K$. Applying (II) yields the claim. □

REMARK 3.2.8. Suppose that Γ_K is a uniform lattice in L. Then, by Lemma 3.1.2, there exists a relatively compact fundamental domain for Γ_K which we will denote by Ω_K in the following. Hence every $y \in L$ can be uniquely written in the form $y = mt$ where $m \in \Gamma_K$ and $t \in \Omega_K$. Let $k \in K$. This implies, using (II) and Lemma 3.2.7, that each $y \in L$ can be uniquely written in the form $y = m_kt_k$ where $m_k \in \Gamma_K$ and $t_k \in \tau_k^{(1)}(\Omega_K \times \{e\})$. Thus, by Proposition 3.2.6, for all $f \in L^2(G)$, the map $Zf : G \times L \to \mathbb{C}$ is uniquely determined by its values on $S_K \times \Omega_K$.

In the abelian case it is important for many applications that the Zak transform is an isometry. Under certain conditions the Zak transform is isometric also in the general situation.

Suppose that G is unimodular and that Γ_K is a uniform lattice in L. Let Ω_K be a relatively compact fundamental domain for Γ_K. Let the Haar measure on G be normalized so that Weil's formula holds, when we take on G/K the normalized G-invariant Radon measure and the counting measure on K. We obtain $|S_K| > 0$ as in the abelian case. The map $\Phi : S_K \to G/K$, $x \mapsto xK$, is a continuous bijection and, for each measurable subset M of S_K, Weil's formula gives

$$|M| = \int_G \chi_M(x)\,dx = \int_{G/K} \left(\sum_{k\in K} \chi_M(xk) \right) d(xK) = |\Phi(M)|.$$

This implies that Φ maps the measure on S_K induced by the Haar measure on G to the chosen measure on G/K.

In addition, let the Haar measure on L be normalized so that Weil's formula holds, when we take on L/Γ_K the normalized Haar measure and the counting measure on Γ_K. Similarly, we see that the induced measure on Ω_K is transformed into the Haar measure on L/Γ_K and $|\Omega_K| = 1$.

PROPOSITION 3.2.9. *Suppose that G is unimodular and that Γ_K is a uniform lattice in L. Further, let Ω_K be a relatively compact fundamental domain* *for* Γ_K. *Then, for almost all* $(x,y) \in S_K \times \Omega_K$,

$$Zf(x,\omega) = \sum_{k\in K} \chi(\tau_k^{(2)}(y,e))f(kx)$$

converges, and the function Zf *belongs to* $L^2(S_K \times \Omega_K)$ *and satisfies* $\|Zf\|_2 = \|f\|_2$.

PROOF. For $k \in K$, define $f_k \in L^2(S_K \times \Omega_K)$ by

$$f_k(x,y) := \chi(\tau_k^{(2)}(y,e))f(kx).$$

Then we have

$$\sum_{k\in K} \|f_k\|_2^2 = \sum_{k\in K} \int_{S_K}\int_{\Omega_K} |f_k(x,y)|^2\,dy\,dx$$

$$= \sum_{k\in K} \int_{S_K} |f(kx)|^2\,dx = \|f\|_2^2.$$

Now let $k, l \in K$ such that $k \neq l$. We claim that $\langle f_k, f_l\rangle = 0$. At first, we have

$$\begin{aligned}&\langle f_k, f_l\rangle \\ &= \int_{S_K}\int_{\Omega_K} f(kx)\overline{f(lx)}\chi(\tau_k^{(2)}(y,e)\tau_l^{(2)}(y^{-1},e))\,dy\,dx \\ &= \int_{S_K} f(kx)\overline{f(lx)}\,dx \int_{\Omega_K} \chi(\tau_k^{(2)}(y,e)\tau_l^{(2)}(y^{-1},e))\,dy.\end{aligned}$$

By [**HR63**, Lemma 23.19], for a compact abelian group C and for a non-trivial character φ of C, $\int_C \varphi(y)dy = 0$. Using the definition of Γ_K, we may apply this to $C = L/\Gamma_K$ and the character φ defined by

$$\varphi(y\Gamma_K) = \chi(\tau_k^{(2)}(y,e)\tau_l^{(2)}(y^{-1},e)), \quad y \in L.$$

Note that φ is non-trivial by (III). We obtain

$$\int_{\Omega_K} \chi(\tau_k^{(2)}(y,e)\tau_l^{(2)}(y^{-1},e))\,dy = \int_{L/\Gamma_K} \varphi(y\Gamma_K)\,d(y\Gamma_K) = 0,$$

and this in turn implies

$$\langle f_k, f_l\rangle = 0.$$

It follows that the series $\sum_{k\in K} f_k$ converges in $L^2(S_K \times \Omega_K)$ and satisfies

$$\|\sum_{k\in K} f_k\|_2^2 = \sum_{k\in K} \|f_k\|_2^2 = \|f\|_2^2.$$

In particular, $Zf(x,\omega)$ exists for almost all $(x,\omega) \in S_K \times \Omega_K$. $\square$

This shows that the Zak transform is defined almost everywhere.

As in the abelian case the Zak transform is even a Hilbert space isomorphism for a large class of locally compact groups. (In Subsection 3.2.2 it will turn out that (I), (II) and (III) and the condition that Γ_K has to be a uniform lattice are not very restrictive.)

THEOREM 3.2.10. *Suppose that G is unimodular. Suppose that Γ_K is a uniform lattice and let Ω_K be a relatively compact fundamental domain for Γ_K. If the set*

$$\{(x,y)\mapsto \chi(\tau^{(2)}_{x^{-1}}(m,e))\chi(\tau^{(2)}_l(y,e)) : (l,m)\in K\times\Gamma_K\}$$

is an orthonormal basis of $L^2(S_K\times\Omega_K)$, then

$$Z : L^2(G)\to L^2(S_K\times\Omega_K)$$

is a Hilbert space isomorphism.

PROOF. By Proposition 3.2.9, $Z : L^2(G)\to L^2(S_K\times\Omega_K)$ is an isometry. Obviously, Z is also linear. Hence, to prove that Z is a Hilbert space isomorphism, it remains to shows that $Z : L^2(G)\to L^2(S_K\times\Omega_K)$ is also surjective.

Let $U\in\mathcal{U}(L^2(G))$ be defined by $Uf(t)=f(t^{-1})$. Now consider the set $M_1\subseteq L^2(G)$, which is defined by

$$\begin{aligned} M_1 := \{\varphi_{l,m} &:= U^{-1}(\rho(l,m,e)(U\chi_{S_K})) \\ &= \chi(\tau^{(2)}_{\cdot^{-1}}(m,e))\cdot L_l\chi_{S_K} : (l,m)\in K\times\Gamma_K\}. \end{aligned}$$

Furthermore, consider the set $M_2\subseteq L^2(S_K\times\Omega_K)$, which is defined by

$$M_2 := \{Z\varphi_{l,m} : (l,m)\in K\times\Gamma_K\}.$$

Now, for $f\in L^2(G)$ and $(x,y)\in G\times L$, $(l,m)\in K\times\Gamma_K$, we obtain

$$\begin{aligned} Z\varphi_{l,m}(x,y) &= \sum_{k\in K}\chi(\tau^{(2)}_k(y,e))\varphi_{l,m}(kx) \\ &= \sum_{k\in K}\chi(\tau^{(2)}_k(y,e))\chi(\tau^{(2)}_{(kx)^{-1}}(m,e))\chi_{S_K}(l^{-1}kx) \\ &\overset{k\mapsto lk}{=} \sum_{k\in K}\chi(\tau^{(2)}_{lk}(y,e))\chi(\tau^{(2)}_{(lkx)^{-1}}(m,e))\chi_{S_K}(kx). \end{aligned}$$

Since every $x\in G$ can be uniquely written in the form $x=ks$ where $k\in K$ and $s\in S_K$, $\chi_{S_K}(kx)\neq 0$ if and only if $k=e$. This implies, using Lemma 3.2.4 (ii) and the definition of Γ_K,

$$\begin{aligned} Z\varphi_{l,m}(x,y) &= \chi(\tau^{(2)}_l(y,e))\chi(\tau^{(2)}_{(lx)^{-1}}(m,e)) \\ &= \chi(\tau^{(2)}_l(y,e))\chi(\tau^{(2)}_{l^{-1}}(m,e))\chi(\tau^{(2)}_{x^{-1}}(\tau^{(1)}_{l^{-1}}(m,e),e)) \\ &= \chi(\tau^{(2)}_l(y,e))\chi(\tau^{(2)}_{x^{-1}}(\tau^{(1)}_{l^{-1}}(m,e),e)). \end{aligned}$$

By Lemma 3.2.7,

$$M_2 = \{\chi(\tau^{(2)}_l(y,e))\chi(\tau^{(2)}_{x^{-1}}(m,e)) : (l,m)\in K\times\Gamma_K\}. \tag{12}$$

Since we supposed M_2 to be an orthonormal basis of $L^2(S_K \times \Omega_K)$, the claim is proven. □

Now we shall investigate when the set

$$\{(x,y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m,e))\chi(\tau^{(2)}_l(y,e)) : (l,m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$. The next proposition gives a necessary and sufficient condition for this.

PROPOSITION 3.2.11. *Suppose that G is unimodular and that Γ_K is a uniform lattice. Further, let Ω_K be a relatively compact fundamental domain for Γ_K. Then the following conditions are equivalent.*

(i) *For all $m \in \Gamma_K$, $m \neq e$,*

$$\int_{S_K} \chi(\tau^{(2)}_{x^{-1}}(m,e))\,dx = 0$$

and the linear subspace

$$\operatorname{span}\{(x,y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m,e))\chi(\tau^{(2)}_l(y,e)) : (l,m) \in K \times \Gamma_K\}$$

is dense in $L^2(S_K \times \Omega_K)$.

(ii) *The set*

$$\{(x,y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m,e))\chi(\tau^{(2)}_l(y,e)) : (l,m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

PROOF. For simplicity, let $\phi_{l,m} \subseteq L^2(S_K \times \Omega_K)$ be defined by

$$\phi_{l,m}(x,y) := \chi(\tau^{(2)}_{x^{-1}}(m,e))\chi(\tau^{(2)}_l(y,e))$$

for all $(x,y) \in S_K \times \Omega_K$ and $(l,m) \in K \times \Gamma_K$. Now, obviously,

$$\|\phi_{l,m}\|_2 = 1$$

for all $(l,m) \in K \times \Gamma_K$. Next, for $(l_1,m_1),(l_2,m_2) \in K \times \Gamma_K$, we obtain

$$\begin{aligned}
&\langle \phi_{l_1,m_1}, \phi_{l_2,m_2} \rangle \\
&= \int_{S_K}\int_{\Omega_K} \chi(\tau^{(2)}_{x^{-1}}(m_1,e)\tau^{(2)}_{l_1}(y,e)) \\
&\qquad \cdot\overline{\chi(\tau^{(2)}_{x^{-1}}(m_2,e)\tau^{(2)}_{l_2}(y,e))}\,dy\,dx \\
&= \int_{S_K} \chi(\tau^{(2)}_{x^{-1}}(m_1,e)\tau^{(2)}_{x^{-1}}(m_2^{-1},e))\,dx \\
&\qquad \cdot \int_{\Omega_K} \chi(\tau^{(2)}_{l_1}(y,e)\tau^{(2)}_{l_2}(y^{-1},e))\,dy.
\end{aligned}$$

Using [**HR63**, Lemma 23.19] (compare the proof of Proposition 3.2.9), we obtain

$$\int_{\Omega_K} \chi(\tau^{(2)}_{l_1}(y,e)\tau^{(2)}_{l_2}(y^{-1},e))\,dy = \delta_{l_1,l_2}.$$

Moreover, for $l_1 = l_2$,

$$\int_{S_K} \chi(\tau^{(2)}_{x^{-1}}(m_1, e)\tau^{(2)}_{x^{-1}}(m_2^{-1}, e))\, dx = \int_{S_K} \chi(\tau^{(2)}_{x^{-1}}(m_1 m_2^{-1}, e))\, dx$$

This proves the equivalence of (i) and (ii). □

3.2.2. Some special cases. Here we focus on some classes of examples of locally compact groups whose Zak transform is a Hilbert space isomorphism.

3.2.2.1. *The case* $\tau^{(1)}_x(\cdot, e) = \mathrm{Id}_L$ *for all* $x \in G$. One important class of groups are those locally compact groups G, for which the associated action τ (compare (II)) satisfies $\tau^{(1)}_x(\cdot, e) = \mathrm{Id}_L$ for all $x \in G$. It will turn out that, if we want (III) to be satisfied, this condition forces G to be abelian. Moreover, we introduce conditions which are easier to check and which imply that the associated Zak transform is a Hilbert space isomorphism.

Now, for the remainder of this subsection, let G be a locally compact group such that (I) holds. Further, suppose that there exist locally compact abelian groups L and Z and some action $\tau : G \to \mathrm{Aut}(L \times Z)$ such that $Z(G \ltimes_\tau (L \times Z)) = Z$ and such that $\tau^{(1)}_x(\cdot, e) = \mathrm{Id}_L$ for all $x \in G$. Then, obviously, the map $y \mapsto \tau^{(1)}_k(y, e)$, $L \to L$, is an isomorphism for all $k \in K$. This implies (II).

The following basic lemma will be used often throughout the next proofs.

LEMMA 3.2.12. *For each* $y \in L$*, the map*

$$x \mapsto \tau^{(2)}_x(y, e), \quad G \to Z,$$

is a homomorphism.

PROOF. This follows immediately from Lemma 3.2.4 (ii). □

The next proposition shows that in this case the action is constant on cosets of the commutator subgroup of G. Thus it is indeed an action of G modulo its commutator subgroup, hence of an abelian group.

PROPOSITION 3.2.13. *Let* $[G, G]$ *denote the commutator subgroup of* G. *Then*

$$\tau : G/[G, G] \to \mathrm{Aut}(L \times Z).$$

PROOF. Let $y \in L$ and let $h_y : G \to Z$ be defined by

$$h_y(x) = \tau^{(2)}_x(y, e).$$

By Lemma 3.2.12, h_y is a homomorphism. Since Z is abelian, we obtain $h_y(x) = e$ for all $x \in [G, G]$. Using Lemma 3.2.4 (i), this implies

$$\tau_x(y, z) = (y, \tau^{(2)}_x(y, z)) = (y, z) \quad \text{for all } x \in [G, G], (y, z) \in L \times Z.$$

This proves the claim. □

COROLLARY 3.2.14. *Let* $\chi \in \widehat{Z}$ *be such that* (III) *holds. Then* G *is abelian.*

Proof. Towards a contradiction, assume that G is non-abelian. This implies that $[G, G]$, the commutator subgroup of G, is not trivial. By Proposition 3.2.13,

$$\chi(\tau_x^{(2)}(y, e)) = 1$$

for all $x \in [G, G]$ and $y \in L$. This contradicts (III). □

In the special situation discussed here it is easy to check whether condition (III) holds.

Proposition 3.2.15. *Let $\chi \in \widehat{Z}$. Then the following conditions are equivalent.*

(i) (III) *holds.*
(ii) $S_{(1,\chi)} = \{e\}$.

Proof. Note that, for $\chi \in \widehat{Z}$,

$$S_{(1,\chi)} = \{x \in G : \chi(\tau_x^{(2)}(y, e)) = 1 \text{ for all } y \in L\}.$$

Now the claim follows from Lemma 3.2.12. □

Finally, we investigate when the set

$$\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

Proposition 3.2.16. *Let $\chi \in \widehat{Z}$ be such that* (III) *holds. Further, suppose that Γ_K is a uniform lattice in L. Then the following conditions are equivalent.*

(i) *The set*

$$\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

(ii) *For each $m \in \Gamma_K$, $m \neq e$, the character on G defined by*

$$x \mapsto \chi(\tau_x^{(2)}(m, e))$$

is non-trivial and we have

$$\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}$$

$$= A(K, \widehat{G}) \times A(\Gamma_K, \widehat{L}).$$

Proof. Let $m \in \Gamma_K$, $m \neq e$. Notice that, by Corollary 3.2.14, G is abelian. By Lemma 3.2.12, the character

$$x \mapsto \chi(\tau_x^{(2)}(m, e)), \quad G \to \mathbb{T},$$

is indeed a character on G/K. Hence, using [**HR63**, Lemma 23.19] (compare the proof of Lemma 3.1.3) and the normalization of the measure on S_K for Lemma 3.1.3, we obtain

$$\int_{S_K} \chi(\tau_{x^{-1}}^{(2)}(m, e))\, dx = 0$$

if and only if the character mentioned above is non-trivial. This implies that the set in question is an orthonormal system in $L^2(S_K \times \Omega_K)$ if and only if, for each $m \in \Gamma_K$, $m \neq e$, the character

$$x \mapsto \chi(\tau_x^{(2)}(m,e))$$

is non-trivial (compare Proposition 3.2.11 and its proof).

Now the map

$$(x,y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m,e))\chi(\tau_l^{(2)}(y,e)), \quad G \times L \to \mathbb{T},$$

is a character on $G \times L$ by Lemma 3.2.12 and belongs to $A(K,\widehat{G}) \times A(\Gamma_K,\widehat{L})$. $A(K,\widehat{G}) \times A(\Gamma_K,\widehat{L})$ is an orthonormal basis of $L^2(S_K \times \Omega_K)$. Applying Proposition 3.2.11 once more finishes the proof. $\square$

3.2.2.2. *Locally compact abelian groups.* Let G be a locally compact abelian group which contains a uniform lattice K. We define L, Z and τ by $L := \widehat{G}$, $Z := \mathbb{T}$ and $\tau : G \to \mathrm{Aut}(\widehat{G} \times \mathbb{T})$, $\tau_x(\omega,z) := (\omega, z\omega(x))$. By Lemma 3.1.2, (I) is satisfied. Let $\chi \in \widehat{\mathbb{T}} = \mathbb{Z}$ be defined by $\chi := 1$. Since the situation discussed here is a special case of the one studied in Subsection 3.2.2.1, it is easily checked that also (II) and (III) are fulfilled. Hence the canonical Zak transform for a locally compact abelian group dealt with in Section 3.1 is a Zak transform in the sense of Definition 3.2.3. Furthermore, we obtain

$$\begin{aligned}\Gamma_K &= \{\gamma \in \widehat{G} : \chi(\tau_k^{(2)}(\gamma,e)) = 1 \quad \text{for all } k \in K\}\\ &= \{\gamma \in \widehat{G} : \gamma(k)^\chi = 1 \quad \text{for all } k \in K\}\\ &= \{\gamma \in \widehat{G} : \gamma(k) = 1 \quad \text{for all } k \in K\}.\end{aligned}$$

This implies that in this case the set Γ_K is just the annihilator of K in $\widehat{G}$.

3.2.2.3. *Connected and simply connected 2-step nilpotent Lie groups.* In this subsection we give a Zak transform for all connected and simply connected 2-step nilpotent Lie groups whose Lie algebra admits a basis with respect to which the structure constants are rational. As a general reference to the theory of connected and simply connected 2-step nilpotent Lie groups we mention [**HN91**] and [**CG90**].

Concerning the existence of uniform lattices in connected and simply connected nilpotent Lie groups, the following result from [**Rag72**] gives a complete answer.

THEOREM 3.2.17. ([**Rag72**, Theorem 2.12]) *Let G be a simply connected nilpotent Lie group and let $\mathfrak{g}$ denote its Lie algebra. Then the following conditions are equivalent.*

(i) *G admits a uniform lattice.*
(ii) *$\mathfrak{g}$ admits a basis with respect to which the structure constants are rational.*

Now let G be connected and simply connected 2-step nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ admits a basis with respect to which the structure constants are rational. Furthermore, let K be a discrete subgroup of G with the property that G/K is compact. Then there exists a relatively compact fundamental domain S_K for K. Thus (I) is fulfilled.

In the following we give an example of locally compact abelian groups L and Z and some action τ such that the associated Zak transform is a Hilbert space isomorphism.

First, recall that the Baker-Campbell-Hausdorff formula defines a group-multiplication $*$ on $\mathfrak{g}$. Since G is a simply connected nilpotent Lie group, G is isomorphic to $(\mathfrak{g}, *)$. Next let $\{X_1, \dots, X_n\}$ be a fixed basis of $\mathfrak{g}$ with respect to which the structure constants are rational. Clearly, $\{X_1^*, \dots, X_n^*\}$ is a basis of $\mathfrak{g}^*$. Now we may identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ with respect to this basis. Further, let $\exp : \mathfrak{g} \to G$ denote the exponential map.

For the remainder of this subsection, let $L = \mathbb{R}^n$ and $Z = \mathbb{R}$. Furthermore, let Ad^* denote the coadjoint map. Recalling the preceding paragraph, we may define $\tau : G \to \mathrm{Aut}(L \times Z)$ by

$$\tau_x(y, z) = (\frac{1}{2}(y + \mathrm{Ad}_x^*(y)), z + \langle x, y \rangle).$$

LEMMA 3.2.18. *In the above situation* (II) *and* (III) *hold.*

PROOF. Note that, for proving (II), it remains to show that the map $y \mapsto \tau_k^{(1)}(y, e)$, $\mathbb{R}^n \to \mathbb{R}^n$, is an isomorphism for all $k \in K$. For all $X, Y \in \mathfrak{g}$ and $l \in \mathfrak{g}^*$, we obtain

$$((\mathrm{Ad}_{\exp X}^* l)(Y)) = l(\mathrm{Ad}_{(\exp X)^{-1}} Y) = l(-X * Y * X) = l(Y + [Y, X]).$$

This implies

$$\frac{1}{2}(l + \mathrm{Ad}_{\exp X}^* l)(Y) = l(Y + \frac{1}{2}[Y, X]).$$

Next, fix some $X \in \mathfrak{g}$ and define $\Phi_l, \Psi_l \in \mathfrak{g}^*$ for $l \in \mathfrak{g}^*$ by

$$\Phi_l(W) = l(W + \frac{1}{2}[W, X]) \quad \text{and} \quad \Psi_l(W) = l(W - \frac{1}{2}[W, X]).$$

It is easily checked that the map $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*, l \mapsto \Psi_l$, is the inverse of $\Phi : \mathfrak{g}^* \to \mathfrak{g}^*, l \mapsto \Phi_l$. Moreover, both maps are continuous. Thus Φ is an isomorphism. By definition of τ, this proves (II).

Moreover, notice that

$$x \mapsto e^{2\pi i \chi \langle x, \cdot \rangle},\ G = (\mathfrak{g}, *) \to \widehat{\mathbb{R}^n},$$

is injective for all $\chi \in \mathbb{R}$, $\chi \neq 0$. Thus also (III) holds. □

Next we consider the subgroup Γ_K of $\mathbb{R}^n$.

LEMMA 3.2.19. *Γ_K is a uniform lattice in $\mathbb{R}^n$.*

PROOF. First, the following equivalences hold.

$A(K, \widehat{\mathbb{R}^n})$ is discrete.

$\iff$ There do not exist elements $u, v \in \mathbb{R}^n$, $v \neq 0$, such that $\langle u, k \rangle + \lambda \langle v, k \rangle \in \mathbb{Z}$ for all $\lambda \in \mathbb{R}$ and $k \in K$.

$\iff$ There does not exist some $v \in \mathbb{R}^n$, $v \neq 0$, such that $\langle v, k \rangle = 0$ for all $k \in K$.

$\iff$ K contains a basis of $\mathbb{R}^n$.

Secondly, we obtain:

$A(K, \widehat{\mathbb{R}^n})$ is cocompact.

$\iff$ There exists a basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ such that $\langle v_i, k \rangle \in \mathbb{Z}$ for all $1 \leq i \leq n$ and $k \in K$.

Now since $A(\{v_1, \dots, v_n\}, \widehat{\mathbb{R}^n}) = A(\langle v_1, \dots, v_n \rangle_{\mathbb{Z}}, \widehat{\mathbb{R}^n})$, we obtain:

$A(K, \widehat{\mathbb{R}^n})$ is cocompact.

$\iff$ There exists a uniform lattice J in $\mathbb{R}^n$ such that $\langle x, k \rangle \in \mathbb{Z}$ for all $x \in J$ and $k \in K$.

$\iff$ There exists a uniform lattice J in $\mathbb{R}^n$ such that $K \subseteq A(J, \widehat{\mathbb{R}^n})$.

The claim follows from [**Rag72**, Theorem 2.12]. $\square$

It remains to check whether the set

$$\{(x, y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m, e))\chi(\tau^{(2)}_l(y, e)) : (l, m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

LEMMA 3.2.20. *The following conditions are equivalent.*

(i) *The set*

$$\{(x, y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m, e))\chi(\tau^{(2)}_l(y, e)) : (l, m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

(ii) K *is a uniform lattice in* $(\mathfrak{g}, +)$.

PROOF. Let $\tilde{K} := A(\Gamma_K, \widehat{\mathbb{R}^n})$. By Lemma 3.2.19, Γ_K is a uniform lattice in $\mathbb{R}^n$, hence also $\tilde{K}$ is a uniform lattice in $\mathbb{R}^n$. Further, let $S_{\tilde{K}}$ be some relatively compact fundamental domain for $\tilde{K}$. Notice that, by the proof of Lemma 3.2.19, we know $K \subseteq \tilde{K}$. Moreover, it is well-known that

$$\{(x, y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m, e))\chi(\tau^{(2)}_l(y, e)) : (l, m) \in \tilde{K} \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_{\tilde{K}} \times \Omega_K)$. Thus the set

$$\{(x, y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m, e))\chi(\tau^{(2)}_l(y, e)) : (l, m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$ if and only if $K = \tilde{K}$. This is equivalent to (ii). $\square$

The previous results yield the following theorem. In the following $K < \mathfrak{g}$ always means $K < (\mathfrak{g}, +)$.

THEOREM 3.2.21. (i) *The Zak transform* $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ *associated with* K *is an isometry.*
(ii) *Suppose that* $K < \mathfrak{g}$. *Then the Zak transform* $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ *associated with* K *is a Hilbert space isomorphism.*

PROOF. This follows immediately from Lemma 3.2.18, Lemma 3.2.19 and Lemma 3.2.20 applied to Proposition 3.2.9 and Theorem 3.2.10. ☐

Now we have to ask, whether there exist uniform lattices in G which are subgroups of $\mathfrak{g}$. The next proposition shows that there indeed exist both uniform lattices which are subgroups of $\mathfrak{g}$ and which are not.

PROPOSITION 3.2.22. *There exist uniform lattices* K_1 *and* K_2 *in* G *such that* $K_1 \not< \mathfrak{g}$ *and* $K_2 < \mathfrak{g}$.

PROOF. Let $\{X_1, \dots, X_n\}$ be a basis of $\mathfrak{g}$ with respect to which the structure constants are rational. Therefore, there exist $p_k^{ij} \in \mathbb{Z}$, $q_k^{ij} \in \mathbb{N}$, $1 \le i, j, k \le n$, such that

$$[X_i, X_j] = \sum_{k=1}^{n} \frac{p_k^{ij}}{q_k^{ij}} X_k \quad (1 \le i, j \le n).$$

Define $r_k \in \mathbb{Q}$, $1 \le k \le n$, by

$$r_k := \frac{\gcd\{p_k^{ij} : 1 \le i, j \le n\}}{\operatorname{lcm}\{q_k^{ij} : 1 \le i, j \le n\}}.$$

Further, let $Y_1, \dots, Y_s \in \{\sum_{k=1}^{n} l_k r_k X_k : l_k \in \mathbb{Z}\}$, $1 \le s \le n$, be a basis of $\mathfrak{g}^1$, the commutator algebra of $\mathfrak{g}$, such that

$$\langle Y_1, \dots, Y_s \rangle_{\mathbb{Z}} = \{\sum_{k=1}^{n} l_k r_k X_k : l_k \in \mathbb{Z}\} \cap \mathfrak{g}^1.$$

Without loss of generality we can assume that $\{X_1, \dots, X_{n-s}, Y_1, \dots, Y_s\}$ is a basis of $\mathfrak{g}$.

We define $K_1 \subseteq G$ by

$$\begin{aligned} K_1 \;:=\; &\{\sum_{i=1}^{n-s} m_i X_i + \sum_{j=1}^{s} l_j Y_j + \frac{1}{2} \sum_{\substack{i,j=1\\ i<j}}^{n-s} m_i m_j [X_i, X_j] : m_i, l_j \in \mathbb{Z} \\ &\text{for all } 1 \le i \le n-s, 1 \le j \le s\}. \end{aligned}$$

First, we are going to prove that $K_1 < (\mathfrak{g}, *)$. For this, let $m_i, m_i', l_j, l_j' \in \mathbb{Z}$ for all $1 \leq i \leq n-s, 1 \leq j \leq s$. Then

$$\begin{aligned}
&\left(\sum_{i=1}^{n-s} m_i X_i + \sum_{j=1}^{s} l_j Y_j + \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{n-s} m_i m_j [X_i, X_j]\right) \\
* &\left(\sum_{i=1}^{n-s} m_i' X_i + \sum_{j=1}^{s} l_j' Y_j + \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{n-s} m_i' m_j' [X_i, X_j]\right) \\
= &\sum_{i=1}^{n-s}(m_i + m_i')X_i + \sum_{j=1}^{s}(l_j + l_j')Y_j + \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{n-s}(m_i m_j + m_i' m_j')[X_i, X_j] \\
&+\frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{n-s}(m_i m_j' - m_i' m_j)[X_i, X_j] \\
= &\sum_{i=1}^{n-s}(m_i + m_i')X_i + \sum_{j=1}^{s}(l_j + l_j')Y_j - \sum_{\substack{i,j=1\\ i<j}}^{n-s} m_i' m_j [X_i, X_j] \\
&+\frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{n-s}(m_i + m_i')(m_j + m_j')[X_i, X_j] \\
\in \ &K_1,
\end{aligned}$$

since

$$\sum_{\substack{i,j=1\\ i<j}}^{n-s} m_i' m_j [X_i, X_j] = \sum_{k=1}^{n}\left[\sum_{\substack{i,j=1\\ i<j}}^{n-s} m_i' m_j \frac{p_k^{ij}}{q_k^{ij}}\right] X_k$$

$$\in \{\sum_{k=1}^{n} l_k r_k X_k : l_k \in \mathbb{Z}\} \cap \mathfrak{g}^1 = \langle Y_1, \ldots, Y_s\rangle_{\mathbb{Z}}.$$

Obviously, this is even a uniform lattice in G.

Now assume, towards a contradiction, that $K_1 < \mathfrak{g}$. Let $u, v \in \{1, \ldots, n\}$ be arbitrary but fixed. This implies that $X_u + X_v$ belongs to K_1, since $X_u, X_v \in K_1$. Hence there exist $m_i, l_j \in \mathbb{Z}$ for all $1 \leq i \leq n-s, 1 \leq j \leq s$, such that

$$X_u + X_v = \sum_{i=1}^{n-s} m_i X_i + \sum_{j=1}^{s} l_j Y_j + \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{n-s} m_i m_j [X_i, X_j].$$

Since $\{X_1, \dots, X_{n-s}, Y_1, \dots, Y_s\}$ is a basis of $\mathfrak{g}$ and $\{Y_1, \dots, Y_s, [X_i, X_j] : 1 \leq i, j \leq n\} \subseteq \mathfrak{g}^1$, we obtain

$$m_u = m_v = 1 \text{ and } m_i = 0 \text{ for all } 1 \leq i \leq n-s, i \neq u, v.$$

Thus there exist $l_j \in \mathbb{Z}$ for all $1 \leq j \leq s$, such that

$$\sum_{j=1}^{s} l_j Y_j + \frac{1}{2}[X_u, X_v] = 0.$$

But, by the choice of r_k, there exist $u, v \in \{1, \dots, n\}$ such that

$$\sum_{k=1}^{n} \frac{1}{2} \frac{p_k^{uv}}{q_k^{uv}} X_k \notin \langle Y_1, \dots, Y_s \rangle_{\mathbb{Z}},$$

a contradiction.

Now we turn to the set $K_2 \subseteq G$ defined by

$$K_2 := \{\sum_{i=1}^{n-s} m_i X_i + \frac{1}{2} \sum_{j=1}^{s} l_j Y_j : m_i, l_j \in \mathbb{Z} \text{ for all } 1 \leq i \leq n-s, 1 \leq j \leq s\}.$$

For $m_i, m_i', l_j, l_j' \in \mathbb{Z}$, $1 \leq i \leq n-s, 1 \leq j \leq s$,

$$\begin{aligned} &\left(\sum_{i=1}^{n-s} m_i X_i + \frac{1}{2}\sum_{j=1}^{s} l_j Y_j\right) * \left(\sum_{i=1}^{n-s} m_i' X_i + \frac{1}{2}\sum_{j=1}^{s} l_j' Y_j\right) \\ = \;& \sum_{i=1}^{n-s}(m_i + m_i')X_i + \frac{1}{2}\sum_{j=1}^{s}(l_j + l_j')Y_j + \frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^{n-s}(m_i m_j' - m_i' m_j)[X_i, X_j] \\ \in \;& K_2 \end{aligned}$$

by the same argument as above. This implies $K_2 < G$. Moreover, K_2 is also a uniform lattice in G. Notice that K_2 coincides with the lattice in $\mathfrak{g}$ generated by $\{X_1, \dots, X_{n-s}, \frac{1}{2}Y_1, \dots, \frac{1}{2}Y_s\}$. Thus $K_2 < \mathfrak{g}$. □

REMARK 3.2.23. Let $G = \mathbb{R}$ and $K = \mathbb{Z}$. Then the action τ defined here yields the classical Zak transform.

The action τ seems to be defined by chance, but its definition is quite natural as we will see in a moment. Recall that the classical Heisenberg group can be constructed using the so-called position and momentum operator. These operators together with the identity operator generate a Lie algebra and then the classical Heisenberg group is defined to be the associated Lie group ([**Fol89**]). This construction may be applied to a more general setting, namely when $\mathbb{R}$ is replaced by $H(\mathbb{R})$. This was done by Folland ([**Fol89**, p. 90]). Here we generalize this approach so that we may apply it to an arbitrary connected and simply connected 2-step nilpotent Lie group G.

For this, let $\{X_1, \dots, X_n\}$ be a basis of $\mathfrak{g}$. Moreover, let $c_k^{ij} \in \mathbb{R}$, $1 \leq i, j, k \leq n$, be defined by

$$[X_i, X_j] = \sum_{k=1}^{n} c_k^{ij} X_k \quad \text{for all } 1 \leq i, j \leq n.$$

As mentioned in the beginning of this subsection we may identify G with $(\mathfrak{g}, *)$. Let $\mathcal{S}(G)$ denote the class of Schwartz functions on G. The replacement of the position operator will be the operators $Q_1, \dots, Q_n : \mathcal{S}(G) \to \mathcal{S}(G)$ defined by

$$Q_i f(x_1, \dots, x_n) = x_i f(x_1, \dots, x_n),\ i = 1, \dots, n.$$

The following operators will play the role of the momentum operator. Let $P_1, \dots, P_n : \mathcal{S}(G) \to \mathcal{S}(G)$ be defined by

$$P_i f(x_1, \dots, x_n) = \frac{\partial f}{\partial x_i}(x_1, \dots, x_n) + \frac{1}{2} \sum_{k=1}^{n} \left(\sum_{j=1}^{n} c_k^{ji} x_j \right) \frac{\partial f}{\partial x_k}(x_1, \dots, x_n).$$

It is easily checked that these operators are derivations, which are left-invariant, that means

$$P_i(L_x f) = L_x(P_i f) \quad (x \in G).$$

Further, let $I : \mathcal{S}(G) \to \mathcal{S}(G)$ be defined by

$$I = \mathrm{Id}_{\mathcal{S}(G)}.$$

The operators $\{Q_i, P_i, I : 1 \leq i \leq n\}$ generate a Lie algebra $\mathfrak{h}$. It is again a straightforward but long calculation to check that

$$\begin{aligned}
[P_i, P_j] &= \sum_{k=1}^{n} c_k^{ij} P_k \quad \text{for all } 1 \leq i, j \leq n, \\
[P_i, Q_i] &= I + \frac{1}{2} \sum_{k=1}^{n} c_i^{ki} Q_k \quad \text{for all } 1 \leq i \leq n \quad \text{and} \\
[P_i, Q_j] &= \frac{1}{2} \sum_{k=1}^{n} c_j^{ki} Q_k \quad \text{for all } 1 \leq i, j \leq n, i \neq j.
\end{aligned}$$

Hence we constructed a 3-step nilpotent Lie algebra with

$$\begin{aligned}
\mathfrak{h}^{(1)} &= \langle I, \sum_{k=1}^{n} c_j^{ki} Q_k (1 \leq i, j \leq n), \sum_{k=1}^{n} c_k^{ij} P_k (1 \leq i, j \leq n) \rangle, \\
\mathfrak{h}^{(2)} &= \langle I \rangle, \\
\mathfrak{h}^{(3)} &= \{0\}.
\end{aligned}$$

Notice that $\mathfrak{g} = \langle P_1, \dots, P_n \rangle$ is a subalgebra of $\mathfrak{h}$. Moreover, $\mathfrak{h}$ is even a semi-direct product, where $\mathfrak{g}$ acts on $\langle Q_1, \dots, Q_n, I \rangle$. By the Baker-Campbell-Hausdorff formula, the Lie group associated with $\mathfrak{h}$ is isomorphic to $\mathfrak{h}$ as a set endowed with the following group multiplication

$$\left(\sum_{k=1}^{n} x_k P_k + \sum_{k=1}^{n} y_k Q_k + zI\right) * \left(\sum_{k=1}^{n} x'_k P_k + \sum_{k=1}^{n} y'_k Q_k + z'I\right)$$
$$= \sum_{k=1}^{n}(x_k + x'_k + \frac{1}{2}\sum_{i,j=1}^{n} x_i x'_j c_k^{ij})P_k + \sum_{k=1}^{n}(y_k + y'_k + \frac{1}{4}\sum_{i,j=1}^{n}(x_i y'_j - y_j x'_i)c_j^{ki})Q_k$$
$$+(z + z' + \frac{1}{2}\sum_{i=1}^{n}(x_i y'_i - y_i x'_i) + \frac{1}{8}\sum_{i,j,k=1}^{n}(y'_k - y_k)x_i x'_j c_k^{ij})I.$$

Let $x_1, \ldots, x_n, y_1, \ldots, y_n, z \in \mathbb{R}$ and define $X, Y \in \mathfrak{h}$ by $X = x_1P_1 + \ldots + x_nP_n$, $Y = y_1Q_1 + \ldots + y_nQ_n + zI$. Then the action τ of $\mathfrak{g}$ on $\langle Q_1, \ldots, Q_n, I\rangle$ is given by

$$\begin{aligned}\tau_X(Y) &= X * Y * (-X) \\ &= \sum_{k=1}^{n}(y_k + \frac{1}{2}\sum_{i,j=1}^{n} x_i y_j c_j^{ki})Q_k + (z + \langle x, y\rangle)I.\end{aligned}$$

We claim that

$$\frac{1}{2}(y + \mathrm{Ad}_x^*(y)) = (y_k + \frac{1}{2}\sum_{i,j=1}^{n} x_i y_j c_j^{ki})_{1\le k\le n}$$

for all $x \in G$, $y \in \mathbb{R}^n$.

For this, let $X = x_1P_1 + \ldots + x_nP_n, W = w_1P_1 + \ldots + w_nP_n \in \mathfrak{g}$ and let $l = y_1P_1^* + \ldots + y_nP_n^* \in \mathfrak{g}^*$. Then, by using the proof of Lemma 3.2.18,

$$\begin{aligned}\frac{1}{2}(l + \mathrm{Ad}_{\exp X}^* l)(W) &= l(W + \frac{1}{2}[W, X]) \\ &= l(\sum_{k=1}^{n}(w_k + \frac{1}{2}\sum_{i,j=1}^{n} c_k^{ij} w_i x_j)P_k) \\ &= \sum_{k=1}^{n}(w_k y_k + \frac{1}{2}\sum_{i,j=1}^{n} c_k^{ij} y_k w_i x_j) \\ &= \sum_{k=1}^{n}(y_k + \frac{1}{2}\sum_{i,j=1}^{n} x_i y_j c_j^{ki})w_k \\ &= \tilde{l}(W),\end{aligned}$$

where

$$\tilde{l} = \sum_{k=1}^{n}(y_k + \frac{1}{2}\sum_{i,j=1}^{n} x_i y_j c_j^{ki})P_k^*.$$

CHAPTER 4

Linear independence of generalized time-frequency shifts

In [**HRT96**] Heil, Ramanathan and Topiwala raised the question whether there exist non-trivial subsets $D \subseteq \mathbb{R} \times \mathbb{R}$ such that, for all $f \in L^2(\mathbb{R})$, $f \neq 0$, and for all finite subsets Λ of D, the set $\{\rho_{\mathbb{R}}(x, y, 1)f : (x, y) \in \Lambda\}$ is linearly independent. This chapter is concerned with extending and generalizing some of the results therein.

In the second section we improve one result in [**HRT96**] by extending their method of proof.

Furthermore, we generalize the question mentioned above to the situation of locally compact abelian groups and establish necessary and sufficient conditions under which we can give a positive answer to the question. This is done in the third section. For this, we need some results about automorphisms of the Heisenberg group associated with a locally compact abelian group. These are given in the first section, since they are also used in the second one.

4.1. Automorphisms of the Heisenberg group

Let G be a locally compact abelian group. In this section we establish some properties of automorphisms of $H(G)$. These will be used in the remainder of this chapter.

At first, consider the case $G = \mathbb{R}$. For the classical Heisenberg group (compare Chapter 1) all automorphisms are well-known. They are compositions of symplectic maps, inner automorphisms, dilations and inversions ([**Fol89**, Theorem 1.22]). Now the group $H(G)$ is a generalization of the reduced Heisenberg group $H(\mathbb{R})$. We will just see that also in this situation we can give a complete characterization (Proposition 4.1.3). For this, we have to restrict our attention to a special class of locally compact abelian groups. However, it will turn out that there exists an abundance of groups belonging to this class.

First, we give some properties of automorphisms of Heisenberg groups associated with arbitrary locally compact abelian groups.

PROPOSITION 4.1.1. *Let G be a locally compact abelian group and α a topological automorphism of $H(G)$. Then the following is true.*

(i) *We have*

$$\textit{either} \quad \alpha(e, 1, z) = (e, 1, z) \quad \textit{or} \quad \alpha(e, 1, z) = (e, 1, \bar{z})$$

for all $z \in \mathbb{T}$.

(ii) *There exists a topological automorphism* φ *of* $G \times \widehat{G}$ *such that, for all* $(x, \omega, z) \in H(G)$,

$$\alpha(x, \omega, z) = (\varphi(x, \omega), \alpha_3(x, \omega, z)).$$

PROOF. At first, we prove (i). Since $\alpha \in \mathrm{Aut}(H(G))$, α induces a topological automorphism of the center of $H(G)$, which is $\{e\} \times \{1\} \times \mathbb{T}$. This shows (i).

Now, by (i), we obtain

$$\alpha(x, \omega, z) = \alpha(x, \omega, 1)\alpha(e, 1, z) = (\alpha_1(x, \omega, 1), \alpha_2(x, \omega, 1), \alpha_3(x, \omega, z))$$

for all $(x, \omega, z) \in H(G)$. Define $\varphi : G \times \widehat{G} \to G \times \widehat{G}$ by

$$\varphi(x, \omega) = (\alpha_1(x, \omega, 1), \alpha_2(x, \omega, 1)).$$

Then, for all $(x, \omega, z) \in H(G)$,

$$\alpha(x, \omega, z) = (\varphi(x, \omega), \alpha_3(x, \omega, z)).$$

It remains to show that φ is a topological isomorphism. But this follows immediately from the fact that φ is the topological automorphism of $H(G)/\mathbb{T}$ induced by α. This finishes the proof of (ii). □

In the following we will restrict to a special class of locally compact abelian groups.

DEFINITION 4.1.2. Let G be a locally compact abelian group. G is called *2-complete*, if the equation $x^2 = y$ has a unique solution for each $y \in G$.

There are lots of examples of groups satisfying this property, the easiest are $\mathbb{R}^q, \mathbb{Q}^r$, $q, r \in \mathbb{N}$ and $\mathbb{Z}_p$, where p is a prime and $p > 2$. The set of 2-complete groups is closed under direct products and inductive limits. An abundance of 2-complete groups can be constructed as follows. Let G be 2-complete, H an arbitrary locally compact abelian group and $\psi : G \to H$ a surjective homomorphism. Then the subgroup $\{(x, \psi(x)) : x \in G\} \subseteq G \times H$ is 2-complete.

In Proposition 4.1.1 we just proved the existence of a topological automorphism of $G \times \widehat{G}$ associated to $\alpha \in \mathrm{Aut}(H(G))$. In the following we give a more detailed characterization of these topological automorphisms.

PROPOSITION 4.1.3. *Let* G *be a locally compact abelian group such that at least one of the groups* G *and* $\widehat{G}$ *is 2-complete. Let* $\varphi : G \times \widehat{G} \to G \times \widehat{G}$ *be a topological automorphism. Then the following conditions are equivalent.*

(i) *For all* $(x, \omega), (x', \omega') \in G \times \widehat{G}$,

$$\varphi_2(x, \omega)(\varphi_1(x', \omega')) \cdot \overline{\varphi_2(x', \omega')(\varphi_1(x, \omega))} = \omega(x')\overline{\omega'(x)}.$$

(ii) *There exists a topological automorphism* α *of* $H(G)$ *such that, for all* $x \in G, \omega \in \widehat{G}, z \in \mathbb{T}$,

(a) $\alpha(x, \omega, z) = (\varphi(x, \omega), \alpha_3(x, \omega, z))$ *and*

(b) $\alpha(e, 1, z) = (e, 1, z)$.

Further, the following conditions are equivalent.

(i') *For all* $(x,\omega),(x',\omega') \in G \times \widehat{G}$,

$$\varphi_2(x,\omega)(\varphi_1(x',\omega')) \cdot \overline{\varphi_2(x',\omega')(\varphi_1(x,\omega))} = \overline{\omega(x')}\omega'(x).$$

(ii') *There exists a topological automorphism* α *of* $H(G)$ *such that, for all* $x \in G, \omega \in \widehat{G}, z \in \mathbb{T}$,

(a') $\alpha(x,\omega,z) = (\varphi(x,\omega), \alpha_3(x,\omega,z))$ *and*

(b') $\alpha(e,1,z) = (e,1,\bar{z})$.

Proof. We only prove the equivalence of (i) and (ii). The equivalence of (i') and (ii') is proven in an analogous way.

Suppose first that (i) holds and that $\widehat{G}$ is 2-complete. If G is 2-complete we can use a similar argument for the proof. We can define $\alpha : H(G) \to H(G)$ by

$$\alpha(x,\omega,z) := (\varphi(x,\omega), \alpha_3(x,\omega,z)),$$

where, for $(x,\omega,z) \in H(G)$,

$$\alpha_3(x,\omega,z) := z\, \overline{\omega^{\frac{1}{2}}(x)} \left(\varphi_2(x,\omega)^{\frac{1}{2}}(\varphi_1(x,\omega))\right).$$

Then properties (a) and (b) are obvious. It remains to show that α is a topological isomorphism. For every $(x,\omega),(x',\omega') \in G \times \widehat{G}$, we obtain that

$$\begin{aligned}
&\alpha_3(xx',\omega\omega',1)\\
&= \overline{\omega^{\frac{1}{2}}(x)}\,\overline{\omega^{\frac{1}{2}}(x')}\,\overline{\omega'^{\frac{1}{2}}(x)}\,\overline{\omega'^{\frac{1}{2}}(x')} \cdot (\varphi_2(x,\omega)^{\frac{1}{2}}(\varphi_1(x,\omega)))\\
&\quad \cdot(\varphi_2(x,\omega)^{\frac{1}{2}}(\varphi_1(x',\omega'))) \cdot (\varphi_2(x',\omega')^{\frac{1}{2}}(\varphi_1(x,\omega)))\\
&\quad \cdot(\varphi_2(x',\omega')^{\frac{1}{2}}(\varphi_1(x',\omega')))\\
&\overset{\text{(i)}}{=} \alpha_3(x,\omega,1)\alpha_3(x',\omega',1) \cdot \left(\overline{\omega^{\frac{1}{2}}(x')}\,\overline{\omega'^{\frac{1}{2}}(x)} \left(\varphi_2(x',\omega')^{\frac{1}{2}}(\varphi_1(x,\omega))\right.\right.\\
&\quad \left.\cdot\, \omega^{\frac{1}{2}}(x')\overline{\omega'^{\frac{1}{2}}(x)}\right) \left.\left(\varphi_2(x',\omega')^{\frac{1}{2}}(\varphi_1(x,\omega))\right)\right)\\
&= \alpha_3(x,\omega,1)\alpha_3(x',\omega',1)\overline{\omega'(x)}\varphi_2(x',\omega')(\varphi_1(x,\omega)).
\end{aligned}$$

Hence α is a homomorphism. Furthermore, α is continuous. In addition, α is bijective, since φ is a topological isomorphism, and its inverse is also continuous. This establishes (ii).

To prove the converse direction, suppose that α is a topological automorphism of $H(G)$ satisfying properties (a) and (b). Using condition (a), we calculate

$$\begin{aligned}
&(\varphi(x,\omega)\varphi(x',\omega'), \alpha_3(xx',\omega\omega',zz'\omega'(x)))\\
&= \alpha((x,\omega,z)(x',\omega',z'))\\
&= \alpha(x,\omega,z)\alpha(x',\omega',z')\\
&= (\varphi(x,\omega)\varphi(x',\omega'), \alpha_3(x,\omega,z)\alpha_3(x',\omega',z')\varphi_2(x',\omega')(\varphi_1(x,\omega))).
\end{aligned}$$

Moreover, using condition (b), we obtain $\alpha_3(x,\omega,z) = z\,\alpha_3(x,\omega,1)$. This implies that

$$\alpha_3(xx',\omega\omega',1) = \alpha_3(x,\omega,1)\alpha_3(x',\omega',1)\left(\overline{\omega'(x)}\varphi_2(x',\omega')(\varphi_1(x,\omega))\right) \tag{13}$$

for all $(x,\omega),(x',\omega') \in G \times \widehat{G}$. From (13) we obtain that

$$\overline{\omega'(x)}\varphi_2(x',\omega')(\varphi_1(x,\omega)) = \overline{\omega(x')}\varphi_2(x,\omega)(\varphi_1(x',\omega')).$$

This finishes the proof of (ii) ⇒ (i). □

Notice that the implication (ii) ⇒ (i) and (ii') ⇒ (i') also hold if none of G and $\widehat{G}$ is supposed to be 2-complete.

Let $\varphi : G\times\widehat{G} \to G\times\widehat{G}$ be a topological automorphism satisfying property (i) or (i') of Proposition 4.1.3. Then a suitable topological automorphism α of $H(G)$ with (ii) or (ii') can explicitly be given. Here, for $\omega \in \widehat{G}$ and $x \in G$, $(\omega(x))^{\frac{1}{2}}$ means $\omega^{\frac{1}{2}}(x)$, if $\widehat{G}$ is 2-complete, otherwise it means $\omega(x^{\frac{1}{2}})$.

COROLLARY 4.1.4. *Let G be a locally compact abelian group such that at least one of the groups G and $\widehat{G}$ is 2-complete. Further, let $\varphi : G\times\widehat{G} \to G\times\widehat{G}$ be a topological automorphism.*

(i) *Suppose that, for all $(x,\omega),(x',\omega') \in G\times\widehat{G}$,*

$$\varphi_2(x,\omega)(\varphi_1(x',\omega'))\cdot\overline{\varphi_2(x',\omega')(\varphi_1(x,\omega))} = \omega(x')\overline{\omega'(x)}.$$

Then $\alpha : H(G) \to H(G)$ defined by

$$\alpha(x,\omega,z) = (\varphi(x,\omega), z\,(\overline{\omega(x)})^{\frac{1}{2}}\,(\varphi_2(x,\omega)(\varphi_1(x,\omega)))^{\frac{1}{2}})$$

is a topological automorphism.

(ii) *Suppose that, for all $(x,\omega),(x',\omega') \in G\times\widehat{G}$,*

$$\varphi_2(x,\omega)(\varphi_1(x',\omega'))\cdot\overline{\varphi_2(x',\omega')(\varphi_1(x,\omega))} = \overline{\omega(x')}\omega'(x).$$

Then $\alpha : H(G) \to H(G)$ defined by

$$\alpha(x,\omega,z) = (\varphi(x,\omega), \bar{z}\,(\overline{\omega(x)})^{\frac{1}{2}}\,(\varphi_2(x,\omega)(\varphi_1(x,\omega)))^{\frac{1}{2}})$$

is a topological automorphism.

PROOF. (i) and (ii) follow immediately from the proof of Proposition 4.1.3. □

At the end of this section let us focus on the case $G = \mathbb{R}$. The following result appears partly in [**FG97**] and [**HRT96**] without proof.

REMARK 4.1.5. Let α be a topological automorphism of $H(\mathbb{R})$. Then, by Proposition 4.1.1, there exists $M \in GL(2,\mathbb{R})$ such that α is of the form

$$\alpha(x,y,z) = (M(x,y), z\,\alpha_3(x,y,z)) \quad ((x,y,z)\in H(\mathbb{R})), \tag{14}$$

or

$$\alpha(x,y,z) = (M(x,y), \bar{z}\,\alpha_3(x,y,z)) \quad ((x,y,z)\in H(\mathbb{R})). \tag{15}$$

By Proposition 4.1.3, we have $\det M = 1$ in the case (14) and $\det M = -1$ in the case (15). Thus $|\det M| = 1$.

Conversely, suppose $M = (m_{i,j})_{1 \le i,j \le 2} \in \{A \in M(2,\mathbb{R}) : |\det A| = 1\}$. Then, by Corollary 4.1.4, $\alpha : H(\mathbb{R}) \to H(\mathbb{R})$ defined by

$$\alpha(x,y,z) = \big(M(x,y), z\, e^{\pi i[-yx+(m_{2,1}x+m_{2,2}y)(m_{1,1}x+m_{1,2}y)]}\big)$$

is a topological automorphism in the case that $\det M = 1$. And in the case that $\det M = -1$, $\alpha : H(\mathbb{R}) \to H(\mathbb{R})$ defined by

$$\alpha(x,y,z) = \big(M(x,y), \bar{z}\, e^{\pi i[-yx+(m_{2,1}x+m_{2,2}y)(m_{1,1}x+m_{1,2}y)]}\big)$$

is a topological automorphism.

4.2. Linear independence in the case $G = \mathbb{R}$

In 1996, Heil, Ramanathan and Topiwala [**HRT96**] considered the question under which conditions the subset $\{\rho_{\mathbb{R}}(x,\omega,1)f : (x,\omega) \in \Lambda\}$ of $L^2(\mathbb{R})$ is linearly independent for a finite subset $\Lambda \subseteq \mathbb{R} \times \mathbb{R}$ and every function $f \in L^2(\mathbb{R})$, $f \neq 0$. They conjectured that the set $\{\rho_{\mathbb{R}}(x,\omega,1)f : (x,\omega) \in \Lambda\}$ is linearly independent for any finite subset $\Lambda \subseteq \mathbb{R} \times \mathbb{R}$ and any function $f \in L^2(\mathbb{R})$, $f \neq 0$. They did not succeed to prove it, but instead [**HRT96**] contains several partial results. For applications, the most important cases of finite subsets of $\mathbb{R} \times \mathbb{R}$ are subsets of lattices in $\mathbb{R} \times \mathbb{R}$. Now naturally the question arises whether, for any finite subset Λ of some lattice in $\mathbb{R} \times \mathbb{R}$, the set $\{\rho_{\mathbb{R}}(x,\omega,1)f : (x,\omega) \in \Lambda\}$ is always linearly independent for all $f \in L^2(\mathbb{R})$, $f \neq 0$. For finite subsets of unit lattices, this has been proven in [**HRT96**]. The proof requires only Gabor-analytic methods and relies, in particular, on the Zak transform. In this context unit lattice means that the parallelogram spanned by the generating vectors of the lattice has an area of measure 1. Later, Linnell [**Lin99**] extended this result to finite subsets of arbitrary lattices in $\mathbb{R} \times \mathbb{R}$. This is proven using von-Neumann algebra theory, that means, in particular, with methods from different areas in mathematics. Now it is very interesting to investigate to which extent this result could be proven using only Gabor-analytic methods. It turns out that it is possible to cover a large class of discrete subgroups of $\mathbb{R} \times \mathbb{R}$ which includes the subgroups investigated by Heil, Ramanathan and Topiwala.

We need some preparations before we can state the theorem.

LEMMA 4.2.1. *Let K be a uniform lattice in $\mathbb{R} \times \mathbb{R}$. Furthermore, let $\{(a,b),(c,d)\}$, $a,b,c,d \in \mathbb{R}$, be a generating system of K. Then the value of $m(K) := |ad - bc|$ is independent of the choice of the generating system.*

PROOF. Let $E_1 := \{(a,b),(c,d)\}$, $a,b,c,d \in \mathbb{R}$, and $E_2 := \{(a',b'),(c',d')\}$, $a',b',c',d' \in \mathbb{R}$, be generating systems of K. In particular, they are bases of $\mathbb{R}^2$, since K is a uniform lattice. Let T denote the linear map which maps E_1 onto E_2, hence

$$T = \frac{1}{ad-bc}\begin{pmatrix} a'd - bc' & ac' - a'c \\ b'd - bd' & ad' - b'c \end{pmatrix}.$$

Since (a', b') and (c', d') belong to K, they can be represented as a linear combination of E_1 with integer coefficients. Hence there exist integers m_1, m_2, m_3 and m_4 such that

$$\begin{aligned}(a', b') &= m_1(a, b) + m_2(c, d), \\ (c', d') &= m_3(a, b) + m_4(c, d).\end{aligned}$$

Now let $\tilde{T}$ be the matrix defined by

$$\tilde{T} := \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix}.$$

Furthermore, let the transformation matrix of the basis transformation ((standard basis of $\mathbb{R}^2$) $\mapsto E_1$) be denoted by S, in particular, $S(a, b) = (1, 0)$ and $S(c, d) = (0, 1)$. It is easily checked that

$$T = S^{-1}\tilde{T}S.$$

Notice that $\det \tilde{T} \in \mathbb{Z}$. Hence $\det T \in \mathbb{Z}$. In a similar way we can prove that $\det T^{-1} \in \mathbb{Z}$. This implies

$$|\det T| = 1.$$

Moreover, it is easily checked that

$$\det T = \frac{a'd' - b'c'}{ad - bc}.$$

This finishes the proof. □

The value of $m(K)$ has the following meaning. Let K be a uniform lattice in $\mathbb{R} \times \mathbb{R}$ and let $\{(a, b), (c, d)\}$, $a, b, c, d \in \mathbb{R}$, be a generating system of K. Then

$$\|(0, 0, m(K))\|_2 = \|(a, b, 0) \times (c, d, 0)\|_2,$$

where $\times$ denotes the vector product. Thus $m(K) = |ad - bc|$ is the measure of the area of the parallelogram spanned by (a, b) and (c, d) and hence the measure of the fundamental area of K.

Let K_1 and K_2 be uniform lattices in $\mathbb{R}\times\mathbb{R}$ and let K_3 be a uniform lattice in $H(\mathbb{R})$ such that $K_2 = K_3\mathbb{T}/\mathbb{T}$. It is often important to know whether there exists a topological automorphism α of $H(\mathbb{R})$ which maps $K_1 \times \{1\}$ onto K_3. Since then K_1 and K_2 "belong to the same class" in the sense of the problem in [**HRT96**]. The following lemma is also needed for the next theorem to reduce the problem from a lattice in $\mathbb{R} \times \mathbb{R}$ to a lattice of the form $r\mathbb{Z} \times \mathbb{Z}$, $r \in \mathbb{R}$.

LEMMA 4.2.2. *Let K be a uniform lattice in $\mathbb{R} \times \mathbb{R}$. Further, let $r \in \mathbb{R}$. Then the following conditions are equivalent.*

(i) *There exists a topological automorphism α of $H(\mathbb{R})$ such that*

$$(\alpha(K \times \{1\})\mathbb{T})/\mathbb{T} = r\mathbb{Z} \times \mathbb{Z}.$$

(ii) $|r| = m(K)$.

PROOF. Let $\{(a,b),(c,d)\}$, $a,b,c,d \in \mathbb{R}$, be a generating system of K. Suppose first that (ii) holds. Let $M \in M(2,\mathbb{R})$ be defined by

$$M := \begin{pmatrix} d & -c \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}.$$

It it easily checked that $\det M = 1$. Now define $\alpha : H(\mathbb{R}) \to H(\mathbb{R})$ by

$$\alpha(x,y,z) = (M(x,y), ze^{\pi i[-yx+\frac{(-bx+ay)(dx-cy)}{ad-bc}]}).$$

By Remark 4.1.5, $\alpha \in \mathrm{Aut}(H(\mathbb{R}))$. Moreover, we obtain

$$\alpha(a,b,1) = (ad-bc, 0, e^{-\pi iab}) \quad \text{and} \quad \alpha(c,d,1) = (0,1,e^{-\pi icd}).$$

Combining these facts gives

$$(\alpha(K \times \{1\})\mathbb{T})/\mathbb{T} = m(K)\mathbb{Z} \times \mathbb{Z} = r\mathbb{Z} \times \mathbb{Z}.$$

Conversely, suppose that (i) holds. By the first part of the proof, there exists a topological automorphism β of $H(\mathbb{R})$ such that

$$(\beta(K \times \{1\})\mathbb{T})/\mathbb{T} = m(K)\mathbb{Z} \times \mathbb{Z}.$$

Without loss of generality we may assume that there exist $z_1, \tilde{z}_1 \in \mathbb{T}$ such that

$$\beta(a,b,1) = (m(K),0,z_1) \quad \text{and} \quad \beta(c,d,1) = (0,1,\tilde{z}_1).$$

By hypothesis, there exists a topological automorphism α of $H(\mathbb{R})$ such that

$$(\alpha(K \times \{1\})\mathbb{T})/\mathbb{T} = r\mathbb{Z} \times \mathbb{Z}.$$

Hence there exist integers m_1, m_2, n_1 and n_2 and complex numbers z_2 and $\tilde{z}_2$ such that

$$\alpha(a,b,1) = (m_1 r, n_1, z_2) \quad \text{and} \quad \alpha(c,d,1) = (m_2 r, n_2, \tilde{z}_2).$$

Since α is a topological automorphism, the set $\{(m_1 r, n_1), (m_2 r, n_2)\}$ is a generating system of $r\mathbb{Z} \times \mathbb{Z}$. Note that also $\{(r,0),(0,1)\}$ is a generating system of $r\mathbb{Z} \times \mathbb{Z}$. Then, by Lemma 4.2.1,

$$|m_1 r n_2 - n_1 m_2 r| = m(r\mathbb{Z} \times \mathbb{Z}) = |r|.$$

This implies that

$$|m_1 n_2 - n_1 m_2| = 1. \tag{16}$$

Now consider the composition $\alpha \circ \beta^{-1} : H(\mathbb{R}) \to H(\mathbb{R})$. Obviously, this is a topological automorphism with the properties

$$\begin{aligned} \alpha \circ \beta^{-1}(m(K),0,z_1) &= (m_1 r, n_1, z_2) \quad \text{and} \\ \alpha \circ \beta^{-1}(0,1,\tilde{z}_1) &= (m_2 r, n_2, \tilde{z}_2). \end{aligned} \tag{17}$$

By Remark 4.1.5, there exists $M \in \{A \in M(2,\mathbb{R}) : |\det A| = 1\}$ such that $\alpha \circ \beta^{-1}$ is of the form

$$\alpha \circ \beta^{-1}(x,y,z) = (M(x,y), (\alpha \circ \beta^{-1})_3(x,y,z)).$$

Now (17) implies that

$$M = \begin{pmatrix} \frac{m_1 r}{m(K)} & m_2 r \\ \frac{n_1}{m(K)} & n_2 \end{pmatrix}.$$

Using (16), we obtain

$$|\det M| = \left|\frac{r}{m(K)}\right| |m_1 n_2 - n_1 m_2| = \left|\frac{r}{m(K)}\right|.$$

Hence $\left|\frac{r}{m(K)}\right| = 1$ and (ii) holds. □

In the following theorem we extend a result of Heil, Ramanathan and Topiwala [**HRT96**, Proposition 2] to finite subsets Λ of uniform lattices $K < \mathbb{R} \times \mathbb{R}$ satisfying the property that $m(K) \in \mathbb{Q}$. Lemma 4.2.2 shows that this cannot be achieved by just applying a topological automorphism of $H(\mathbb{R})$ to a unit lattice. Moreover, notice that our proof only uses Gabor-analytic methods, in particular, the Zak transform, in contrast to the proof of Linnell [**Lin99**, Theorem 1.2]. The idea of using the Zak transform already appears in [**HRT96**, Proposition 2]. Here we extend this method.

THEOREM 4.2.3. *Let K be a uniform lattice in $\mathbb{R}\times\mathbb{R}$ such that $m(K) \in \mathbb{Q}$. Further, let Λ be a finite subset of K. Then, for all $f \in L^2(\mathbb{R})$, $f \neq 0$, the subset of $L^2(\mathbb{R})$,*

$$\{\rho_{\mathbb{R}}(x,y,1)f : (x,y) \in \Lambda\},$$

is linearly independent.

PROOF. By Lemma 4.2.2, there exists a topological automorphism α of $H(\mathbb{R})$ such that

$$(\alpha(K \times \{1\})\mathbb{T})/\mathbb{T} = m(K)\mathbb{Z} \times \mathbb{Z}. \tag{18}$$

Moreover, Proposition 4.1.1 shows that without loss of generality we may assume that $\alpha(0,0,z) = (0,0,z)$ for all $z \in \mathbb{T}$. Now define $(\rho_{\mathbb{R}})_\alpha : H(\mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}))$ by

$$(\rho_{\mathbb{R}})_\alpha := \rho_{\mathbb{R}} \circ \alpha.$$

Notice that, for all $z \in \mathbb{T}$ and $f \in L^2(\mathbb{R})$, $(\rho_{\mathbb{R}})_\alpha(0,0,z)f = \rho_{\mathbb{R}}(\alpha(0,0,z))f = zf$. By the Stone-von Neumann theorem, this implies that $(\rho_{\mathbb{R}})_\alpha$ is unitarily equivalent to $\rho_{\mathbb{R}}$. Hence there exists a $U \in \mathcal{U}(L^2(\mathbb{R}))$ such that $\rho_{\mathbb{R}} = U \circ (\rho_{\mathbb{R}})_\alpha \circ U^{-1}$. Thus $\{\rho_{\mathbb{R}}(x,y,1)f : (x,y) \in \Lambda\}$ is linearly independent for all $f \in L^2(\mathbb{R})$, $f \neq 0$, if and only if this is true for the set $\{U((\rho_{\mathbb{R}})_\alpha(x,y,1)(U^{-1}f)) : (x,y) \in \Lambda\}$. Hence, for proving the theorem, it suffices to show that the set $\{\rho_{\mathbb{R}}(\alpha(x,y,1))f : (x,y) \in \Lambda\}$ is linearly independent for all $f \in L^2(\mathbb{R})$, $f \neq 0$. By Remark 4.1.5, the elements $(\alpha_1(x,y,1), \alpha_2(x,y,1))$, $(x,y) \in \Lambda$, are pairwise different. Moreover, we have $\rho_{\mathbb{R}}(x,y,z) = \rho_{\mathbb{R}}(x,y,1) \circ \rho_{\mathbb{R}}(0,0,z)$ for all $(x,y,z) \in H(\mathbb{R})$ and $\rho_{\mathbb{R}}(0,0,z)f = zf$ for all $z \in \mathbb{T}$, $f \in L^2(\mathbb{R})$. Thus the component of $\mathbb{T}$ plays no role. Combining these facts, it suffices to show that the set

$$\{\rho_{\mathbb{R}}(x,y,1)f : (x,y) \in (\alpha(\Lambda \times \{1\})\mathbb{T})/\mathbb{T}\}$$

is linearly independent for all $f \in L^2(\mathbb{R})$, $f \neq 0$.

Now, by hypothesis, there exist coprime integers p, q such that $\frac{p}{q} = m(K)$. Further, using (18), there exist some $N \in \mathbb{N}$ and $m_j, n_j \in \mathbb{Z}$, $1 \leq j \leq N$, such that the elements (m_j, n_j), $1 \leq j \leq N$, are pairwise different and such that

$$(\alpha(\Lambda \times \{1\})\mathbb{T})/\mathbb{T} = \{(\frac{1}{q}m_j, n_j) : 1 \leq j \leq N\}.$$

Note that it remains to prove the linear independence of the set

$$\{\rho_{\mathbb{R}}(\frac{1}{q}m_j, n_j, 1)f : 1 \leq j \leq N\}$$

for all $f \in L^2(\mathbb{R})$, $f \neq 0$.

For this, let $f \in L^2(\mathbb{R})$, $f \neq 0$, and let $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ be such that

$$\sum_{j=1}^{N} \lambda_j \rho_{\mathbb{R}}(\frac{1}{q}m_j, n_j, 1)f = 0. \tag{19}$$

Now we may apply the Zak transform $Z : L^2(\mathbb{R}) \to L^2([0,1)^2)$ (compare Definition 3.1.4) to (19). We obtain that, for almost all $(x,y) \in [0,1)^2$,

$$\begin{aligned} 0 &= Z\left(\sum_{j=1}^{N} \lambda_j \rho_{\mathbb{R}}(\frac{1}{q}m_j, n_j, 1)f\right)(x,y) \\ &= \sum_{j=1}^{N} \lambda_j \sum_{k\in\mathbb{Z}} e^{2\pi iky} e^{2\pi i n_j(x+k)} f(x+k+\frac{1}{q}m_j) \\ &= \sum_{j=1}^{N} \lambda_j e^{2\pi i n_j x} \sum_{k\in\mathbb{Z}} e^{2\pi iky} f(x+k+\frac{1}{q}m_j). \end{aligned}$$

Now we define $I_l \subseteq \{1, \ldots, N\}$, $0 \leq l \leq q-1$, by

$$I_l := \{j \in \{1, \ldots, N\} : m_j \equiv l (\operatorname{mod} q)\}.$$

Then, by using the quasi-periodicity of the Zak transform (Lemma 3.1.6),

$$\begin{aligned} &\sum_{j=1}^{N} \lambda_j e^{2\pi i n_j x} \sum_{k\in\mathbb{Z}} e^{2\pi iky} f(x+k+\frac{1}{q}m_j) \\ &= \sum_{l=0}^{q-1} \sum_{j\in I_l} \lambda_j e^{2\pi i n_j x} \sum_{k\in\mathbb{Z}} e^{2\pi iky} f(x+(k+\frac{m_j-l}{q})+\frac{l}{q}) \\ &= \sum_{l=0}^{q-1} \sum_{j\in I_l} \lambda_j e^{2\pi i [n_j x - (\frac{m_j-l}{q})y]} Zf(x+\frac{l}{q}, y). \end{aligned}$$

Next we substitute x by $x+\frac{r}{q}$, $0 \le r \le q-1$, and use again the quasi-periodicity relation of Z. This yields a system of q equations. Hence we obtain, for all $0 \le r \le q-1$ and for almost all $(x,y) \in [0,1)^2$,

$$\sum_{l=0}^{q-r-1} \sum_{j \in I_l} \lambda_j e^{2\pi i n_j \frac{r}{q}} e^{2\pi i [n_j x - \left(\frac{m_j - l}{q}\right) y]} Zf(x + \frac{l+r}{q}, y) \tag{20}$$

$$+ \sum_{l=q-r}^{q-1} \sum_{j \in I_l} \lambda_j e^{2\pi i n_j \frac{r}{q}} e^{2\pi i [n_j x - \left(\frac{m_j + q - l}{q}\right) y]} Zf(x + \frac{l+r-q}{q}, y) = 0.$$

In order to simplify the following equations, for the remainder of this proof we will denote $p_{r,l}$, $0 \le r, l \le q-1$, by

$$p_{r,l}(x,y) := \begin{cases} \sum_{j \in I_l} \lambda_j e^{2\pi i n_j \frac{r}{q}} e^{2\pi i [n_j x - \left(\frac{m_j - l}{q}\right) y]} & : \quad 0 \le l \le q-r-1, \\ \sum_{j \in I_l} \lambda_j e^{2\pi i n_j \frac{r}{q}} e^{2\pi i [n_j x - \left(\frac{m_j + q - l}{q}\right) y]} & : \quad q-r \le l \le q-1 \end{cases}$$

for almost all $(x,y) \in [0,1)^2$. Note that the functions $p_{r,l}$ are trigonometric polynomials. This implies that the functions $p_{r,l}$ are non-zero almost everywhere. Moreover, $Zf \ne 0$, since $f \ne 0$. Hence there exists a set of positive measure $M \subseteq [0,1)^2$ such that, for all $0 \le r, l \le q-1$ and for all $(x,y) \in M$, we obtain

$$Zf(x,y) \ne 0 \quad \text{and} \quad p_{r,l}(x,y) \ne 0.$$

Note that, for all $(x,y) \in M$, the system of linear equations (20) can be written in the form

$$\underbrace{\begin{pmatrix} p_{0,0}(x,y) & p_{0,1}(x,y) & p_{0,2}(x,y) & \cdots & p_{0,q-1}(x,y) \\ p_{1,q-1}(x,y) & p_{1,0}(x,y) & p_{1,1}(x,y) & \cdots & p_{1,q-2}(x,y) \\ p_{2,q-2}(x,y) & p_{2,q-1}(x,y) & p_{2,0}(x,y) & \cdots & p_{2,q-3}(x,y) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{q-1,1}(x,y) & p_{q-1,2}(x,y) & p_{q-1,3}(x,y) & \cdots & p_{q-1,0}(x,y) \end{pmatrix}}_{=:P(x,y)}$$

$$\cdot \underbrace{\begin{pmatrix} Zf(x,y) \\ Zf(x+\frac{1}{q},y) \\ Zf(x+\frac{2}{q},y) \\ \vdots \\ Zf(x+\frac{q-1}{q},y) \end{pmatrix}}_{=:z(x,y)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By the definition of M, for all $(x,y) \in M$, we obtain $\det P(x,y) = 0$, since $z(x,y) \neq 0$ for all $(x,y) \in M$. Thus, for all $(x,y) \in M$,

$$\begin{aligned} 0 &= \det P(x,y) \\ &= \sum_{\sigma \in S_q} \operatorname{sign}(\sigma)\, p_{0,(\sigma(1)-1)(\mathrm{mod}\, q)}(x,y)\, p_{1,(\sigma(2)-2)(\mathrm{mod}\, q)}(x,y) \cdot \ldots \\ &\quad \cdot p_{q-1,(\sigma(q)-q)(\mathrm{mod}\, q)}(x,y) \\ &= \sum_{\sigma \in S_q} \operatorname{sign}(\sigma) \sum_{j_1 \in I_{(\sigma(1)-1)(\mathrm{mod}\, q)}} \sum_{j_2 \in I_{(\sigma(2)-2)(\mathrm{mod}\, q)}} \cdots \sum_{j_q \in I_{(\sigma(q)-q)(\mathrm{mod}\, q)}} \lambda_{j_1} \cdot \ldots \cdot \lambda_{j_q} \\ &\quad \cdot e^{2\pi i [n_{j_2}\frac{1}{q} + n_{j_3}\frac{2}{q} + \ldots + n_{j_q}\frac{q-1}{q}]} e^{2\pi i [(n_{j_1}+\ldots+n_{j_q})x - \left(\frac{m_{j_1}+\ldots+m_{j_q}}{q}\right)y]} \\ &\quad \cdot e^{2\pi i \frac{1}{q}\left[\sum_{s=1}^{q}(\sigma(s)-s)(\mathrm{mod}\, q)\right] y} e^{-2\pi i k y}. \end{aligned}$$

Here S_q denotes the symmetric group of $\{1,\ldots,q\}$ and $k \in \mathbb{N}$ is defined by

$$\begin{aligned} k &:= |\{r \in \{0,\ldots,q-1\} : q - r \leq (\sigma(r+1) - (r+1))(\mathrm{mod}\, q)\}| \\ &= |\{r \in \{0,\ldots,q-1\} : \sigma(r+1) < r+1\}|. \end{aligned}$$

Moreover, define $J \subseteq \{1,\ldots,q\}$ by

$$J := \{s \in \{1,\ldots,q\} : \sigma(s) < s\}.$$

Then

$$\begin{aligned} \sum_{s=1}^{q}(\sigma(s)-s)(\mathrm{mod}\, q) &= \sum_{s \in J}(q + \sigma(s) - s) + \sum_{s \in \{1,\ldots,q\}\setminus J}(\sigma(s)-s) \\ &= kq + \sum_{s=1}^{q}(\sigma(s)-s) \\ &= kq. \end{aligned}$$

For all $(x,y) \in M$, this implies that

$$\sum_{\sigma \in S_q} \operatorname{sign}(\sigma) \sum_{j_1 \in I_{(\sigma(1)-1)(\mathrm{mod}\, q)}} \sum_{j_2 \in I_{(\sigma(2)-2)(\mathrm{mod}\, q)}} \cdots \sum_{j_q \in I_{(\sigma(q)-q)(\mathrm{mod}\, q)}} \lambda_{j_1} \cdot \ldots \cdot \lambda_{j_q}$$
$$\cdot e^{2\pi i \frac{1}{q}[n_{j_2} + 2n_{j_3} + \ldots + (q-1)n_{j_q}]} e^{2\pi i [(n_{j_1}+\ldots+n_{j_q})x - \left(\frac{m_{j_1}+\ldots+m_{j_q}}{q}\right)y]} = 0.$$

Now let $\sim$ be the equivalence relation on $\{1,\ldots,N\}^q$ defined by

$$\begin{aligned} &(k_1,\ldots,k_q) \sim (l_1,\ldots,l_q) \\ :\Longleftrightarrow\ & n_{k_1} + \ldots + n_{k_q} = n_{l_1} + \ldots + n_{l_q} \quad \text{and} \\ & m_{k_1} + \ldots + m_{k_q} = m_{l_1} + \ldots + m_{l_q}. \end{aligned}$$

Let $\mathcal{R} \subseteq \{1, \dots, N\}^q$ be a corresponding representation system. Moreover, we define $\mathcal{K} \subseteq \{1, \dots, N\}^q$ by

$$\mathcal{K} := \Big\{(j_1, \dots, j_q) \in \{1, \dots, N\}^q : \text{ There exists some } \sigma \in S_q \text{ such that}$$

$$j_s \in I_{(\sigma(s)-s)(\text{mod } q)} \text{ for all } 1 \leq s \leq q\Big\}.$$

Let $(j_1, \dots, j_q) \in \mathcal{K}$. Further, let $\sigma, \pi \in S_q$ be such that, for all $1 \leq s \leq q$, $j_s \in I_{(\sigma(s)-s)(\text{mod } q)}$ and $j_s \in I_{(\pi(s)-s)(\text{mod } q)}$. For all $1 \leq s \leq q$, this implies

$$m_{j_s} \equiv ((\sigma(s) - s)(\text{mod } q))(\text{mod } q) \quad \text{and}$$

$$m_{j_s} \equiv ((\pi(s) - s)(\text{mod } q))(\text{mod } q).$$

Hence

$$(\sigma(s) - s)(\text{mod } q) = (\pi(s) - s)(\text{mod } q).$$

Thus $\sigma = \pi$. Let $\sigma_{(j_1,\dots,j_q)}$ denote this uniquely determined element of S_q. For the remainder of the proof the equivalence class corresponding to $\sim$ of an element $(k_1, \dots, k_q) \in \{1, \dots, N\}^q$ is denoted by $[(k_1, \dots, k_q)]_\sim$. Thus, for all $(x, y) \in M$,

$$\sum_{(k_1,\dots,k_q)\in\mathcal{R}\cap\mathcal{K}} \Bigg[\sum_{\substack{(l_1,\dots,l_q) \\ \in[(k_1,\dots,k_q)]_\sim\cap\mathcal{K}}} \text{sign}(\sigma_{(l_1,\dots,l_q)})\lambda_{l_1} \cdot \ldots \cdot \lambda_{l_q}$$

$$\cdot e^{2\pi i\frac{1}{q}[n_{l_2}+2n_{l_3}+\ldots+(q-1)n_{l_q}]}\Bigg] e^{2\pi i[(n_{k_1}+\ldots+n_{k_q})x-\left(\frac{m_{k_1}+\ldots+m_{k_q}}{q}\right)y]} = 0.$$

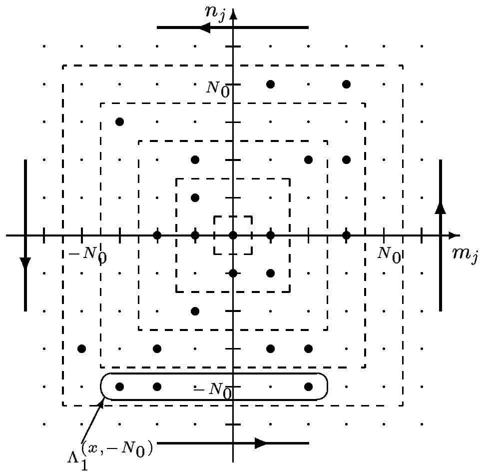
Since trigonometric polynomials are non-zero almost everywhere, we obtain, for all elements $(k_1, \dots, k_q)$ of $\mathcal{R} \cap \mathcal{K}$,

$$\sum_{\substack{(l_1,\dots,l_q) \\ \in[(k_1,\dots,k_q)]_\sim\cap\mathcal{K}}} \text{sign}(\sigma_{(l_1,\dots,l_q)})\lambda_{l_1} \cdot \ldots \cdot \lambda_{l_q} e^{2\pi i\frac{1}{q}[n_{l_2}+2n_{l_3}+\ldots+(q-1)n_{l_q}]} = 0. \tag{21}$$

We are now going to prove that this implies $\lambda_1 = \ldots = \lambda_N = 0$.

The set $\Lambda_1 := \{(m_j, n_j) : 1 \leq j \leq N\}$ is finite. Hence there exists some minimal $N_0 \in \mathbb{N}$ such that $\Lambda_1 \subseteq [-N_0, N_0]^2 \cap \mathbb{Z}^2$. The following steps are illustrated in Figure 4.1.

In the first step we prove that $\lambda_j = 0$ if $|m_j| = N_0$ or $|n_j| = N_0$. We first consider those elements (m_j, n_j), $j \in \{1, \dots, N\}$, which satisfy $n_j = -N_0$ and denote this set by $\Lambda_1^{(x,-N_0)}$. If this set is empty, we jump to the next step. Otherwise, let $(m_{j_0}, -N_0)$ be the unique element in Λ_1 such that $m_{j_0} = \min\{m_j : (m_j, -N_0) \in \Lambda_1^{(x,-N_0)}\}$. Note that, by the definition of j_0,



• : elements of Λ_1

Here:
$N_0 = 4$ and $|\Lambda_1| = 22$

FIGURE 4.1. Illustration of the second part of the proof of Theorem 4.2.3.

$$\begin{aligned}[(j_0, \ldots, j_0)]_\sim &= \Big\{(k_1, \ldots, k_q) \in \{1, \ldots, N\}^q : -qN_0 = n_{k_1} + \ldots + n_{k_q} \\ &\quad \text{and } qm_{j_0} = m_{k_1} + \ldots + m_{k_q}\Big\} \\ &= \{(j_0, \ldots, j_0)\}.\end{aligned}$$

Now let $l_0 \in \{0, \ldots, q-1\}$ be such that $j_0 \in I_{l_0}$. Further, let $\sigma \in S_q$ be defined by

$$\sigma(s) = \left\{ \begin{array}{ccc} l_0 + s & : & l_0 + s \leq q, \\ l_0 + s - q & : & l_0 + s > q \end{array} \right\}, \quad 1 \leq s \leq q.$$

Then, for all $1 \leq s \leq q$, we obtain $j_0 \in I_{l_0} = I_{(\sigma(s)-s)(\mathrm{mod}\, q)}$. This implies $(j_0, \ldots, j_0) \in \mathcal{K}$. Hence, by (21),

$$\mathrm{sign}(\sigma_{(j_0,\ldots,j_0)})\lambda_{j_0}^q e^{2\pi i \frac{1}{q} n_{j_0} \sum_{j=1}^{q-1} j} = 0.$$

Thus $\lambda_{j_0} = 0$. Next let $(m_{j_1}, -N_0)$ be the unique element in Λ_1 such that $m_{j_1} = \min\{m_j : (m_j, -N_0) \in \Lambda_1^{(x,-N_0)} \text{ and } m_j \neq m_{j_0}\}$. Then we calculate

$$\begin{aligned}&[(j_1, \ldots, j_1)]_\sim \\ &\quad = \Big\{(k_1, \ldots, k_q) \in \{1, \ldots, N\}^q : -qN_0 = n_{k_1} + \ldots + n_{k_q} \\ &\qquad \text{and } qm_{j_1} = m_{k_1} + \ldots + m_{k_q}\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \{(j_1,\dots,j_1)\} \cup \Big\{(k_1,\dots,k_q) \in \{1,\dots,N\}^q : \text{ There exists} \\
&\quad \text{some } s \in \{1,\dots,q\} \text{ such that } k_s = j_0,\ -qN_0 = n_{k_1} + \ldots + n_{k_q} \\
&\quad \text{and } qm_{j_1} = m_{k_1} + \ldots + m_{k_q}\Big\} \\
&=: \{(j_1,\dots,j_1)\} \cup M.
\end{aligned}
$$

By the same arguments as before, we obtain $(j_1,\dots,j_1) \in \mathcal{K}$. Since $\lambda_{j_0} = 0$ was shown before and since each vector belonging to M contains at least one component equal to j_0, (21) implies

$$\operatorname{sign}(\sigma_{(j_1,\dots,j_1)})\lambda_{j_1}^q e^{2\pi i \frac{1}{q} n_{j_1} \sum_{j=1}^{q-1} j} = 0.$$

Thus $\lambda_{j_1} = 0$. We may repeat this argument for each element of $\Lambda_1^{(x,-N_0)}$. Thus, for all $j \in \{k \in \{1,\dots,N\} : (m_k, -N_0) \in \Lambda_1^{(x,-N_0)}\}$, $\lambda_j = 0$.

Next we consider the elements of the set $\Lambda_1^{(N_0,y)} := \{(N_0,n) : (N_0,n) \in \Lambda_1\}$. As before, we start with the unique element $(N_0, n_{j_2}) \in \Lambda_1$ determined by $n_{j_2} = \min\{n_j : (N_0,n_j) \in \Lambda_1^{(N_0,y)}\}$. The same arguments as before yield $\lambda_{j_2} = 0$. We deal with the elements of $\Lambda_1^{(N_0,y)}$ step-by-step in a similar way as above. In an analogous way we treat the elements of the sets $\{(m,N_0) : (m,N_0) \in \Lambda_1\}$ and $\{(-N_0,n) : (-N_0,n) \in \Lambda_1\}$. Summarizing all facts, we obtain

$$\lambda_j = 0 \quad \text{for all } j \in \{k \in \{1,\dots,N\} : |m_k| = N_0 \text{ or } |n_k| = N_0\}.$$

During the next large step, we remove the outer frame from Λ_1 and consider the set $\Lambda_1 \backslash \{(m,n) \in \Lambda_1 : |m| = N_0 \text{ or } |n| = N_0\}$. In analogy to the first large step we prove

$$\lambda_j = 0 \quad \text{for all } j \in \{k \in \{1,\dots,N\} : |m_k| = N_0 - 1 \text{ or } |n_k| = N_0 - 1\}.$$

At the latest after the N_0^{th} step we obtain

$$\lambda_1 = \ldots = \lambda_N = 0,$$

since the connection $(m_j,n_j) \leftrightarrow \lambda_j$ is bijective. This completes the proof. □

4.3. Linear independence in the case of a locally compact abelian group

The problem, which has been raised in [**HRT96**], admits a canonical generalization to locally compact abelian groups. Thus from now on, this chapter focuses on the following question.

Let G be a locally compact abelian group. For which finite subsets Λ of $G \times \widehat{G}$ and for which functions $f \in L^2(G)$, $f \neq 0$, is the subset $\{\rho_G(x,\omega,1)f : (x,\omega) \in G \times \widehat{G}\}$ of $L^2(G)$ linearly independent?

But we shall examine this problem in a slightly more general setting, considering finite subsets Λ of $H(G)$ instead of finite subsets of $G \times \widehat{G}$. Note that this is more natural, since the Schrödinger representation is defined on $H(G)$.

In the remainder of this chapter we will extend several results shown by Heil, Ramanathan and Topiwala for $G = \mathbb{R}$ to our general situation. These results give us a big advantage, since on the one hand they include all cases important for applications, for example $G = \mathbb{R}^p$, $\mathbb{Z}^q$, $\mathbb{T}^r$, $p, q, r \in \mathbb{N}$, and finite abelian groups, and on the other hand the general setting helps to understand the problem of linear independence in greater generality as it was pointed out in [**HRT96**].

Heil, Ramanathan and Topiwala conjectured in [**HRT96**], that the subset $\{\rho_{\mathbb{R}}(x, y, 1)f : (x, y) \in \Lambda\}$ of $L^2(\mathbb{R})$ is linearly independent for all finite subsets Λ of $\mathbb{R}^2$ and for all functions $f \in L^2(\mathbb{R})$, $f \neq 0$. It turns out (see Theorem 4.3.4, Theorem 4.3.8, Corollary 4.3.11 and Theorem 4.3.15) that the appropriate conjecture for locally compact abelian groups should be the following one.

CONJECTURE. *Let G be a locally compact abelian group and let Λ be a finite subset of $H(G)$. Then the following conditions are equivalent.*

(I) *The subset $\{\rho_G(x, \omega, z)f : (x, \omega, z) \in \Lambda\}$ of $L^2(G)$ is linearly independent for all $f \in L^2(G)$, $f \neq 0$.*
(II) *The elements $(x, \omega, z)H(G)^c$, $(x, \omega, z) \in \Lambda$, are pairwise different.*

In Subsection 4.3.1 we give a positive answer to the question whether (I) implies (II). Unfortunately, we did not succeed to prove the converse direction in general. The problem remains even open for $G = \mathbb{R}$ (compare [**HRT96**]). But we will prove the equivalence of (I) and (II) in several cases, which are especially important for applications.

In Subsection 4.3.2.1 we prove the equivalence of (I) and (II) when Λ is a finite subset of $K \times A(K, \widehat{G}) \times \mathbb{T}$, where K is a uniform lattice in G. In the next subsection we extend this result by applying metaplectic transforms to the finite subsets Λ dealt with in Subsection 4.3.2.1. This result generalizes Proposition 2 from [**HRT96**]. Finally, we prove the equivalence of (I) and (II) for finite, collinear subsets $\Lambda \subseteq G \times \widehat{G}$. This a generalization of Proposition 1 from [**HRT96**].

The following results, which will be used for simplifying the proofs of Theorem 4.3.4, Theorem 4.3.8 and Theorem 4.3.15, show that it is equivalent to consider finite subsets Λ of $H(G)$ or of $G \times \widehat{G} \times \{1\}$.

First, we will study conditions of the form (I).

LEMMA 4.3.1. *Let G be a locally compact abelian group and let $q : H(G) \to H(G)/\mathbb{T} = G \times \widehat{G}$ denote the quotient map. Further, let Λ be a finite subset of $H(G)$ such that $q|_{\Lambda}$ is injective. Then, for each function $f \in L^2(G)$, $f \neq 0$, the following conditions are equivalent.*

(i) *$\{\rho_G(x, \omega, z)f : (x, \omega, z) \in \Lambda\}$ is linearly independent.*
(ii) *$\{\rho_G(x, \omega, 1)f : (x, \omega) \in q(\Lambda)\}$ is linearly independent.*

PROOF. Note that this is an immediate consequence of the definition of the Schrödinger representation. Indeed, for all $(x, \omega, z) \in H(G)$, we have $\rho_G(x, \omega, z) = \rho_G(e, 1, z)\rho_G(x, \omega, 1)$. Moreover, $(\rho_G(e, 1, z)f)(t) = zf(t)$ for all $f \in L^2(G)$ and for all $z \in \mathbb{T}$, $t \in G$. □

The next lemma is well-known. It is needed for simplifying conditions of the form (II).

LEMMA 4.3.2. *Let G be a locally compact group, let K be a compact, normal subgroup of G and let $q : G \to G/K$ denote the quotient homomorphism. Then, for all $x, y \in G$, the following conditions are equivalent.*

(i) $xG^c \neq yG^c$.
(ii) $q(x)(G/K)^c \neq q(y)(G/K)^c$.

PROOF. This follows immediately from $G^c = q^{-1}((G/K)^c)$. □

Summarizing these two results, we obtain the following proposition.

PROPOSITION 4.3.3. *Let G be a locally compact abelian group, let Λ be a finite subset of $H(G)$ and let $q : H(G) \to H(G)/\mathbb{T}$ denote the quotient homomorphism. Let* (i), (ii), (iii) *and* (iv) *be the conditions given by*

(i) $\{\rho_G(x, \omega, 1)f : (x, \omega) \in q(\Lambda)\}$ *is linearly independent for all* $f \in L^2(G)$, $f \neq 0$.
(ii) *The elements* $(x, \omega)(G^c \times (\widehat{G})^c)$, $(x, \omega) \in q(\Lambda)$, *are pairwise different.*
(iii) $\{\rho_G(x, \omega, z)f : (x, \omega, z) \in \Lambda\}$ *is linearly independent for all* $f \in L^2(G)$, $f \neq 0$.
(iv) *The elements* $(x, \omega, z)H(G)^c$, $(x, \omega, z) \in \Lambda$, *are pairwise different.*

Then we obtain the following.

(a) (i) $\Rightarrow$ (ii) *implies* (iii) $\Rightarrow$ (iv).
(b) (i) $\Leftarrow$ (ii) *implies* (iii) $\Leftarrow$ (iv).

PROOF. First, we prove (a). Suppose that we have (i) $\Rightarrow$ (ii) and that (iii) holds. This implies that $q|_\Lambda$ is injective. Hence, by Lemma 4.3.1, (i) holds. By hypothesis, this implies (ii) and hence, by Lemma 4.3.2, property (iv). This proves (a).

Conversely, suppose that we have (i) $\Leftarrow$ (ii) and that (iv) holds. Lemma 4.3.2 implies (ii) and hence, by hypothesis, we obtain property (i). Moreover, (iv) implies that $q|_\Lambda$ is injective. So we may apply Lemma 4.3.1 and obtain (iii). Thus (b) is shown. □

4.3.1. Necessity of the condition. We may ask in which cases condition (I) of our conjecture implies condition (II). This is important, since we are interested in finite subsets $\Lambda \subseteq H(G)$ such that $\{\rho_G(x, \omega, z)f : (x, \omega, z) \in \Lambda\}$ is linearly independent for all $f \in L^2(G)$, $f \neq 0$. The necessity of (II) tells us to which extent we have to restrict our choice of the group G and of the finite subset Λ. In fact, we prove the implication (I) $\Rightarrow$ (II) for all locally compact abelian groups and for all finite subsets Λ of $H(G)$ (Theorem 4.3.4). Notice that for $G = \mathbb{R}$ condition (II) plays no role (compare Remark 4.3.6).

THEOREM 4.3.4. *Let G be a locally compact abelian group and let Λ be a finite subset of $H(G)$. Suppose that, for each $f \in C_c(G)$, $f \neq 0$, the subset*

$$\{\rho_G(x,\omega,z)f : (x,\omega,z) \in \Lambda\}$$

of $L^2(G)$ is linearly independent. Then the elements

$$(x,\omega,z)H(G)^c, \quad (x,\omega,z) \in \Lambda,$$

are pairwise different.

We will begin with the special case of a compactly generated locally compact abelian Lie group.

PROPOSITION 4.3.5. *Let G be a compactly generated locally compact abelian Lie group. Then the conclusion of Theorem 4.3.4 holds.*

PROOF. Let Λ be a finite subset of $G \times \widehat{G}$. Further, suppose that the set $\{\rho_G(x,\omega,1)f : (x,\omega) \in \Lambda\}$ is linearly independent for each $f \in L^2(G)$, $f \neq 0$. Towards a contradiction, assume that there exist two different elements $(x,\omega), (x',\omega') \in \Lambda$ such that

$$(x,\omega)(G^c \times (\widehat{G})^c) = (x',\omega')(G^c \times (\widehat{G})^c).$$

To prove the claim, by Proposition 4.3.3 (a), it suffices to show that this yields a contradiction.

The structure theorem for compactly generated locally compact abelian Lie groups implies that G is of the form $G = \mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^r \times F$, where F is a finite group and $p, q, r \in \mathbb{N}_0$. Hence the assumption says that $x_1 = x_1'$, $x_2 = x_2'$, $\omega_1 = \omega_1'$ and $\omega_3 = \omega_3'$.

Now we have to construct a function $f \in C_c(G)$, $f \neq 0$, such that the functions $\rho_G(x,\omega,1)f$ and $\rho_G(x',\omega',1)f$ are linearly dependent. As mentioned above, this yields a contradiction. For that, fix $f_1 \in C_c(\mathbb{R}^p)$, $f_1 \neq 0$, and let $f_4 : F \to \mathbb{C}$, $f_4 \neq 0$, be arbitrary. f_4 will be specified later. Then consider a function $f \in C_c(G)$, $f \neq 0$, of the form

$$f(t) := \begin{cases} f_1(t_1)f_4(t_4) & : \quad t_2 = 0, \\ 0 & : \quad t_2 \neq 0. \end{cases}$$

Let $\lambda \in \mathbb{C}^2$, $\lambda \neq 0$. Using the definition of the representation ρ_G and the assumption that $x_1 = x_1'$, $x_2 = x_2'$, $\omega_1 = \omega_1'$ and $\omega_3 = \omega_3'$, we obtain that the condition

$$\lambda_1(\rho_G(x,\omega,1)f)(t) + \lambda_2(\rho_G(x',\omega',1)f)(t) = 0 \quad \text{for almost all } t \in G$$

is equivalent to

$$\begin{aligned} &\lambda_1\omega_2(t_2)\omega_4(t_4)f(x_1+t_1, x_2+t_2, x_3t_3, x_4t_4) \\ &+ \lambda_2\omega_2'(t_2)\omega_4'(t_4)f(x_1+t_1, x_2+t_2, x_3't_3, x_4't_4) = 0 \quad \text{for all } t \in G. \end{aligned} \tag{22}$$

By the construction of f and the fact that $f_1 \neq 0$, (22) holds if and only if, for all $t_4 \in F$,

$$\lambda_1\overline{\omega_2(x_2)}\omega_4(t_4)f_4(x_4t_4) + \lambda_2\overline{\omega_2'(x_2)}\omega_4'(t_4)f_4(x_4't_4) = 0. \tag{23}$$

In order to simplify this equation, replace t_4 by ${x'_4}^{-1}t_4$ and define $\tilde{\lambda} \in \mathbb{C}^2$ by $\tilde{\lambda}_1 := \lambda_1\overline{\omega_2(x_2)}\,\overline{\omega_4(x'_4)}$ and $\tilde{\lambda}_2 := \lambda_2\overline{\omega'_2(x_2)}\,\overline{\omega'_4(x'_4)}$. Then (23) is equivalent to the fact that, for all $t_4 \in F$,

$$\tilde{\lambda}_1\omega_4(t_4)f_4((x_4{x'_4}^{-1})t_4) + \tilde{\lambda}_2\omega'_4(t_4)f_4(t_4) = 0.$$

Thus, towards a contradiction, it remains to show that, for arbitrary $x \in F$ and $\omega \in \widehat{F}$, there exist some $\mu \in \mathbb{C}$, $\mu \neq 0$, and a function $g : F \to \mathbb{C}$, $g \neq 0$, such that, for all $t \in F$,

$$g(xt) = \mu\,\omega(t)g(t). \tag{24}$$

Since F is finite, there exists a minimal $N \in \mathbb{N}$ such that $x^N = e$. Then $H := \{x^n : 0 \leq n \leq N-1\}$ is a subgroup of F. Now define $\mu \in \mathbb{C}$ such that

$$\mu^N\omega(x)^{\frac{N(N-1)}{2}} = 1,$$

and define $g : F \to \mathbb{C}$ by

$$g(t) := \begin{cases} \mu^n\omega(x)^{\frac{n(n-1)}{2}} & : \quad t = x^n \in H,\ 0 \leq n \leq N-1, \\ 0 & : \quad t \in F\backslash H. \end{cases}$$

Then (24) holds for all $t \in F\backslash H$, because this implies $xt \in F\backslash H$. If, on the other hand, $t = x^n \in H$, we have that

$$g(x^{n+1}) = \mu^{n+1}\omega(x)^{\frac{n(n+1)}{2}} = \mu\,\omega(x^n)g(x^n)$$

in the case $0 \leq n < N-1$. Moreover, in the case $n = N-1$ we obtain

$$g(x^N) = g(x^0) = 1 = \mu^N\omega(x)^{\frac{N(N-1)}{2}} = \mu\,\omega(x^{N-1})g(x^{N-1}).$$

Thus equation (24) is fulfilled, which finishes the proof. □

Now we can prove Theorem 4.3.4 using Proposition 4.3.5.

PROOF OF THEOREM 4.3.4. Let G be an arbitrary locally compact abelian group and let Λ be a finite subset of $G \times \widehat{G}$. Further, suppose that the set $\{\rho_G(x,\omega,1)f : (x,\omega) \in \Lambda\}$ is linearly independent for each $f \in C_c(G)$, $f \neq 0$. Towards a contradiction, assume that there exist two different elements $(x,\omega), (x',\omega') \in \Lambda$ such that

$$(x,\omega)(G^c \times (\widehat{G})^c) = (x',\omega')(G^c \times (\widehat{G})^c).$$

Note that it suffices to show that this implies a contradiction (Proposition 4.3.3 (a)).

This will be achieved by first reducing to locally compact abelian Lie groups and then to compactly generated locally compact abelian Lie groups, in which situation Proposition 4.3.5 applies.

Since there exists a compact subgroup K of G such that G/K is a Lie group and since each compact abelian group is a projective limit of Lie groups (compare [**HR70**, 28.61 (c)]), also G is a projective limit of Lie groups. Therefore, there exists a system $\mathcal{H}$ of compact subgroups H of G, $\mathcal{H}$ downwards directed and $\bigcap_{H\in\mathcal{H}} H = \{e\}$, such that G/H is a Lie group for every $H \in \mathcal{H}$.

Since $\widehat{G} = \bigcup_{H\in\mathcal{H}} \widehat{G/H}$, there exists $H \in \mathcal{H}$ such that $\omega, \omega' \in \widehat{G/H}$. Let $\pi : G \to G/H$ denote the quotient map. Then $xx'^{-1} \in G^c = \pi^{-1}((G/H)^c)$ implies $\pi(x)\pi(x')^{-1} \in (G/H)^c$. Similarly, $\omega\overline{\omega'} \in (\widehat{G})^c \cap \widehat{G/H} = (\widehat{G/H})^c$. So we obtain that

$$(\pi(x), \omega)\left((G/H)^c \times (\widehat{G/H})^c\right) = (\pi(x'), \omega')\left((G/H)^c \times (\widehat{G/H})^c\right).$$

Suppose that we have found $g \in C_c(G/H)$, $g \neq 0$, such that $\rho_{G/H}(\pi(x), \omega, 1)g$ and $\rho_{G/H}(\pi(x'), \omega', 1)g$ are linearly dependent. Let $f := g\circ\pi$. Then $f \in C_c(G)$, $f \neq 0$, and, for all $(y, \chi) \in \{(x, \omega), (x', \omega')\}$,

$$(\rho_{G/H}(\pi(y), \chi, 1)g) \circ \pi = \rho_G(y, \chi, 1)f.$$

Hence we may assume that G is a Lie group.

For the next step let L be an open, compactly generated subgroup of G, with the property that x, $x' \in L$. Then $xx'^{-1} \in G^c$ implies that $xx'^{-1} \in L^c$. We also have that $\omega|_L\overline{\omega'|_L} \in (\widehat{L})^c$, because the restriction map $\widehat{G} \to \widehat{L}$ is continuous. Hence

$$(x, \omega|_L)(L^c \times (\widehat{L})^c) = (x', \omega'|_L)(L^c \times (\widehat{L})^c).$$

Now we may apply Proposition 4.3.5 to this situation. This yields a function $g \in C_c(L)$, $g \neq 0$, and $\lambda \in \mathbb{C}^2$, $\lambda \neq 0$, such that

$$\lambda_1\rho_L(x, \omega|_L, 1)g + \lambda_2\rho_L(x', \omega'|_L, 1)g = 0. \tag{25}$$

Let $f \in C_c(G)$ be the function which equals g on L and is zero on $G\backslash L$. Then $\mathrm{supp} f \subseteq L$. Since x and x' are elements in L, for every $t \in G$, we have that $xt, x't \in L$ if and only if $t \in L$. So it follows from (25) that

$$\lambda_1\rho_G(x, \omega, 1)f + \lambda_2\rho_G(x', \omega', 1)f = 0,$$

a contradiction. □

Note that this theorem implies the necessity of condition (II) even if we only consider functions of $C_c(G)$.

REMARK 4.3.6. Now consider the case $G = \mathbb{R}$ and let Λ be a finite subset of $H(\mathbb{R})$. Let $q : H(\mathbb{R}) \to H(\mathbb{R})/\mathbb{T}$ denote the quotient homomorphism. Obviously, $\mathbb{R}^c = \{0\}$. Hence, by Lemma 4.3.2, the following two conditions are equivalent.

(i) The elements $(x, \omega, z)H(\mathbb{R})^c$, $(x, \omega, z) \in \Lambda$, are pairwise different.
(ii) The elements (x, ω), $(x, \omega) \in q(\Lambda)$, are pairwise different.

Thus, if we consider a finite subset $\Lambda \subseteq H(\mathbb{R})$ such that $q|_\Lambda$ is injective and study linear independence of the set of functions $\{\rho_{\mathbb{R}}(x, \omega, 1)f : (x, \omega) \in q(\Lambda)\}$, $f \in L^2(\mathbb{R})$, $f \neq 0$, as has been done in [**HRT96**], condition (II) is always satisfied.

4.3.2. Sufficiency of the condition. Let G be a locally compact abelian group. We now prove the implication (II) $\Rightarrow$ (I) for several important classes of finite subsets Λ of $H(G)$. Then, by the last subsection, (I) and (II) are equivalent for these classes.

4.3.2.1. *Uniform lattices.* Let $K \subseteq \mathbb{R} \times \mathbb{R}$ be a uniform lattice satisfying $m(K) = 1$ (see Lemma 4.2.1 for the definition of $m(K)$) and let Λ be a finite subset of K. Then the set $\{\rho_{\mathbb{R}}(x, y, 1)f : (x, y) \in \Lambda\}$ is linearly independent for all functions $f \in L^2(G)$, $f \neq 0$ ([**HRT96**, Proposition 2]). An important special case are the uniform lattices of the form $K = p\mathbb{Z} \times \frac{1}{p}\mathbb{Z}$, $p \in \mathbb{R}^*$. In [**HRT96**] the result was proven first for these uniform lattices and afterwards extended by a metaplectic transform to uniform lattices which satisfy $m(K) = 1$. Here we also use this idea. Hence we first restrict to a suitable generalization of uniform lattices of the form $K = p\mathbb{Z} \times \frac{1}{p}\mathbb{Z}$, $p \in \mathbb{R}^*$. Afterwards, in Subsection 4.3.2.2 we extend this result by using a metaplectic transform.

It is known that the uniform lattices in $\mathbb{R}$ are precisely the subgroups K of the form $K = p\mathbb{Z}$, $p \in \mathbb{R}^*$. In particular, also $\frac{1}{p}\mathbb{Z}$ is a uniform lattice in $\mathbb{R} = \widehat{\mathbb{R}}$. Furthermore, note that $\frac{1}{p}\mathbb{Z} = A(p\mathbb{Z}, \widehat{\mathbb{R}})$. Now let G be a locally compact abelian group. Then the canonical generalization would be choosing a uniform lattice $K < G$ and considering the subgroup $K \times A(K, \widehat{G}) < G \times \widehat{G}$. By Remark 1.0.2, this is also a uniform lattice. Hence a natural generalization would be to consider finite subsets $\Lambda \subseteq K \times A(K, \widehat{G}) \times \mathbb{T} \subseteq H(G)$. In this situation we will prove that condition (II) implies (I).

The following lemma establishes some important properties of trigonometric polynomials on locally compact abelian groups which will be very useful for the next theorem.

LEMMA 4.3.7. *Let G be a locally compact abelian group and let Λ be a finite subset of $\widehat{G}$. Further, let the elements*

$$\omega(\widehat{G})^c,\ \omega \in \Lambda,$$

be pairwise different. Then, for all $(\lambda_\omega)_{\omega\in\Lambda} \subseteq \mathbb{C}$, $(\lambda_\omega)_{\omega\in\Lambda} \neq 0$,

$$\sum_{\omega\in\Lambda} \lambda_\omega \omega(x) \neq 0 \quad \text{for almost all } x \in G.$$

PROOF. At first, we restrict to G being a compactly generated locally compact abelian Lie group. Towards a contradiction, assume that there exist a set $W \subseteq G$ of positive measure and some $(\lambda_\omega)_{\omega\in\Lambda} \subseteq \mathbb{C}$ such that

$$\sum_{\omega\in\Lambda} \lambda_\omega \omega(x) = 0 \quad \text{for all } x \in W.$$

Now, by the structure theorem for compactly generated locally compact abelian Lie groups, G is of the form $G = \mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^r \times F$, where F is a finite group and p, q, r are positive integers or zero. Applying Fubini's theorem yields the existence of elements $k_0 \in \mathbb{Z}^q$ and $m_0 \in F$ and a measurable set $U_1 \subseteq \mathbb{R}^p \times \mathbb{T}^r$ such that $U_1 \times \{k_0\} \times \{m_0\} \subseteq W$ and $\mu_G(U_1 \times \{k_0\} \times \{m_0\}) > 0$. Hence, for all $(x_1, x_3) \in U_1$,

$$\sum_{\omega \in \Lambda} \lambda_\omega \omega_1(x_1)\omega_2(k_0)\omega_3(x_3)\omega_4(m_0) = 0.$$

To simplify this equation, for all $\omega \in \Lambda$, define $\tilde{\lambda}_\omega \in \mathbb{C}$ by

$$\tilde{\lambda}_\omega := \lambda_\omega \omega_2(k_0)\omega_4(m_0).$$

Thus, for all $(x_1, x_3) \in U_1$,

$$\sum_{\omega \in \Lambda} \tilde{\lambda}_\omega \omega_1(x_1)\omega_3(x_3) = 0.$$

Now decompose the set Λ into maximal disjoint subsets Λ_m, $1 \leq m \leq s$, such that $\omega_1 = \omega'_1$ for all $\omega, \omega' \in \Lambda_m$. This allows us to write the above equation in a more convenient form:

$$\sum_{m=1}^{s} \left[\sum_{\omega \in \Lambda_m} \tilde{\lambda}_\omega \omega_3(x_3) \right] \omega_1(x_1) = 0$$

for all $(x_1, x_3) \in U_1$. Now fix the element x_3 and regard the left side of the above equation as a function depending on x_1. By Fubini's theorem, the measurable set

$$U_2 := \{x_3 \in \mathbb{T}^r : \mu_{\mathbb{R}^p}(\{x_1 \in \mathbb{R}^p : (x_1, x_3) \in U_1\}) > 0\}$$

has positive measure. Now a trigonometric polynomial on $\mathbb{R}^p$ which is zero on a set of positive measure has all coefficients equal to zero. Hence, for all $x_3 \in U_2$, $1 \leq m \leq s$,

$$\sum_{\omega \in \Lambda_m} \tilde{\lambda}_\omega \omega_3(x_3) = 0.$$

By hypothesis, the elements $\omega(\widehat{G})^c$, $\omega \in \Lambda$, are all pairwise different. Since $\widehat{G} = \widehat{\mathbb{R}^p} \times \widehat{\mathbb{Z}^q} \times \widehat{\mathbb{T}^r} \times \widehat{F} = \mathbb{R}^p \times \mathbb{T}^q \times \mathbb{Z}^r \times F$ and hence $(\widehat{G})^c = \mathbb{T}^q \times F$, this implies that the elements $(\omega_1, \omega_3) \in \mathbb{R}^p \times \mathbb{Z}^r$, $\omega \in \Lambda$, are pairwise different. Hence, for all $\omega, \omega' \in \Lambda_m$, $1 \leq m \leq s$, we have $\omega_3 \neq \omega'_3$. Recall that a trigonometric polynomial on $\mathbb{T}^r$ which is zero on a set of positive measure can be lifted to a trigonometric polynomial on $\mathbb{R}^r$ which is zero on a set of positive measure. So again the coefficients are equal to zero. This implies $\tilde{\lambda}_\omega = 0$ and hence $\lambda_\omega = 0$ for all $\omega \in \Lambda$.

Now let G be an arbitrary locally compact abelian group. Recall that locally compact abelian groups are projective limits of Lie groups. Therefore, there exists a downwards directed system $\mathcal{H}$ of compact subgroups H of G with $\bigcap_{H \in \mathcal{H}} H = \{e\}$, such that G/H is a Lie group for every $H \in \mathcal{H}$. Since $\widehat{G} = \bigcup_{H \in \mathcal{H}} \widehat{G/H}$, there exists $H \in \mathcal{H}$ such that $\omega \in \widehat{G/H}$ for all $\omega \in \Lambda$. Since

$(\widehat{G})^c \cap \widehat{G/H} = (\widehat{G/H})^c$, for all $\omega, \omega' \in \Lambda$, we obtain

$$\omega(\widehat{G/H})^c = \omega'(\widehat{G/H})^c \quad \text{if and only if} \quad \omega(\widehat{G})^c = \omega'(\widehat{G})^c.$$

Thus the elements

$$\omega(\widehat{G/H})^c, \quad \omega \in \Lambda,$$

are pairwise different. Now, towards a contradiction, assume that there exist a set $W \subseteq G$ of positive measure and some $(\lambda_\omega)_{\omega\in\Lambda} \subseteq \mathbb{C}$ such that

$$\sum_{\omega\in\Lambda} \lambda_\omega \omega(x) = 0 \quad \text{for all } x \in W.$$

Let $\pi : G \to G/H$ denote the quotient map. Then $\pi(W) \subseteq G/H$ is a set of positive measure and, for all $\pi(x) \in \pi(W)$,

$$\sum_{\omega\in\Lambda} \lambda_\omega \omega(\pi(x)) = \sum_{\omega\in\Lambda} \lambda_\omega \omega(x) = 0.$$

Hence we may assume that G is a Lie group.

For the last step let L be an open, compactly generated subgroup of G. Recall that $(\widehat{G})^c = \bigcup_{M<G,\, M\,\text{open}} \widehat{G/M}$. Thus we can assume that L also fulfills the property that $\omega|_L\overline{\omega'|_L} \notin (\widehat{L})^c$ for all $\omega, \omega' \in \Lambda$. Hence the elements

$$\omega|_L(\widehat{L})^c, \quad \omega \in \Lambda,$$

are also pairwise different. As before, we again assume, towards a contradiction, that there exist a set $W \subseteq G$ of positive measure and some $(\lambda_\omega)_{\omega\in\Lambda} \subseteq \mathbb{C}$ such that

$$\sum_{\omega\in\Lambda} \lambda_\omega \omega(x) = 0 \quad \text{for all } x \in W.$$

Without loss of generality we may assume that $L \cap W \subseteq L$ is of positive measure. Moreover, we have

$$\sum_{\omega\in\Lambda} \lambda_\omega \omega(x) = 0 \quad \text{for all } x \in L \cap W.$$

Now we may apply the first part of the proof to this situation. Thus $\lambda_\omega = 0$ for all $\omega \in \Lambda$. □

The following theorem proves the equivalence of (I) and (II) for a class of finite subsets Λ especially important for applications. The idea of using the Zak transform already appears in [**HRT96**].

THEOREM 4.3.8. *Let G be a locally compact abelian group and let K be a uniform lattice in G. Further, let Λ be a finite subset of $K \times A(K, \widehat{G}) \times \mathbb{T} \subseteq H(G)$. Then the following conditions are equivalent.*

(i) *The subset $\{\rho_G(x, \omega, z)f : (x, \omega, z) \in \Lambda\}$ of $L^2(G)$ is linearly independent for all $f \in L^2(G)$, $f \neq 0$.*
(ii) *The elements $(x, \omega, z)H(G)^c$, $(x, \omega, z) \in \Lambda$, are pairwise different.*

PROOF. The implication (i) $\Rightarrow$ (ii) was already proven in Theorem 4.3.4. Thus it remains to show that (ii) implies (i).

Now suppose that (ii) holds. Let Λ be a finite subset of $K \times A(K, \widehat{G})$ such that the elements $(x, \omega) \in (G^c \times (\widehat{G})^c)$, $(x, \omega) \in \Lambda$, are pairwise different. By Proposition 4.3.3 (b), it suffices to show that this implies that the subset $\{\rho_G(x, \omega, 1)f : (x, \omega) \in \Lambda\}$ of $L^2(G)$ is linearly independent for all $f \in L^2(G)$, $f \neq 0$.

For this, let $f \in L^2(G)$, $f \neq 0$. Further, let $(\lambda_{(x,\omega)})_{(x,\omega)\in\Lambda} \subseteq \mathbb{C}$ be such that

$$\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}(\rho_G(x, \omega, 1)f) = 0 \tag{26}$$

on G. By Lemma 3.1.2, there exists a relatively compact fundamental domain S_K for K in G. Since the subgroup $A(K, \widehat{G})$ is a uniform lattice in $\widehat{G}$ (Remark 1.0.2), there similarly exists a relatively compact fundamental domain Ω_K for $A(K, \widehat{G})$ in $\widehat{G}$. Now we may apply the Zak transform $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ (compare Definition 3.1.4) to (26). We obtain that, for almost all $(y, \chi) \in S_K \times \Omega_K$,

$$Z\left(\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}\rho_G(x, \omega, 1)f\right)(y, \chi) = 0.$$

Using the quasi-periodicity relations which Z satisfies (Lemma 3.1.6), we obtain

$$\left(\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}\omega(y)\overline{\chi(x)}\right) Zf(y, \chi) = 0 \text{ for almost all } (y, \chi) \in S_K \times \Omega_K.$$

Clearly, since $f \neq 0$ and since Z is an isometry (Lemma 3.1.3), the set $W := \operatorname{supp}(Zf) \subseteq S_K \times \Omega_K$ has positive measure. Since the elements in $\widehat{G}$ are continuous, it follows that

$$\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}\omega(y)\overline{\chi(x)} = 0 \text{ for all } (y, \chi) \in W.$$

Hence there exists a set of positive measure $U \subseteq \widehat{G} \times G$ such that

$$\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}(\hat{x}, \omega)(\chi, y) = 0 \text{ for all } (\chi, y) \in U.$$

In addition, by hypothesis, the elements

$$(\hat{x}, \omega)(\widehat{\widehat{G} \times G})^c = (x, \omega)(G \times \widehat{G})^c, \quad (x, \omega) \in \Lambda,$$

are pairwise different. Now we may apply Lemma 4.3.7. Thus $\lambda_{(x,\omega)} = 0$ for all $(x, \omega) \in \Lambda$. $\square$

4.3.2.2. *Metaplectic transforms.* As said before in the beginning of Subsection 4.3.2.1, in this subsection we will give a generalization of Proposition 2 of [**HRT96**]. For this, let G be a locally compact abelian group such that at least one of the groups G and $\widehat{G}$ is 2-complete (compare Definition 4.1.2). Let K be a uniform lattice therein and let $q : H(G) \to H(G)/\mathbb{T}$ denote the quotient map. We consider finite subsets $\Lambda \subseteq H(G)$ which satisfy $\Phi(q(\Lambda)) \subseteq K \times A(K, \widehat{G})$ and prove the equivalence of (I) and (II) in this situation. Here Φ is supposed to be a metaplectic transform $\Phi : G \times \widehat{G} \to G \times \widehat{G}$. In [**HRT96**] and [**FG97**] these transforms are used to prove results for finite subsets of $\mathbb{R} \times \mathbb{R}$, which are not necessarily contained in $\mathbb{Z} \times \mathbb{Z}$.

We first have to extend the definition of a metaplectic transform to locally compact abelian groups in an appropriate way. We start by considering the special case $G = \mathbb{R}$. Let M be a 2×2 matrix with determinant one and let $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$. Then the map

$$(x, y) \mapsto M(x, y) + (x_0, y_0), \quad \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R},$$

is called a *metaplectic transform.* We see that $\det M = 1$ is equivalent to M leaving the symplectic form $[(x, y), (x', y')] = yx' - y'x$ invariant. Thus we may identify M with some linear map $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ which satisfies

$$\varphi_2(x, y)\varphi_1(x', y') - \varphi_2(x', y')\varphi_1(x, y) = yx' - y'x \tag{27}$$

for all $(x, y), (x', y') \in \mathbb{R} \times \mathbb{R}$. Hence the classical definition of metaplectic transform admits a natural generalization to locally compact abelian groups.

Definition 4.3.9. Let G be a locally compact abelian group and let φ be a topological automorphism of $G \times \widehat{G}$ such that

$$\varphi_2(x, \omega)(\varphi_1(x', \omega')) \cdot \overline{\varphi_2(x', \omega')(\varphi_1(x, \omega))} = \omega(x')\overline{\omega'(x)}$$

for all $(x, \omega), (x', \omega') \in G \times \widehat{G}$. Let $(x_0, \omega_0) \in G \times \widehat{G}$. Then

$$\Phi : G \times \widehat{G} \to G \times \widehat{G}, \ (x, \omega) \mapsto \varphi(x, \omega)(x_0, \omega_0),$$

is called a *metaplectic transform.*

It it easy to check that this definition coincides with the classical definition for $G = \mathbb{R}$ using the canonical identification of $\mathbb{R}$ with $\widehat{\mathbb{R}}$ (compare Remark 1.0.1).

Now we may give the generalization of Proposition 2 of [**HRT96**], which is formulated here as a corollary of Theorem 4.3.8. We need the following lemma for the proof of this result.

Lemma 4.3.10. *Let G be a locally compact abelian group such that at least one of the groups G and $\widehat{G}$ is 2-complete, and let $\Phi : G \times \widehat{G} \to G \times \widehat{G}$ be a metaplectic transform. Furthermore, let Λ be a finite subset of $G \times \widehat{G}$. Then the following conditions are equivalent.*

(i) $\{\rho_G(x, \omega, 1)f : (x, \omega) \in \Lambda\}$ *is linearly independent for every* $f \in L^2(G)$, $f \neq 0$.

(ii) $\{\rho_G(\Phi(x,\omega),1)f : (x,\omega) \in \Lambda\}$ *is linearly independent for every* $f \in L^2(G)$, $f \neq 0$.

PROOF. By Definition 4.3.9, there exist $\varphi \in \mathrm{Aut}(G \times \widehat{G})$ and $(x_0, \omega_0) \in G \times \widehat{G}$ such that $\Phi(x,\omega) = \varphi(x,\omega)(x_0,\omega_0)$. Applying Proposition 4.1.3 yields the existence of a topological automorphism $\alpha : H(G) \to H(G)$ such that $\alpha(x,\omega,z) = (\varphi(x,\omega), \alpha_3(x,\omega,z))$ and $\alpha(e,1,z) = (e,1,z)$ for all $x \in G$, $\omega \in \widehat{G}$ and $z \in \mathbb{T}$. Thus, for all $f \in L^2(G)$ and $z \in \mathbb{T}$, $t \in G$,

$$\rho_G(\alpha(e,1,z))f(t) = zf(t). \tag{28}$$

We may regard $H(G)$ as the semidirect product $G \ltimes (\widehat{G} \times \mathbb{T})$, where the action is given by $\tau : G \to \mathrm{Aut}(\widehat{G} \times \mathbb{T})$, $\tau_x(\omega, z) = (\omega, z\,\omega(x))$. Let $(y,m) \in (\widehat{G} \times \mathbb{T})^\wedge = G \times \mathbb{Z}$. Let $S_{(y,m)}$ denote the stabilizer and let $O(y,m)$ denote the orbit of (y,m). It is easily checked that

$$S_{(y,m)} = \{x \in G : x^m = e\} \text{ and } O(y,m) = y \cdot \{x^m : x \in G\} \times \{m\}.$$

Hence in the case $m = 1$ the orbit is closed and the natural map $xS_{(y,1)} \mapsto (y,1) \circ \tau_x$ from $G/S_{(y,1)}$ to $O(y,1)$ is a homeomorphism. Therefore, by equation (28) and by Mackey's theory [**Rie79**], we obtain that $\rho_G \simeq \rho_G \circ \alpha$. Hence (compare the beginning of the proof of Theorem 4.2.3) we obtain the equivalence of (i) and

(i') $\{\rho_G(\alpha(x,\omega,1))f : (x,\omega) \in \Lambda\}$ is linearly independent for every $f \in L^2(G), f \neq 0$.

Furthermore, by the special structure of α and by Lemma 4.3.1 (the hypotheses are fulfilled, since φ is a topological isomorphism), condition (i') holds if and only if

(i") $\{\rho_G(\varphi(x,\omega),1)f : (x,\omega) \in \Lambda\}$ is linearly independent for every $f \in L^2(G), f \neq 0$.

It remains to prove the equivalence of (i") and (ii). But this is an immediate consequence of the fact that

$$\rho_G(\varphi(x,\omega)(x_0,\omega_0),1) = \rho_G(\varphi(x,\omega),1)\rho_G(e,1,\overline{\omega_0(\varphi_1(x,\omega))})\rho_G(x_0,\omega_0,1)$$

for all $(x,\omega) \in \Lambda$ and that $\rho_G(e,1,z)f(t) = zf(t)$ for all $z \in \mathbb{T}, t \in G$. □

COROLLARY 4.3.11. *Let G be a locally compact abelian group such that at least one of the groups G and $\widehat{G}$ is 2-complete, let K be a uniform lattice in G and let $\Phi : G \times \widehat{G} \to G \times \widehat{G}$ be a metaplectic transform. Furthermore, let Λ be a finite subset of $H(G)$ such that $\Phi(q(\Lambda)) \subseteq K \times A(K, \widehat{G})$, where $q : H(G) \to H(G)/\mathbb{T}$ denotes the quotient homomorphism. Then the following conditions are equivalent.*

(i) *The subset $\{\rho_G(x,\omega,z)f : (x,\omega,z) \in \Lambda\}$ of $L^2(G)$ is linearly independent for every $f \in L^2(G)$, $f \neq 0$.*
(ii) *The elements $(x,\omega,z)H(G)^c$, $(x,\omega,z) \in \Lambda$, are pairwise different.*

PROOF. Applying Lemma 4.3.1 and Lemma 4.3.10 to Theorem 4.3.8 yields the result. □

4.3.2.3. *Collinear points.* In this subsection we intend to generalize Proposition 1 of [**HRT96**] to locally compact abelian groups. In this proposition Heil, Ramanathan and Topiwala proved that for a finite collinear set $\Lambda \subseteq \mathbb{R} \times \mathbb{R}$, the subset $\{\rho_{\mathbb{R}}(x, y, 1)f : (x, y) \in \Lambda\}$ of $L^2(\mathbb{R})$ is linearly independent for all $f \in L^2(\mathbb{R})$, $f \neq 0$.

First, we have to extend the definition of a collinear set to locally compact abelian groups. It is easy to rewrite the definition of a collinear set in $\mathbb{R} \times \mathbb{R}$ using a metaplectic transform. We claim that a set $M \subseteq \mathbb{R} \times \mathbb{R}$ is collinear if and only if there exists a metaplectic transform $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that $\Phi(M) \subseteq \mathbb{R} \times \{0\}$.

For this, let M be a collinear subset of $\mathbb{R} \times \mathbb{R}$. Then there exists a line $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$, $a, b \in \mathbb{R}$, such that the graph of f contains all elements of M. Now define $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by

$$\Phi(x, y) = \begin{pmatrix} \cos(\arctan(a)) & \sin(\arctan(a)) \\ -\sin(\arctan(a)) & \cos(\arctan(a)) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -b\sin(\arctan(a)) \\ -b\cos(\arctan(a)) \end{pmatrix}.$$

Then, for all $x \in \mathbb{R}$,

$$\Phi(x, ax + b) \in \mathbb{R} \times \{0\},$$

hence, for all $(x, y) \in M$,

$$\Phi(x, y) \in \mathbb{R} \times \{0\}.$$

Conversely, suppose that there exists a metaplectic transform $\Phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that $\Phi(M) \subseteq \mathbb{R} \times \{0\}$. This implies the existence of $A \in M(2, \mathbb{R})$ and $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ such that $\det A = 1$ and $\Phi(x, y) = A(x, y) + (x_0, y_0)$. Write $A^{-1} = (b_{i,j})_{1 \leq i,j \leq 2}$. Since Φ is invertible, we obtain

$$\begin{aligned} M &\subseteq \Phi^{-1}(\mathbb{R} \times \{0\}) \\ &= \{A^{-1}((x, 0) - (x_0, y_0)) : x \in \mathbb{R}\} \\ &= \{(b_{1,1}x, b_{2,1}x) - A^{-1}(x_0, y_0) : x \in \mathbb{R}\}. \end{aligned}$$

Obviously, this is a set of collinear points.

For $G = \mathbb{R}$, it is equivalent to write $\Phi(M) \subseteq \mathbb{R} \times \{0\}$ or $\Phi(M) \subseteq \{0\} \times \mathbb{R}$, since there exists the metaplectic transform $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $\Psi(x, y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. In the general case of a locally compact abelian group G, the existence of a metaplectic transform $\Psi : G \times \widehat{G} \to G \times \widehat{G}$ which satisfies $\Psi(G \times \{1\}) \subseteq \{e\} \times \widehat{G}$ is not always guaranteed. For example, consider $G = \mathbb{T}$. Assume, towards a contradiction, that there exists a metaplectic transform

$\Psi : \mathbb{T} \times \mathbb{Z} \to \mathbb{T} \times \mathbb{Z}$ such that $\Psi(\mathbb{T} \times \{0\}) \subseteq \{1\} \times \mathbb{Z}$. Note that $\mathbb{T} \times \{0\}$ is open in $\mathbb{T} \times \mathbb{Z}$ but $\{1\} \times \mathbb{Z}$ is not open, a contradiction.

Hence we give the definition of a collinear subset of $G \times \widehat{G}$ in the following form.

DEFINITION 4.3.12. Let G be a locally compact abelian group. A subset M of $G \times \widehat{G}$ is called *collinear*, if there exists a metaplectic transform $\Phi : G \times \widehat{G} \to G \times \widehat{G}$ such that either

$$\Phi(x, \omega) \in G \times \{1\} \quad \text{for all } (x, \omega) \in M$$

or

$$\Phi(x, \omega) \in \{e\} \times \widehat{G} \quad \text{for all } (x, \omega) \in M.$$

By the previous considerations, this definition reduces to the classical definition for $G = \mathbb{R}$.

Let G be a locally compact abelian group and let Λ be a finite subset of $H(G)$ such that $q(\Lambda) \subseteq G \times \widehat{G}$ is collinear, where $q : H(G) \to H(G)/\mathbb{T}$ denotes the quotient map. We intend to prove the equivalence of (I) and (II) in this situation. First, we have to investigate two important special cases of collinear points in $G \times \widehat{G}$. In addition to the two usual conditions coming from our conjecture, we add a third equivalent condition concerning trigonometric polynomials, which sometimes may be easier to check.

THEOREM 4.3.13. *Let G be a locally compact abelian group and let Λ be a finite subset of $\widehat{G}$. Then the following conditions are equivalent.*

(i) $\{\rho_G(e, \omega, 1)f : \omega \in \Lambda\}$ *is linearly independent for every* $f \in L^2(G)$, $f \neq 0$.

(ii) *The elements* $\omega(\widehat{G})^c$, $\omega \in \Lambda$, *are pairwise different.*

(iii) *For each* $(\lambda_\omega)_{\omega \in \Lambda} \subseteq \mathbb{C}$, $(\lambda_\omega)_{\omega \in \Lambda} \neq 0$, *we have*

$$\sum_{\omega \in \Lambda} \lambda_\omega \omega(x) \neq 0 \quad \textit{for almost all } x \in G.$$

PROOF. By Theorem 4.3.4, (i) implies (ii). Moreover, using Lemma 4.3.7 yields (ii) $\Rightarrow$ (iii).

Now suppose that (iii) holds. Let $f \in L^2(G)$, $f \neq 0$. Further, let $(\lambda_\omega)_{\omega \in \Lambda} \subseteq \mathbb{C}$, $(\lambda_\omega)_{\omega \in \Lambda} \neq 0$, be such that

$$\sum_{\omega \in \Lambda} \lambda_\omega (\rho_G(e, \omega, 1)f)(x) = f(x) \sum_{\omega \in \Lambda} \lambda_\omega \omega(x) = 0 \quad \text{for almost all } x \in G.$$

Since $f \neq 0$, there exists a set $W \subseteq G$ of positive measure such that $f(x) \neq 0$ for almost all $x \in W$. Then, for almost all $x \in W$,

$$\sum_{\omega \in \Lambda} \lambda_\omega \omega(x) = 0.$$

Now, by (iii), $\lambda_\omega = 0$ for all $\omega \in \Lambda$. Thus (iii) implies (i). This finishes the proof. $\square$

COROLLARY 4.3.14. *Let G be a locally compact abelian group and let Λ be a finite subset of G. Then the following conditions are equivalent.*

(i) $\{\rho_G(x,1,1)f : x \in \Lambda\}$ *is linearly independent for every* $f \in L^2(G)$, $f \neq 0$.

(ii) *The elements* xG^c, $x \in \Lambda$, *are pairwise different.*

(iii) *For each* $(\lambda_x)_{x\in\Lambda} \subseteq \mathbb{C}$, $(\lambda_x)_{x\in\Lambda} \neq 0$, *we have*

$$\sum_{x\in\Lambda} \lambda_x \omega(x) \neq 0 \quad \textit{for almost all } \omega \in \widehat{G}.$$

PROOF. Recall that the Plancherel transform $\mathcal{P}$ is an isometry and we have, for all $x \in G$ and $\omega \in \widehat{G}$,

$$\mathcal{P}(\rho_G(x,1,1)f)(\omega) = \mathcal{P}(L_{x^{-1}}f)(\omega) = \omega(x)(\mathcal{P}f)(\omega).$$

Then, for $(\lambda_x)_{x\in\Lambda} \subseteq \mathbb{C}$,

$$\begin{aligned} & \sum_{x\in\Lambda} \lambda_x(\rho_G(x,1,1)f)(t) = 0 && \text{for almost all } t \in G \\ \Leftrightarrow\quad & \left[\sum_{x\in\Lambda} \lambda_x\omega(x)\right](\mathcal{P}f)(\omega) = 0 && \text{for almost all } \omega \in \widehat{G} \\ \Leftrightarrow\quad & \sum_{x\in\Lambda} \lambda_x(\rho_{\widehat{G}}(1,\hat{x},1)(\mathcal{P}f))(\omega) = 0 && \text{for almost all } \omega \in \widehat{G}. \end{aligned}$$

Note that we may regard Λ as a subset of $\widehat{\widehat{G}}$. Applying Theorem 4.3.13 to $\widehat{G}$ and Λ yields the claim. □

Now we extend Proposition 1 of [**HRT96**] to locally compact abelian groups. The idea of reducing the problem first to $\Lambda \subseteq G \times \{1\}$ or $\Lambda \subseteq \{e\} \times \widehat{G}$ already appears in [**HRT96**].

THEOREM 4.3.15. *Let G be a locally compact abelian group and let $q : H(G) \to H(G)/\mathbb{T}$ denote the quotient map. Further, let Λ be a finite subset of $H(G)$ such that $q(\Lambda)$ is collinear in $G \times \widehat{G}$. Then the following conditions are equivalent.*

(i) *The subset* $\{\rho_G(x,\omega,z)f : (x,\omega,z) \in \Lambda\}$ *of* $L^2(G)$ *is linearly independent for every* $f \in L^2(G)$, $f \neq 0$.

(ii) *The elements* $(x,\omega,z)H(G)^c$, $(x,\omega,z) \in \Lambda$, *are pairwise different.*

PROOF. By Theorem 4.3.4, (i) implies (ii). It remains to prove the converse direction. For this, let Λ be a finite, collinear subset of $G \times \widehat{G}$ such that the elements $(x,\omega)(G^c \times (\widehat{G})^c)$, $(x,\omega) \in \Lambda$, are pairwise different. By Proposition 4.3.3 (b), it suffices to show that this implies the linear independence of $\{\rho_G(x,\omega,1)f : (x,\omega) \in \Lambda\}$ for all $f \in L^2(G)$, $f \neq 0$.

Note that without loss of generality we may assume that there exists a metaplectic transform $\Phi : G \times \widehat{G} \to G \times \widehat{G}$ such that, for all $(x,\omega) \in \Lambda$, $\Phi(x,\omega) \in \{e\} \times \widehat{G}$. In the case $\Phi(x,\omega) \in G \times \{1\}$ for all $(x,\omega) \in \Lambda$, using Corollary 4.3.14 instead of Theorem 4.3.13, the proof remains almost the same.

Now let $p_{\widehat{G}} : G \times \widehat{G} \to \widehat{G}$ denote the projection onto the second component, and define $\tilde{\Lambda} \subseteq \widehat{G}$ by $\tilde{\Lambda} := p_{\widehat{G}}(\Phi(\Lambda))$. By Lemma 4.3.10, the set $\{\rho_G(x,\omega,1)f : (x,\omega) \in \Lambda\}$ is linearly independent for all $f \in L^2(G)$, $f \neq 0$, if and only if

$$(\mathrm{i}')\{\rho_G(e,\omega,1)f : \omega \in \tilde{\Lambda}\} \quad \text{is linearly independent for all } f \in L^2(G),\ f \neq 0.$$

Since Φ is a topological isomorphism, the elements $(x,\omega)(G^c \times (\widehat{G})^c)$, $(x,\omega) \in \Lambda$, are pairwise different if and only if the elements $\omega(\widehat{G})^c$, $\omega \in \tilde{\Lambda}$, are pairwise different. Applying Theorem 4.3.13 yields the claim. □

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