THE EFFECT OF PERTURBATIONS OF FRAME SEQUENCES AND FUSION FRAMES ON THEIR DUALS

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Abstract. Fusion frames, and, more generally, operator-valued frame sequences are generalizations of classical frames, which are today a standard notion when redundant, yet stable sequences are required. However, the question of stability with respect to perturbations has not been satisfactorily answered. In this paper, we quantitatively measure this stability by considering the associated deviations of the canonical and alternate dual sequences from the original ones. It is proven that operator-valued frame sequences are indeed stable in this sense. Along the way, we also introduce a novel notion of a fusion frame dual. It is designed to satisfy a list of prespecified desiderata, in particular, constituting again a fusion frame, thereby improving on previously known definitions.

1. Introduction

Introduced in 1952 by Duffin and Schaeffer [13], frames as an extension of the concept of orthonormal bases allowing for redundancy, while still maintaining stability properties, are today a standard notion in mathematics and engineering. Applications range from more theoretical problems such as the Kadison-Singer Problem [9] and tensor decomposition [25] over questions inspired by applications such as (sparse) approximation theory [22] to real-world problems such as wireless communication and coding theory [27], quantum mechanics [14], and inverse scattering problems [23]. Recently, due to both necessities from applications and also theoretical goals, generalizations of this framework have been developed, first fusion frames [7], then operator-valued frames [20] and G-frames [28].

Reconstruction of the original vector from frame, fusion frame, or more general measurements, is typically achieved by using a so-called (alternate or canonical) dual system. While having ensured stability of this measurement process already in the definition, the question of stability with respect to perturbations of the frame or generalizations of this concept is highly delicate. The most natural approach to quantitatively measure this stability is by considering the associated deviations of the dual systems. However, already in the fusion frame setting, not even the question of a suitable notion of a dual fusion frame is entirely

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settled, with questions related to stability with respect to perturbations being completely open so far. These are the problems we tackle in this paper.

1.1. Frames, Fusion Frames, and Beyond

The two key properties of frames are redundancy and stability, which can easily be seen by their definition. A (not necessarily orthogonal) sequence $\Phi = (\varphi_i)_{i\in I}$ in a Hilbert space $\mathcal{H}$ forms a *frame*, if it exhibits a norm equivalence $\|(\langle\,\cdot\,,\varphi_i\rangle)_{i\in I}\|_{\ell^2(I)} \asymp \|\cdot\|_{\mathcal{H}}$. The associated analysis operator – allowing the analysis of a vector – is defined by $T_\Phi : \mathcal{H} \to \ell^2(I)$, $x \mapsto (\langle x, \varphi_i\rangle)_{i\in I}$.

An analysis operator can be regarded as a one-dimensional projection, leading in a natural way to so-called fusion frames [7] as a generalization of frames, serving, in particular, applications under distributed processing requirements. As before, redundancy in combination with stability are key to their success. In fact, a *fusion frame* is a sequence $\mathcal{W} = ((W_i, c_i))_{i\in I}$ of pairs of closed subspaces and weights, again exhibiting a norm equivalence $\|(c_i P_{W_i}(\cdot))_{i\in I}\|_{\bigoplus_{i\in I} W_i} \asymp \|\cdot\|_{\mathcal{H}}$ with P_{W_i} denoting the respective orthogonal projection and analysis operator being given by $T_{\mathcal{W}} : \mathcal{H} \to \bigoplus_{i\in I} W_i$, $x \mapsto (c_i P_{W_i}(\cdot))_{i\in I}$. Increasing the flexibility level once more and aiming for a thorough theoretical understanding, the orthogonal projections were replaced by general operators $(A_i)_{i\in I}$ with $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $i \in I$, $\mathcal{K}$ a Hilbert space, leading to *operator-valued frames* [20] and to the equivalent notion of *G-frames* [28].

1.2. Reconstruction, Expansion, and Duals

At the heart of frame and fusion frame theory as well as their extensions is the problem of reconstruction of the original vector after its analysis, i.e., after applying the analysis operator, which depending on the application can also be regarded as a measurement operator or sampling operator. In frame theory, the reconstruction formula takes the shape of

$$x = \sum_{i\in I} \langle x, \varphi_i\rangle \tilde{\varphi}_i = \sum_{i\in I} \langle x, \tilde{\varphi}_i\rangle \varphi_i, \quad x \in \mathcal{H},$$

with $(\tilde{\varphi}_i)_{i\in I}$ the so-called *(alternate) dual frame*. As can be seen, the second part of this formula even allows an expansion into the frame with a closed form sequence of coefficients. Notice that this is not self-evident due to the redundancy of the frame. The *canonical dual frame* is a specific alternate dual frame, which exhibits a closed form expression.

In the fusion frame setting, one certainly aims for a sequence with similarly advantageous properties, which one might combine in the following list of desiderata for fusion frame duals:

- **(D1)** Reconstruction of any $x \in \mathcal{H}$ from $T_{\mathcal{W}}(x)$ possible.
- **(D2)** Proper generalization of alternate dual frames.
- **(D3)** Constitute a fusion frame themselves.
- **(D4)** Proper generalization of the canonical dual frame.

Recent approaches to introduce fusion frame duals [16, 19, 18] have so far only achieved to accomodate some of those desirable properties, which already shows the delicacy of the fusion frame setting – noting that the definition for a dual frame is in fact quite simple.

Interestingly, the definition of a dual in the setting of operator-valued frames or the slightly more general setting of operator-valued frame sequences, while also still being open, is not that delicate. The reason for this phenomenon is that (D3) does not impose such a strong condition, since it is now only required that the dual shall constitute an operator-valued frame sequence.

1.3. Analysis of Stability

While frames and their extensions are by definition stable in the sense of their analysis operator being continuous, applications require stronger forms of stability. Of particular importance is robustness with respect to perturbations which is typically regarded as being encoded in terms of the associated analysis operator. More precisely, in the frame setting, say, one would refer to a frame $\Psi = (\psi_i)_{i\in I}$ as being a *μ-perturbation* of Φ, if $\|T_\Phi - T_\Psi\| \le \mu$. In the situation of frames, this is a well-studied subject (see, e.g., [2, 11, 4, 8, 17, 15]).

As discussed before, exact reconstruction or expansion is a crucial property of frames and their extensions, which in turn depends heavily on (alternate or canonial) dual sequences. Thus, in this paper, we study the effect of perturbations on those dual sequences again measured in terms of (μ-)perturbations. Not only will this provide a very clear picture of this type of stability, but also allow us a deep understanding of the relation of the dual sequences with the original fusion frames or operator-valued frame sequences, say.

Perturbations of sequences beyond the frame setting have hardly been studied before. There have been works of W. Sun [29] as well as A.A. Arefijamaal and S. Ghasemi [1] on perturbations of g-frames and their duals. Concerning fusion frames, the only paper in this direction analyzes erasures of subspaces [6], which already required a quite subtle treatment.

1.4. Our Contribution

Our main contributions are two-fold. First, based on the previous definition from [19], we introduce a novel definition of a dual fusion frame (Def. 3.9), which now satisfies all of the previously discussed desiderata **(D1)-(D4)**. It should be emphasized that this is not just a simple restriction of the definition of a dual sequence with respect to an operator-valued frame sequence – which we also introduce though more straightforwardly –, but requires a much more delicate handling. We expect that this new notion will become a useful ingredient in fusion frame theory due to its rational design.

Second, we provide a comprehensive perturbation analysis of both fusion frames and general operator-valued frame sequences in terms of their effect on their (alternate and canonical) duals. In Theorems 4.5, 4.9 as well as 5.6, we show that indeed stability can be achieved in these situations and derive precise error estimates.

1.5. Outline

The paper is organized as follows. Section 2 is devoted to the introduction of operator-valued frame (Bessel) sequences extending and slightly deviating from [20], and discuss vector frames and fusion frames as special cases. Canonical and alternate duals are then introduced in Section 3 and their key properties analyzed. First, in Subsection 3.1 duals are defined and studied in the general setting of operator-valued frame sequences. This is followed by the introduction of a novel notion of dual for fusion frames, and proving a list of desired properties for those (see Subsection 3.2). The last two sections focus on the impact of perturbations of the initial sequences on their duals, both in the general setting (Section 4) and in fusion frame setting (Section 5).

1.6. Notation

We close this introduction by fixing the notation we will use. The set of all bounded and everywhere defined linear operators between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ will be denoted by $\mathcal{B}(\mathcal{H},\mathcal{K})$. As ususal, we set $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. The norm on $\mathcal{B}(\mathcal{H},\mathcal{K})$ will be the usual operator norm, i.e.

$$\|T\| := \sup\left\{\|Tx\| : x \in \mathcal{H},\ \|x\| = 1\right\}.$$

We denote the range (i.e., the image) and the kernel (i.e., the null space) of $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ by $\operatorname{ran} T$ and $\ker T$, respectively. The restriction of an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ to a subspace $V \subset \mathcal{H}$ will be denoted by $T|V$. If V is closed, by P_V we denote the orthogonal projection onto V in $\mathcal{H}$ and by I_V the identity operator on V.

Throughout this paper, $I \subset \mathbb{N}$ stands for a finite or countable index set and $\mathcal{H}$ and $\mathcal{K}$ always denote Hilbert spaces. Recall that the space of $\mathcal{H}$-valued ℓ^2-sequences over I, defined by

$$\ell^2(I,\mathcal{H}) := \left\{ (x_i)_{i\in I} : x_i \in \mathcal{H}\, \forall i \in I,\ \sum_{i\in I} \|x_i\|^2 < \infty \right\},$$

is a Hilbert space with scalar product

$$\big\langle (x_i)_{i\in I}, (y_i)_{i\in I} \big\rangle = \sum_{i\in I} \langle x_i, y_i \rangle, \quad (x_i)_{i\in I}, (y_i)_{i\in I} \in \ell^2(I,\mathcal{H}).$$

We shall often denote

$$\mathfrak{H} := \ell^2(I,\mathcal{H}) \quad \text{and} \quad \mathfrak{K} := \ell^2(I,\mathcal{K}).$$

An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is called *bounded below* if there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in \mathcal{H}$. In the sequel, we will frequently make use of the following well known operator theoretical lemma.

Lemma 1.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then for $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ the following statements are equivalent.*

(i) *T is injective and* $\operatorname{ran} T$ *is closed.*
(ii) *T^* is surjective.*
(iii) *T is bounded below.*

2. Operator-valued Sequences

This section is devoted to recalling the definition of the main object we shall work with, namely *operator-valued frame sequences*. We shall introduce the associated operators (analysis, synthesis, and frame operator) and discuss how vector frames and fusion frames fit into this framework.

2.1. Operator-valued Frame Sequences

As mentioned before, the concept of operator-valued frames was first introduced and intensively studied in [20]. Here, we generalize this notion to operator-valued Bessel sequences.

Definition 2.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A sequence of operators $\mathcal{A} = (A_i)_{i\in I}$ with $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $i \in I$, is said to be an* operator-valued (*or* $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued) Bessel sequence *if there exists $\beta > 0$ such that*

$$\sum_{i\in I} \|A_i x\|^2 \leq \beta \|x\|^2 \quad \text{for all } x \in \mathcal{H}. \tag{2.1}$$

The bound β is said to be a Bessel bound *of $\mathcal{A}$.*

The following characterization of operator-valued Bessel sequences is easily proved by using [26, Chapter VII, p. 263].

Lemma 2.2. *Let $\mathcal{A} = (A_i)_{i\in I}$ be a sequence of operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$ and $\beta > 0$. Then the following are equivalent.*

(i) *$\mathcal{A}$ is an operator-valued Bessel sequence with Bessel bound β.*

(ii) *The series*

$$\sum_{i\in I} A_i^* A_i$$

converges in the strong operator topology[1] *to a bounded non-negative self-adjoint operator with norm* $\le \beta$.

For any sequence of operators $\mathcal{A} = (A_i)_{i\in I} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ we set

$$\mathcal{H}_{\mathcal{A}} := \overline{\operatorname{span}}\,\{\operatorname{ran} A_i^* : i \in I\} \qquad \text{and} \qquad P_{\mathcal{A}} := P_{\mathcal{H}_{\mathcal{A}}}. \tag{2.2}$$

Given an operator-valued Bessel sequence $\mathcal{A} = (A_i)_{i\in I}$, we define its associated *analysis operator* $T_{\mathcal{A}} : \mathcal{H} \to \mathfrak{K}$ by

$$T_{\mathcal{A}} x := (A_i x)_{i\in I}, \quad x \in \mathcal{H}, \tag{2.3}$$

where, we recall,

$$\mathfrak{K} := \ell^2(I,\mathcal{K}).$$

The relation (2.1) ensures that $T_{\mathcal{A}}$ is well-defined and that it is an element of $\mathcal{B}(\mathcal{H},\mathfrak{K})$ with $\|T_{\mathcal{A}}\| \le \sqrt{\beta}$. The adjoint operator $T_{\mathcal{A}}^*$ of $T_{\mathcal{A}}$ is called the *synthesis operator* of $\mathcal{A}$, and it is easily seen that

$$T_{\mathcal{A}}^*(z_i)_{i\in I} = \sum_{i\in I} A_i^* z_i, \quad (z_i)_{i\in I} \in \mathfrak{K}. \tag{2.4}$$

For this, we only have to show that the series $\sum_{i\in I} A_i^* z_i$ converges in $\mathcal{H}$. But this is (in the case $I = \mathbb{N}$) seen from

$$\begin{aligned}\left\|\sum_{i=m+1}^{n} A_i^* z_i\right\|^2 &= \sup_{\|x\|=1}\left|\left\langle x, \sum_{i=m+1}^{n} A_i^* z_i\right\rangle\right|^2 \le \left(\sup_{\|x\|=1}\sum_{i=m+1}^{n} |\langle A_i x, z_i\rangle|\right)^2\\ &\le \sup_{\|x\|=1}\left(\sum_{i=m+1}^{n}\|A_i x\|^2\right)\left(\sum_{i=m+1}^{n}\|z_i\|^2\right) \le \beta\left(\sum_{i=m+1}^{n}\|z_i\|^2\right).\end{aligned}$$

Lemma 2.3. *Let* $\mathcal{A} = (A_i)_{i\in I} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ *be an operator-valued Bessel sequence. Then*

$$\overline{\operatorname{ran} T_{\mathcal{A}}^*} = \mathcal{H}_{\mathcal{A}} \quad \textit{and} \quad \ker T_{\mathcal{A}} = \mathcal{H}_{\mathcal{A}}^{\perp}. \tag{2.5}$$

Proof. Indeed, we have

$$\mathcal{H}_{\mathcal{A}}^{\perp} = \left(\overline{\operatorname{span}}\,\{\operatorname{ran} A_i^* : i \in I\}\right)^{\perp} = \bigcap_{i\in I} (\operatorname{ran} A_i^*)^{\perp} = \bigcap_{i\in I}\ker A_i = \ker T_{\mathcal{A}}.$$

The first relation follows from this. □

The operator $S_{\mathcal{A}} := T_{\mathcal{A}}^* T_{\mathcal{A}} = \sum_{i\in I} A_i^* A_i$ (the series converging in the strong operator topology) is called the *frame operator* corresponding to $\mathcal{A}$. It follows from Lemma 2.2 that $S_{\mathcal{A}}$ is a bounded non-negative self-adjoint operator in $\mathcal{H}$. Moreover, (2.5) implies that $\mathcal{H}_{\mathcal{A}}$ is invariant under $S_{\mathcal{A}}$ and that

$$\langle S_{\mathcal{A}} x, x\rangle = \|T_{\mathcal{A}} x\|^2 = \sum_{i\in I}\|A_i x\|^2 > 0 \quad \text{for all } x \in \mathcal{H}_{\mathcal{A}} \setminus \{0\}.$$

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[1] A sequence $(T_i)_{i\in I} \subset \mathcal{B}(\mathcal{H})$ is said to converge in the strong operator topology to $T \in \mathcal{B}(\mathcal{H})$ if $(T_i x)_{i\in I}$ converges to Tx for every $x \in \mathcal{H}$.

Definition 2.4. *A sequence* $\mathcal{A} = (A_i)_{i\in I}$ *of operators in* $\mathcal{B}(\mathcal{H},\mathcal{K})$ *is called an* operator-valued (*or* $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued) frame sequence *if there exist* $\alpha, \beta > 0$ *such that*

$$\alpha\|x\|^2 \,\le\, \sum_{i\in I}\|A_i x\|^2 \,\le\, \beta\|x\|^2 \quad \text{for all } x \in \mathcal{H}_{\mathcal{A}}. \tag{2.6}$$

The constants α *and* β *are called* lower *and* upper frame bound *of* $\mathcal{A}$*, respectively. If* $\alpha = \beta$ *is possible,* $\mathcal{A}$ *is said to be* tight*. If even* $\alpha = \beta = 1$*, then* $\mathcal{A}$ *is called an operator-valued* Parseval *frame sequence. If* $\mathcal{H}_{\mathcal{A}} = \mathcal{H}$ *we say that* $\mathcal{A}$ *is an* operator-valued frame *for* $\mathcal{H}$*.*

An operator-valued frame sequence $\mathcal{A} = (A_i)_{i\in I}$ with upper frame bound β is an operator-valued Bessel sequence with Bessel bound β since for $x \in \mathcal{H}$ we have $A_i(I - P_{\mathcal{A}})x = 0$ for each $i \in I$ (see (2.5)) and hence

$$\sum_{i\in I}\|A_i x\|^2 = \sum_{i\in I}\|A_i P_{\mathcal{A}} x\|^2 \le \beta\|P_{\mathcal{A}}x\|^2 \le \beta\|x\|^2.$$

Hence, $S_{\mathcal{A}}$ is defined, and (2.6) is equivalent to $\alpha I_{\mathcal{H}_{\mathcal{A}}} \le S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}} \le \beta I_{\mathcal{H}_{\mathcal{A}}}$. In particular, $S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}}$ is boundedly invertible.

For each $j \in I$ we define the canonical embedding $\mathfrak{E}_j : \mathcal{K} \to \mathfrak{K}$ by

$$\mathfrak{E}_j z := (\delta_{ij} z)_{i\in I}, \quad z \in \mathcal{K}.$$

Its adjoint $\mathfrak{E}_j^* : \mathfrak{K} \to \mathcal{K}$ is given by

$$\mathfrak{E}_j^*(z_i)_{i\in I} = z_j, \quad (z_i)_{i\in I} \in \mathfrak{K}.$$

Hence, for an operator-valued Bessel sequence $\mathcal{A} = (A_i)_{i\in I} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ we have

$$\mathfrak{E}_j^* T_{\mathcal{A}} = A_j.$$

From this, it follows that $T_{\mathcal{A}}$ determines the operator-valued Bessel sequence $\mathcal{A}$ uniquely. Therefore, we will often identify an operator-valued Bessel sequence and its analysis operator. Even more: each $T \in \mathcal{B}(\mathcal{H},\mathfrak{K})$ is the analysis operator of a $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued Bessel sequence. Indeed, for each $i \in I$, define $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$ by $A_i := \mathfrak{E}_i^* T$. Then, we have $A_i x = \mathfrak{E}_i^* T x = (Tx)_i$, $i \in I$, and thus

$$\sum_{i\in I}\|A_i x\|^2 = \|Tx\|^2 \le \|T\|^2\|x\|^2, \quad x \in \mathcal{H},$$

which implies that $\mathcal{A} = (A_i)_{i\in I}$ is an operator-valued Bessel sequence. Moreover, for $x \in \mathcal{H}$ we have $Tx = (A_i x)_{i\in I} = T_{\mathcal{A}}x$, hence, $T = T_{\mathcal{A}}$. In particular, the (linear) mapping $\mathcal{A} \mapsto T_{\mathcal{A}}$ between the space of operator-valued Bessel sequences $\mathfrak{B}(I,\mathcal{H},\mathcal{K})$, indexed by I, and $\mathcal{B}(\mathcal{H},\mathfrak{K})$ is bijective. With the norm $\|\mathcal{A}\| := \|T_{\mathcal{A}}\|$ on $\mathfrak{B}(I,\mathcal{H},\mathcal{K})$ it even becomes unitary.

The next lemma is the analogue of Corollary 5.5.3. in [10].

Lemma 2.5. *Let* $T \in \mathcal{B}(\mathcal{H},\mathfrak{K})$*. Then the following statements hold:*

(i) *T is the analysis operator of an operator-valued frame sequence if and only if* $\operatorname{ran} T^*$ *is closed.*

(ii) *T is the analysis operator of an operator-valued frame if and only if T^* is surjective.*

Proof. Due to the above discussion, we have $T = T_{\mathcal{A}}$, where $\mathcal{A} := (\mathfrak{E}_i^* T)_{i\in I}$. Moreover, by the closed range theorem (see, e.g., [21, Theorem IV-5.13]), $\operatorname{ran} T$ is closed if and only if $\operatorname{ran} T^*$ is closed.

(i). By definition, $\mathcal{A}$ is an operator-valued frame sequence if and only if $\hat{T} := T_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}}$ is bounded below. By Lemma 1.1 this is the case if and only if $\hat{T}$ is injective and $\operatorname{ran}\hat{T}$ is closed. By (2.5), $\hat{T}$ is always injective and $\operatorname{ran}\hat{T} = \operatorname{ran} T_{\mathcal{A}} = \operatorname{ran} T$. Hence, $\mathcal{A}$ is an operator-valued frame sequence if and only if $\operatorname{ran} T$ is closed.

(ii). By (i) and (2.5), $\mathcal{A}$ is an operator-valued frame if and only if $\operatorname{ran} T_{\mathcal{A}}$ is closed and $\ker T_{\mathcal{A}} = \{0\}$. Due to Lemma 1.1, this holds if and only if $T^* = T_{\mathcal{A}}^*$ is surjective. □

Remark 2.6. *(a) The definition of the analysis operator in* [20] *slightly differs from the one we give here, since it is defined to be an operator in* $\mathcal{B}(\mathcal{H}, \ell^2(I) \otimes \mathcal{K})$ *(see* [20, Proposition 2.3]*). However, it can be seen that the spaces* $\ell^2(I) \otimes \mathcal{K}$ *and* $\mathfrak{K}$ *are isometrically isomorphic through the mapping* $\Upsilon : \ell^2(I) \otimes \mathcal{K} \to \mathfrak{K}$, $\Upsilon(e_j \otimes z) = \mathfrak{E}_j(z)$ *where* $(e_i)_{i\in I}$ *is the standard basis of* $\ell^2(I)$*. Moreover, for the (analysis) operator* $\theta_{\mathcal{A}}$ *defined in* [20, Eq. (6)]*, one has* $\Upsilon\theta_{\mathcal{A}} = T_{\mathcal{A}}$*. Here, we prefer to work with the analysis operator as defined in* (2.3) *since it is more suitable for our purposes and also it is a natural extension of the analysis operator associated to a vector frame.*

(b) In [28]*, Sun introduced the slightly more general concept of G-frames. G-frames are sequences of operators* $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$ *between Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}_i$*, satisfying* (2.6) *for every* $x \in \mathcal{H}$*. In principle,* $\mathcal{K}_i$ *could be different from* $\mathcal{K}_j$ *for* $i \neq j$*. However, in* [28]*, Sun pointed out that for any sequence of Hilbert spaces* $\mathcal{K}_i$ *one can always find a Hilbert space* $\mathcal{K}$ *containing all* $\mathcal{K}_i$*, namely* $\mathcal{K} = \bigoplus_{i\in I} \mathcal{K}_i$*. In this sense, every G-frame is also an operator-valued frame.*

2.2. Vector Frames as Special Operator-valued Frames

Recall that a sequence (of vectors) $\Phi = (\varphi_i)_{i\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *Bessel sequence* if there exists $\beta > 0$ such that

$$\sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \le \beta \|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

The *analysis operator* $T_\Phi : \mathcal{H} \to \ell^2(I)$ corresponding to a Bessel sequence $\Phi = (\varphi_i)_{i\in I}$ is defined by

$$T_\Phi x := (\langle x, \varphi_i\rangle)_{i\in I}, \quad x \in \mathcal{H}.$$

It is well known that T_Φ is bounded with norm $\|T_\Phi\| \le \sqrt{\beta}$. The adjoint $T_\Phi^* : \ell^2(I) \to \mathcal{H}$ of T_Φ is called the *synthesis operator* corresponding to Φ and it is given by

$$T_\Phi^* c = \sum_{i\in I} c_i \varphi_i, \quad c \in \ell^2(I).$$

The operator $S_\Phi := T_\Phi^* T_\Phi$ is called the *frame operator* corresponding to Φ and it is a non-negative bounded selfadjoint operator. A sequence $\Phi = (\varphi_i)_{i\in I}$ in $\mathcal{H}$ is called a *frame sequence* in $\mathcal{H}$ if there exist $\alpha, \beta > 0$ such that

$$\alpha\|x\|^2 \le \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \le \beta \|x\|^2 \quad \text{for all } x \in \mathcal{H}_\Phi,$$

where

$$\mathcal{H}_\Phi = \overline{\operatorname{span}}\,\{\varphi_i : i \in I\}.$$

A frame sequence Φ in $\mathcal{H}$ is called a *frame for* $\mathcal{H}$ if $\mathcal{H}_\Phi = \mathcal{H}$. Consequently, a vector sequence is a frame sequence if and only if it is a frame for its closed linear span.

Given a Bessel sequence $\Phi = (\varphi_i)_{i\in I}$ in $\mathcal{H}$, for every $i \in I$ we define an operator $A_i \in \mathcal{B}(\mathcal{H}, \mathbb{C})$ by $A_i x := \langle x, \varphi_i\rangle$, $x \in \mathcal{H}$. Thus, it is clear that $\mathcal{A} = (A_i)_{i\in I}$ is an operator-valued Bessel sequence with Bessel bound β. Noticing that $A_i^* c = c\varphi_i$ for $c \in \mathbb{C}$ and that $\ell^2(I, \mathbb{C}) = \ell^2(I)$, we have that the analysis operator associated with $\mathcal{A}$ coincides with the usual analysis operator corresponding to Φ:

$$T_{\mathcal{A}} x = (A_i x)_{i\in I} = (\langle x, \varphi_i\rangle)_{i\in I} = T_\Phi x, \quad x \in \mathcal{H}.$$

Consequently, we also have that $T_{\mathcal{A}}^* = T_\Phi^*$ and $S_{\mathcal{A}} = S_\Phi$. It is also clear that

$$\mathcal{H}_{\mathcal{A}} = \overline{\operatorname{span}}\,\{\operatorname{ran} A_i^* : i \in I\} = \overline{\operatorname{span}}\,\{\varphi_i : i \in I\} = \mathcal{H}_\Phi.$$

Therefore, the Bessel sequences in $\mathcal{H}$ are exactly the $\mathcal{B}(\mathcal{H}, \mathbb{C})$-valued Bessel sequences, so that operator-valued Bessel sequences naturally extend the notion of (vector) Bessel sequences. The analogous correspondence holds for frame sequences/frames and $\mathcal{B}(\mathcal{H}, \mathbb{C})$-valued frame sequences/frames.

On the other hand, as it was noticed in [20], an operator-valued frame sequence $\mathcal{A} = (A_i)_{i\in I} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ with $\dim \operatorname{ran} A_i = 1$ for all $i \in I$ defines a (vector) frame sequence in $\mathcal{H}$. Indeed, for each $i \in I$, let $e_i \in \mathcal{K}$ be a unit vector such that $\operatorname{ran} A_i = \operatorname{span}\{e_i\}$. By the Riesz Representation Theorem, for every $i \in I$ there exists $\varphi_i \in \mathcal{H}$ such that $A_i x = \langle x, \varphi_i\rangle e_i$ for all $x \in \mathcal{H}$. Since $A_i^* = \langle \cdot, e_i\rangle \varphi_i$, we have that $\mathcal{H}_\Phi = \mathcal{H}_{\mathcal{A}}$, where $\Phi = (\varphi_i)_{i\in I}$. Thus, (2.6) immediately yields that Φ is a frame sequence in $\mathcal{H}$.

2.3. Fusion Frames as Special Operator-valued Frames

Let $(W_i)_{i\in I}$ be a sequence of closed subspaces of $\mathcal{H}$ and $(c_i)_{i\in I}$ a sequence of non-negative real numbers such that for all $i \in I$ we have

$$W_i = \{0\} \quad\Longleftrightarrow\quad c_i = 0. \tag{2.7}$$

We shall call the sequence of pairs $((W_i, c_i))_{i\in I}$ a *fusion sequence*. A *fusion frame* for $\mathcal{H}$ is a fusion sequence $((W_i, c_i))_{i\in I}$ for which there exist $\alpha, \beta > 0$ such that

$$\alpha\|x\|^2 \,\le\, \sum_{i\in I} c_i^2 \|P_{W_i} x\|^2 \,\le\, \beta\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

Fusion frames appeared for the first time in the literature in 2004 (cf. [5]) as "frames of subspaces" and were later renamed in [7]. Obviously, a fusion sequence $\mathcal{W} = ((W_i, c_i))_{i\in I}$ can be identified with the sequence of operators $\mathcal{A} := (c_i P_{W_i})_{i\in I}$ which is a $\mathcal{B}(\mathcal{H})$-valued frame for $\mathcal{H}$ if and only if $\mathcal{W}$ is a fusion frame for $\mathcal{H}$. Therefore, the set of fusion frames for $\mathcal{H}$ can be considered as a special (proper) subset of the $\mathcal{B}(\mathcal{H})$-valued frames for $\mathcal{H}$. We shall also call $\mathcal{W}$ a *Bessel fusion sequence* (*fusion frame sequence*) whenever $\mathcal{A}$ is a $\mathcal{B}(\mathcal{H})$-valued Bessel sequence (frame sequence, resp.). The *analysis operator* and the *fusion frame operator* of the Bessel fusion sequence $\mathcal{W}$ are then defined by $T_{\mathcal{W}} := T_{\mathcal{A}}$ and $S_{\mathcal{W}} := S_{\mathcal{A}}$, respectively. In accordance with (2.2), we also define

$$\mathcal{H}_{\mathcal{W}} := \overline{\operatorname{span}}\,\{W_i : i \in I\} \qquad\text{and}\qquad P_{\mathcal{W}} := P_{\mathcal{H}_{\mathcal{W}}}.$$

At this point we would like to remark that in previous works (see, e.g., [5, 7]) the analysis operator $T_{\mathcal{W}}$ was not considered as an operator from $\mathcal{H}$ to $\mathfrak{H} = \ell^2(I, \mathcal{H})$ but from $\mathcal{H}$ to $\bigoplus_{i\in I} W_i \subset \mathfrak{H}$.

3. Duals of Operator-valued Sequences and Fusion Frames

In this section we describe and investigate the concept of duals of operator-valued sequences and provide a useful parametrization for the set of all duals of a given operator-valued frame sequence. As we shall see, operator-valued duals of fusion frames are in general not fusion frames. Therefore, we will introduce and discuss a new notion of duality for fusion frames which is inspired by that of Heineken et al. in [19].

3.1. Duals of Operator-valued Frame Sequences

Let us first define duals of operator-valued frame sequences.

Definition 3.1. *Let $\mathcal{A} = (A_i)_{i\in I} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ be an operator-valued frame sequence and $\widetilde{\mathcal{A}} = (\widetilde{A}_i)_{i\in I} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ an operator-valued Bessel sequence. We say that $\widetilde{\mathcal{A}}$ is a* dual operator-valued frame sequence *(or simply a* dual *) of $\mathcal{A}$ if*

$$\mathcal{H}_{\widetilde{\mathcal{A}}} \subset \mathcal{H}_{\mathcal{A}} \qquad \text{and} \qquad \sum_{i\in I} \widetilde{A}_i^* A_i x = x \quad \text{for all } x \in \mathcal{H}_{\mathcal{A}}. \tag{3.1}$$

It is immediately seen that (3.1) is equivalent to

$$\operatorname{ran} T_{\widetilde{\mathcal{A}}}^* \subset \mathcal{H}_{\mathcal{A}} \qquad \text{and} \qquad T_{\widetilde{\mathcal{A}}}^* T_{\mathcal{A}} = P_{\mathcal{A}}.$$

And as the above inclusion can equivalently be replaced by an equality, it follows from Lemma 2.5 that a dual is itself an operator-valued frame sequence.

By $\mathcal{D}(\mathcal{A})$ we denote the set of all duals of the operator-valued frame sequence $\mathcal{A}$ (which we identify with their analysis operators), that is,

$$\mathcal{D}(\mathcal{A}) := \{T \in \mathcal{B}(\mathcal{H},\mathfrak{K}) : T^* T_{\mathcal{A}} = P_{\mathcal{A}},\ \operatorname{ran} T^* \subset \mathcal{H}_{\mathcal{A}}\}.$$

Among all duals of $\mathcal{A}$ there is the so-called *canonical dual* which will play a special role in the sequel. It is defined by

$$\big(A_i (S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} P_{\mathcal{A}}\big)_{i\in I}. \tag{3.2}$$

It is easily seen that this is indeed a dual of $\mathcal{A}$ and that its analysis operator is given by

$$T_{\mathcal{A}} (S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} P_{\mathcal{A}}.$$

In analogy to the vector frame sequence case, we call the remaining duals of $\mathcal{A}$ *alternate duals*.

The following lemma provides a characterization of the duals of $\mathcal{A}$ in terms of their analysis operators. It can be viewed as an operator theoretical variant of [10, Theorem 5.6.5], see also [24].

Lemma 3.2. *Let $\mathcal{A} = (A_i)_{i\in I} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ be an operator-valued frame sequence. Then*

$$\mathcal{D}(\mathcal{A}) = \big\{(T_{\mathcal{A}}(S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} + L)P_{\mathcal{A}} \,:\, L \in \mathcal{B}(\mathcal{H}_{\mathcal{A}},\mathfrak{K}),\ L^* T_{\mathcal{A}} = 0\big\}.$$

Proof. Let $T = (T_{\mathcal{A}}(S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} + L)P_{\mathcal{A}}$, where $L \in \mathcal{B}(\mathcal{H}_{\mathcal{A}},\mathfrak{K})$ is such that $L^* T_{\mathcal{A}} = 0$. Then $T^* = (S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} T_{\mathcal{A}}^* + L^*$, which implies $\operatorname{ran} T^* \subset \mathcal{H}_{\mathcal{A}}$, and

$$T^* T_{\mathcal{A}} = ((S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} T_{\mathcal{A}}^* + L^*)T_{\mathcal{A}} = (S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} T_{\mathcal{A}}^* T_{\mathcal{A}} = (S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} S_{\mathcal{A}} = P_{\mathcal{A}},$$

which proves $T \in \mathcal{D}(\mathcal{A})$.

Conversely, let $T \in \mathcal{D}(\mathcal{A})$, i.e. $\operatorname{ran} T^* \subset \mathcal{H}_{\mathcal{A}}$ and $T^* T_{\mathcal{A}} = P_{\mathcal{A}}$. Define the operator $L := (T|\mathcal{H}_{\mathcal{A}}) - T_{\mathcal{A}}(S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} \in \mathcal{B}(\mathcal{H}_{\mathcal{A}},\mathfrak{K})$. Then we have

$$L^* T_{\mathcal{A}} = P_{\mathcal{A}} T^* T_{\mathcal{A}} - (S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} T_{\mathcal{A}}^* T_{\mathcal{A}} = P_{\mathcal{A}} - P_{\mathcal{A}} = 0,$$

and $T|\mathcal{H}_{\mathcal{A}} = T_{\mathcal{A}}(S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} + L$. Since $\operatorname{ran} T^* \subset \mathcal{H}_{\mathcal{A}}$ implies $\mathcal{H}_{\mathcal{A}}^{\perp} \subset \ker T$, we find that $T = (T_{\mathcal{A}}(S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1} + L)P_{\mathcal{A}}$. □

Corollary 3.3. *The set of duals $\mathcal{D}(\mathcal{A})$ of a $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued frame sequence is a closed affine subspace of $\mathcal{B}(\mathcal{H},\mathfrak{K})$.*

Proof. By Lemma 3.2, $\mathcal{D}(\mathcal{A}) = T_{\mathcal{A}}(S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1}P_{\mathcal{A}} + \mathcal{L}_{\mathcal{A}}$, where

$$\mathcal{L}_{\mathcal{A}} = \{L P_{\mathcal{A}} : L \in \mathcal{B}(\mathcal{H}_{\mathcal{A}},\mathfrak{K}),\ L^* T_{\mathcal{A}} = 0\} \subset \mathcal{B}(\mathcal{H},\mathfrak{K}). \tag{3.3}$$

This proves the claim as $\mathcal{L}_{\mathcal{A}}$ is obviously a closed linear subspace of $\mathcal{B}(\mathcal{H},\mathfrak{K})$. □

Remark 3.4. *If $\mathcal{H}$ is finite-dimensional, I is finite, and $\mathcal{A} = (A_i)_{i\in I} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ is an operator-valued frame for $\mathcal{H}$, we have*

$$\mathcal{L}_\mathcal{A} = \{L \in \mathcal{B}(\mathcal{H},\mathfrak{K}) : L^*T_\mathcal{A} = 0\}.$$

Note that $\mathfrak{K} = \mathcal{K}^{|I|}$ in this case. The space $\mathcal{L}_\mathcal{A}$ is then perpendicular to $T_\mathcal{A}S_\mathcal{A}^{-1}$ with respect to the Hilbert-Schmidt scalar product

$$\langle X, Y\rangle_{\rm HS} = \operatorname{Tr}(Y^*X), \quad X, Y \in \mathcal{B}(\mathcal{H},\mathfrak{K}),$$

since for $L \in \mathcal{L}_\mathcal{A}$ we have $\langle T_\mathcal{A}S_\mathcal{A}^{-1}, L\rangle_{\rm HS} = \operatorname{Tr}(L^*T_\mathcal{A}S_\mathcal{A}^{-1}) = 0$. However, this observation cannot be generalized to the infinite-dimensional situation since $T_\mathcal{A}S_\mathcal{A}^{-1}$ cannot be Hilbert-Schmidt in this case. Indeed, if $T_\mathcal{A}S_\mathcal{A}^{-1}$ was Hilbert-Schmidt, then the operator $(T_\mathcal{A}S_\mathcal{A}^{-1})^*T_\mathcal{A}S_\mathcal{A}^{-1} = S_\mathcal{A}^{-1}$ would be compact, implying that $S_\mathcal{A}$ is not bounded. A contradiction.

If $L \in \mathcal{B}(\mathcal{H}_\mathcal{A},\mathfrak{K})$ such that $L^*T_\mathcal{A} = 0$, we denote by $\widetilde{\mathcal{A}}(L)$ that dual of $\mathcal{A}$ with the analysis operator

$$T = \left(T_\mathcal{A}(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} + L\right)P_\mathcal{A}. \tag{3.4}$$

Thus, $\widetilde{\mathcal{A}}(0)$ is the canonical dual. Note that the mapping $L \mapsto \widetilde{\mathcal{A}}(L)$ is one-to-one and therefore parametrizes the duals of $\mathcal{A}$.

Corollary 3.5. *If $\mathcal{A}$ is an operator-valued frame sequence and $L \in \mathcal{B}(\mathcal{H}_\mathcal{A},\mathfrak{K})$ is such that $L^*T_\mathcal{A} = 0$ then the frame operator of $\widetilde{\mathcal{A}}(L)$ is given by*

$$S_{\widetilde{\mathcal{A}}(L)} = ((S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} + L^*L)P_\mathcal{A}.$$

Proof. As $L^*T_\mathcal{A} = 0$, and thus also $T_\mathcal{A}^*L = 0$, we have

$$\begin{aligned}
S_{\widetilde{\mathcal{A}}(L)} &= \left((T_\mathcal{A}(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} + L)P_\mathcal{A}\right)^*(T_\mathcal{A}(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} + L)P_\mathcal{A}\\
&= P_\mathcal{A}\left((S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1}T_\mathcal{A}^* + L^*\right)(T_\mathcal{A}(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} + L)P_\mathcal{A}\\
&= P_\mathcal{A}\left[(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1}T_\mathcal{A}^*T_\mathcal{A}(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} + L^*L\right]P_\mathcal{A}\\
&= P_\mathcal{A}\left((S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} + L^*L\right)P_\mathcal{A}\\
&= ((S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} + L^*L)P_\mathcal{A},
\end{aligned}$$

which proves the claim. □

Remark 3.6. *The frame operator of the canonical dual $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(0)$ of $\mathcal{A}$ is given by $(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1}P_\mathcal{A}$. This implies that if $0 < \alpha \le \beta$ are frame bounds for $\mathcal{A}$, then $0 < \beta^{-1} \le \alpha^{-1}$ are frame bounds for $\widetilde{\mathcal{A}}$. In particular, this last fact gives*

$$\left\|T_\mathcal{A}(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1}P_\mathcal{A}\right\| \le \frac{1}{\sqrt{\alpha}}, \tag{3.5}$$

which we will use frequently below.

In Subsection 2.3, it was shown that fusion frames can be regarded as special operator-valued frames. However, as the next proposition shows, the operator-valued duals of fusion frames might themselves not correspond to fusion frames. We shall say that two operator-valued Bessel sequences $\mathcal{A}$ and $\mathcal{B}$ are orthogonal if $\mathcal{H}_\mathcal{A} \perp \mathcal{H}_\mathcal{B}$.

Proposition 3.7. *For a fusion frame sequence $\mathcal{W} = ((W_i, c_i))_{i\in I}$ in $\mathcal{H}$ the following statements are equivalent:*

(i) *The canonical dual of $(c_iP_{W_i})_{i\in I}$ is a fusion frame sequence.*
(ii) *$\mathcal{W}$ is a union of mutually orthogonal tight fusion frame sequences.*

Proof. The canonical dual of $(c_iP_{W_i})_{i\in I}$ is given by $(c_iP_{W_i}(S_\mathcal{W}|\mathcal{H}_\mathcal{W})^{-1}P_\mathcal{W})_{i\in I}$ (cf. (3.2)). It obviously corresponds to a fusion frame sequence if and only if for each $i\in I$ the operator $P_{W_i}(S_\mathcal{W}|\mathcal{H}_\mathcal{W})^{-1}$ coincides with a positive multiple of an orthogonal projection in $\mathcal{H}_\mathcal{W}$. As the range of this operator coincides with W_i, (i) is satisfied if and only if for each $i\in I$ there exists $d_i>0$ such that $P_{W_i}(S_\mathcal{W}|\mathcal{H}_\mathcal{W})^{-1}=d_iP_{W_i}|\mathcal{H}_\mathcal{W}$, and, by adjunction,

$$(S_\mathcal{W}|\mathcal{H}_\mathcal{W})^{-1}(P_{W_i}|\mathcal{H}_\mathcal{W})=d_i(P_{W_i}|\mathcal{H}_\mathcal{W})\quad \forall i\in I.$$

This is equivalent to

$$S_\mathcal{W}P_{W_i}=d_i^{-1}P_{W_i}\quad \forall i\in I. \tag{3.6}$$

Hence, (i) holds if and only if there exists $(d_i)_{i\in I}\subset(0,\infty)$ such that (3.6) holds.

(i)$\Rightarrow$(ii). Let $(d_i)_{i\in I}\subset(0,\infty)$ be as in (3.6). Define an equivalence relation $\sim$ on I by $i_1\sim i_2 :\Longleftrightarrow\ d_{i_1}=d_{i_2}$, $i_1,i_2\in I$. Let $J\subset I$ be a set of representatives of all the cosets in $I/\sim$, and for $j\in J$ put $I_j:=[j]_\sim$. Then, $(I_j)_{j\in J}$ is a partition of I. For $j\in J$ we further define $\lambda_j:=d_i^{-1}$ if $i\in I_j$ and $V_j:=\overline{\operatorname{span}}\{W_i : i\in I_j\}$. Then (3.6) implies that $V_j\subset\ker(S_\mathcal{W}-\lambda_jId)$, and since eigenspaces of self-adjoint operators are mutually orthogonal, we have that $V_j\perp V_k$ for $j\neq k$, $j,k\in J$. Hence, $((W_i,c_i))_{i\in I_j}$ and $((W_i,c_i))_{i\in I_k}$ are orthogonal for $j\neq k$, $j,k\in J$. It remains to show that for each $j\in J$ the sequence $\mathcal{W}_j:=((W_i,c_i))_{i\in I_j}$ is a tight fusion frame sequence. For this, let $j\in J$ and $x\in V_j$. Then the tightness is seen by

$$S_{\mathcal{W}_j}x=\sum_{i\in I_j}c_i^2P_{W_i}x=\sum_{k\in J}\sum_{i\in I_k}c_i^2P_{W_i}x=\sum_{i\in I}c_i^2P_{W_i}x=S_\mathcal{W}x=\lambda_jx,$$

where in the second equality we use that $(V_j)_{j\in J}$ are mutually orthogonal.

(ii)$\Rightarrow$(i). Due to (ii), there exist a partition $I=\bigcup_{j\in J}I_j$ of I and $(\alpha_j)_{j\in J}\subset(0,\infty)$ such that $\mathcal{W}_j:=((W_i,c_i))_{i\in I_j}$ is an α_j-tight fusion frame sequence for each $j\in J$ and $\mathcal{W}_j$ and $\mathcal{W}_k$ are orthogonal for $j\neq k$. Put $V_j:=\mathcal{H}_{\mathcal{W}_j}$. Then, by the tightness of the $\mathcal{W}_j$, for $x\in\mathcal{H}$ we have

$$S_\mathcal{W}x=\sum_{j\in J}\sum_{i\in I_j}c_i^2P_{W_i}x=\sum_{j\in J}S_{\mathcal{W}_j}x=\sum_{j\in J}\alpha_jP_{V_j}x.$$

For $i\in I$, let $j(i)\in J$ be such that $i\in I_{j(i)}$. Then the mutual orthogonality of the $\mathcal{W}_j$ gives

$$S_\mathcal{W}P_{W_i}=\sum_{j\in J}\alpha_jP_{V_j}P_{W_i}=\alpha_{j(i)}P_{W_i},$$

which is (3.6) for $d_i=\alpha_{j(i)}^{-1}$. □

Remark 3.8. *Note that* (ii) *in Proposition* 3.7 *implies that each W_i is a subspace of some eigenspace of $S_\mathcal{W}$ (cf.* (3.6)*).*

3.2. Duals of Fusion Frames

Since their introduction in 2004 (see [5]), fusion frames have been extensively studied. However, there have only been two approaches yet to define duals of fusion frames (see [16, 19], and also [18] for the finite-dimensional case). In fact, this is a delicate task since it quickly turns out that one has to give up on certain analogues to the vector frames case. For example, if we desired a dual of a fusion frame to be a dual of its operator-valued version, we would have to set aside that the dual is also a fusion frame (cf. Proposition 3.7). Therefore, it seems convenient to us to enumerate a few desiderata which duals of a fusion frame $\mathcal{W}=((W_i,c_i))_{i\in I}$ should satisfy:

(D1) They should allow for reconstructing signals $x \in \mathcal{H}$ from their fusion frame measurements $c_i P_{W_i} x$, $i \in I$.

(D2) They should properly generalize the notion of dual (vector) frames, that is:

(D2a) If $(\psi_i)_{i\in I}$ is a dual of the frame $(\varphi_i)_{i\in I}$ for $\mathcal{H}$, then $((\operatorname{span}\{\psi_i\}, \|\psi_i\|))_{i\in I}$ is a dual fusion frame of $((\operatorname{span}\{\varphi_i\}, \|\varphi_i\|))_{i\in I}$.

(D2b) If $\dim W_i \in \{0,1\}$ for each $i \in I$ and $\mathcal{V} = ((V_i, d_i))_{i\in I}$ is a dual fusion frame of $\mathcal{W}$ with $\dim V_i \in \{0,1\}$ for each $i \in I$, then there exist vectors $\varphi_i \in W_i$ and $\psi_i \in V_i$ with $\|\varphi_i\| = c_i$ and $\|\psi_i\| = d_i$, $i \in I$, such that $(\psi_i)_{i\in I}$ is a dual frame of $(\varphi_i)_{i\in I}$.

(D3) If $\mathcal{V}$ is a dual fusion frame of $\mathcal{W}$ then it should itself be a fusion frame and also $\mathcal{W}$ should be a dual fusion frame of $\mathcal{V}$.

(D4) The fusion sequence $((S_{\mathcal{W}}^{-1} W_i, c_i\|S_{\mathcal{W}}^{-1}|W_i\|))_{i\in I}$ should be a dual fusion frame of $\mathcal{W}$ (the canonical dual).

Whereas **(D1)**–**(D3)** are evident requirements, the choice of the weights of the canonical dual fusion frame in **(D4)** might not be clear a priori. To explain our choice, consider the canonical dual $(S_\Phi^{-1}\varphi_i)_{i\in I}$ of a (vector) frame $\Phi = (\varphi_i)_{i\in I}$. Translated to the fusion frame setting, we have $W_i = \operatorname{span}\{\varphi_i\}$ and $c_i = \|\varphi_i\|$ as well as $V_i = \operatorname{span}\{S_\Phi^{-1}\varphi_i\} = S_\Phi^{-1} W_i$ and $d_i = \|S_\Phi^{-1}\varphi_i\|$. Thus, if $c_i \neq 0$, the weights of the canonical dual are $d_i = c_i\|S_\Phi^{-1}(\varphi_i/\|\varphi_i\|)\| = c_i\|S_\Phi^{-1}|W_i\|$. The same trivially holds for $c_i = 0$.

At first glance, one might want to directly generalize the duality definition for vector frames by requiring that $T_{\mathcal{V}}^* T_{\mathcal{W}} = I$ for a dual $\mathcal{V}$ of $\mathcal{W}$. But this definition violates **(D4)**. A simple example for this is given by

$$\mathcal{H} = \mathbb{C}^2,\ W_1 = \operatorname{span}\{e_1\},\ W_2 = \operatorname{span}\{e_1 + e_2\},\ c_1 = c_2 = 1,$$

where $\{e_1, e_2\}$ is the canonical standard basis of $\mathbb{C}^2$.

There are two further approaches towards a definition of fusion frame duals. The first was made by P. Găvruţa in [16]. He calls a Bessel Fusion sequence $\mathcal{V} = ((V_i, d_i))_{i\in I}$ a dual of a fusion frame $\mathcal{W} = ((W_i, c_i))_{i\in I}$ if

$$\sum_{i\in I} c_i d_i P_{V_i} S_{\mathcal{W}}^{-1} P_{W_i} x = x \quad \text{for all } x \in \mathcal{H}.$$

Here, we shall call these fusion sequences *Găvruţa duals*. Unfortunately, these do not even satisfy desideratum **(D3)**. Although it is proven in [16] that a Găvruţa dual $\mathcal{V}$ of $\mathcal{W}$ is itself a fusion frame, it is in general not true that, conversely, $\mathcal{W}$ is a Găvruţa dual of $\mathcal{V}$. A simple counterexample is the following:

$$\mathcal{H} = \mathbb{C}^2, \quad \mathcal{W} = ((\operatorname{span}\{e_i\}, 1))_{i=1}^2, \quad \mathcal{V} = ((\mathbb{C}^2, 1))_{i=1}^2.$$

In fact, we have $S_{\mathcal{W}} = I$ and $S_{\mathcal{V}} = 2I$, and thus

$$\sum_{i=1}^2 c_i d_i P_{V_i} S_{\mathcal{W}}^{-1} P_{W_i} = I, \quad \text{whereas} \quad \sum_{i=1}^2 c_i d_i P_{W_i} S_{\mathcal{V}}^{-1} P_{V_i} = \frac{1}{2} I.$$

Recently, in [19] (see also [18]), a Bessel fusion sequence $\mathcal{V} = ((V_i, d_i))_{i\in I}$ was called a dual fusion frame of $\mathcal{W} = ((W_i, c_i))_{i\in I}$ if there exists a bounded operator

$$Q : \bigoplus_{i\in I} W_i \to \bigoplus_{i\in I} V_i$$

such that

$$x = \sum_{i\in I} d_i \big(Q(c_j P_{W_j} x)_{j\in I}\big)_i, \quad x \in \mathcal{H}. \tag{3.7}$$

However, given a fusion frame for $\mathcal{H}$, *every* fusion frame for $\mathcal{H}$ is a corresponding dual fusion frame in this sense. Indeed, let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ and $\mathcal{V} = ((V_i, d_i))_{i\in I}$ be fusion frames for $\mathcal{H}$. Define

$$Q := (T_{\mathcal{V}} S_{\mathcal{V}}^{-1} S_{\mathcal{W}}^{-1} T_{\mathcal{W}}^*)\Big|\bigoplus_{i\in I} W_i.$$

Note that $\operatorname{ran} Q \subset \bigoplus_{i\in I} V_i$ so that Q can be seen as an operator in $\mathcal{B}(\bigoplus_i W_i, \bigoplus_i V_i)$. Now, we have (note that $(c_j P_{W_j} x)_{j\in I} = T_{\mathcal{W}} x$)

$$\begin{aligned}\sum_{i\in I} d_i\big(Q(c_j P_{W_j} x)_{j\in I}\big)_i &= \sum_{i\in I} d_i\big(T_{\mathcal{V}} S_{\mathcal{V}}^{-1} S_{\mathcal{W}}^{-1} T_{\mathcal{W}}^* T_{\mathcal{W}} x\big)_i = \sum_{i\in I} d_i \left(T_{\mathcal{V}} S_{\mathcal{V}}^{-1} x\right)_i \\ &= \sum_{i\in I} d_i^2 P_{V_i} S_{\mathcal{V}}^{-1} x = S_{\mathcal{V}}^{-1} \sum_{i\in I} d_i^2 P_{V_i} x = S_{\mathcal{V}}^{-1} S_{\mathcal{V}} x = x.\end{aligned}$$

Thus, $\mathcal{V}$ is a dual of $\mathcal{W}$ in the sense of [19].

This shows that there is too much freedom in the choice of the operator Q in this definition. In addition, the reconstruction formula (3.7) – in this general form – seems to be of hardly any use in applications. However, when restricting the set of "admissible" operators Q to diagonal operators, formula (3.7) becomes much simpler:

$$x = \sum_{i\in I} c_i d_i Q_i P_{W_i} x, \quad x \in \mathcal{H},$$

where $Q_i \in \mathcal{B}(W_i, V_i)$ for each $i \in I$. Since in the present paper we prefer to work with $\mathfrak{H} = \ell^2(I, \mathcal{H})$ instead of $\bigoplus_{i\in I} W_i$ and $\bigoplus_{i\in I} V_i$, we allow $Q_i \in \mathcal{B}(\mathcal{H})$ for each $i \in I$ and ask for the validity of

$$x = \sum_{i\in I} c_i d_i P_{V_i} Q_i P_{W_i} x, \quad x \in \mathcal{H}. \tag{3.8}$$

Since the d_i's are somewhat arbitrary in this version (if $(Q_i)_{i\in I}$ and $(d_i)_{i\in I}$ satisfy (3.8), also $(\alpha_i Q_i)_{i\in I}$ and $(\alpha_i^{-1} d_i)_{i\in I}$ do for every bounded sequence $(\alpha_i)_{i\in I}$), we shall furthermore require that $\|Q_i\| = 1$. Moreover, since in (3.8) only the action of Q_i on W_i is important and, in addition, only that part being mapped to V_i, we shall also require that $W_i^\perp \subset \ker Q_i$ and $\operatorname{ran} Q_i \subset V_i$. Then (3.8) reduces to

$$x = \sum_{i\in I} c_i d_i Q_i x, \quad x \in \mathcal{H}, \tag{3.9}$$

since in this case we have $Q_i = P_{V_i} Q_i = Q_i P_{W_i}$.

For two fusion sequences $\mathcal{V} = ((V_i, d_i))_{i\in I}$ and $\mathcal{W} = ((W_i, c_i))_{i\in I}$ in $\mathcal{H}$ we define (see (2.7))

$$I_0(\mathcal{V}, \mathcal{W}) := \{i \in I : V_i = \{0\} \text{ or } W_i = \{0\}\} = \{i \in I : c_i = 0 \text{ or } d_i = 0\}.$$

Definition 3.9. *Let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ be a fusion frame for $\mathcal{H}$. A Bessel fusion sequence $\mathcal{V} = ((V_i, d_i))_{i\in I}$ will be called a* dual fusion frame (*or a* fusion frame dual *or, shortly, a* FF-dual) *of $\mathcal{W}$ if there exists a sequence $(Q_i)_{i\in I} \subset \mathcal{B}(\mathcal{H})$ satisfying*

$$W_i^\perp \subset \ker Q_i, \quad \operatorname{ran} Q_i \subset V_i, \quad \textit{and} \ \ \|Q_i\| = 1 \ \textit{if}\ i \notin I_0(\mathcal{V}, \mathcal{W}) \tag{3.10}$$

for each $i \in I$ such that (3.9) *holds.*

Remark 3.10. *(a) If $V_i = \{0\}$ or $W_i = \{0\}$ the first two conditions in* (3.10) *imply that $Q_i = 0$.*

(b) A dual fusion frame $((V_i, d_i))_{i\in I}$ as defined in [19] *was called* component-preserving *if there exists a bounded sequence $(Q_i)_{i\in I} \subset \mathcal{B}(\mathcal{H})$ satisfying* (3.8) *and $P_{V_i} Q_i W_i = V_i$ for each $i \in I$. Here, we shall not further study this subclass.*

(c) If $\mathcal{V} = ((V_i, d_i))_{i\in I}$ is a Găvruţa dual of $\mathcal{W} = ((W_i, c_i))_{i\in I}$ with the property that $P_{V_i} S_{\mathcal{W}}^{-1} P_{W_i} = 0 \implies i \in I_0(\mathcal{V}, \mathcal{W})$ for every $i \in I$, then the fusion sequence $((V_i, \|P_{V_i} S_{\mathcal{W}}^{-1} P_{W_i}\| d_i))_{i\in I}$ is a fusion frame dual of $\mathcal{W}$. Indeed, if for $i \in I$ we put $A_i := P_{V_i} S_{\mathcal{W}}^{-1} P_{W_i}$ as well as $Q_i := A_i/\|A_i\|$ if $A_i \neq 0$ and $Q_i := 0$ otherwise, then $(Q_i)_{i\in I}$ satisfies (3.10)*, and for $x \in \mathcal{H}$ we have $x = \sum_{i\in I} c_i d_i A_i x = \sum_{i\in I} c_i(\|A_i\| d_i) Q_i x$.*

Proposition 3.11. *The desiderata* **(D1)**–**(D4)** *are satisfied for the notion of fusion frame duals defined as in Definition* 3.9.

Proof. It is clear that the definition satisfies **(D1)** (see (3.8)). Moreover, (3.8) is equivalent to $T_{\mathcal{V}}^* Q T_{\mathcal{W}} = I$, where $Q = \bigoplus_{i\in I} Q_i$. This identity yields both $\operatorname{ran} T_{\mathcal{V}}^* = \mathcal{H}$ and $T_{\mathcal{W}}^* Q^* T_{\mathcal{V}} = I$. Therefore, $\mathcal{V}$ is a fusion frame for $\mathcal{H}$ (cf. Lemma 2.5), and $\mathcal{W}$ is a FF-dual of $\mathcal{V}$, meaning that **(D3)** is satisfied. To prove that **(D4)** holds, we note that $((S_{\mathcal{W}}^{-1} W_i, c_i))_{i\in I}$ is a Găvruţa dual of $\mathcal{W}$ by [16]. It has the property in Remark 3.10 (c). Hence, $((S_{\mathcal{W}}^{-1} W_i, c_i \|S_{\mathcal{W}}^{-1}|W_i\|))_{i\in I}$ is a FF-dual of $\mathcal{W}$ since $\|P_{S_{\mathcal{W}}^{-1} W_i} S_{\mathcal{W}}^{-1} P_{W_i}\| = \|S_{\mathcal{W}}^{-1} P_{W_i}\| = \|S_{\mathcal{W}}^{-1}|W_i\|$.

Let us see that also **(D2)** is satisfied. For this, let $(\psi_i)_{i\in I}$ be a dual of the frame $(\varphi_i)_{i\in I}$ for $\mathcal{H}$ as in (D2a). For $i \in I$, we put $W_i := \operatorname{span}\{\varphi_i\}$, $V_i := \operatorname{span}\{\psi_i\}$, and

$$Q_i x := \begin{cases} 0 & \text{if } \varphi_i = 0 \text{ or } \psi_i = 0 \\ \left\langle x, \frac{\varphi_i}{\|\varphi_i\|} \right\rangle \frac{\psi_i}{\|\psi_i\|} & \text{otherwise} \end{cases}, \qquad x \in \mathcal{H}.$$

Then $(Q_i)_{i\in I}$ satisfies (3.10). Moreover, we have

$$\sum_{i\in I} \|\varphi_i\|\|\psi_i\| Q_i x = \sum_{i\in I,\, \psi_i\neq 0, \varphi_i\neq 0} \|\varphi_i\|\|\psi_i\| \left\langle x, \frac{\varphi_i}{\|\varphi_i\|} \right\rangle \frac{\psi_i}{\|\psi_i\|} = \sum_{i\in I} \langle x, \varphi_i\rangle \psi_i = x.$$

Hence, $((V_i, \|\psi_i\|))_{i\in I}$ is a FF-dual of $((W_i, \|\varphi_i\|))_{i\in I}$, as desired in (D2a).

For (D2b), let $(Q_i)_{i\in I} \subset \mathcal{B}(\mathcal{H})$ be a sequence as in Definition 3.9. Choose $\varphi_i \in W_i$ with $\|\varphi_i\| = c_i$, $i \in I$. We put $\psi_i := c_i^{-1} d_i Q_i \varphi_i$ if $c_i \neq 0$. If $c_i = 0$ we choose an arbitrary $\psi_i \in V_i$ with $\|\psi_i\| = d_i$. Let us see that $\|\psi_i\| = d_i$ for each $i \in I$. This is clear if $c_i = 0$. Let $c_i \neq 0$. If $Q_i \neq 0$ then $\|Q_i\| = 1$ and hence $\|Q_i \varphi_i\| = c_i$, i.e., $\|\psi_i\| = d_i$. If $Q_i = 0$ then $i \in I_0(\mathcal{V}, \mathcal{W})$, implying that $V_i = \{0\}$ as $c_i \neq 0$ yields $W_i \neq \{0\}$. But then $d_i = 0$ and thus $\|\psi_i\| = 0 = d_i$. For arbitrary $x \in \mathcal{H}$ we have

$$\sum_{i\in I} |\langle x, \psi_i\rangle|^2 = \sum_{d_i\neq 0} |\langle x, \psi_i\rangle|^2 = \sum_{d_i\neq 0} d_i^2 \|P_{V_i} x\|^2 = \sum_{i\in I} d_i^2 \|P_{V_i} x\|^2 = \|T_{\mathcal{V}} x\|^2.$$

This implies that $(\psi_i)_{i\in I}$ is a Bessel sequence in $\mathcal{H}$. Finally,

$$\begin{aligned} \sum_{i\in I} \langle x, \varphi_i\rangle \psi_i &= \sum_{c_i\neq 0} c_i^{-1} d_i \langle x, \varphi_i\rangle Q_i \varphi_i = \sum_{c_i\neq 0} c_i^{-1} d_i Q_i\big(\langle x, \varphi_i\rangle \varphi_i\big) \\ &= \sum_{c_i\neq 0} c_i d_i Q_i P_{W_i} x = \sum_{i\in I} c_i d_i Q_i x = x. \end{aligned}$$

Hence, $(\psi_i)_{i\in I}$ is a dual frame of $(\varphi_i)_{i\in I}$. □

The following theorem provides a characterization of fusion frame duals.

Theorem 3.12. *Let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ be a fusion frame for $\mathcal{H}$ and $\mathcal{V} = ((V_i, d_i))_{i\in I}$ a Bessel fusion sequence. Then the following statements are equivalent:*

(i) *$\mathcal{V}$ is a fusion frame dual of $\mathcal{W}$.*
(ii) *There exists a sequence $(Q_i)_{i\in I} \subset \mathcal{B}(\mathcal{H})$ satisfying* (3.10) *such that $(d_i Q_i^*)_{i\in I}$ is a $\mathcal{B}(\mathcal{H})$-valued dual of $(c_i P_{W_i})_{i\in I}$.*

(iii) *There exists a Bessel sequence* $\mathcal{L} = (L_i)_{i\in I} \subset \mathcal{B}(\mathcal{H})$ *with* $T_{\mathcal{L}}^* T_{\mathcal{W}} = 0$ *such that for the operators* $A_i := (c_i S_{\mathcal{W}}^{-1} + L_i^*) P_{W_i}$, $i \in I$, *we have* $\operatorname{ran} A_i \subset V_i$ *and, if* $i \notin I_0(\mathcal{V}, \mathcal{W})$, $\|A_i\| = d_i$.

Proof. (i)⇒(ii). Let $(Q_i)_{i\in I} \subset \mathcal{B}(\mathcal{H})$ be as in Definition 3.9. Then this sequence satisfies (3.10) and $x = \sum_{i\in I} c_i d_i Q_i x$ for all $x \in \mathcal{H}$. Since $Q_i = P_{V_i} Q_i$ and $\|Q_i\| \le 1$ for each $i \in I$, we have for $x \in \mathcal{H}$:

$$\sum_{i\in I} \|d_i Q_i^* x\|^2 = \sum_{i\in I} d_i^2 \|Q_i^* P_{V_i} x\|^2 \le \sum_{i\in I} d_i^2 \|P_{V_i} x\|^2 = \|T_{\mathcal{V}} x\|^2,$$

which shows that $\mathcal{B} := (d_i Q_i^*)_{i\in I}$ is a Bessel sequence. Moreover, for $x \in \mathcal{H}$, $T_{\mathcal{B}}^* T_{\mathcal{W}} x = \sum_{i\in I} d_i Q_i (c_i P_{W_i} x) = \sum_{i\in I} c_i d_i Q_i x = x$. This proves (ii).

(ii)⇒(iii). As before, put $\mathcal{B} := (d_i Q_i^*)_{i\in I}$. By Lemma 3.2, there exists some $L \in \mathcal{B}(\mathcal{H}, \mathfrak{H})$ with $L^* T_{\mathcal{W}} = 0$ such that $T_{\mathcal{B}} = T_{\mathcal{W}} S_{\mathcal{W}}^{-1} + L$. Put $L_i := \mathfrak{E}_i^* L$, $i \in I$. Then $\mathcal{L} := (L_i)_{i\in I}$ is a $\mathcal{B}(\mathcal{H})$-valued Bessel sequence with $T_{\mathcal{L}} = L$. From $T_{\mathcal{B}} = T_{\mathcal{W}} S_{\mathcal{W}}^{-1} + L$ we conclude that $d_i Q_i^* = c_i P_{W_i} S_{\mathcal{W}}^{-1} + L_i$ for $i \in I$, that is, $d_i Q_i = c_i S_{\mathcal{W}}^{-1} P_{W_i} + L_i^*$. And since $Q_i = Q_i P_{W_i}$, we obtain $d_i Q_i = (c_i S_{\mathcal{W}}^{-1} + L_i^*) P_{W_i} = A_i$, $i \in I$. Therefore, $\operatorname{ran} A_i \subset V_i$ and $\|A_i\| = d_i$ if $i \notin I_0(\mathcal{V}, \mathcal{W})$.

(iii)⇒(i). For $i \in I$, define $Q_i := d_i^{-1} A_i$ if $d_i \neq 0$ and $Q_i := 0$ otherwise. Then $\operatorname{ran} Q_i \subset V_i$ and $W_i^\perp \subset \ker Q_i$, $i \in I$. If $d_i \neq 0$ then $\|Q_i\| = 1$ for $i \notin I_0(\mathcal{V}, \mathcal{W})$. If $d_i = 0$ then $V_i = \{0\}$, i.e., $i \in I_0(\mathcal{V}, \mathcal{W})$. Hence, $(Q_i)_{i\in I}$ satisfies (3.10). Moreover, for $x \in \mathcal{H}$ we have

$$\begin{aligned}\sum_{i\in I} c_i d_i Q_i x &= \sum_{d_i \neq 0} c_i A_i x = \sum_{i\in I} c_i A_i x = \sum_{i\in I} \left(c_i^2 S_{\mathcal{W}}^{-1} P_{W_i} x + c_i L_i^* P_{W_i} x\right) \\ &= S_{\mathcal{W}}^{-1} S_{\mathcal{W}} x + T_{\mathcal{L}}^* T_{\mathcal{W}} x = x,\end{aligned}$$

and (i) follows. □

As desired in **(D4)** and proved in Proposition 3.11, the Bessel fusion sequence $((S_{\mathcal{W}}^{-1} W_i, c_i \|S_{\mathcal{W}}^{-1}|W_i\|))_{i\in I}$ is always a FF-dual of $\mathcal{W} = ((W_i, c_i))_{i\in I}$ (and therefore itself a fusion frame). In analogy with the vector frame setting, we call

$$\widetilde{\mathcal{W}} := \left((S_{\mathcal{W}}^{-1} W_i, c_i \|S_{\mathcal{W}}^{-1}|W_i\|)\right)_{i\in I} \tag{3.11}$$

the *canonical* fusion frame dual of $\mathcal{W} = ((W_i, c_i))_{i\in I}$. Unfortunately, it is not clear at all whether for a given fusion frame there exist fusion frame duals other than the canonical one. However, in the finite-dimensional situation it follows from Theorem 3.12 that there are many of these. The next corollary – which is an immediate consequence of Theorem 3.12 – states this more precisely.

Corollary 3.13. *Let I be finite, and let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ be a fusion frame for the finite-dimensional Hilbert space $\mathcal{H}$. Then for each $L \in \mathcal{B}(\mathcal{H}, \mathfrak{H})$ with $\operatorname{ran} L \subset \ker T_{\mathcal{W}}^*$, the sequence $((\operatorname{ran} A_i, \|A_i\|))_{i\in I}$, where*

$$A_i := (c_i S_{\mathcal{W}}^{-1} + L^* \mathfrak{E}_i) P_{W_i}, \quad i \in I,$$

is a fusion frame dual of $\mathcal{W}$.

4. Perturbations of Operator-valued Frame Sequences

In this section, we prove that – in some sense and under certain conditions – duals of operator-valued frames and frame sequences are stable under small perturbations. For this, we generalize the notion of μ-perturbation from [17] to operator-valued Bessel sequences.

Definition 4.1. *Let $\mu > 0$, and let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued Bessel sequences. We say that $\mathcal{B}$ is a μ-*perturbation *of $\mathcal{A}$ (and vice versa) if*

$$\|T_{\mathcal{A}} - T_{\mathcal{B}}\| \le \mu.$$

Remark 4.2. *The term μ-perturbation was originally introduced in [17] for vector sequences which might not be Bessel sequences. However, two vector Bessel sequences being μ-perturbations of one another in the sense of [17] means that the difference of their synthesis operators has a norm which does not exceed μ. Therefore, in the case $\mathcal{K} = \mathbb{C}$, the above definition coincides with the one in [17] (for Bessel sequences). Furthermore, we mention that $\|T_{\mathcal{A}} - T_{\mathcal{B}}\| \le \mu$ implies $\|A_i - B_i\| \le \mu$ for every $i \in I$ since*

$$\|A_i - B_i\| = \|\mathfrak{E}_i^* T_{\mathcal{A}} - \mathfrak{E}_i^* T_{\mathcal{B}}\| \le \|T_{\mathcal{A}} - T_{\mathcal{B}}\| \le \mu.$$

In order to treat perturbations of operator-valued frame sequences, we will utilize the notion of the gap between subspaces of a Hilbert space.

4.1. The Gap between Subspaces

For two closed subspaces V and W of $\mathcal{H}$ the *gap from V to W* is defined by

$$\delta(V,W) := \sup\{\|v - P_W v\| : v \in V,\ \|v\| = 1\} = \|(I - P_W)|V\| = \|P_{W^\perp}|V\|.$$

We remark that in [17], $\delta(V,W)$ was called the gap *between* V and W. Here, we agree to follow, e.g., [12], and choose a different term in order to emphasize the order of V and W in $\delta(V,W)$. It is worth noting that

$$\delta(W^\perp, V^\perp) = \|P_V|W^\perp\| = \|(P_{W^\perp}|V)^*\| = \|P_{W^\perp}|V\| = \delta(V,W). \tag{4.1}$$

Instead of the gap, some authors prefer to work with the *infimum cosine angle* $R(V,W)$ from V to W which is given by

$$R(V,W) := \inf\{\|P_W v\| : v \in V,\ \|v\| = 1\}.$$

It is easy to see that

$$\delta(V,W) = \sqrt{1 - R(V,W)^2}. \tag{4.2}$$

As δ is not a metric, Kato (see [21, §IV.2]) defines the *gap between V and W* by

$$\Delta(V,W) := \max\{\delta(V,W), \delta(W,V)\},$$

and shows that

$$\Delta(V,W) = \|P_V - P_W\|.$$

The next lemma is well known (see, e.g., [3, 12] or [21, Theorem I-6.34]). For the sake of completeness, we provide a short proof here.

Lemma 4.3. *The following statements hold.*

(i) *If $\delta(V,W) < 1$, then $V \cap W^\perp = \{0\}$, and $P_W|V \in \mathcal{B}(V,W)$ is bounded below.*
(ii) *If $\Delta(V,W) < 1$ then $\delta(V,W) = \delta(W,V)$, and the operators $P_W|V \in \mathcal{B}(V,W)$ and $P_V|W \in \mathcal{B}(W,V)$ are isomorphisms.*

Proof. (i). From (4.2), we see that $R(V,W) > 0$ which implies that $P_W|V$ is bounded below. In particular, $V \cap W^\perp = \ker(P_W|V) = \{0\}$.

(ii). By (i), $P_W|V$ and $P_V|W$ are bounded below. And since $P_V|W = (P_W|V)^*$, we conclude from Lemma 1.1 that these operators are bijective. Finally,

$$R(V,W)^{-1} = \|(P_W|V)^{-1}\| = \|((P_W|V)^*)^{-1}\| = \|(P_V|W)^{-1}\| = R(W,V)^{-1}$$

proves $\delta(V,W) = \delta(W,V)$. □

Lemma 4.4. *Let $\mathcal{A}$ be an $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued frame sequence with lower frame bound α, and $\mathcal{B}$ an $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued Bessel sequence. Then*

$$\delta(\mathcal{H}_\mathcal{A},\mathcal{H}_\mathcal{B}) \,\le\, \frac{\|T_\mathcal{A} - T_\mathcal{B}\|}{\sqrt{\alpha}}.$$

Proof. Since the restriction $T_\mathcal{A}|\mathcal{H}_\mathcal{A}$ is bounded below, its inverse $J : \operatorname{ran} T_\mathcal{A} \to \mathcal{H}_\mathcal{A}$ exists. It satisfies $T_\mathcal{A}^* J^* = I_{\mathcal{H}_\mathcal{A}}$. And since $(I - P_\mathcal{B})T_\mathcal{B}^* = 0$, it follows that

$$(I - P_\mathcal{B})|\mathcal{H}_\mathcal{A} = (I - P_\mathcal{B})(T_\mathcal{A}^* - T_\mathcal{B}^*)J^*.$$

For $y \in \operatorname{ran} T_\mathcal{A}$ we have $\|Jy\|^2 \le \alpha^{-1}\|T_\mathcal{A} Jy\|^2 = \alpha^{-1}\|y\|^2$. Therefore,

$$\delta(\mathcal{H}_\mathcal{A},\mathcal{H}_\mathcal{B}) = \|(I - P_\mathcal{B})|\mathcal{H}_\mathcal{A}\| \,\le\, \|(T_\mathcal{A}^* - T_\mathcal{B}^*)J^*\| \,\le\, \frac{\|T_\mathcal{A} - T_\mathcal{B}\|}{\sqrt{\alpha}},$$

and the lemma is proven. □

4.2. The Effect on the Canonical Dual

In [17], it was studied how perturbations of a vector frame sequence affect the canonical dual. In the following, we analyze the analogous problem for operator-valued frame sequences. The following theorem generalizes the results in [17] and even improves them.

Theorem 4.5. *Let $\mathcal{A}$ be an $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued frame sequence with frame bounds α and β, and let $\mathcal{B}$ be an $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued Bessel sequence such that it is a μ-perturbation of $\mathcal{A}$. Then we have*

$$\delta(\mathcal{H}_\mathcal{A},\mathcal{H}_\mathcal{B}) \,\le\, \frac{\mu}{\sqrt{\alpha}}. \tag{4.3}$$

Moreover, if $\mu < \sqrt{\alpha}$ and $\Delta := \Delta(\mathcal{H}_\mathcal{A},\mathcal{H}_\mathcal{B}) < 1$, then the following statements hold:

(i) *$\mathcal{B}$ is an operator-valued frame sequence with frame bounds*

$$\big(\sqrt{\alpha} - \mu\big)^2 \quad \textit{and} \quad \big(\delta(\mathcal{H}_\mathcal{B},\mathcal{H}_\mathcal{A}^\perp)\sqrt{\beta} + \mu\big)^2. \tag{4.4}$$

(ii) *For the canonical duals $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ of $\mathcal{A}$ and $\mathcal{B}$, respectively, we have*

$$\big\|T_{\widetilde{\mathcal{A}}} - T_{\widetilde{\mathcal{B}}}\big\| \le \frac{(\alpha + 2\beta + \sqrt{\beta}\mu)\mu \,+\, (\alpha + (\sqrt{\alpha} - \mu)^2)\Delta}{\alpha(\sqrt{\alpha} - \mu)^2}. \tag{4.5}$$

(iii) *For the gap $\Delta' := \Delta(\operatorname{ran} T_\mathcal{A}, \operatorname{ran} T_\mathcal{B})$ between the closed subspaces $\operatorname{ran} T_\mathcal{A}$ and $\operatorname{ran} T_\mathcal{B}$ we have*

$$\Delta' \le \big\|T_{\widetilde{\mathcal{A}}} - T_{\widetilde{\mathcal{B}}}\big\|\sqrt{\beta} + \frac{\mu}{\sqrt{\alpha} - \mu}.$$

Proof. The relation (4.3) follows directly from Lemma 4.4. Assume now that $\mu < \sqrt{\alpha}$ and $\Delta := \Delta(\mathcal{H}_\mathcal{A},\mathcal{H}_\mathcal{B}) < 1$.

(i). For the upper frame bound of $\mathcal{B}$, let $x \in \mathcal{H}_\mathcal{B}$. Then

$$\|T_\mathcal{B} x\| \le \|(T_\mathcal{B} - T_\mathcal{A})x\| + \|T_\mathcal{A}(P_\mathcal{A}|\mathcal{H}_\mathcal{B})x\| \le \Big(\mu + \sqrt{\beta}\|P_\mathcal{A}|\mathcal{H}_\mathcal{B}\|\Big)\,\|x\|.$$

For the lower frame bound, let $x \in \mathcal{H}_\mathcal{A}$. Then we have (see (2.5))

$$\|T_\mathcal{B} P_\mathcal{B} x\| = \|T_\mathcal{B} x\| \ge \|T_\mathcal{A} x\| - \|(T_\mathcal{A} - T_\mathcal{B})x\| \ge (\sqrt{\alpha} - \mu)\|x\| \ge (\sqrt{\alpha} - \mu)\|P_\mathcal{B} x\|.$$

And as (due to $\Delta < 1$) $P_\mathcal{B}$ maps $\mathcal{H}_\mathcal{A}$ bijectively onto $\mathcal{H}_\mathcal{B}$, a lower frame bound of $\mathcal{B}$ is $(\sqrt{\alpha} - \mu)^2$.

(ii). First of all, we observe that

$$\begin{aligned}
&(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} P_\mathcal{A} - (S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B} \\
&\quad = (S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} P_\mathcal{A} P_\mathcal{B} - P_\mathcal{A}(S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B} + (S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} P_\mathcal{A}(I - P_\mathcal{B}) \\
&\qquad\qquad - (I - P_\mathcal{A})(S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B} \\
&\quad = (S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} P_\mathcal{A}(S_\mathcal{B} - S_\mathcal{A})(S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B} + (S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} P_\mathcal{A}(I - P_\mathcal{B}) \\
&\qquad\qquad - (I - P_\mathcal{A}) P_\mathcal{B} (S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B}.
\end{aligned}$$

Since

$$\|S_\mathcal{B} - S_\mathcal{A}\| \le \|T_\mathcal{B}^*(T_\mathcal{B} - T_\mathcal{A})\| + \|(T_\mathcal{B}^* - T_\mathcal{A}^*)T_\mathcal{A}\| \le (2\sqrt{\beta} + \mu)\mu \tag{4.6}$$

and

$$\|(I - P_\mathcal{B})P_\mathcal{A}\| = \|(I - P_\mathcal{B})|\mathcal{H}_\mathcal{A}\| = \Delta = \|(I - P_\mathcal{A})|\mathcal{H}_\mathcal{B}\| = \|(I - P_\mathcal{A})P_\mathcal{B}\|,$$

we obtain

$$\left\|(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} P_\mathcal{A} - (S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B}\right\| \le \frac{(2\sqrt{\beta} + \mu)\mu}{\alpha(\sqrt{\alpha} - \mu)^2} + \frac{\Delta}{\alpha} + \frac{\Delta}{(\sqrt{\alpha} - \mu)^2}. \tag{4.7}$$

Finally, this give

$$\begin{aligned}
\left\|T_{\widetilde{\mathcal{A}}} - T_{\widetilde{\mathcal{B}}}\right\| &= \left\|T_\mathcal{A}(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} P_\mathcal{A} - T_\mathcal{B}(S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B}\right\| \\
&\le \left\|T_\mathcal{A}\left[(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} P_\mathcal{A} - (S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B}\right]\right\| + \left\|(T_\mathcal{A} - T_\mathcal{B})(S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B}\right\| \\
&\le \sqrt{\beta}\left(\frac{(2\sqrt{\beta} + \mu)\mu}{\alpha(\sqrt{\alpha} - \mu)^2} + \frac{\Delta}{\alpha} + \frac{\Delta}{(\sqrt{\alpha} - \mu)^2}\right) + \frac{\mu}{(\sqrt{\alpha} - \mu)^2},
\end{aligned}$$

which is (4.5).

(iii). We first note that $P_{\operatorname{ran} T_\mathcal{A}} = T_\mathcal{A}(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} T_\mathcal{A}^*$ and $P_{\operatorname{ran} T_\mathcal{B}} = T_\mathcal{B}(S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} T_\mathcal{B}^*$. Thus, taking into account Remark 3.6

$$\begin{aligned}
\Delta' &= \left\|\left[T_\mathcal{A}(S_\mathcal{A}|\mathcal{H}_\mathcal{A})^{-1} P_\mathcal{A} - T_\mathcal{B}(S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B}\right] T_\mathcal{A}^* + T_\mathcal{B}(S_\mathcal{B}|\mathcal{H}_\mathcal{B})^{-1} P_\mathcal{B}(T_\mathcal{A}^* - T_\mathcal{B}^*)\right\| \\
&\le \left\|T_{\widetilde{\mathcal{A}}} - T_{\widetilde{\mathcal{B}}}\right\|\sqrt{\beta} + \frac{\mu}{\sqrt{\alpha} - \mu},
\end{aligned}$$

and the theorem is proven. □

Remark 4.6. (a) *The condition that* $\Delta(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B}) < 1$ *cannot be omitted in* (i). *A counterexample in the vector sequence case can be found in* [10, Example 15.3.1].

(b) *Note that* Δ *in item* (ii) *of Theorem* 4.5 *tends* (*linearly*) *to zero with* μ (*cf.* (4.3)).

(c) *In the vector sequence case, a similar statement as that of Theorem* 4.5 (ii) *can be found in* [17, Theorem 3.2]. *However, our condition on* μ *is weaker, and the estimate* (4.5) *is stronger than that in* [17].

Let us state Theorem 4.5 especially for the (operator-valued) frame situation (where $\Delta = 0$).

Corollary 4.7. *Let* $\mathcal{A}$ *be an* $\mathcal{B}(\mathcal{H}, \mathcal{K})$*-valued frame with bounds* α *and* β*, and let* $\mathcal{B}$ *be an* $\mathcal{B}(\mathcal{H}, \mathcal{K})$*-valued Bessel sequence such that it is a* μ*-perturbation of* $\mathcal{A}$ *with* $\mu < \sqrt{\alpha}$*. Then the following statements hold.*

(i) *Also $\mathcal{B}$ is an operator-valued frame, and*

$$\left\|T_{\mathcal{A}}S_{\mathcal{A}}^{-1} - T_{\mathcal{B}}S_{\mathcal{B}}^{-1}\right\| \le \frac{\alpha + 2\beta + \sqrt{\beta}\mu}{\alpha(\sqrt{\alpha}-\mu)^2}\,\mu. \tag{4.8}$$

(ii) *For the gap between the closed subspaces* $\operatorname{ran} T_{\mathcal{A}}$ *and* $\operatorname{ran} T_{\mathcal{B}}$ *we have*

$$\Delta(\operatorname{ran} T_{\mathcal{A}}, \operatorname{ran} T_{\mathcal{B}}) \le \frac{(\alpha + 2\beta + \sqrt{\beta}\mu)\sqrt{\beta} + \alpha(\sqrt{\alpha}-\mu)}{\alpha(\sqrt{\alpha}-\mu)^2}\,\mu.$$

Remark 4.8. *We mention that W. Sun proved* (*see* [29, Theorem 4.1]) *that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued frames with bounds $\alpha_j \le \beta_j$ $(j = 1, 2)$ which are μ-perturbations of each other, then*

$$\|T_{\mathcal{A}_1}S_{\mathcal{A}_1}^{-1} - T_{\mathcal{A}_2}S_{\mathcal{A}_2}^{-1}\| \le \frac{\alpha_1 + \beta_1 + \sqrt{\beta_1\beta_2}}{\alpha_1\alpha_2}\,\mu.$$

With $\alpha_2 = (\sqrt{\alpha_1}-\mu)^2$ and $\beta_2 = (\sqrt{\beta_1}+\mu)^2$ (cf. (4.4)*), this becomes the same estimate as* (4.8)*. Hence, Theorem* 4.5 *can be regarded as a generalization of* [29, Theorem 4.1].

4.3. The Effect on the Alternate Duals

In the following, we shall study the effect of perturbations on the alternate duals. In particular, we show that whenever $\mathcal{A}$ is an operator-valued frame sequence, $\widetilde{\mathcal{A}}$ a dual of $\mathcal{A}$, and $\mathcal{B}$ a small perturbation of $\mathcal{A}$, then there is a dual $\widetilde{\mathcal{B}}$ of $\mathcal{B}$ which also is a small perturbation of $\widetilde{\mathcal{A}}$. For the notation $\widetilde{\mathcal{A}}(L)$ with $L \in \mathcal{B}(\mathcal{H}_{\mathcal{A}}, \mathfrak{K})$ we refer the reader to (3.4).

Theorem 4.9. *Let $\mathcal{A}$ be an $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued frame sequence with frame bounds α and β, and let $L \in \mathcal{B}(\mathcal{H}_{\mathcal{A}}, \mathfrak{K})$ with $L^*T_{\mathcal{A}} = 0$, defining the dual $\widetilde{\mathcal{A}}(L)$ of $\mathcal{A}$. Moreover, let $\mathcal{B}$ be an $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued Bessel sequence such that it is a μ-perturbation of $\mathcal{A}$, where $\mu < \sqrt{\alpha}$ and $\Delta := \Delta(\mathcal{H}_{\mathcal{A}}, \mathcal{H}_{\mathcal{B}}) < 1$. Then, $\mathcal{B}$ is an operator-valued frame sequence, and the dual $\widetilde{\mathcal{B}}\big(P_{\ker T_{\mathcal{B}}^*}L(P_{\mathcal{A}}|\mathcal{H}_{\mathcal{B}})\big)$ of $\mathcal{B}$ is a λ-perturbation of $\widetilde{\mathcal{A}}(L)$, where*

$$\lambda = \lambda_0 + \lambda_1\|L\|$$

with

$$\lambda_0 = \frac{(\alpha + 2\beta + \sqrt{\beta}\mu)\mu + (\alpha + (\sqrt{\alpha}-\mu)^2)\Delta}{\alpha(\sqrt{\alpha}-\mu)^2} \quad \text{and} \quad \lambda_1 = \frac{\mu}{\sqrt{\alpha}-\mu} + \Delta. \tag{4.9}$$

Proof. First, note that $\mathcal{B}$ is an operator-valued frame sequence by Theorem 4.5 and that, by Lemma 3.2, $\widetilde{\mathcal{B}}(P_{\ker T_{\mathcal{B}}^*}L(P_{\mathcal{A}}|\mathcal{H}_{\mathcal{B}}))$ is a dual of $\mathcal{B}$. Now, we put $\widetilde{\mathcal{B}} := \widetilde{\mathcal{B}}(P_{\ker T_{\mathcal{B}}^*}L(P_{\mathcal{A}}|\mathcal{H}_{\mathcal{B}}))$ and $\widetilde{\mathcal{A}} := \widetilde{\mathcal{A}}(L)$. Then, by definition, we have

$$\begin{aligned} T_{\widetilde{\mathcal{A}}} - T_{\widetilde{\mathcal{B}}} &= \big[T_{\mathcal{A}}(S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1}P_{\mathcal{A}} + LP_{\mathcal{A}}\big] - \big[T_{\mathcal{B}}(S_{\mathcal{B}}|\mathcal{H}_{\mathcal{B}})^{-1}P_{\mathcal{B}} + P_{\ker T_{\mathcal{B}}^*}LP_{\mathcal{A}}P_{\mathcal{B}}\big] \\ &= \big[T_{\mathcal{A}}(S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1}P_{\mathcal{A}} - T_{\mathcal{B}}(S_{\mathcal{B}}|\mathcal{H}_{\mathcal{B}})^{-1}P_{\mathcal{B}}\big] + \big[LP_{\mathcal{A}} - P_{\ker T_{\mathcal{B}}^*}LP_{\mathcal{A}}P_{\mathcal{B}}\big]. \end{aligned}$$

Due to Theorem 4.5 (ii),

$$\|T_{\mathcal{A}}(S_{\mathcal{A}}|\mathcal{H}_{\mathcal{A}})^{-1}P_{\mathcal{A}} - T_{\mathcal{B}}(S_{\mathcal{B}}|\mathcal{H}_{\mathcal{B}})^{-1}P_{\mathcal{B}}\| \le \lambda_0,$$

where λ_0 is as in (4.9) above. On the other hand, taking into consideration that $P_{\operatorname{ran} T_{\mathcal{B}}} = T_{\mathcal{B}}(S_{\mathcal{B}}|\mathcal{H}_{\mathcal{B}})^{-1}T_{\mathcal{B}}^*$ and $T_{\mathcal{A}}^*L = 0$, we obtain

$$\begin{aligned} LP_{\mathcal{A}} - P_{\ker T_{\mathcal{B}}^*}LP_{\mathcal{A}}P_{\mathcal{B}} &= (I - P_{\ker T_{\mathcal{B}}^*})LP_{\mathcal{A}} + P_{\ker T_{\mathcal{B}}^*}LP_{\mathcal{A}}(I - P_{\mathcal{B}}) \\ &= P_{\operatorname{ran} T_{\mathcal{B}}}LP_{\mathcal{A}} + P_{\ker T_{\mathcal{B}}^*}LP_{\mathcal{A}}(I - P_{\mathcal{B}}) \\ &= T_{\mathcal{B}}(S_{\mathcal{B}}|\mathcal{H}_{\mathcal{B}})^{-1}P_{\mathcal{B}}(T_{\mathcal{B}}^* - T_{\mathcal{A}}^*)LP_{\mathcal{A}} + P_{\ker T_{\mathcal{B}}^*}LP_{\mathcal{A}}(I - P_{\mathcal{B}}). \end{aligned}$$

This leads to

$$\begin{aligned}\|LP_{\mathcal{A}} - P_{\ker T_{\mathcal{B}}^*} LP_{\mathcal{A}} P_{\mathcal{B}}\| &\le \left\|T_{\mathcal{B}}(S_{\mathcal{B}}|\mathcal{H}_{\mathcal{B}})^{-1} P_{\mathcal{B}}\right\| \mu\|L\| + \|L\|\|P_{\mathcal{A}}(I - P_{\mathcal{B}})\| \\ &\le \frac{\mu\|L\|}{\sqrt{\alpha} - \mu} + \|L\|\Delta = \lambda_1\|L\|\end{aligned}$$

with λ_1 as defined in (4.9). This proves the theorem. □

For the case of operator-valued frames we derive the following result.

Corollary 4.10. *Let $\mathcal{A}$ be a $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued frame with frame bounds α and β, and let $L \in \mathcal{B}(\mathcal{H},\mathfrak{K})$ with $L^*T_{\mathcal{A}} = 0$, defining the dual frame $\widetilde{\mathcal{A}}(L)$ of $\mathcal{A}$. Moreover, let $\mathcal{B}$ be an $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued Bessel sequence which is a μ-perturbation of $\mathcal{A}$, where $\mu < \sqrt{\alpha}$. Then $\mathcal{B}$ is an operator-valued frame, and the dual $\widetilde{\mathcal{B}}(P_{\ker T_{\mathcal{B}}^*}L)$ of $\mathcal{B}$ is a λ-perturbation of $\widetilde{\mathcal{A}}(L)$, where*

$$\lambda = \frac{\alpha + 2\beta + \sqrt{\beta}\mu}{\alpha(\sqrt{\alpha} - \mu)^2}\,\mu + \frac{\mu\|L\|}{\sqrt{\alpha} - \mu}.$$

Remark 4.11. *Towards the end of our studies on the subject, we came across the paper* [1] *where the authors prove the following:* Given an operator-valued frame $\mathcal{A}$, a dual $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ and $\mu > 0$ sufficiently small, then for every operator-valued frame $\mathcal{B}$ being a μ-perturbation of $\mathcal{A}$ there exist $C > 0$ and a dual $\widetilde{\mathcal{B}}$ of $\mathcal{B}$ which is a $C\mu$-perturbation of $\widetilde{\mathcal{A}}$. *The constant C developed in the proof depends on the frame bounds of $\mathcal{A}$, $\mathcal{B}$, and $\widetilde{\mathcal{A}}$, and on $\|L\|$, where L is such that $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(L)$. The main and essential difference to Corollary* 4.10 *is that in* [1] *it is required that both $\mu < \|L\|^{-1}$ and that $\mathcal{B}$ is an operator-valued frame, whereas in Corollary* 4.10 *we only need $\mu < \sqrt{\alpha}$ to ensure that $\mathcal{B}$ is an operator-valued frame. In fact, in* [1] *the authors require that $W := I + T_{\mathcal{B}}^*L$ is invertible (which is the case when $\mu < \|L\|^{-1}$) and define $\widetilde{\mathcal{B}}$ by $T_{\widetilde{\mathcal{B}}} := (T_{\mathcal{B}}S_{\mathcal{B}}^{-1} + L)W^{-1}$.*

Theorem 4.9 shows that, given two operator-valued frame sequences $\mathcal{A}$ and $\mathcal{B}$ which are close to each other, then for any dual $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ there exists a special dual $\widetilde{\mathcal{B}}$ of $\mathcal{B}$ which is close to $\widetilde{\mathcal{A}}$. The following proposition now states that under certain conditions the mapping $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is one-to-one and onto.

Proposition 4.12. *Let $\mathcal{A}$ be an $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued frame sequence with bounds α and β, and let $\mathcal{B}$ be a ($\mathcal{B}(\mathcal{H},\mathcal{K})$-valued) μ-perturbation of $\mathcal{A}$ such that $\Delta(\mathcal{H}_{\mathcal{A}},\mathcal{H}_{\mathcal{B}}) < 1$ and $\mu < \sqrt{\alpha}$. Denote the canonical duals of $\mathcal{A}$ and $\mathcal{B}$ by $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$, respectively. If $\left\|T_{\widetilde{\mathcal{A}}} - T_{\widetilde{\mathcal{B}}}\right\|\sqrt{\beta} + \frac{\mu}{\sqrt{\alpha}-\mu} < 1$, then the (affine) mapping*

$$\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}), \qquad \widetilde{\mathcal{A}}(L) \mapsto \widetilde{\mathcal{B}}\left(P_{\ker T_{\mathcal{B}}^*}L\,(P_{\mathcal{A}}|\mathcal{H}_{\mathcal{B}})\right)$$

is bijective.

Proof. It obviously suffices to prove that the linear mapping

$$\mathcal{R} : \mathcal{L}_{\mathcal{A}} \to \mathcal{L}_{\mathcal{B}}, \qquad \mathcal{R}X := P_{\ker T_{\mathcal{B}}^*}XP_{\mathcal{B}}, \quad X \in \mathcal{L}_{\mathcal{A}},$$

is bijective. For this, we observe that Theorem 4.5 (iii) implies $\Delta(\operatorname{ran} T_{\mathcal{A}}, \operatorname{ran} T_{\mathcal{B}}) < 1$. Let $X = LP_{\mathcal{A}} \in \mathcal{L}_{\mathcal{A}}$, $L \in \mathcal{B}(\mathcal{H}_{\mathcal{A}},\mathfrak{K})$, be such that $\mathcal{R}X = P_{\ker T_{\mathcal{B}}^*}LP_{\mathcal{A}}P_{\mathcal{B}} = 0$. Then, since $T_{\mathcal{A}}^*L = 0$, we obtain

$$\operatorname{ran}(LP_{\mathcal{A}}P_{\mathcal{B}}) \subset (\ker T_{\mathcal{B}}^*)^\perp \cap \ker T_{\mathcal{A}}^* = \operatorname{ran} T_{\mathcal{B}} \cap (\operatorname{ran} T_{\mathcal{A}})^\perp = \{0\},$$

where the last equality is due to Lemma 4.3 (i). Hence, we conclude $LP_{\mathcal{A}}P_{\mathcal{B}} = 0$. And as $P_{\mathcal{A}}$ maps $\mathcal{H}_{\mathcal{B}}$ surjectively onto $\mathcal{H}_{\mathcal{A}}$, it follows that $L = 0$, i.e., $X = 0$, and $\mathcal{R}$ is injective.

In order to prove that $\mathcal{R}$ is also surjective, we define $K : \ker T_{\mathcal{A}}^* \to \ker T_{\mathcal{B}}^*$ by $K := P_{\ker T_{\mathcal{B}}^*}|\ker T_{\mathcal{A}}^*$. Due to Lemma 4.3 and (4.2), the operator K is an isomorphism. Now, let $Y \in \mathcal{L}_{\mathcal{B}}$, i.e., $Y = L_1 P_{\mathcal{B}}$ with $L_1 \in \mathcal{B}(\mathcal{H}_{\mathcal{B}}, \mathfrak{K})$ such that $T_{\mathcal{B}}^* L_1 = 0$. The latter means that $\operatorname{ran} L_1 \subset \ker T_{\mathcal{B}}^*$. Hence, the operator $L := K^{-1} L_1 (P_{\mathcal{A}}|\mathcal{H}_{\mathcal{B}})^{-1} \in \mathcal{B}(\mathcal{H}_{\mathcal{A}}, \mathfrak{K})$ is well defined, and $X := L P_{\mathcal{A}}$ is an element of $\mathcal{L}_{\mathcal{A}}$. Finally, we have

$$\mathcal{R}X = P_{\ker T_{\mathcal{B}}^*} X P_{\mathcal{B}} = P_{\ker T_{\mathcal{B}}^*} K^{-1} L_1 (P_{\mathcal{A}}|\mathcal{H}_{\mathcal{B}})^{-1} P_{\mathcal{A}} P_{\mathcal{B}} = K K^{-1} L_1 P_{\mathcal{B}} = Y,$$

and the proposition is proven. □

5. Perturbations of Fusion Frames

This section is devoted to studying the behavior of the canonical FF-dual (as defined in (3.11)) under perturbations. We consider perturbations of fusion frames in the same sense as of operator-valued frames. More precisely:

Definition 5.1. *Let $\mu > 0$, and let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ and $\mathcal{V} = ((V_i, d_i))_{i\in I}$ be two Bessel fusion sequences in $\mathcal{H}$. We say that $\mathcal{V}$ is a μ-*perturbation *of $\mathcal{W}$ (and vice versa) if $(d_i P_{V_i})_{i\in I}$ is a μ-perturbation of $(c_i P_{W_i})_{i\in I}$ in the sense of Definition 4.1, that is, when $\|T_{\mathcal{W}} - T_{\mathcal{V}}\| \le \mu$.*

Remark 5.2. (a) *If $\mathcal{V} = ((V_i, d_i))_{i\in I}$ is a μ-perturbation of $\mathcal{W} = ((W_i, c_i))_{i\in I}$, then for each $i \in I$ we have*

$$\|c_i P_{W_i} - d_i P_{V_i}\| = \|(T_{\mathcal{W}}^* - T_{\mathcal{V}}^*)\mathfrak{E}_i\| \le \mu. \tag{5.1}$$

In particular, if $c_i = d_i = 1$, $i \in I$, then $\Delta(W_i, V_i) \le \mu$ for all $i \in I$. Since $c_i = \|c_i P_{W_i}\| \le \|c_i P_{W_i} - d_i P_{V_i}\| + d_i$ and $d_i \le \|c_i P_{W_i} - d_i P_{V_i}\| + c_i$, relation (5.1) *implies*

$$|c_i - d_i| \le \|c_i P_{W_i} - d_i P_{V_i}\| \le \mu.$$

(b) *In* [7]*, a different notion of perturbation for fusion frames was considered (see* [7, Definition 5.1]*) and it was proven that fusion frames are stable under these perturbations (see* [7, Proposition 5.8]*). When $\mathcal{W} = ((W_i, c_i))_{i\in I}$ and $\mathcal{V} = ((V_i, c_i))_{i\in I}$ are Bessel fusion sequences and the sequence of weights $c := \{c_i\}_{i\in I}$ belongs to $\ell^2(I)$, the notion of perturbation in* [7] *implies that $\mathcal{V}$ is a μ-perturbation of $\mathcal{V}$ for $\mu = (\lambda_1 + \lambda_2 + \varepsilon)\|c\|_{\ell^2(I)}$, with $\lambda_1, \lambda_2 \ge 0$ and $\varepsilon > 0$ being the perturbation parameters of* [7, Definition 5.1]*. On the other hand, by* (5.1)*, Definition 5.1 implies* [7, Definition 5.1] *only when the weights satisfy $\inf_{i\in I} c_i > 0$. However, in the finite-dimensional setting, both notions of perturbation are equivalent.*

In the following, we will show that the canonical FF-dual of a μ-perturbation of a fusion frame $\mathcal{W}$ will be a $C\mu$-perturbation of the canonical FF-dual $\widetilde{\mathcal{W}}$ of $\mathcal{W}$, where $C > 0$ depends on μ and $\mathcal{W}$. For this, we shall exploit the following two lemmas.

Lemma 5.3. *Let P and Q be orthogonal projections in $\mathcal{H}$ and $c, d > 0$. Then*

$$\|P - Q\| \le \sqrt{\frac{1}{c^2} + \frac{1}{d^2}}\,\|cP - dQ\|.$$

Proof. Let $x \in \mathcal{H}$, $\|x\| = 1$. Then we have

$$\begin{aligned}\|cPx - dQx\|^2 &= \|cQPx + c(I-Q)Px - dQx\|^2\\ &= \|Q(cPx - dx)\|^2 + c^2\|(I-Q)Px\|^2\\ &\ge c^2\|(I-Q)Px\|^2.\end{aligned}$$

Analogously, one obtains $\|cPx - dQx\|^2 \ge d^2\|(I-P)Qx\|^2$. Thus, we have

$$\|(I-Q)P\| \le \frac{1}{c}\|cP - dQ\| \quad \text{and} \quad \|(I-P)Q\| \le \frac{1}{d}\|cP - dQ\|.$$

Hence, also $\|Q(I-P)\| = \|((I-P)Q)^*\| = \|(I-P)Q\| \le \frac{1}{d}\|cP - dQ\|$. Since, for $x \in \mathcal{H}$,

$$\|(P-Q)x\|^2 = \|QPx + (I-Q)Px - Qx\|^2 = \|Q(I-P)x\|^2 + \|(I-Q)Px\|^2,$$

the claim follows from the above inequalities. □

Lemma 5.4. *Let $W \subset \mathcal{H}$ be a closed subspace and $A \in \mathcal{B}(\mathcal{H})$ boundedly invertible. Then, for every $\lambda > 0$, the operator*

$$R(\lambda) := AP_W + \lambda A^{-*}P_{W^\perp}$$

where $A^{-*} = (A^{-1})^*$, is boundedly invertible and we have

$$P_{AW} = R(\lambda)^{-*}P_W A^*. \tag{5.2}$$

Moreover, if $c, d > 0$ are such that $c\|x\| \le \|Ax\| \le d\|x\|$ for $x \in \mathcal{H}$ then

$$d^{-1}\min\{1, \lambda^{-1}cd\}\|x\| \le \|R(\lambda)^{-1}x\| \le c^{-1}\max\{1, \lambda^{-1}cd\}\|x\|. \tag{5.3}$$

As a consequence, we obtain

$$d^{-1}\|P_W A^* x\| \le \|P_{AW}x\| \le c^{-1}\|P_W A^* x\|. \tag{5.4}$$

Proof. First of all, we note that $(AW)^\perp = A^{-*}W^\perp$. From this, it immediately follows that $R(\lambda)$ is boundedly invertible and that $P_{AW}R(\lambda) = AP_W$. The latter implies $P_{AW} = AP_W R(\lambda)^{-1}$. Adjoining this gives (5.2). For the proof of (5.3) let $x \in \mathcal{H}$. Then we obtain

$$\begin{aligned}\|R(\lambda)x\|^2 &= \|AP_W x\|^2 + \lambda^2\|A^{-*}P_{W^\perp}x\|^2 \ge c^2\|P_W x\|^2 + \lambda^2 d^{-2}\|P_{W^\perp}x\|^2\\ &\ge \min\{c^2, \lambda^2 d^{-2}\}\|x\|^2,\end{aligned}$$

as well as

$$\begin{aligned}\|R(\lambda)x\|^2 &= \|AP_W x\|^2 + \lambda^2\|A^{-*}P_{W^\perp}x\|^2 \le d^2\|P_W x\|^2 + \lambda^2 c^{-2}\|P_{W^\perp}x\|^2\\ &\le \max\{d^2, \lambda^2 c^{-2}\}\|x\|^2.\end{aligned}$$

This implies (5.3). Setting $\lambda = cd$ and using (5.2) yields (5.4). □

We briefly remark that Lemma 5.4 immediately implies the following corollary which was already proved in [16, Theorem 2.4].

Corollary 5.5. *Let $((W_i, c_i))_{i\in I}$ be a fusion frame for $\mathcal{H}$ with bounds $\alpha \le \beta$ and let $A \in \mathcal{B}(\mathcal{H})$ be boundedly invertible. Then also $((AW_i, c_i))_{i\in I}$ is a fusion frame for $\mathcal{H}$ with bounds $\alpha\gamma^{-2} \le \beta\gamma^2$, where $\gamma = \|A\|\|A^{-1}\|$.*

We are now ready to prove our main result in this section.

Theorem 5.6. *Let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds $\alpha \le \beta$ and let $\mathcal{V} = ((V_i, d_i))_{i\in I}$ be a μ-perturbation of $\mathcal{W}$, where $0 < \mu < \sqrt{\alpha}$. If both sequences $(c_i)_{i\in I}$ and $(d_i)_{i\in I}$ are bounded from below by some $\tau > 0$ then the canonical FF-dual $\widetilde{\mathcal{V}}$ of $\mathcal{V}$, is a $C\mu$-perturbation of the canonical FF-dual $\widetilde{\mathcal{W}}$ of $\mathcal{W}$, where*

$$C = \frac{c^2 + d^2}{\alpha}\left[\frac{1 + (\alpha^{-1} + \beta)^2}{\sqrt{\alpha}}\left(\frac{\sqrt{2}}{\tau} + cd^2\right) + d^2\left(1 + c^2 d^2\right)\right]$$

with $c := 2\sqrt{\beta} + \mu$ and $d := (\sqrt{\alpha} - \mu)^{-1}$.

Proof. For $i \in I$ we define the operators

$$R_{W_i} := S_{\mathcal{W}}^{-1} P_{W_i} + S_{\mathcal{W}} P_{W_i^\perp} \qquad \text{and} \qquad R_{V_i} := S_{\mathcal{V}}^{-1} P_{V_i} + S_{\mathcal{V}} P_{V_i^\perp}.$$

By Lemma 5.4, these are boundedly invertible and we have

$$P_{S_{\mathcal{W}}^{-1} W_i} = R_{W_i}^{-*} P_{W_i} S_{\mathcal{W}}^{-1} \qquad \text{and} \qquad P_{S_{\mathcal{V}}^{-1} V_i} = R_{V_i}^{-*} P_{V_i} S_{\mathcal{V}}^{-1}.$$

Furthermore,

$$\|R_{W_i}^{-1}\| \le \max\{\beta, \alpha^{-1}\} \le \alpha^{-1} + \beta \quad \text{and} \quad \|R_{V_i}^{-1}\| \le \max\{c^2, d^2\} \le c^2 + d^2.$$

Put $\hat{c}_i := \|S_{\mathcal{W}}^{-1}|W_i\| c_i$ and $\hat{d}_i := \|S_{\mathcal{V}}^{-1}|V_i\| d_i$. Then $\hat{c}_i \le \alpha^{-1} c_i$ and $\hat{d}_i \le (\sqrt{\alpha} - \mu)^{-2} d_i = d^2 d_i$. For $x \in \mathcal{H}$ define

$$\Delta_i(x) := \left\| \hat{c}_i P_{S_{\mathcal{W}}^{-1} W_i} x - \hat{d}_i P_{S_{\mathcal{V}}^{-1} V_i} x \right\| = \left\| \hat{c}_i R_{W_i}^{-*} P_{W_i} S_{\mathcal{W}}^{-1} x - \hat{d}_i R_{V_i}^{-*} P_{V_i} S_{\mathcal{V}}^{-1} x \right\|.$$

Since $\|T_{\widetilde{\mathcal{W}}} x - T_{\widetilde{\mathcal{V}}} x\|^2 = \sum_{i \in I} \Delta_i^2(x)$, it is our aim to estimate $\Delta_i(x)$. We have

$$\begin{aligned}
\Delta_i(x) &\le \left\| \hat{c}_i (R_{W_i}^{-*} - R_{V_i}^{-*}) P_{W_i} S_{\mathcal{W}}^{-1} x \right\| + \left\| R_{V_i}^{-*} \left(\hat{c}_i P_{W_i} S_{\mathcal{W}}^{-1} x - \hat{d}_i P_{V_i} S_{\mathcal{V}}^{-1} x \right) \right\| \\
&\le \left\| R_{W_i}^{-1} - R_{V_i}^{-1} \right\| \alpha^{-1} c_i \|P_{W_i} S_{\mathcal{W}}^{-1} x\| + \left\| R_{V_i}^{-1} \right\| \left\| \hat{c}_i P_{W_i} S_{\mathcal{W}}^{-1} x - \hat{d}_i P_{V_i} S_{\mathcal{W}}^{-1} x \right\| \\
&\qquad + \left\| R_{V_i}^{-1} \right\| d^2 d_i \left\| P_{V_i} \left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1} \right) x \right\|.
\end{aligned}$$

Since $R_{W_i}^{-1} - R_{V_i}^{-1} = R_{W_i}^{-1} (R_{V_i} - R_{W_i}) R_{V_i}^{-1}$ and $\|R_{V_i}^{-1}\| \le c^2 + d^2$ by Lemma 5.4, with

$$\Delta_i^{(1)} := \alpha^{-1} \left\| R_{W_i}^{-1} \right\| \|R_{W_i} - R_{V_i}\| \qquad \text{and} \qquad \Delta_i^{(2)} := \left| \left\| S_{\mathcal{W}}^{-1} | W_i \right\| - \left\| S_{\mathcal{V}}^{-1} | V_i \right\| \right|$$

we obtain

$$\begin{aligned}
\frac{\Delta_i(x)}{c^2 + d^2} &\le \Delta_i^{(1)} c_i \left\| P_{W_i} S_{\mathcal{W}}^{-1} x \right\| + d^2 d_i \left\| P_{V_i} \left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1} \right) x \right\| + \Delta_i^{(2)} c_i \left\| P_{W_i} S_{\mathcal{W}}^{-1} x \right\| \\
&\qquad + \left\| S_{\mathcal{V}}^{-1} | V_i \right\| \left\| \left(c_i P_{W_i} - d_i P_{V_i} \right) S_{\mathcal{W}}^{-1} x \right\| \\
&\le \left(\Delta_i^{(1)} + \Delta_i^{(2)} \right) c_i \left\| P_{W_i} S_{\mathcal{W}}^{-1} x \right\| + d^2 d_i \left\| P_{V_i} \left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1} \right) x \right\| \\
&\qquad + d^2 \left\| \left(c_i P_{W_i} - d_i P_{V_i} \right) S_{\mathcal{W}}^{-1} x \right\|.
\end{aligned}$$

Let us start with estimating $\Delta_i^{(2)}$. From $\|S_{\mathcal{W}}^{-1}|W_i\| = \|S_{\mathcal{W}}^{-1} P_{W_i}\|$, Lemma 5.3, and Remark 5.2 it follows that

$$\begin{aligned}
\Delta_i^{(2)} &= \left| \left\| S_{\mathcal{W}}^{-1} P_{W_i} \right\| - \left\| S_{\mathcal{V}}^{-1} P_{V_i} \right\| \right| \;\le\; \left\| S_{\mathcal{W}}^{-1} P_{W_i} - S_{\mathcal{V}}^{-1} P_{V_i} \right\| \\
&\le \left\| S_{\mathcal{W}}^{-1} \left(P_{W_i} - P_{V_i} \right) \right\| + \left\| S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1} \right\| \\
&\le \alpha^{-1} \sqrt{\frac{1}{c_i^2} + \frac{1}{d_i^2}} \; \|c_i P_{W_i} - d_i P_{V_i}\| + \frac{(2\sqrt{\beta} + \mu)\mu}{\alpha(\sqrt{\alpha} - \mu)^2} \\
&\le \frac{\sqrt{2}\mu}{\tau\alpha} + \frac{(2\sqrt{\beta} + \mu)\mu}{\alpha(\sqrt{\alpha} - \mu)^2} = M\mu,
\end{aligned}$$

where

$$M = \frac{1}{\alpha} \left(\frac{\sqrt{2}}{\tau} + \frac{2\sqrt{\beta} + \mu}{(\sqrt{\alpha} - \mu)^2} \right) = \frac{1}{\alpha} \left(\frac{\sqrt{2}}{\tau} + c d^2 \right).$$

Here, we have estimated $\|S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1}\|$ as in (4.7). In order to estimate $\Delta_i^{(1)}$, we observe that (see (4.6))

$$\begin{aligned}
\|R_{W_i} - R_{V_i}\| &= \left\|S_{\mathcal{W}}^{-1}P_{W_i} + S_{\mathcal{W}}P_{W_i^\perp} - S_{\mathcal{V}}^{-1}P_{V_i} - S_{\mathcal{V}}P_{V_i^\perp}\right\| \\
&\le M\mu + \left\|S_{\mathcal{W}}P_{W_i^\perp} - S_{\mathcal{V}}P_{V_i^\perp}\right\| \\
&\le M\mu + \left\|S_{\mathcal{W}}\left(P_{W_i^\perp} - P_{V_i^\perp}\right)\right\| + \left\|\left(S_{\mathcal{W}} - S_{\mathcal{V}}\right)P_{V_i^\perp}\right\| \\
&\le \left(\frac{1}{\alpha}\left(\frac{\sqrt{2}}{\tau} + cd^2\right) + \frac{\sqrt{2}\beta}{\tau} + c\right)\mu \\
&= \left(\alpha^{-1} + \beta\right)\left(\frac{\sqrt{2}}{\tau} + cd^2 \cdot \frac{1 + \alpha d^{-2}}{1 + \alpha\beta}\right)\mu \\
&\le \left(\alpha^{-1} + \beta\right)\left(\frac{\sqrt{2}}{\tau} + cd^2\right)\mu = \alpha\left(\alpha^{-1} + \beta\right)M\mu,
\end{aligned}$$

where in the last inequality we have used that $d^{-2} \le \beta$. Thus, we have

$$\Delta_i^{(1)} + \Delta_i^{(2)} \;\le\; \left(\alpha^{-1} + \beta\right)^2 M\mu + M\mu = \left(1 + \left(\alpha^{-1} + \beta\right)^2\right)M\mu.$$

Now, define the functionals

$$\begin{aligned}
r_i(x) &:= \left(1 + \left(\alpha^{-1} + \beta\right)^2\right)M\mu c_i\left\|P_{W_i}S_{\mathcal{W}}^{-1}x\right\|, \\
s_i(x) &:= d^2 d_i\left\|P_{V_i}\left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1}\right)x\right\|, \\
t_i(x) &:= d^2\left\|\left(c_iP_{W_i} - d_iP_{V_i}\right)S_{\mathcal{W}}^{-1}x\right\|.
\end{aligned}$$

as well as

$$R(x) := \sqrt{\sum_{i\in I} r_i^2(x)}, \qquad S(x) := \sqrt{\sum_{i\in I} s_i^2(x)}, \qquad \text{and} \qquad T(x) := \sqrt{\sum_{i\in I} t_i^2(x)}.$$

Then $(c^2+d^2)^{-1}\Delta_i(x) \le r_i(x) + s_i(x) + t_i(x)$ and, by applying the triangle inequality on $\ell^2(I)$, one obtains

$$\frac{1}{(c^2+d^2)^2}\sum_{i\in I}\Delta_i^2(x) \le \sum_{i\in I}\left(r_i(x) + s_i(x) + t_i(x)\right)^2 \;\le\; \left(R(x) + S(x) + T(x)\right)^2.$$

We have (see Remark 3.6)

$$\begin{aligned}
R^2(x) &= \left(1 + \left(\alpha^{-1} + \beta\right)^2\right)^2 M^2\mu^2\left\|T_{\mathcal{W}}S_{\mathcal{W}}^{-1}x\right\|^2 \le \left(\frac{1 + \left(\alpha^{-1} + \beta\right)^2}{\sqrt{\alpha}}\right)^2 M^2\mu^2\|x\|^2, \\
S^2(x) &= d^4\left\|T_{\mathcal{V}}\left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1}\right)x\right\|^2 \le \alpha^{-2}c^4d^8\mu^2\|x\|^2, \\
T^2(x) &= d^4\left\|\left(T_{\mathcal{W}} - T_{\mathcal{V}}\right)S_{\mathcal{W}}^{-1}x\right\|^2 \le \alpha^{-2}d^4\mu^2\|x\|^2.
\end{aligned}$$

That is,

$$\frac{1}{c^2+d^2}\sqrt{\sum_{i\in I}\Delta_i^2(x)} \;\le\; \left[\frac{1 + \left(\alpha^{-1} + \beta\right)^2}{\sqrt{\alpha}}M + \alpha^{-1}c^2d^4 + \alpha^{-1}d^2\right]\mu\|x\|.$$

This shows that $\|T_{\widetilde{\mathcal{W}}} - T_{\widetilde{\mathcal{V}}}\| \le C\mu$. □

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