Error Correction for Erasures of Quantized Frame Coefficients

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Abstract:

In this paper we investigate an algorithm for the suppression of errors caused by quantization of frame coefficients and by erasures in their subsequent transmission. The erasures are assumed to happen independently, modeled by a Bernoulli experiment. The algorithm for error correction in this study embeds check bits in the quantization of frame coefficients, causing a possible, but only temporary quantizer overload. The overload can be controlled similarly as in the case of the sigma-delta algorithm without an insertion of check bits. Assuming the existence of suitable codes yields that the erasure-averaged reconstruction error after error correction exhibits a power-law decay as sigma-delta quantization without erasures.

1. Extended Abstract

The power of redundant systems, in particular frames, has been demonstrated by their resilience to erasures and by their usefulness to suppress quantization errors. In the context of finite frames, the statistical error estimates by Goyal, Kovačević, Vetterli and Kelner [7, 6] were to the authors' knowledge the first instance of a combined analysis of erasures and quantization.

In recent years, the robustness of finite frames against erasures has been more extensively studied, for instance, in [5] or [2]. These studies typically provide estimates for the (average and worst case) blind reconstruction error, meaning all erased (unknown) coefficients are set to zero and the reconstruction relies on a fixed synthesis operator. It is well-known that if the frame vectors related to the non-erased coefficients still form a complete set, then the frame operator of those can be inverted, leading to perfect reconstruction. However, the latency caused by the wait until all coefficients have been transmitted and the computational cost of inverting the frame operator make perfect reconstruction less practicable.

On the other hand, Benedetto, Powell and Yilmaz [1] investigated an easily implementable, active error correction for the compensation of quantization errors with so-called sigma-delta algorithms, which provide highly accurate reconstruction.

Recently, Boufounos, Oppenheim and Goyal [4] introduced an erasure correction scheme with strong similarities to quantization-noise shaping, offering the possibility of a combined treatment of both types of errors. The idea of pre-compensation and error-forward projection deserves to be explored further, but the algorithm by Boufounos, Oppenheim and Goyal is computationally still more costly than a simple application of sigma-delta quantization.

The need for results on low-complexity quantization-and-erasure correcting algorithms motivated the present study, which investigates a rather simple strategy for error compensation, a combination of sigma-delta algorithm and check bits. The error correction algorithm we present allows precise bounds on quantization errors and also on the effect of erasures from unreliable transmissions of frame coefficients.

1.1 PCM quantization and blind reconstruction

We first revisit erasure-averaged error bounds for PCM quantization of frame coefficients and blind reconstruction after transmission.

Definition. Let $\mathcal{H}$ be a d-dimensional Hilbert space. A frame $\mathcal{F} = \{f_1, f_2, \dots f_N\}$ for $\mathcal{H}$ is a spanning set. If all vectors in the frame have the same norm, we call $\mathcal{F}$ equal-norm. If $x = \frac{1}{A}\sum_{j=1}^{N}\langle x, f_j\rangle f_j$ for all $x \in \mathcal{H}$, then we say that $\mathcal{F}$ is A-tight.

Quantizing frame coefficients simply means mapping them to a finite set.

Definition. A function Q on $\mathbb{R}$ is called *a quantizer with accuracy* $\epsilon > 0$ *on the interval* $[-L, +L]$ if it has a finite range $\mathbb{A}$ and for any $x \in [-L, +L]$, $Q(x)$ satisfies $|x - Q(x)| \le \epsilon$. The range $\mathbb{A}$ of the quantizer Q is also called the alphabet. If this alphabet consists of all integer multiples of a fixed step-size δ contained in the interval $[-L - \delta/2, +L + \delta/2]$ and the quantizer assigns to $x \in [-L, +L]$ the unique value $m\delta, m \in \mathbb{Z}$, satisfying $(m - \frac{1}{2})\delta < x \le (m + \frac{1}{2})\delta$ then we call Q the *uniform mid-tread quantizer with step-size* δ [3]. Alternatively, if the alphabet is $\mathbb{A} = (\mathbb{Z} + \frac{1}{2})\delta \cap [-L - \delta/2, +L + \delta/2]$ and if Q assigns to $x \in [-L, +L]$ the value $(m + \frac{1}{2})\delta$ such that $m\delta < x \le (m + 1)\delta$, then we speak of the so-called *uniform mid-riser quantizer with step-size* δ. In the latter part of this study, we focus on the single-bit mid-riser quantizer which rounds to $\mathbb{A} = \{-\delta/2, +\delta/2\}$.

We want to apply this quantizer to frame coefficients.

Definition. Given a quantizer Q, the PCM quantization of a vector x in a real Hilbert space $\mathcal{H}$ of dimension $\dim(\mathcal{H}) = d$, equipped with an A-tight frame $\mathcal{F} =$

$\{f_i\}_{i=1}^N$, is defined by

$$Q_{\mathcal{F}}(x) = \frac{1}{A}\sum_{j=1}^{N} Q(\langle x, f_j\rangle) f_j\,.$$

Remark. We recall that the PCM quantization error resulting from a uniform quantizer Q with accuracy $\epsilon > 0$ on $[-L, +L]$, and a N/d-tight equal-norm frame $\mathcal{F}$ applied to any input vector $x \in \mathcal{H}$ satisfying $\|x\| \le L$ is in norm bounded by

$$\|Q_{\mathcal{F}}(x) - x\| \le \sqrt{d}\epsilon\,.$$

This is in contrast to erasures, where the bound on the reconstruction error depends on the norm of the input vector.

Definition. Given a probability measure $\mathbb{P}$ on the set of erasures, and the analysis operator V belonging to an A-tight frame, we define the erasure-averaged reconstruction error to be

$$e(V,\mathbb{P}) = \mathbb{E}[\|\frac{1}{A}V^*E(\omega)V - I\|]\,.$$

Hereby, $\mathbb{E}[\cdot]$ is the expectation with respect to the probability measure $\mathbb{P}$ on $\Omega = \{0,1\}^N$, and $E : \Omega \to \mathbb{R}^{N\times N}$ is a random diagonal matrix with entries $E_{j,j} = \omega_j$.

Theorem. Let $\mathcal{H}$ be a real Hilbert space of dimension d, equipped with an A-tight equal-norm frame F. If all the frame coefficients are erased with a probability $0 \le p \le 1$, independently of each other, then the erasure-averaged reconstruction error is bounded by

$$p \le \mathbb{E}[\|\frac{1}{A}V^*EV - I\|] \le pd\,.$$

Thus, for a vector x for which $p\|x\|$ is bigger than $(\delta/2)\sqrt{d}$, the bound on the worst case error due to erasures dominates that of PCM quantization.

A similar phenomenon happens when the quantization is obtained with first and higher-order sigma delta quantization. For sufficiently large N, the bound for the worst-case quantization error, see e.g. [3], is smaller than the worst-case erasures error. This motivates investigating active error correction for erasures.

1.2 Sigma-delta quantization with embedded check bits

Our main goal is to make the two error bounds for erasures and quantization comparable. To this end, we use systematic binary error-correcting codes for packets of quantized coefficients, and replace a portion of the output from the sigma-delta quantizer by the check bits.

Definition. A binary (n,k)-code is an invertible map

$$C : \mathbb{Z}_2^k \to \mathbb{Z}_2^n\,.$$

The *minimum distance* of this code is the minimal number of bits by which any two code words (elements in the range of C) differ. Note that we do not require C to be a linear map over $\mathbb{Z}_2$. A *systematic* (n,k)*-code* simply appends check bits, meaning $q = (q_1, q_2, \dots q_k)$ maps to $C(q) = (q_1', q_2', \dots q_n')$ such that $q_j' = q_j$ for all $j \in \{1, 2, \dots k\}$.

The relevance of this definition is that among any block of n transmitted bits, the minimum distance is the number of bit erasures that cannot be corrected any more.

As already mentioned, we will exploit a particular accompanying quantization strategy, which we briefly explain.

Definition. Let Q be the binary mid-riser quantizer with stepsize $\delta > 0$ and let $\mathcal{F} = \{f_1, f_2, \dots f_N\}$ be an N/d-tight frame for a d-dimensional real Hilbert space $\mathcal{H}$. Also, assume that C is a binary (n,k)-code. Given an input vector $x \in \mathcal{H}$, then the C*-embedded first-order sigma-delta quantized sequence* $\{q_j\}_{j=1}^\infty$ associated with the initialization value $u_0 \in \mathbb{R}$ is defined by

$$q_{m+j} := \begin{cases} Q(\langle x, f_{m+j}\rangle + u_{m+j-1}), & 1 \le j \le k\,,\\ C((q_{m+1}, q_{m+2}), \dots q_{m+k}))_j, & \text{else}\,, \end{cases}$$

for any $m \in \{0, n, 2n, \dots\}$, and $j \in \{1, 2, \dots n\}$, where the map for updating the internal variable is

$$u_{m+j} := \langle x, f_{m+j}\rangle - q_{m+j} + u_{m+j-1}\,.$$

From now on, we choose the convention $u_0 = 0$ and abbreviate $Q_{\mathcal{F},C}(x) = \frac{d}{N}\sum_{j=1}^N q_j f_j$.

Our first main theorem is the stability of this modified sigma-delta algorithm.

Theorem. Let Q be a binary mid-riser quantizer with stepsize $\delta > 0$, let $\mathcal{F} = \{f_1, f_2, \dots f_N\}$ be an N/d-tight frame for a d-dimensional real Hilbert space $\mathcal{H}$, and let C be a systematic binary (n,k)-code, such that n divides N. If $\|x\| < \delta/2$ and

$$k \ge \frac{n}{2} + \frac{n\|x\|}{\delta}$$

then in the course of the C-embedded first-order sigma-delta quantization, the internal variable is bounded by

$$|u_j| \le \frac{\delta}{2}(n-k+1) + (n-k)\|x\|$$

for all $j \in \{1, 2, \dots N\}$.

Similarly as in [1] and [3], we conclude an error estimate. The relevant quantity in this estimate is derived from the frame geometry, as in [3],

$$T(\mathcal{F}) = \|f_1 \pm f_2 \pm \dots \pm f_N\|\,.$$

We define the maximal error caused by quantization to be

$$eq(V,\delta) = \max_x \|Q_{\mathcal{F},C}(x) - x\|\,,$$

where V is the analysis operator of the frame $\mathcal{F}$ and the maximum is taken over vectors of fixed norm $\|x\| < \delta/2$.

Theorem. Under the same assumptions as in the preceding theorem,

$$eq(V,\delta) \le \frac{d}{N}(\frac{\delta}{2}(n-k+1) + (n-k)\|x\|)T(\mathcal{F})\,.$$

In comparison with the unmodified first order sigma-delta quantization, we have a bound that is worse by roughly a factor of $2(n-k)$. However, the advantage of the embedded check bits is the ability to correct erasures in each block.

Assume the initial probability measure applies an erasure with a probability of p to each coefficient. Assume that the code C has minimal distance $np + t$ with $t > 0$. Let $\mathbb{P}'$ denote the probability measure governing the erasures remaining after the error correction has been applied in each block of length n.

Definition. The combination of quantization, erasures and error correction gives the reconstruction error

$$ec(V, \delta, \mathbb{P}') = \mathbb{E}[\max_x \| \frac{1}{A} \sum_j \omega_j q_j f_j - x \|] \, ,$$

where the maximum is taken over x with a fixed norm $\|x\| < \delta/2$ and $\omega_j = 0$ means that the j-th coefficient is erased.

The following lemma helps bound the probability of erasures remaining, if the weight of the code is larger than the expected number of erasures before correction.

Lemma. By Hoeffding's inequality, the probability p' of an individual coefficient being erased after the error correction is applied is bounded by

$$p' \leq \exp(-2t^2/n) \, .$$

Now we can combine the two error estimates for quantization and erasures.

Theorem. Let $t > 0$, assume C has minimal distance $np + t$. Let $\mathbb{P}'$ be the probability measure governing the erasures after the error correction has been applied. Under the additional assumptions of the preceding theorem,

$$ec(V, \delta, \mathbb{P}') \leq eq(V, \delta) + d\delta \exp(-2t^2/n) \, .$$

Corollary. If $n = mN^\alpha$ for some fixed m, $0 < \alpha < 1$, and if the minimum distance $np + t$ with $t > \sqrt{(n \ln n)/2(1-\alpha)}$ is achievable for a sequence of codes as N diverges, then

$$ec(V, \delta, \mathbb{P}') \leq \frac{d\delta}{N^{\alpha-1}}(m + m^{\alpha-1}) \, ,$$

and thus both terms, the quantization error and the remaining erasures contribute with the same asymptotic behavior in the limit of large N.

The remaining question is which values of α can be achieved with a suitable sequence of codes. We will investigate this elsewhere.

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References:

[1] John J. Benedetto, Alexander M. Powell, and Ozgur Yilmaz, Sigma-Delta quantization and finite frames, *IEEE Trans. Inform. Theory* 52:1990–2005, 2006.

[2] Bernhard G. Bodmann and Vern I. Paulsen. Frames, graphs and erasures. *Linear Algebra Appl.* 404:118–146, 2005.

[3] Bernhard G. Bodmann and Vern I. Paulsen. Frame Paths and Error Bounds for Sigma-Delta Quantization. *Appl. Comput. Harmon. Anal.* 22:176–197, 2007.

[4] Petros Boufounos, Alan V. Oppenheim, and Vivek K. Goyal. Causal Compensation for Erasures in Frame Representations. *IEEE Trans. Signal Proc.* 3:1071–1082, 2008.

[5] Peter Casazza and Jelena Kovačević, Equal-norm tight frames with erasures. (English summary) Frames. *Adv. Comput. Math.* 18:387–430, 2003.

[6] Vivek K. Goyal, Jelena Kovačević, and Jonathan A. Kelner. Quantized frame expansions with erasures. *Appl. Comp. Harm. Anal.*, 10:203–233, 2001.

[7] Vivek K. Goyal, Jelena Kovačević, and Martin Vetterli. Quantized frame expansions as source-channel codes for erasure channels. In: Proc. Data Compr. Conf., Snowbird, UT, Mar. 1999.