A Theoretical Analysis of Deep Neural Networks and Parametric PDEs

Gitta Kutyniok [*†] Philipp Petersen[‡] Mones Raslan[*] Reinhold Schneider[*]

March 31, 2019

Abstract

We derive upper bounds on the complexity of ReLU neural networks approximating the solution maps of parametric partial differential equations. In particular, without any knowledge of its concrete shape, we use the inherent low-dimensionality of the solution manifold to obtain approximation rates which are significantly superior to those provided by classical approximation results. We use this low dimensionality to guarantee the existence of a reduced basis. Then, for a large variety of parametric partial differential equations, we construct neural networks that yield approximations of the parametric maps not suffering from a curse of dimension and essentially only depending on the size of the reduced basis.

Keywords: deep neural networks, parametric PDEs, approximation rates, curse of dimension, reduced basis method

Mathematical Subject Classification: 35A35, 35J99, 41A25, 41A46, 68T05, 65N30

1 Introduction

In this work, we analyze the suitability of deep neural networks (DNNs) for the numerical solution of parametric problems. Such problems connect a parameter space with a solution state space via a so-called *parametric map*, [53]. One special case of such a parametric problem arises when the parametric map results from solving a partial differential equation (PDE) and the parameters describe physical or geometrical constraints of the PDE such as, for example, the shape of the physical domain, boundary conditions, or a source term. Applications that lead to these problems include modeling unsteady and steady heat and mass transfer, acoustics, fluid mechanics, or electromagnetics, [34].

Solving a parametric PDE for every point in the parameter space of interest individually typically leads to two types of problems. First, if the number of parameters of interest is excessive—a scenario coined many-query application—then the associated computational complexity could be unreasonably high. Second, if the computation time is severely limited, such as in real-time applications, then solving even a single PDE might be too costly.

A core assumption to overcome the two issues outlined above is that the solution manifold, i.e., the set of all admissible solutions associated with the parameter space, is inherently low dimensional. This assumption forms the foundation for the so-called reduced basis method (RBM). A reduced basis discretization is then a (Galerkin) projection on a low-dimensional approximation space that is built from snapshots of the parametrically induced manifold, [62].

[*]Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany, e-mail: {kutyniok, raslan, schneidr}@math.tu-berlin.de

[†]Institut für Informatik und Elektrotechnik, Technische Universität Berlin

[‡]Mathematical Institute, University of Oxford, Andrew Wiles Building, Woodstock Road, Oxford, OX2 6GG, UK, e-mail: Philipp.Petersen@maths.ox.ac.uk

Constructing the low-dimensional approximation spaces is typically computationally expensive because it involves solving the PDEs for multiple instances of parameters. These computations take place in a so-called *offline* phase—a step of pre-computation, where one assumes to have access to sufficiently powerful computational resources. Once a suitable low-dimensional space is found, the cost of solving the associated PDEs for a new parameter value is significantly reduced and can be performed quickly and *online*, i.e., with limited resources, [5, 58]. We will give a more thorough introduction to RBMs in Section 2. An extensive survey of works on RBMs, which can be traced back to the seventies and eighties of the last century (see for instance [23, 51, 52]), is beyond the scope of this paper. We refer, for example, to [34, Chapter 1.1], [59, 17, 29] and [12, Chapter 1.9] for (historical) studies of this topic.

In this work, we show that the low-dimensionality of the solution manifold also enables an efficient approximation of the parametric map by DNNs. In this context, the RBM will be, first and foremost, a tool to model this low-dimensionality by acting as a blueprint for the construction of the DNNs.

1.1 Statistical Learning Problems

The motivation to study the approximability of parametric maps by DNNs stems from the following similarities between parametric problems and *statistical learning problems*: Assume that we are given a *domain set* $X \subset \mathbb{R}^n$, $n \in \mathbb{N}$ and a *label set* $Y \subset \mathbb{R}^k$, $k \in \mathbb{N}$. Further assume that there exists an unknown probability distribution ρ on $X \times Y$.

Given a *loss function* $\mathcal{L}\colon Y \times Y \to \mathbb{R}^+$, the goal of a statistical learning problem is to find a function f, which we will call *prediction rule*, from a hypothesis class $H \subset \{h\colon X \to Y\}$ such that the *expected loss* $\mathbb{E}_{(\mathbf{x},\mathbf{y})\sim\rho}\mathcal{L}(f(\mathbf{x}),\mathbf{y})$ is minimized, [15]. Since the probability measure ρ is unknown, we have no direct access to the expected loss. Instead, we assume that we are given a set of training data, i.e. pairs $(\mathbf{x}_i,\mathbf{y}_i)_{i=1}^N$, $N \in \mathbb{N}$, which were drawn independently with respect to ρ. Then one finds f by minimizing the so-called *empirical loss*

$$\sum_{i=1}^{N} \mathcal{L}(f(\mathbf{x}_i),\mathbf{y}_i) \tag{1.1}$$

over H. We will call optimizing the empirical loss the *learning procedure*.

In view of PDEs, the approach proposed above can be rephrased in the following way. We are aiming to produce a function from a parameter set to a state space based on a few snapshots only. This function should satisfy the involved PDEs as precisely as possible, and the evaluation of this function should be very efficient even though the construction of it can potentially be computationally expensive.

In the above-described sense, a parametric PDE problem almost perfectly matches the definition of a statistical learning problem. Indeed, the PDEs and the metric on the state space correspond to a (deterministic) distribution ρ and a loss function. Moreover, the snapshots are construed as the training data, and the offline phase mirrors the learning procedure. Finally, the parametric map is the prediction rule.

One of the most efficient learning methods nowadays is deep learning. This method describes a range of learning procedures to solve statistical learning problems where the hypothesis class H is taken to be a set of DNNs, [41, 24]. These methods outperform virtually all classical machine learning techniques in sufficiently complicated tasks from speech recognition to image classification. Strikingly, training DNNs is a computationally very demanding task that is usually performed on highly parallelized machines. Once a DNN is fully trained, however, its application to a given input is orders of magnitudes faster than the training process. This observation again reflects the common offline-online phase distinction that is common in RBM approaches.

Based on the overwhelming success of these techniques and the apparent similarities of learning problems and parametric problems it appears natural to apply methods from deep learning to statistical learning problems in the sense of (partly) replacing the parameter-dependent map by a DNN. Very successful advances in this direction have been reported in [40, 35, 43, 72, 60].

1.2 Our Contribution

In the applications [40, 35, 43, 72, 60] mentioned above, the combination of DNNs and parametric problems seems to be remarkably efficient. In this paper, we present a theoretical justification of this approach. We address the question to what extent the hypothesis class of DNNs is sufficiently broad to approximately represent the associated parametric maps. Concretely, we aim at understanding the necessary number of parameters of DNNs required to allow a sufficiently accurate approximation. We will demonstrate that depending on the target accuracy the required number of parameters of DNNs essentially only scales with the intrinsic dimension of the solution manifold, in particular, according to its Kolmogorov N-widths. We outline our results in Subsection 1.2.1. Then, we present a simplified exposition of our argument leading to the main results in Subsection 1.2.2.

1.2.1 Approximation Theoretical Results

The main contributions of this work are given by two approximation results with DNNs based on ReLU activation functions. In both cases, we aim to learn a variation of the parametric map

$$\mathcal{Y} \ni y \mapsto u_y \in \mathcal{H},$$

where $\mathcal{Y}$ is the parameter space and $\mathcal{H}$ is a Hilbert space. In our case, the parameter space will be a compact subset of $\mathbb{R}^p$ for some fixed, but possibly large $p \in \mathbb{N}$, i.e. we consider the case of finitely supported parameter vectors.

1. *Approximation of a discretized solution:* In our first result, we assume that there exists a sufficiently large basis of a high-fidelity discretization of $\mathcal{H}$. Let $\mathbf{u}_y$ be the coefficient vector of u_y with respect to the high-fidelity discretization. Moreover, we assume that there exists a RB approximating u_y sufficiently accurately for every $y \in \mathcal{Y}$.

   Theorem 4.3 then states that, under some technical assumptions, there exists a DNN that approximates the map
   $$\mathcal{Y} \ni y \mapsto \mathbf{u}_y$$
   up to a uniform error of $\epsilon > 0$, while having a size that is polylogarithmically in ϵ, cubic in the size of the reduced basis, and at most linear in the size of the high-fidelity basis.

2. *Approximation of the continuous solution:* Next, we study the approximation of the non-discretized parametric map. To be able to make any approximation theoretical statements, we assume that $\mathcal{H}$ is a Hilbert space of functions. Concretely, we assume $\mathcal{H} \subset \{h\colon \Omega \to \mathbb{R}^k\}$, $\Omega \subset \mathbb{R}^n$, $n, k \in \mathbb{N}$. Under the assumption that there exists a RB the elements of which can be well approximated by DNNs, we demonstrate in Theorem 4.5 that there exists a DNN approximating
   $$\mathcal{Y} \times \Omega \ni (y, \mathbf{x}) \mapsto u_y(\mathbf{x})$$
   up to a uniform error of $\epsilon > 0$ and the size of which depends polylogarithmically on ϵ, whereas the dependence on the size of the RB is cubic.

These results highlight the common observation that, if a low-dimensional structure is present in a problem, then DNNs are able to identify it and use it advantageously. Concretely, our results show that a DNN is sufficiently flexible to benefit from the existence of a reduced basis in the sense that its size in the complex task of solving a parametric PDE does not or only weakly depend on the high-fidelity discretization and mainly on the size of the reduced basis.

At this point we also highlight that we do not make any concrete assumptions on the shape of the reduced basis. In particular, we do not assume that this basis is made from polynomial chaos functions as in [64].

The task of finding the optimal DNNs the existence of which we prove in this work will not be further analyzed. It is, to some extent, conventional wisdom that DNNs can be trained efficiently with stochastic gradient descent methods. In this work, we operate under the assumption that given sufficient computational resources, this optimization can be carried out successfully, which was empirically established in the aforementioned works [40, 35, 43, 72, 60].

1.2.2 Simplified Presentation of the Argument

In this section, we present a simplified outline of the arguments leading to the two approximation results described in Subsection 1.2.1. In this simplified setup, we think of a ReLU neural network (ReLU NN) as a function

$$\mathbb{R}^n \to \mathbb{R}^k,\ \mathbf{x} \mapsto T_L \varrho(T_{L-1}\varrho(\dots \varrho(T_1(\mathbf{x})))), \tag{1.2}$$

where $L \in \mathbb{N}$, $T_1, \dots, T_L$ are affine maps, and $\varrho : \mathbb{R} \to \mathbb{R}$, $\varrho(x) := \max\{0, x\}$ is the ReLU activation function which is applied coordinate-wise in (1.2). We call L the number of layers of the NN. Since T_ℓ are affine linear maps, we have for all $\mathbf{x} \in \operatorname{dom} T_\ell$ that $T_\ell(\mathbf{x}) = \mathbf{A}_\ell(\mathbf{x}) + \mathbf{b}_\ell$ for a matrix $\mathbf{A}_\ell$ and a vector $\mathbf{b}_\ell$. We define the size of the NN as number of non-zero entries of all $\mathbf{A}_\ell$ and $\mathbf{b}_\ell$ for $\ell \in \{1, \dots, L\}$. This definition will later be sharpened and extended in Definition 3.1.

1. As a first step, we recall the construction of a *scalar multiplication operator by ReLU NNs* due to [73]. This construction is based on two observations. First, defining $g\colon [0,1] \to [0,1]$, $g(x) := \min\{2x, 2-2x\}$, we see that g is a hat function. Moreover, multiple compositions of g with itself produce saw-tooth functions. We set, for $s \in \mathbb{N}$, $g_1 := g$ and $g_{s+1} := g \circ g_s$. It was demonstrated in [73] that

   $$x^2 = \lim_{n\to\infty} f_n(x) := \lim_{n\to\infty} x - \sum_{s=1}^{n} \frac{g_s(x)}{2^{2s}}, \qquad \text{for all } x \in [0,1]. \tag{1.3}$$

   The second observation for establishing an approximation of a scalar multiplication by NNs is that we can write $g(x) = 2\varrho(x) - 4\varrho(x - 1/2) + 2\varrho(x-2)$ and therefore g_s can be exactly represented by a ReLU NN. Given that g_s is bounded by 1, it is not hard to see that f_n converges to the square function exponentially fast for $n \to \infty$. Moreover, f_n can be implemented exactly as a ReLU NN by previous arguments. Finally, the parallelogram identity, $xz = 1/4((x+z)^2 - (x-z)^2)$ for $x, z \in \mathbb{R}$, demonstrates how an approximate realization of the square function by ReLU NNs yields an approximate realization of scalar multiplication by ReLU NNs.

   It is intuitively clear from the exponential convergence in (1.3) and proved in [73, Proposition 3] that the size of a NN approximating the scalar multiplication on $[-1,1]^2$ up to an error of $\epsilon > 0$ is $\mathcal{O}(\log_2(1/\epsilon))$.

2. As a next step we use the approximate scalar multiplication to approximate a *multiplication operator for matrices by ReLU NNs*. A matrix multiplication of two matrices of size $d \times d$ can be performed using d^3 scalar multiplications. Of course, as famously shown in [67], a more efficient matrix multiplication can also be carried out with less than d^3 multiplications. However, for simplicity, we focus here on the most basic implementation of matrix multiplication. Hence, the approximate multiplication of two matrices with entries bounded by 1 can be performed by NN of size $\mathcal{O}(d^3 \log_2(1/\epsilon))$ with accuracy $\epsilon > 0$. We make this precise in Proposition 3.7. Along the same lines, we can demonstrate how to construct a *NN emulating matrix-vector multiplications*.

3. Concatenating multiple matrix multiplications, we can implement *matrix polynomials by ReLU NNs*. In particular, for $\mathbf{A} \in \mathbb{R}^{d\times d}$ such that $\|\mathbf{A}\|_2 \le 1 - \delta$ for some $\delta \in (0,1)$, the map $\mathbf{A} \mapsto \sum_{s=1}^m \mathbf{A}^s$ can be approximately implemented by a ReLU NN with an accuracy of $\epsilon > 0$ and which has a size of $\mathcal{O}(m \log_2^2(m) d^3 \cdot (\log(1/\epsilon) + \log_2(m))$, where the additional $\log_2$ term in m inside the brackets appears since each of the approximations of the sum needs to be performed with accuracy ϵ/m. It is well known, that the *Neumann series* $\sum_{s=1}^m \mathbf{A}^s$ converges exponentially fast to $(\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1}$ for $m \to \infty$. Therefore, under suitable conditions on the matrix $\mathbf{A}$, we can construct a NN $\Phi^{\mathrm{inv}}_\epsilon$ that *approximates the inversion operator*, i.e. the map $\mathbf{A} \mapsto \mathbf{A}^{-1}$ up to accuracy $\epsilon > 0$. This NN has size $\mathcal{O}(d^3 \log_2^q(1/\epsilon))$ for a constant $q > 0$. This is made precise in Theorem 3.8.

4. The existence of $\Phi^{\mathrm{inv}}_\epsilon$ and the emulation of approximate matrix-vector multiplications yield that there exists a NN that for a given matrix and right-hand side approximately solves the associated linear system. Next, we make two assumptions that are satisfied in many applications as we demonstrate in Subsection 4.2:

- The map from the parameters to the associated stiffness matrices of the Galerkin discretization of the parametric PDE with respect to a reduced basis can be well approximated by NNs.
- The map from the parameters to the right-hand side of the parametric PDEs discretized according to the reduced basis can be well approximated by NNs.

From these assumptions and the existence of $\Phi_\epsilon^{\mathrm{inv}}$ and a ReLU NN emulating a matrix-vector multiplication, it is not hard to see that there is a NN that *approximately implements the map from a parameter to the associated discretized solution with respect to the reduced basis*. If the reduced basis has size d and the implementations of the map yielding the stiffness matrix and the right-hand side are sufficiently efficient then, by the construction of $\Phi_\epsilon^{\mathrm{inv}}$, the resulting NN has size $\mathcal{O}(d^3 \log_2^q(1/\epsilon))$. We call this NN $\Phi_\epsilon^{\mathrm{rb}}$.

5. Finally, we build on the construction of $\Phi_\epsilon^{\mathrm{rb}}$ to establish the two results from Section 1.2.1. First of all, let D be the size of the high-fidelity basis. If D is sufficiently large, then every element from the reduced basis can be approximately represented in the high-fidelity basis. Therefore, one can perform an *approximation to a change of bases* by applying a linear map $\mathbf{V} \in \mathbb{R}^{D \times d}$ to a vector with respect to the reduced basis. The first statement of Subsection 1.2.1 now follows directly by considering the NN $\mathbf{V} \circ \Phi_\epsilon^{\mathrm{rb}}$. Through this procedure, the size of the NN is increased to $\mathcal{O}(d^3 \log_2^q(1/\epsilon)) + dD)$. The full argument is presented in the proof of Theorem 4.3.

   Concerning the second statement in Subsection 1.2.1 we additionally *assume that NNs can approximate each element of the reduced basis accurately*. Using scalar multiplications again, we can multiply each coefficient of the output of the solution map with the associated approximate implementation of the basis function to obtain an approximate implementation of the map $\mathcal{Y} \times \Omega \ni (y, \mathbf{x}) \mapsto u_y(\mathbf{x})$. This yields the second statement of Subsection 1.2.1. The details are presented in Theorem 4.5.

1.3 Potential Impact and Extensions

We believe that the results of this article have the potential to significantly impact the research on NNs and parametric problems in the following ways:

- *Theoretical foundation:* We offer a theoretical underpinning for the empirical success of NNs for parametric problems which was observed in, e.g., [40, 35, 43, 72, 60]. Indeed, our results, Theorem 4.3 and Theorem 4.5, indicate that properly trained NNs are as efficient in solving parametric PDEs as RBMs if the complexity of NNs is measured in terms of free parameters. On a broader level, linking deep learning techniques for parametric PDE problems with approximation theory opens the field up to a new direction of thorough mathematical analysis.

- *Understanding the lack of curse of dimension:* It has been repeatedly observed that NNs seem to offer approximation rates of high-dimensional functions that do not deteriorate exponentially with increasing dimension, [47, 24]. In this sense, NNs appear to alleviate the so-called curse of dimension. One possible explanation for this observation is that these systems can very efficiently adapt to implicit low-dimensional structures, such as compositionality, [50, 57], or invariances, [47, 56]. This article gives another instance of such a phenomenon, where an unspecified low-dimensional structure can be used to yield approximation rates that are virtually independent of the ambient dimension.

- *Identifying suitable architectures:* One question in applications is how to choose the right NN architectures for the associated problem. Our results show that NNs of sufficient depth and size are able to yield very efficient approximations. Nonetheless, it needs to be mentioned that our results do not produce a lower bound on the number of layers and thus it is not clear whether deep NNs are indeed necessary.

This work is a step towards establishing a theory of deep learning-based solutions of parametric problems. However, given the complexity of this field, it is clear that many more steps need to follow. We outline a couple of natural further questions of interest below:

- *General parametric problems:* Below we restrict ourselves to coercive, symmetric, and linear parametric problems with finitely many parameters. There exist many extensions to, e.g. noncoercive, nonsymmetric, or nonlinear problems, [70, 25, 10, 39, 11, 74], or to infinite parameter spaces, see e.g. [3, 1]. It would be interesting to see if the methods proposed in this work can be generalized to these more challenging situations.

- *Bounding the number of snapshots:* The interpretation of the parametric problem as a statistical learning problem has the convenient side-effect that various techniques have been established to bound the number of necessary samples N, such that the empirical loss (1.1) is very close to the expected loss. In other words, the generalization error of the minimizer of the learning procedure is small, meaning that the prediction rule performs well on unseen data. (Here, the error is measured in a norm induced by the loss function and the underlying probability distribution.). Using these techniques, it is possible to bound the number of snapshots required for the offline phase to achieve a certain fidelity in the online phase. Estimates of the generalization error in the context of high-dimensional PDEs have been deduced in, e.g., [19, 26, 7, 21, 61].

- *Special NN architectures:* This article studies the feasibility of standard feed-forward NNs. In practice, one often uses special architectures that have proved efficient in applications. First and foremost, almost all NNs used in applications are convolutional neural networks (CNNs), [42]. Hence a relevant question is to what extent the results of this work also hold for such architectures. It was demonstrated in [55] that there is a direct correspondence between the approximation rates of CNNs and that of standard NNs. Thus we expect that the results of this work translate to CNNs.

  Another successful architecture is that of residual neural networks (ResNets), [33]. These neural networks also admit skip-connections, i.e., do not only connect neurons in adjacent layers. This architecture is by design more powerful than a standard NN and hence inherits all approximation properties of standard NNs.

- *Necessary properties of neural networks:* In this work, we demonstrate the attainability of certain approximation rates by NNs. It is not clear if the presented results are optimal or if there are specific necessary assumptions on the architectures, such as a minimal depth, a minimal number of parameters, or a minimal number of neurons per layer. For approximation results of classical function spaces such lower bounds on specifications of NNs have been established for example in [9, 28, 56, 73]. It is conceivable that the techniques in these works can be transferred to the approximation tasks described in this work.

- *General matrix polynomials:* As outlined in Subsection 1.2.2, our results are based on the approximate implementation of matrix polynomials. Naturally, this construction can be used to define and construct a ReLU NN based functional calculus. In other words, for any $d \in \mathbb{N}$ and every continuous function f that can be well approximated by polynomials, we can construct a ReLU NN which approximates the map $\mathbf{A} \mapsto f(\mathbf{A})$ for any appropriately bounded matrix $\mathbf{A}$.

  A special instance of such a function of interest is given by $f(\mathbf{A}) := e^{t\mathbf{A}},\ t > 0$, which is analytic and plays an important role in the treatment of initial value problems.

1.4 Related Work

In this section, we give an extensive overview of works related to this paper. In particular, for completeness, we start by giving a review of approximation theory of NNs without an explicit connection to PDEs. Afterward, we will see how NNs have been employed for the solution of PDEs. Finally, we examine relationships between NNs and tensors, which constitute another well-established method for the solution of high-dimensional PDEs.

1.4.1 Review of Approximation Theory of Neural Networks

The first and most fundamental results on the approximation capabilities of NNs were universality results. These results claim that NNs with at least one hidden layer can approximate any continuous function on a bounded domain to arbitrary accuracy if they have sufficiently many neurons, [36, 16]. However, these results do not quantify the required sizes of NNs to achieve these rates. One of the first results in this direction was given in [6]. There, a bound on the sufficient size of NNs with sigmoidal activation functions approximating a function with finite Fourier moments is presented. Further results describe approximation rates for various smoothness classes by sigmoidal or even more general activation functions, [49, 48, 44, 46].

For the non-differentiable activation function ReLU, first rates of approximation were identified in [73] for classes of smooth functions, in [56] for piecewise smooth functions, and in [27] for oscillatory functions. Moreover, NNs mirror the approximation rates of various dictionaries such as wavelets, [65], general affine systems, [9], linear finite elements, [32], and higher-order finite elements, [54].

1.4.2 Neural Networks and PDEs

A well-established line of research is that of solving high-dimensional PDEs by NNs assuming that the NN is the solution of the underlying PDE, e.g., [66, 7, 31, 38, 61, 37, 19, 21]. In this regime, it is often possible to bound the size of the involved NNs in a way that does not depend exponentially on the underlying dimension. In that way, these results are quite related to our approaches. Our results do not seek to represent the solution of a PDE as a NN, but a parametric map. Moreover, we analyze the low complexity of the solution manifold in terms of Kolmogorov N-widths. Finally, the underlying spatial dimension of the involved PDEs in our case would usually be moderate. However, the dimension of the parameter space could be immense.

In [64], the approximation rates of NNs for polynomial chaos functions based on the analyticity of the solution map $y \mapsto u_y$ are established. Polynomial chaos functions are one particular basis, which can often be chosen to be reasonably small in many parametric problems. Hence, these results prove that NNs can solve parametric problems depending only on an intrinsic dimension. Our results allow a wider range of reduced bases and remain valid if the parametric map is not necessarily analytic.

Finally, we mention the works [40, 35, 43, 72, 60] which apply NNs in one way or another to parametric problems. These approaches study the topic of learning a parametric problem but do not offer a theoretical analysis of the required sizes of the involved NNs. These results form our motivation to study the constructions of this paper.

1.4.3 Neural Networks and Hierarchical Tensor Formats

Neural networks form a parametrized set of functions where the parametrization via the weights of the network is non-linear. Another successful approach is to approximate the parametric maps by multi-linear maps on the parameter space and the physical domain, so-called tensors [2, 3, 30]. Similarly to NNs, tensors can be efficiently parametrized by, for example, hierarchical (Tucker) tensor (HT) representations and tree-based tensor formats, see [30, 4] for an overview. Due to the prominence of these methods, it interesting to point out the similarities between HTs and NNs. An example of the application of HTs for the solution of parametric PDEs can be found for instance in [20].

We proceed by shortly describing the construction of HTs on tensor product spaces $\mathcal{V} = \bigotimes_{j \in J} \mathcal{V}_j$. A tensor is a multilinear map on $\mathcal{V}$, i.e., it is linear with respect to the input from each of the spaces $\mathcal{V}_j$. For each $j \in J$, we start with appropriate approximation spaces (feature spaces), each spanned by a reduced basis $(v^1_{\ell_j})_{\ell_j \in \{1,\dots,L_j\}} \subset \mathcal{V}_j$, $L_j \in \mathbb{N}$. In the next step one iterates this concept of choosing optimized subspaces in a hierarchical way for tensor product spaces. The basis for the optimized subspace $\mathcal{V}_{j_1,j_2} \subset \mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2}$, hopefully with $\dim(\mathcal{V}_{j_1,j_2}) \ll \dim(\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2})$ is represented with respect to the basis $(v^1_{\ell_{j_1}} \otimes v^1_{\ell_{j_2}})_{\ell_{j_1} \in \{1,\dots L_{j_1}\}, \ell_{j_2} \in \{1,\dots,L_{j_2}\}}$ of $\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2}$. In particular, we have for the basis vectors v^2_m of $\mathcal{V}_{j_1,j_2}$, where $m = 1, \dots, \dim(\mathcal{V}_{j_1,j_2})$, that

$$v^2_m = \sum_{\ell_{j_1} \in \{1,\dots,L_{j_1}\}, \ell_{j_2} \in \{1,\dots,L_{j_2}\}} \mathbf{c}(m, \ell_{j_1}, \ell_{j_2}) v^1_{\ell_{j_1}} \otimes v^1_{\ell_{j_2}} \in \mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2}, \tag{1.4}$$

for some coefficients $\mathbf{c}(m, \ell_{j_1}, \ell_{j_2})$. Proceeding in this hierarchical way ultimately leads to a hierarchical representation of a tensor through the coefficients $\mathbf{c}$. In the hierarchical tensor calculus, one ultimately only stores the coefficients $\mathbf{c}$. We refer to [30, 4] for a detailed exposition.

In regards to the solution of parametric PDEs with parameter space $\mathcal{Y}$ and domain Ω, one assumes a suitable (approximate) decomposition of the space of parametric maps as $\bigotimes_{j\in J} \mathcal{V}_j \subset L^2(\mathcal{Y} \times \Omega)$. Typical examples for the sets $\mathcal{V}_j$ are orthogonal univariate polynomials, or appropriately chosen subspaces of polynomials (Tucker basis), see [30]. Since HTs are compositions of bilinear operations, low-rank matrix techniques can be used, see [30, 4, 45]. For an analysis of the approximation theory of these bases, we refer to [63].

Let us now illuminate the connections between HTs or tree-based tensors and NNs. First of all, inspection of (1.4) shows that a HT is formed by repeated applications of linear parametrized operations and a nonlinearity given by the tensor product—closely resembling the functionality of a NN.

In [14] the authors interpret the tensor product $\mathbf{x} \otimes \mathbf{y}$ for vectors $\mathbf{x}, \mathbf{y}$ as a generalized pooling operation and view specific convolutional NNs as HTs with the tropical multiplication given by the max pooling $x * z := \max\{x, z\}$. In this context, they establish that special convolutional NNs correspond to a generalized form of a tree-based tensor format. In other words, certain NNs can be considered a special case of tensors. Furthermore, it has been established in [13] that shallow networks correspond to tensors in the canonical format. As a result, deep networks have an exponentially larger power of expressivity than shallow ones.

Additionally, by the parallelogram identity, we can exactly represent the binary product operation $\mathbb{R}^2 \ni (x, z) \mapsto x \cdot z$ if we can exactly represent the square function. Choosing $\varrho(x) = x^2$ in (1.2), we can therefore exactly implement HTs as NNs. Besides, it transpires from our analysis in Section 1.2.2 that all HTs can be approximated by ReLU NNs with sizes logarithmic in the inverse approximation error.

These relations of HTs or tree-based tensor formats and NNs offer an alternative approach to the approximation of parametric maps in the regime where specialized HT-based methods work well. Indeed, by reapproximating the elements of the optimized subspaces and using the approximate emulation of hierarchical representations by NNs, we can construct an approximation of the parametric map by realizations of NNs. In the case of bases of polynomial chaos functions, this observation is the foundation of [64].

1.5 Outline

In Section 2, we describe the type of parametric PDEs we are considering in this paper, and we recall the theory of RBs. Section 3 introduces a NN calculus which is the basis for all constructions in this work. There we will also construct the NN that maps a matrix to its approximate inverse in Theorem 3.8. In Section 4, we construct NNs approximating parametric maps. First, in Theorem 4.1, we approximate the parametric maps after a high-fidelity discretization. Afterward, in Subsection 4.2, we list two broad examples where all assumptions which we imposed are satisfied. Finally, in Theorem 4.5, we demonstrate a construction of a NN approximating the non-discretized parametric map. To not interrupt the flow of reading, we have deferred all auxiliary results and proofs to the appendices.

1.6 Notation

We denote by $\mathbb{N} = \{1, 2, ...\}$ the set of all *natural numbers* and define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, for $a \in \mathbb{R}$ we set $\lfloor a \rfloor := \max\{b \in \mathbb{Z} \colon\ b \le a\}$ and $\lceil a \rceil := \min\{b \in \mathbb{Z} \colon\ b \ge a\}$. Let $n, l \in \mathbb{N}$. Let $\mathbf{Id}_{\mathbb{R}^n}$ be the *identity* and $\mathbf{0}_{\mathbb{R}^n}$ be the *zero vector* on $\mathbb{R}^n$. Moreover, for $\mathbf{A} \in \mathbb{R}^{n\times l}$, we denote by $\mathbf{A}^T$ its *transpose,* by $\sigma(\mathbf{A})$ the *spectrum of* $\mathbf{A}$, by $\|\mathbf{A}\|_2$ its *spectral norm* and by $\|\mathbf{A}\|_0 := \#\{(i, j) \colon \mathbf{A}_{i,j} \neq 0\}$, where $\#V$ denotes the cardinality of a set V, the *number of non-zero entries* of $\mathbf{A}$. Moreover, for $\mathbf{v} \in \mathbb{R}^n$ we denote by $|\mathbf{v}|$ its *Euclidean norm.* Let V be a vector space. Then we say that $X \subset^{\mathrm{s}} V$, if X is a *linear subspace* of V. Moreover, if $(V, \|\cdot\|_V)$ is a normed vector space, X is a subset of V and $v \in V$, we denote by $\operatorname{dist}(v, X) := \inf\{\|x - v\|_V \colon\ x \in X\}$ the *distance* between v, X and by $(V^*, \|\cdot\|_{V^*})$ the *topological dual space of* V*,* i.e. the set of all scalar-valued, linear, continuous functions equipped with the *operator norm.* For a compact set $\Omega \subset \mathbb{R}^n$ we denote by $C^r(\Omega)$, $r \in \mathbb{N}_0 \cup \{\infty\}$, the spaces of r *times continuously differentiable functions*, by $L^p(\Omega, \mathbb{R}^n), p \in [1, \infty]$ the $\mathbb{R}^n$*-valued Lebesgue spaces*, where we set $L^p(\Omega) := L^p(\Omega, \mathbb{R})$ and by $H^1(\Omega) := W^{1,2}(\Omega)$ the *first-order Sobolev space.*

2 Parametric PDEs and Reduced Basis Methods

In this section, we introduce the type of parametric problems that we study in this paper. A parametric problem in its most general form is based on a map $\mathcal{P}\colon \mathcal{Y} \to \mathcal{Z}$, where $\mathcal{Y}$ is the *parameter space* and $\mathcal{Z}$ is called *solution state space* $\mathcal{Z}$. In the case of parametric PDEs, $\mathcal{Y}$ describes certain parameters of a partial differential equation, $\mathcal{Z}$ is a function space or a discretization thereof, and $\mathcal{P}(y) \in \mathcal{Z}$ is found by solving a PDE with parameter y.

We will place several assumptions on the PDEs underlying $\mathcal{P}$ and the parameter spaces $\mathcal{Y}$ in Section 2.1. Afterward, we give an abstract overview of Galerkin methods in Section 2.2 before recapitulating some basic facts about RBs in Section 2.3.

2.1 Parametric Partial Differential Equations

In the following, we will consider parameter-dependent equations given in the variational form

$$b_y\left(u_y, v\right) = f_y(v), \qquad \text{for all } y \in \mathcal{Y},\ v \in \mathcal{H}, \tag{2.1}$$

where

(i) $\mathcal{Y}$ is the *parameter set* specified in Assumption 2.1,

(ii) $\mathcal{H}$ is a Hilbert space,

(iii) $b_y\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a *continuous bilinear form*, which fulfills certain well-posedness conditions specified in Assumption 2.1,

(iv) $f_y \in \mathcal{H}^*$ is the *parameter-dependent right-hand side of* (2.1),

(v) $u_y \in \mathcal{H}$ is the *solution of* (2.1).

Assumption 2.1. *Throughout this paper, we impose the following assumptions on Equation* (2.1).

- ***The parameter set*** $\mathcal{Y}$: *We assume that $\mathcal{Y}$ is a compact subset of $\mathbb{R}^p$, where $p \in \mathbb{N}$ is fixed and potentially large.*

  Remark. *In [12, Section 1.2], it has been justified that if $\mathcal{Y}$ is a compact subset of some Banach space V, then one can describe every element in $\mathcal{Y}$ by a sequence of real numbers in an affine way. To be more precise, there exist $(v_i)_{i=0}^{\infty} \subset V$ such that for every $y \in \mathcal{Y}$ and some coefficient sequence $\mathbf{c}_y$ whose elements can be bounded in absolute value by 1 there holds $y = v_0 + \sum_{i=1}^{\infty} (\mathbf{c}_y)_i v_i$, implying that $\mathcal{Y}$ can be completely described by the collection of sequences $\mathbf{c}_y$. In this paper, we assume these sequences $\mathbf{c}_y$ to be finite with a fixed, but possibly large support size.*

- ***Symmetry, uniform continuity, and coercivity of the bilinear forms:*** *We assume that for all $y \in \mathcal{Y}$ the bilinear forms b_y are* symmetric, *i.e.*

  $$b_y(u, v) = b_y(v, u), \qquad \textit{for all } u, v \in \mathcal{H}.$$

  Moreover, we assume that the bilinear forms b_y are uniformly continuous *in the sense that there exists a constant $C_{\mathrm{cont}} > 0$ with*

  $$|b_y(u, v)| \leq C_{\mathrm{cont}} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \qquad \textit{for all } u \in \mathcal{H}, v \in \mathcal{H}, y \in \mathcal{Y}.$$

  Finally, we assume that the involved bilinear forms are uniformly coercive *in the sense that there exists a constant $C_{\mathrm{coer}} > 0$ such that*

  $$\inf_{u \in \mathcal{H}\setminus\{0\}} \frac{b_y(u, u)}{\|u\|_{\mathcal{H}}^2} \geq C_{\mathrm{coer}}, \qquad \textit{for all } u \in \mathcal{H}, y \in \mathcal{Y}.$$

  Hence, by the Lax-Milgram lemma (see [59, Lemma 2.1]), Equation (2.1) *is* well-posed, *i.e. for every $y \in \mathcal{Y}$ and every $f_y \in \mathcal{H}^*$ there exists exactly one $u_y \in \mathcal{H}$ such that* (2.1) *is satisfied and u_y depends continuously on f_y.*

- ***Uniform boundedness of the right-hand side:*** *We assume that there exists a constant* $C_{\text{RHS}} > 0$ *such that*

$$\|f_y\|_{\mathcal{H}^*} \leq C_{\text{RHS}}, \qquad \textit{for all } y \in \mathcal{Y}.$$

- ***Compactness of the solution manifold:*** *We assume that the* solution manifold

$$S(\mathcal{Y}) := \{u_y\colon u_y \textit{ is the solution of } (2.1),\ y \in \mathcal{Y}\}$$

is compact in $\mathcal{H}$.

Remark. *The assumption that* $S(\mathcal{Y})$ *is compact follows immediately if the solution map* $y \mapsto u_y$ *is continuous. This condition is true (see [59, Proposition 5.1, Corollary 5.1]), if for all* $u, v \in \mathcal{H}$ *the maps* $y \mapsto b_y(u, v)$ *as well as* $y \to f_y(v)$ *are Lipschitz continuous. In fact, there exists a multitude of parametric PDEs, for which the maps* $y \mapsto b_y(u, v)$ *and* $y \to f_y(v)$ *are even in* C^r *for some* $r \in \mathbb{N} \cup \{\infty\}$. *In this case,* $\{(y, u_y) : y \in \mathcal{Y}\} \subset \mathbb{R}^p \times \mathcal{H}$ *is a* p*-dimensional manifold of class* C^r *(see [59, Proposition 5.2, Remark 5.4]). Moreover, we refer to [59, Remark 5.2] and the references therein for a discussion under which circumstances it is possible to turn a discontinuous parameter dependency into a continuous one ensuring the compactness of* $S(\mathcal{Y})$.

2.2 High-Fidelity Approximations

In practice, one cannot hope to solve (2.1) exactly for every $y \in \mathcal{Y}$. Instead, if we assume for the moment that y is fixed, a common approach towards the calculation of an approximate solution of (2.1) is given by the *Galerkin method*, which we will describe shortly following [34, Appendix A] and [59, Chapter 2.4]. In this framework, instead of solving (2.1), one solves a discrete scheme of the form

$$b_y\left(u_y^{\text{disc}}, v\right) = f_y(v) \qquad \text{for all } v \in U^{\text{disc}}, \tag{2.2}$$

where $U^{\text{disc}} \subset^{\text{s}} \mathcal{H}$ is a subspace of $\mathcal{H}$ with $\dim\left(U^{\text{disc}}\right) < \infty$ and $u_y^{\text{disc}} \in U^{\text{disc}}$ is the solution of (2.2). For the solution u_y^{disc} of (2.2) we have that

$$\left\|u_y^{\text{disc}}\right\|_{\mathcal{H}} \leq \frac{1}{C_{\text{coer}}} \|f_y\|_{\mathcal{H}^*}.$$

Moreover, up to a constant, we have that u_y^{disc} is a best approximation of the solution u_y of (2.1) by elements in U^{disc}. To be more precise, by *Cea's Lemma*, [59, Lemma 2.2.],

$$\left\|u_y - u_y^{\text{disc}}\right\|_{\mathcal{H}} \leq \frac{C_{\text{cont}}}{C_{\text{coer}}} \inf_{w \in U^{\text{disc}}} \|u_y - w\|_{\mathcal{H}}. \tag{2.3}$$

Let us now assume that U^{disc} is given. Moreover, if $N := \dim\left(U^{\text{disc}}\right)$, let $(\varphi_i)_{i=1}^N$ be a basis for U^{disc}. Then the matrix

$$\mathbf{B}_y := \left(b_y\left(\varphi_j, \varphi_i\right)\right)_{i,j=1}^N$$

is non-singular and positive definite. The solution u_y^{disc} of (2.2) satisfies

$$u_y^{\text{disc}} = \sum_{i=1}^N (\mathbf{u}_y)_i \varphi_i,$$

where

$$\mathbf{u}_y := \left(\mathbf{B}_y\right)^{-1} \mathbf{f}_y \in \mathbb{R}^N$$

and $\mathbf{f}_y := \left(f_y\left(\varphi_i\right)\right)_{i=1}^N \in \mathbb{R}^N$. Typically, one starts with a *high-fidelity discretization* of the space $\mathcal{H}$, i.e. one chooses a finite- but potentially high-dimensional subspace for which the computed discretized solutions are sufficiently accurate for any $y \in \mathcal{Y}$. To be more precise, we postulate the following:

Assumption 2.2. *We assume that there exists a finite dimensional space* $U^{\mathrm{h}} \subset^{\mathrm{s}} \mathcal{H}$ *with dimension* $D < \infty$ *and basis* $(\varphi_i)_{i=1}^D$. *This space is called* high-fidelity discretization. *For* $y \in \mathcal{Y}$, *denote by* $\mathbf{B}_y^{\mathrm{h}} := (b_y(\varphi_j, \varphi_i))_{i,j=1}^D \in \mathbb{R}^{D\times D}$ *the stiffness matrix of the high-fidelity discretization, by* $\mathbf{f}_y^{\mathrm{h}} := (f_y(\varphi_i))_{i=1}^D$ *the discretized right-hand side and by* $\mathbf{u}_y^{\mathrm{h}} := \left(\mathbf{B}_y^{\mathrm{h}}\right)^{-1} \mathbf{f}_y^{\mathrm{h}} \in \mathbb{R}^D$ *the coefficient vector of the Galerkin solution with respect to the high-fidelity discretization. Moreover, we denote by* $u_y^{\mathrm{h}} := \sum_{i=1}^D \left(\mathbf{u}_y^{\mathrm{h}}\right)_i \varphi_i$ *the Galerkin solution.*

We assume that, for every $y \in \mathcal{Y}$, $\sup_{y\in\mathcal{Y}} \left\|u_y - u_y^{\mathrm{h}}\right\|_{\mathcal{H}} \leq \hat{\epsilon}$ *for an arbitrarily small, but fixed* $\hat{\epsilon} > 0$. *In the following, similarly as in [17], we will not distinguish between* $\mathcal{H}$ *and* U^{h}, *unless such a distinction matters.*

In practice, following this approach, one often needs to calculate $u_y^{\mathrm{h}} \approx u_y$ for a variety of parameters $y \in \mathcal{Y}$ which in general is a very expensive procedure due to the high-dimensionality of the space U^{h}. In particular, given $(\varphi_i)_{i=1}^D$, one needs to solve high-dimensional systems of linear equations to determine the coefficient vector $\mathbf{u}_y^{\mathrm{h}}$. A well-established remedy to overcome these difficulties is given by methods based on the theory of reduced bases, which we will recapitulate in the upcoming subsection.

Before we proceed, let us fix some notation. We denote by $\mathbf{G} := (\langle\varphi_i, \varphi_j\rangle_{\mathcal{H}})_{i,j=1}^D \in \mathbb{R}^{D\times D}$ the symmetric, positive definite *Gram matrix* of the basis vectors $(\varphi_i)_{i=1}^D$. Then, for any $v \in U^{\mathrm{h}}$ with coefficient vector $\mathbf{v}$ with respect to the basis $(\varphi_i)_{i=1}^D$ we have (see [59, Equation 2.41])

$$|\mathbf{v}|_{\mathbf{G}} := \left|\mathbf{G}^{1/2}\mathbf{v}\right| = \|v\|_{\mathcal{H}}. \tag{2.4}$$

2.3 Theory of Reduced Bases

In this subsection and unless stated otherwise, we follow [59, Chapter 5] and the references therein. The main motivation behind the theory of RBs lies in the fact that under Assumption 2.1 the solution manifold $S(\mathcal{Y})$ is a compact subset of $\mathcal{H}$. This compactness property allows posing the question whether, for every $\tilde{\epsilon} \geq \hat{\epsilon}$, it is possible to construct a finite-dimensional subspace $U_{\tilde{\epsilon}}^{\mathrm{rb}}$ of $\mathcal{H}$ such that $d(\tilde{\epsilon}) := \dim\left(U_{\tilde{\epsilon}}^{\mathrm{rb}}\right) \ll D$ and such that

$$\sup_{y\in\mathcal{Y}} \inf_{w\in U_{\tilde{\epsilon}}^{\mathrm{rb}}} \|u_y - w\|_{\mathcal{H}} \leq \tilde{\epsilon}, \tag{2.5}$$

or, equivalently, if there exist linearly independent vectors $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$ with the property that

$$\left\|\sum_{i=1}^{d(\tilde{\epsilon})} (\mathbf{c}_y)_i \psi_i - u_y\right\|_{\mathcal{H}} \leq \tilde{\epsilon}, \quad \text{for all } y \in \mathcal{Y} \text{ and some coefficient vector } \mathbf{c}_y \in \mathbb{R}^{d(\tilde{\epsilon})}.$$

The starting point of this theory lies in the concept of the Kolmogorov N-width which is defined as follows.

Definition 2.3 ([17])**.** *For* $N \in \mathbb{N}$, *the* Kolmogorov N-width *of a bounded subset* X *of a normed space* V *is defined by*

$$W_N(X) := \inf_{\substack{V_N \subset^{\mathrm{s}} V \\ \dim(V_N)\leq N}} \sup_{x\in X} \operatorname{dist}(x, V_N).$$

This quantity describes the best possible uniform approximation error of X by an at most N-dimensional linear subspace of V. The aim of RBMs is to construct the spaces $U_{\tilde{\epsilon}}^{\mathrm{rb}}$ in such a way that the quantity $\sup_{y\in\mathcal{Y}} \operatorname{dist}\left(u_y, U_{\tilde{\epsilon}}^{\mathrm{rb}}\right)$ is close to $W_{d(\tilde{\epsilon})}(S(\mathcal{Y}))$.

The identification of the basis vectors $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$ of $U_{\tilde{\epsilon}}^{\mathrm{rb}}$ usually happens in an *offline phase* in which one has considerable computational resources available and which is usually based on the determination of high-fidelity discretizations of samples of the parameter set $\mathcal{Y}$. The most common methods are based on (weak) greedy procedures (see for instance [59, Chapter 7] and the references therein) or proper orthogonal decompositions (see for instance [59, Chapter 6] and the references therein). In the last step, an orthogonalization procedure (such as a Gram-Schmidt process) is performed to obtain an orthonormal set of basis vectors $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$.

Afterward, in the *online phase*, one assembles for a given input y the corresponding low-dimensional stiffness matrices and vectors and determines the Galerkin solution by solving a low-dimensional system of linear equations. To ensure an efficient implementation of the online phase, a common assumption which we do *not* require in this paper is the *affine decomposition* of (2.1), which means that there exist $Q_b, Q_f \in \mathbb{N}$, parameter-independent bilinear forms $b^q\colon \mathcal{H}\times\mathcal{H}\to\mathbb{R}$, maps $\theta_q\colon \mathcal{Y}\to\mathbb{R}$ for $q=1,...,Q_b$, parameter-independent $f^{q'}\in\mathcal{H}^*$ as well as maps $\theta^{q'}\colon\mathcal{Y}\to\mathbb{R}$ for $q'=1,...,Q_f$ such that

$$b_y = \sum_{q=1}^{Q_b}\theta_q(y)b^q, \qquad \text{as well as} \qquad f_y = \sum_{q'=1}^{Q_f}\theta^{q'}(y)f^{q'}, \qquad \text{for all } y\in\mathcal{Y}.$$

As has been pointed out in [59, Chapter 5.7], in principal three types of reduced bases generated by RBMs have been established in the literature - the *Lagrange reduced basis*, the *Hermite reduced basis* and the *Taylor reduced basis.* While the most common type, the Lagrange RB, consists of orthonormalized versions of high-fidelity *snapshots* $u^{\mathrm{h}}\left(y^1\right)\approx u\left(y^1\right),...,u^{\mathrm{h}}\left(y^n\right)\approx u\left(y^n\right)$, Hermite RBs consist of snapshots $u^{\mathrm{h}}\left(y^1\right)\approx u\left(y^1\right),...,u^{\mathrm{h}}\left(y^n\right)\approx u\left(y^n\right)$, as well as their first partial derivatives $\frac{\partial u^{\mathrm{h}}}{\partial y_i}(y^j)\approx\frac{\partial u}{\partial y_i}(y^j), i=1,...,p,\ j=1,...,n,$ whereas Taylor RBs are built of derivatives of the form $\frac{\partial^k u^{\mathrm{h}}}{\partial y_i^k}(\overline{y})\approx\frac{\partial^k u}{\partial y_i^k}(\overline{y}),\ i=1,...,p,\ k=0,...,n-1$ around a given expansion point $\overline{y}\in\mathcal{Y}$. In this paper, we will later assume that there exist small RBs $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$ generated by *arbitrary* linear combinations of the high-fidelity elements $(\varphi_i)_{i=1}^D$. Note that all types of RBs discussed above satisfy this assumption.

The next statement gives a (generally sharp) upper bound which relates the possibility of constructing small snapshot RBs directly to the Kolmogorov N-width.

Theorem 2.4 ([8, Theorem 4.1.])**.** *Let $N\in\mathbb{N}$. For a compact subset X of a normed space V, define the* inner N-width of X *by*

$$\overline{W}_N(X) := \inf_{V_N\in\mathcal{M}_N}\sup_{x\in X}\operatorname{dist}(x, V_N),$$

where $\mathcal{M}_N := \left\{V_N\subset^{\mathrm{s}} V\colon\ V_N=\operatorname{span}(x_i)_{i=1}^N, x_1,...,x_N\in X\right\}$*. Then*

$$\overline{W}_N(X)\le(N+1)W_N(X).$$

Translated into our framework, Theorem 2.4 states that for every $N\in\mathbb{N}$, there exist solutions $u^{\mathrm{h}}(y^i)\approx u(y^i),\ \ i=1,...,N$ of (2.1) such that

$$\sup_{y\in\mathcal{Y}}\ \inf_{w\in\operatorname{span}(u^{\mathrm{h}}(y^i))_{i=1}^N}\|u_y-w\|_{\mathcal{H}}\le(N+1)W_N(S(\mathcal{Y})).$$

Remark 2.5. *We note that this bound is sharp for general X,V. However, it is not necessarily optimal for special instances of $S(\mathcal{Y})$. If, for instance, $W_N(S(\mathcal{Y}))$ decays polynomially, then $\overline{W}_N(S(\mathcal{Y}))$ decays with the same rate (see [8, Theorem 3.1.]). Moreover, if $W_N(S(\mathcal{Y}))\le Ce^{-cN^\beta}$ for some $c,C,\beta>0$ then by [18, Corollary 3.3 (iii)] we have $\overline{W}_N(S(\mathcal{Y}))\le\tilde{C}e^{-\tilde{c}N^\beta}$ for some $\tilde{c},\tilde{C}>0$.*

We note that the aforementioned statements concerning the decay of $W_N(S(\mathcal{Y}))$ are not void. In fact, for large subclasses of parametric PDEs one can show exponential decay of the Kolmogorov N-width. One instance (see [53, Theorem 3.1.]) is given by problems (2.1)*, for which the bilinear forms b_y are affinely decomposed with continuous parameter dependency and the right-hand side is parameter-independent. Another example is given by problems, for which the maps $y\mapsto b_y(u,v)$ and $y\mapsto f_y(v)$ are* analytic *for all $u,v\in\mathcal{H}$.*

Taking the discussion above as a justification, we assume from now on that for every $\tilde{\epsilon}\ge\hat{\epsilon}$ there exists a RB space $U_{\tilde{\epsilon}}^{\mathrm{rb}}=\operatorname{span}(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$, which fulfills (2.5), where the linearly independent basis vectors $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$ are linear combinations of the high-fidelity basis vectors $(\varphi_i)_{i=1}^D$ in the sense that there exists a transformation matrix $\mathbf{V}_{\tilde{\epsilon}} \in \mathbb{R}^{D\times d(\tilde{\epsilon})}$ such that

$$(\psi_i)_{i=1}^{d(\tilde{\epsilon})} = \left(\sum_{j=1}^{D} (\mathbf{V}_{\tilde{\epsilon}})_{j,i}\varphi_j \right)_{i=1}^{d(\tilde{\epsilon})}$$

and where $d(\tilde{\epsilon}) \ll D$ is chosen to be as small as possible, at least fulfilling $\operatorname{dist}\left(S(\mathcal{Y}), U_{\tilde{\epsilon}}^{\mathrm{rb}}\right) \leq \overline{W}_{d(\tilde{\epsilon})}(S(\mathcal{Y}))$. In addition, we assume that the vectors $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$ form an orthonormal system in $\mathcal{H}$, which is equivalent to the fact that the columns of $\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}$ are orthonormal (see [59, Remark 4.1]). This in turn implies

$$\left\| \mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}} \right\|_2 = 1, \quad \text{for all } \tilde{\epsilon} \geq \hat{\epsilon} \tag{2.6}$$

as well as

$$\left\| \sum_{i=1}^{d(\tilde{\epsilon})} \mathbf{c}_i \psi_i \right\|_{\mathcal{H}} = |\mathbf{c}|, \quad \text{for all } \mathbf{c} \in \mathbb{R}^{d(\tilde{\epsilon})}. \tag{2.7}$$

For the underlying discretization matrix, one can demonstrate (see for instance [59, Section 3.4.1]) that

$$\mathbf{B}_{y,\tilde{\epsilon}}^{\mathrm{rb}} := (b_y(\psi_j,\psi_i))_{i,j=1}^{d(\tilde{\epsilon})} = \mathbf{V}_{\tilde{\epsilon}}^T \mathbf{B}_{y,\tilde{\epsilon}}^{\mathrm{h}} \mathbf{V}_{\tilde{\epsilon}} \in \mathbb{R}^{d(\tilde{\epsilon})\times d(\tilde{\epsilon})}, \quad \text{for all } y \in \mathcal{Y}.$$

Moreover, due to the symmetry and the coercivity of the underlying bilinear forms combined with the orthonormality of the basis vectors $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$, one can show (see for instance [59, Remark 3.5]) that

$$C_{\mathrm{coer}} \leq \left\| \mathbf{B}_{y,\tilde{\epsilon}}^{\mathrm{rb}} \right\|_2 \leq C_{\mathrm{cont}}, \quad \text{as well as} \quad \frac{1}{C_{\mathrm{cont}}} \leq \left\| \left(\mathbf{B}_{y,\tilde{\epsilon}}^{\mathrm{rb}}\right)^{-1} \right\|_2 \leq \frac{1}{C_{\mathrm{coer}}}, \quad \text{for all } y \in \mathcal{Y}, \tag{2.8}$$

implying that the condition number of the stiffness matrix with respect to the RB remains bounded independently of y and the dimension $d(\tilde{\epsilon})$. Additionally, the discretized right-hand side with respect to the RB is given by

$$\mathbf{f}_{y,\tilde{\epsilon}}^{\mathrm{rb}} := (f_y(\psi_i))_{i=1}^{d(\tilde{\epsilon})} = \mathbf{V}_{\tilde{\epsilon}}^T \mathbf{f}_{y,\tilde{\epsilon}}^{\mathrm{h}} \in \mathbb{R}^{d(\tilde{\epsilon})}$$

and, by the Bessel inequality, we have that $\left|\mathbf{f}_{y,\tilde{\epsilon}}^{\mathrm{rb}}\right| \leq \|f_y\|_{\mathcal{H}^*} \leq C_{\mathrm{rhs}}$. Moreover, let

$$\mathbf{u}_{y,\tilde{\epsilon}}^{\mathrm{rb}} := \left(\mathbf{B}_{y,\tilde{\epsilon}}^{\mathrm{rb}}\right)^{-1} \mathbf{f}_{y,\tilde{\epsilon}}^{\mathrm{rb}}$$

be the coefficient vector of the Galerkin solution with respect to the RB space. Then, the Galerkin solution $u_{y,\tilde{\epsilon}}^{\mathrm{rb}}$ can be written as

$$u_{y,\tilde{\epsilon}}^{\mathrm{rb}} = \sum_{i=1}^{d(\tilde{\epsilon})} \left(\mathbf{u}_{y,\tilde{\epsilon}}^{\mathrm{rb}}\right)_i \psi_i = \sum_{j=1}^{D} \left(\mathbf{V}_{\tilde{\epsilon}} \mathbf{u}_{y,\tilde{\epsilon}}^{\mathrm{rb}}\right)_j \varphi_j,$$

i.e.

$$\tilde{\mathbf{u}}_{y,\tilde{\epsilon}}^{\mathrm{h}} := \mathbf{V}_{\tilde{\epsilon}} \mathbf{u}_{y,\tilde{\epsilon}}^{\mathrm{rb}} \in \mathbb{R}^D$$

is the coefficient vector of the RB solution if expanded with respect to the high-fidelity basis $(\varphi_i)_{i=1}^D$. Finally, as in Equation 2.3, we obtain

$$\sup_{y\in\mathcal{Y}} \left\| u_y - u_{y,\tilde{\epsilon}}^{\mathrm{rb}} \right\|_{\mathcal{H}} \leq \sup_{y\in\mathcal{Y}} \frac{C_{\mathrm{cont}}}{C_{\mathrm{coer}}} \inf_{w \in U_{\tilde{\epsilon}}^{\mathrm{rb}}} \|u_y - w\|_{\mathcal{H}} \leq \frac{C_{\mathrm{cont}}}{C_{\mathrm{coer}}} \tilde{\epsilon}. \tag{2.9}$$

In the following sections we will emulate RBMs with NNs by showing that we are able to construct NNs which approximate the maps $\mathbf{u}_{\cdot,\tilde{\epsilon}}^{\mathrm{rb}}, \tilde{\mathbf{u}}_{\cdot,\tilde{\epsilon}}^{\mathrm{h}}, u_{\cdot,\tilde{\epsilon}}^{\mathrm{rb}}$ avoiding the curse of dimensionality and whose complexity either does not depend on D or at most in a linear way. The key ingredient will be the construction of small NNs implementing an approximate matrix inversion based on Richardson iterations in Section 3. In Section 4 we then proceed with building NNs which approximate the maps $\mathbf{u}_{\cdot,\tilde{\epsilon}}^{\mathrm{rb}}, \tilde{\mathbf{u}}_{\cdot,\tilde{\epsilon}}^{\mathrm{h}}, u_{\cdot,\tilde{\epsilon}}^{\mathrm{rb}}$.

3 Neural Network Calculus

The goal of this chapter is to emulate the matrix inversion by NNs. In Section 3.1, we introduce some basic notions connected to NNs as well as some basic operations one can perform with these. In Section 3.2, we state a result which shows the existence of NNs the ReLU-realizations of which take a matrix $\mathbf{A} \in \mathbb{R}^{d\times d}$, $\|\mathbf{A}\|_2 < 1$ as their input and calculate an approximation of $(\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1}$ based on its Neumann series expansion. The associated proofs can be found in Appendix A.

3.1 Basic Definitions and Operations

We start by introducing a formal definition of a NN. Afterward, we introduce several operations, such as parallelization and concatenation that can be used to assemble complex NNs out of simpler ones. Unless stated otherwise we follow the notion of [56]. First, we introduce a terminology for NNs that allows us to differentiate between a NN as a family of weights and the function implemented by the NN. This implemented function will be called the realization of the NN.

Definition 3.1. *Let $n, L \in \mathbb{N}$. A* NN Φ with input dimension $\dim_{\text{in}}(\Phi) := n$ and L layers *is a sequence of matrix-vector tuples*

$$\Phi = \big((\mathbf{A}_1, \mathbf{b}_1), (\mathbf{A}_2, \mathbf{b}_2), \ldots, (\mathbf{A}_L, \mathbf{b}_L)\big),$$

where $N_0 = n$ and $N_1, \ldots, N_L \in \mathbb{N}$, and where each $\mathbf{A}_\ell$ is an $N_\ell \times N_{\ell-1}$ matrix, and $\mathbf{b}_\ell \in \mathbb{R}^{N_\ell}$.

If Φ is a NN as above, $K \subset \mathbb{R}^n$, and if $\varrho\colon \mathbb{R} \to \mathbb{R}$ is arbitrary, then we define the associated realization of Φ with activation function ϱ over K *(in short, the* ϱ-realization of Φ over K*) as the map $\mathrm{R}_\varrho^K(\Phi)\colon K \to \mathbb{R}^{N_L}$ such that*

$$\mathrm{R}_\varrho^K(\Phi)(\mathbf{x}) = \mathbf{x}_L,$$

where $\mathbf{x}_L$ results from the following scheme:

$$\begin{aligned}
\mathbf{x}_0 &:= \mathbf{x}, \\
\mathbf{x}_\ell &:= \varrho(\mathbf{A}_\ell\, \mathbf{x}_{\ell-1} + \mathbf{b}_\ell), \qquad \text{for } \ell = 1, \ldots, L-1, \\
\mathbf{x}_L &:= \mathbf{A}_L\, \mathbf{x}_{L-1} + \mathbf{b}_L,
\end{aligned}$$

and where ϱ acts componentwise, that is, $\varrho(\mathbf{v}) = (\varrho(\mathbf{v}_1), \ldots, \varrho(\mathbf{v}_m))$ for any $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_m) \in \mathbb{R}^m$.

We call $N(\Phi) := n + \sum_{j=1}^{L} N_j$ the number of neurons of the NN Φ *and $L = L(\Phi)$ the* number of layers. *For $\ell \le L$ we call $M_\ell(\Phi) := \|\mathbf{A}_\ell\|_0 + \|\mathbf{b}_\ell\|_0$ the* number of weights in the ℓ-th layer *and we define $M(\Phi) := \sum_{\ell=1}^{L} M_\ell(\Phi)$, which we call the* number of weights of Φ. *Moreover, we refer to* $\dim_{\text{out}}(\Phi) := N_L$ *as* the output dimension *of Φ.*

First of all, we note that it is possible to concatenate two NNs in the following way.

Definition 3.2. *Let $L_1, L_2 \in \mathbb{N}$ and let $\Phi^1 = \big((\mathbf{A}_1^1, \mathbf{b}_1^1), \ldots, (\mathbf{A}_{L_1}^1, \mathbf{b}_{L_1}^1)\big), \Phi^2 = \big((\mathbf{A}_1^2, \mathbf{b}_1^2), \ldots, (\mathbf{A}_{L_2}^2, \mathbf{b}_{L_2}^2)\big)$ be two NNs such that the input layer of Φ^1 has the same dimension as the output layer of Φ^2. Then, $\Phi^1 \bullet \Phi^2$ denotes the following $L_1 + L_2 - 1$ layer NN:*

$$\Phi^1 \bullet \Phi^2 := \big((\mathbf{A}_1^2, \mathbf{b}_1^2), \ldots, (\mathbf{A}_{L_2-1}^2, \mathbf{b}_{L_2-1}^2), (\mathbf{A}_1^1\mathbf{A}_{L_2}^2, \mathbf{A}_1^1\mathbf{b}_{L_2}^2 + \mathbf{b}_1^1), (\mathbf{A}_2^1, \mathbf{b}_2^1), \ldots, (\mathbf{A}_{L_1}^1, \mathbf{b}_{L_1}^1)\big).$$

We call $\Phi^1 \bullet \Phi^2$ the concatenation of Φ^1, Φ^2.

In general, there is no bound on $M(\Phi^1 \bullet \Phi^2)$ that is linear in $M(\Phi^1)$ and $M(\Phi^2)$.

For the remainder of the paper, let ϱ be given by the ReLU activation function, i.e. $\varrho(x) = \max\{x, 0\}$ for $x \in \mathbb{R}$. We will see in the following, that we are able to introduce an alternative concatenation which helps us to control the number of non-zero weights. Towards this goal, we give the following result which shows that we can construct NNs the ReLU-realization of which is the identity function on $\mathbb{R}^n$.

Lemma 3.3. *For any $L \in \mathbb{N}$ there exists a NN $\Phi^{\mathbf{Id}}_{n,L}$ with input dimension n, output dimension n and at most $2nL$ non-zero, $\{-1,1\}$-valued weights such that*

$$\mathrm{R}_{\varrho}^{\mathbb{R}^n}\left(\Phi^{\mathbf{Id}}_{n,L}\right) = \mathbf{Id}_{\mathbb{R}^n}.$$

We now introduce the sparse concatenation of two NNs.

Definition 3.4. *Let Φ^1, Φ^2 be two NNs such that the output dimension of Φ^2 and the input dimension of Φ^1 equal $n \in \mathbb{N}$. Then the* sparse concatenation of Φ^1 and Φ^2 *is defined as*

$$\Phi^1 \odot \Phi^2 := \Phi^1 \bullet \Phi^{\mathbf{Id}}_{n,1} \bullet \Phi^2.$$

We will see later in Lemma 3.6 the properties of the sparse concatenation of NNs. We proceed with the second operation we can perform with NNs, called parallelization.

Definition 3.5 ([56, 22]). *Let $\Phi^1, ..., \Phi^k$ be NNs which have equal input dimension such that there holds $\Phi^i = \left((\mathbf{A}_1^i, \mathbf{b}_1^i), ..., (\mathbf{A}_L^i, \mathbf{b}_L^i)\right)$ for some $L \in \mathbb{N}$. Then define the* parallelization of $\Phi^1, ..., \Phi^k$ *by*

$$\mathrm{P}\left(\Phi^1, ..., \Phi^k\right) := \left(\left(\begin{pmatrix} \mathbf{A}_1^1 & & & \\ & \mathbf{A}_1^2 & & \\ & & \ddots & \\ & & & \mathbf{A}_1^k \end{pmatrix}, \begin{pmatrix} \mathbf{b}_1^1 \\ \mathbf{b}_1^2 \\ \vdots \\ \mathbf{b}_1^k \end{pmatrix}\right), ..., \left(\begin{pmatrix} \mathbf{A}_L^1 & & & \\ & \mathbf{A}_L^2 & & \\ & & \ddots & \\ & & & \mathbf{A}_L^k \end{pmatrix}, \begin{pmatrix} \mathbf{b}_L^1 \\ \mathbf{b}_L^2 \\ \vdots \\ \mathbf{b}_L^k \end{pmatrix}\right)\right).$$

Now, let Φ be a NN and $L \in \mathbb{N}$ such that $L(\Phi) \leq L$. Then, define the NN

$$E_L(\Phi) := \begin{cases} \Phi, & \text{if } L(\Phi) = L, \\ \Phi^{\mathbf{Id}}_{\dim_{\mathrm{out}}(\Phi), L - L(\Phi)} \odot \Phi, & \text{if } L(\Phi) < L. \end{cases}$$

Finally, let $\tilde{\Phi}^1, ..., \tilde{\Phi}^k$ be NNs which have the same input dimension and let

$$\tilde{L} := \max\left\{L\left(\tilde{\Phi}^1\right), ..., L\left(\tilde{\Phi}^k\right)\right\}.$$

Then define

$$\mathrm{P}\left(\tilde{\Phi}^1, ..., \tilde{\Phi}^k\right) := \mathrm{P}\left(E_{\tilde{L}}\left(\tilde{\Phi}^1\right), ..., E_{\tilde{L}}\left(\tilde{\Phi}^k\right)\right).$$

We call $\mathrm{P}\left(\tilde{\Phi}^1, ..., \tilde{\Phi}^k\right)$ the parallelization of $\tilde{\Phi}^1, ..., \tilde{\Phi}^k$.

The following lemma was established in [22, Lemma 5.4] and examines the properties of the sparse concatenation as well as of the parallelization of NNs.

Lemma 3.6 ([22]). *Let $\Phi^1, ..., \Phi^k$ be NNs.*

(a) If the input dimension of Φ^1, which shall be denoted by n_1, equals the output dimension of Φ^2, and n_2 is the input dimension of Φ^2, then

$$\mathrm{R}_{\varrho}^{\mathbb{R}^{n_1}}\left(\Phi^1\right) \circ \mathrm{R}_{\varrho}^{\mathbb{R}^{n_2}}\left(\Phi^2\right) = \mathrm{R}_{\varrho}^{\mathbb{R}^{n_2}}\left(\Phi^1 \odot \Phi^2\right)$$

and

(i) $L\left(\Phi^1 \odot \Phi^2\right) \leq L\left(\Phi^1\right) + L\left(\Phi^2\right)$,

(ii) $M\left(\Phi^1 \odot \Phi^2\right) \leq M\left(\Phi^1\right) + M\left(\Phi^2\right) + M_1\left(\Phi^1\right) + M_{L(\Phi^2)}\left(\Phi^2\right) \leq 2M\left(\Phi^1\right) + 2M\left(\Phi^2\right)$,

(iii) $M_1\left(\Phi^1 \odot \Phi^2\right) = M_1\left(\Phi^2\right)$,

(iv) $M_{L(\Phi^1 \odot \Phi^2)}\left(\Phi^1 \odot \Phi^2\right) = M_{L(\Phi^1)}\left(\Phi^1\right)$.

(b) *If the input dimension of Φ^i, denoted by n, equals the input dimension of Φ^j, for all i, j, then for the NN $\mathrm{P}\left(\Phi^1, \Phi^2, ..., \Phi^k\right)$ we have*

$$\mathrm{R}_\varrho^{\mathbb{R}^n}\left(\mathrm{P}\left(\Phi^1, \Phi^2, ..., \Phi^k\right)\right) = \left(\mathrm{R}_\varrho^{\mathbb{R}^n}\left(\Phi^1\right), \mathrm{R}_\varrho^{\mathbb{R}^n}\left(\Phi^2\right), \ldots, \mathrm{R}_\varrho^{\mathbb{R}^n}\left(\Phi^k\right)\right)$$

as well as

(i) $L\left(\mathrm{P}\left(\Phi_1, \Phi^2, ..., \Phi^k\right)\right) = \max_{i=1,\ldots,k} L\left(\Phi^i\right)$,

(ii) $M\left(\mathrm{P}\left(\Phi^1, \Phi^2, ..., \Phi^k\right)\right) \leq 2\left(\sum_{i=1}^k M\left(\Phi^i\right)\right) + 4\left(\sum_{i=1}^k \dim_{\mathrm{out}}\left(\Phi^i\right)\right)\max_{i=1,\ldots,k} L\left(\Phi^i\right)$,

(iii) $M\left(\mathrm{P}\left(\Phi^1, \Phi^2, ..., \Phi^k\right)\right) = \sum_{i=1}^k M\left(\Phi^i\right)$, *if* $L\left(\Phi^1\right) = L\left(\Phi^2\right) = \ldots = L\left(\Phi^k\right)$,

(iv) $M_1\left(\mathrm{P}\left(\Phi^1, \Phi^2, ..., \Phi^k\right)\right) = \sum_{i=1}^k M_1\left(\Phi^i\right)$,

(v) $M_{L(\mathrm{P}(\Phi^1,\Phi^2,...,\Phi^k))}\left(\mathrm{P}\left(\Phi^1, \Phi^2, ..., \Phi^k\right)\right) \leq \sum_{i=1}^k \max\left\{2\dim_{\mathrm{out}}\left(\Phi^i\right), M_{L(\Phi^i)}\left(\Phi^i\right)\right\}$,

(vi) $M_{L(\mathrm{P}(\Phi^1,\Phi^2,...,\Phi^k))}\left(\mathrm{P}\left(\Phi^1, \Phi^2, ..., \Phi^k\right)\right) = \sum_{i=1}^k M_{L(\Phi^i)}\left(\Phi^i\right)$, *if* $L\left(\Phi^1\right) = L\left(\Phi^2\right) = \ldots = L\left(\Phi^k\right)$.

3.2 A Neural Network Based Approach Towards Matrix Inversion

The goal of this subsection is to emulate the inversion of square matrices by NNs which are comparatively small in size. In particular, Theorem 3.8 shows that, for $d \in \mathbb{N}$, $\epsilon \in (0, 1/4)$, and $\delta \in (0, 1)$, we are able to construct NNs $\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}$ the ReLU-realization of which approximates the map

$$\left\{\mathbf{A} \in \mathbb{R}^{d\times d} \colon \|\mathbf{A}\|_2 \leq 1-\delta\right\} \to \mathbb{R}^{d\times d},\ \mathbf{A} \mapsto \left(\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A}\right)^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k$$

up to an $\|\cdot\|_2$- error of ϵ and the size of which does not suffer from the curse of dimensionality.

To stay in the classical NN setting, we employ vectorized matrices in the remainder of this paper. Let $\mathbf{A} \in \mathbb{R}^{d\times l}$. We write

$$\mathbf{vec}(\mathbf{A}) \coloneqq (\mathbf{A}_{1,1}, ..., \mathbf{A}_{d,1}, ..., \mathbf{A}_{1,l}, ..., \mathbf{A}_{d,l})^T \in \mathbb{R}^{dl}.$$

Moreover, for a vector $\mathbf{v} = (\mathbf{v}_{1,1}, ..., \mathbf{v}_{d,1}, ..., \mathbf{v}_{1,d}, ..., \mathbf{v}_{d,l})^T \in \mathbb{R}^{dl}$ we set

$$\mathbf{matr}(\mathbf{v}) \coloneqq (\mathbf{v}_{i,j})_{i=1,\ldots,d,\ j=1,\ldots,l} \in \mathbb{R}^{d\times l}.$$

In addition, for $d, n, l \in \mathbb{N}$ and $Z > 0$ we set

$$K^Z_{d,n,l} \coloneqq \left\{(\mathbf{vec}(\mathbf{A}), \mathbf{vec}(\mathbf{B})) \colon (\mathbf{A}, \mathbf{B}) \in \mathbb{R}^{d\times n} \times \mathbb{R}^{n\times l},\ \|\mathbf{A}\|_2, \|\mathbf{B}\|_2 \leq Z\right\}$$

as well as

$$K^Z_d \coloneqq \left\{\mathbf{vec}(\mathbf{A}) \colon \mathbf{A} \in \mathbb{R}^{d\times d},\ \|\mathbf{A}\|_2 \leq Z\right\}.$$

The basic ingredient for the construction of NNs emulating a matrix inversion is the following result about NNs emulating the multiplication of two matrices.

Proposition 3.7. *Let $d, n, l \in \mathbb{N}$, $\epsilon \in (0, 1)$, $Z > 0$. There exists a NN $\Phi^{Z,d,n,l}_{\mathrm{mult};\epsilon}$ with $n\cdot(d+l)$- dimensional input, dl-dimensional output such that, for some absolute constant $C_{\mathrm{mult}} > 0$, the following properties are fulfilled:*

(i) $L\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right) \leq C_{\mathrm{mult}} \cdot \left(\log_2(1/\epsilon) + \log_2\left(n\sqrt{dl}\right) + \log_2(\max\{1, Z\})\right)$,

(ii) $M\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right) \leq C_{\mathrm{mult}} \cdot \left(\log_2(1/\epsilon) + \log_2\left(n\sqrt{dl}\right) + \log_2(\max\{1, Z\})\right) dnl$,

(iii) $M_1\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right) \leq C_{\mathrm{mult}} dnl,$ *as well as* $M_{L\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right)}\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right) \leq C_{\mathrm{mult}} dnl,$

(iv) $\sup_{(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B}))\in K^Z_{d,n,l}} \left\|\mathbf{AB} - \mathbf{matr}\left(\mathrm{R}_\varrho^{K^Z_{d,n,l}}\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\epsilon}\right)(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B}))\right)\right\|_2 \leq \epsilon,$

(v) for any $(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B})) \in K^Z_{d,n,l}$ *we have*

$$\left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K^Z_{d,n,l}}\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right)(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B}))\right)\right\|_2 \leq \epsilon + \|\mathbf{A}\|_2\|\mathbf{B}\|_2 \leq \epsilon + Z^2 \leq 1 + Z^2.$$

Based on Proposition 3.7, we construct in Appendix A.2 NNs emulating the map $\mathbf{A} \mapsto \mathbf{A}^k$ for square matrices $\mathbf{A}$ and $k \in \mathbb{N}$. This construction is then used to prove the following result.

Theorem 3.8. *For* $\epsilon, \delta \in (0,1)$ *define*

$$m(\epsilon,\delta) := \left\lceil \frac{\log_2(0.5\epsilon\delta)}{\log_2(1-\delta)} \right\rceil - 1.$$

There exists a universal constant $C_{\mathrm{inv}} > 0$ *such that for every* $d \in \mathbb{N}$, $\epsilon \in (0,1/4)$ *and every* $\delta \in (0,1)$ *there exists a NN* $\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}$ *with* d^2*-dimensional input,* d^2*-dimensional output and the following properties:*

(i) $L\left(\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}\right) \leq C_{\mathrm{inv}} \log_2(m(\epsilon,\delta)) \cdot (\log_2(1/\epsilon) + \log_2(m(\epsilon,\delta)) + \log_2(d)),$

(ii) $M\left(\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}\right) \leq C_{\mathrm{inv}} m(\epsilon,\delta)\log_2^2(m(\epsilon,\delta)) d^3 \cdot (\log_2(1/\epsilon) + \log_2(m(\epsilon,\delta)) + \log_2(d)),$

(iii) $\sup_{\mathbf{vec}(\mathbf{A})\in K^{1-\delta}_d} \left\|(\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1} - \mathbf{matr}\left(\mathrm{R}_\varrho^{K^{1-\delta}_d}\left(\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon,$

(iv) for any $\mathbf{vec}(\mathbf{A}) \in K^{1-\delta}_d$ *we have*

$$\left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K^{1-\delta}_d}\left(\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon + \left\|(\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1}\right\|_2 \leq \epsilon + \frac{1}{1-\|\mathbf{A}\|_2} \leq \epsilon + \frac{1}{\delta}.$$

Remark 3.9. *In the proof of Theorem 3.8, we approximate the function mapping a matrix to its inverse via the Neumann series and then emulate this construction by NNs. There certainly exist alternative approaches to approximating this inversion function, such as, for example, via Chebyshev matrix polynomials (for an introduction of Chebyshev polynomials, see for instance [68, Chapter 8.2]). In fact, approximation by Chebyshev matrix polynomials is more efficient in terms of the degree of the polynomials required to reach a certain approximation accuracy. However, emulation of Chebyshev matrix polynomials by NNs either requires larger networks than that of monomials or, if they are represented in a monomial basis, coefficients that grow exponentially with the polynomial degree. In the end, the advantage of a smaller degree in the approximation through Chebyshev matrix polynomials does not seem to set off the drawbacks described before.*

4 Neural Networks and Solutions of PDEs Using Reduced Bases

In this section, we invoke the estimates for the approximate matrix inversion from Section 3.2 to approximate the parameter-dependent solution of parametric PDEs by NNs. In other words, for $\tilde{\epsilon} \geq \hat{\epsilon}$, we construct NNs approximating the maps

$$\mathcal{Y} \to \mathbb{R}^D\colon \quad y \mapsto \tilde{\mathbf{u}}^{\mathrm{h}}_{y,\tilde{\epsilon}}, \qquad \mathcal{Y} \to \mathbb{R}^{d(\tilde{\epsilon})}\colon \quad y \mapsto \mathbf{u}^{\mathrm{rb}}_{y,\tilde{\epsilon}}, \quad \text{and} \quad \mathcal{Y} \to \mathcal{H}\colon \quad y \mapsto u_y,$$

respectively. Here, the sizes of the NNs essentially only depend on the approximation fidelity $\tilde{\epsilon}$ and the size of an appropriate RB, but are independent from the dimension of the high-fidelity discretization D except when approximating the map $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}$ where the size depends linearly on D.

We start in Section 4.1 by constructing, under some general assumptions on the parametric problem, a NN emulating the maps $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}$ and $\mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$. In Section 4.2, we verify these assumptions on two broad examples. Finally, in Section 4.3, we demonstrate under which conditions the NN approximation of $\mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$ leads to a construction of a NN that efficiently approximates $y \mapsto u_y$. All proofs can be found in Appendix B.

4.1 Determining the Coefficients of the Solution

Next, we present constructions of NNs the ReLU-realizations of which approximate the maps $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}$ and $\mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$, respectively. In our main result of this subsection, the approximation error of the NN approximation $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}$ will be measured with respect to the $|\cdot|_{\mathbf{G}}$-norm since we can relate this norm directly to the norm on $\mathcal{H}$ via Equation (2.4). In contrast, the approximation error of the NN approximating $\mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$ will be measured with respect to the $|\cdot|$-norm due to Equation 2.7.

As already indicated earlier, the main ingredient of the following arguments is an application of the NN of Theorem 3.8 to the matrix $\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}$. As a preparation, we show in Proposition B.1 in the appendix, that we can rescale the matrix $\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}$ with a constant factor $\alpha := (2\max\{1, C_{\mathrm{cont}}\}^2)^{-1}$ (in particular, independent of y and $d(\tilde{\epsilon})$) so that with $\delta := \alpha C_{\mathrm{cont}} \leq 1/2$

$$\left\|\mathbf{Id}_{\mathbb{R}^{d(\tilde{\epsilon})}} - \alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right\|_2 \leq 1 - \delta < 1.$$

We will fix these values of α and δ for the remainder of the manuscript. Next, we state two abstract assumptions on the approximability of the map $\mathbf{B}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$ which we will later on specify when we consider concrete examples in Subsection 4.2.

Assumption 4.1. *We assume that, for any $\tilde{\epsilon} \geq \hat{\epsilon}, \epsilon > 0$, and for a corresponding RB $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$, there exists a NN $\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon}$ with p-dimensional input and $d(\tilde{\epsilon})^2$-dimensional output such that*

$$\sup_{y \in \mathcal{Y}} \left\|\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon}\right)(y)\right)\right\|_2 \leq \epsilon.$$

We set $B_M(\tilde{\epsilon}, \epsilon) := M\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon}\right) \in \mathbb{N}$ and $B_L(\tilde{\epsilon}, \epsilon) := L\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon}\right) \in \mathbb{N}$.

In addition to Assumption 4.1, we state the following assumption on the approximability of the map $\mathbf{f}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$ which we will later on specify when we consider concrete examples.

Assumption 4.2. *We assume that for every $\tilde{\epsilon} \geq \hat{\epsilon}, \epsilon > 0$, and a corresponding RB $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$ there exists a NN $\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon}$ with p-dimensional input and $d(\tilde{\epsilon})$-dimensional output such that*

$$\sup_{y \in \mathcal{Y}} \left|\mathbf{f}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon}\right)(y)\right| \leq \epsilon.$$

We set $F_L(\tilde{\epsilon}, \epsilon) := L(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon})$ and $F_M(\tilde{\epsilon}, \epsilon) := M\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon}\right)$.

Now we are in a position to construct NNs the ReLU-realizations of which approximate the coefficient maps $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}, \mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$.

Theorem 4.3. *Let $\tilde{\epsilon} \geq \hat{\epsilon}$ and $\epsilon \in (0, \alpha/4 \cdot \min\{1, C_{\mathrm{coer}}\})$. Moreover, define $\epsilon' := \epsilon/\max\{6, C_{\mathrm{rhs}}\}$, $\epsilon'' := \epsilon/3 \cdot C_{\mathrm{coer}}$, $\epsilon''' := 3/8 \cdot \epsilon' \alpha C^2_{\mathrm{coer}}$ and $\kappa := 2\max\{1, C_{\mathrm{rhs}}, 1/C_{\mathrm{coer}}\}$. Additionally, assume that Assumption 4.1 and Assumption 4.2 hold. For the NNs*

$$\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon} := \Phi^{\kappa,d(\tilde{\epsilon}),d(\tilde{\epsilon}),1}_{\mathrm{mult};\frac{\epsilon}{3}} \odot \mathrm{P}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}, \Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right) \quad \text{and} \quad \Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon} := ((\mathbf{V}_{\tilde{\epsilon}}, \mathbf{0}_{\mathbb{R}^D})) \odot \Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon},$$

the following properties hold:

(i) $\sup_{y\in\mathcal{Y}} \left|\mathbf{u}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon}\right)(y)\right| \leq \epsilon,$ *and* $\sup_{y\in\mathcal{Y}} \left|\tilde{\mathbf{u}}^{\mathrm{h}}_{y,\tilde{\epsilon}} - \mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}\right)(y)\right|_{\mathbf{G}} \leq \epsilon,$

(ii) there exists a constant $C^{\mathbf{u}}_L > 0$ *such that*

$$L\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon}\right) \leq L\left(\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}\right) \leq \max\left\{C^{\mathbf{u}}_L \log_2(\log_2(1/\epsilon))\left(\log_2(1/\epsilon) + \log_2(\log_2(1/\epsilon)) + \log_2(d(\tilde{\epsilon}))\right) + B_L(\tilde{\epsilon},\epsilon'''), F_L\left(\tilde{\epsilon},\epsilon''\right)\right\},$$

(iii) there exists a constant $C^{\mathbf{u}}_M > 0$ *such that*

$$M\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon}\right) \leq C^{\mathbf{u}}_M d(\tilde{\epsilon})^2 \cdot \Bigg(d(\tilde{\epsilon})\log_2(1/\epsilon)\log_2^2(\log_2(1/\epsilon))\big(\log_2(1/\epsilon) + \log_2(\log_2(1/\epsilon)) + \log_2(d(\tilde{\epsilon}))\big)... \\ ... + B_L(\tilde{\epsilon},\epsilon''') + F_L\left(\tilde{\epsilon},\epsilon''\right)\Bigg) + 2B_M(\tilde{\epsilon},\epsilon''') + F_M\left(\tilde{\epsilon},\epsilon''\right),$$

(iv) $M\left(\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}\right) \leq 2Dd(\tilde{\epsilon}) + 2M\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon}\right),$

(v) $\sup_{y\in\mathcal{Y}} \left|\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon}\right)(y)\right| \leq \kappa^2 + \frac{\epsilon}{3},$ *and* $\sup_{y\in\mathcal{Y}} \left|\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}\right)(y)\right|_{\mathbf{G}} \leq \kappa^2 + \frac{\epsilon}{3}.$

4.2 Examples of Neural Network Approximation of Parametric Maps without Curse of Dimension

In this subsection, we apply Theorem 4.3 to a variety of concrete examples in which the approximation of the coefficient maps $\mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$, $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}$ can be approximated by comparatively small NNs which do not suffer from the curse of dimensionality by verifying Assumption 4.1 and Assumption 4.2, respectively. We will state the following examples already in their variational formulation and note that they fulfill the requirements of Assumption 2.1. We also remark that the presented examples are only a small excerpt of problems to which our theory is applicable.

4.2.1 Curse of Dimension: Comparison to Direct Approximation

Before we present two examples below, we would like to stress that the resulting approximation rates differ significantly from and are substantially better than alternative approaches. First of all, in all examples below the parametric maps $\mathcal{Y} \to \mathbb{R}^{d(\tilde{\epsilon})}$, $\mathcal{Y} \to \mathbb{R}^D$ are analytic. Therefore, classical approximation rates with NNs (such as those provided by [73, Theorem 1], [56, Theorem 3.1.] or [28, Corollary 4.2.]) promise arbitrarily fast approximation rates. However, in all of these rates, the dimension enters through a constant depending on $d(\tilde{\epsilon}), D$. Moreover, this dependence is typically exponential.

In contrast, we will demonstrate below, that in our approach the approximation rates depend only polylogarithmically on $1/\epsilon$, (up to a log factor) cubically on $d(\tilde{\epsilon})$, are independent from or at worst linear in D, and depend linearly on $B_M(\tilde{\epsilon},\epsilon)$ and $F_M\left(\tilde{\epsilon},\epsilon\right)$.

Another approach is to directly solve the linear systems from the high-fidelity discretization. Without further assumptions on sparsity properties of the matrices, the resulting complexity would be $\mathcal{O}(D^3)$ times dependencies on $d(\tilde{\epsilon})$ and $1/\epsilon$, which is significantly worse than our approximation rate.

4.2.2 Example I: Diffusion Equation

We consider a special case of [59, Chapter 2.3.1] which can be interpreted as a generalized version of the heavily used example $-\mathrm{div}(a\nabla u) = f$, where a is a scalar field (see for instance [13, 64] and the references therein). Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$, be a Lipschitz domain and $\mathcal{H} \coloneqq H^1_0(\Omega) = \left\{u \in H^1(\Omega)\colon\ u|_{\partial\Omega} = 0\right\}$. We assume that the parameter set is given by a compact set $\mathcal{T} \subset L^\infty(\Omega,\mathbb{R}^{n\times n})$ such that for all $\mathbf{T} \in \mathcal{T}$ and almost all $\mathbf{x} \in \Omega$ the matrix $\mathbf{T}(\mathbf{x})$ is symmetric, positive definite with matrix norm that can be bounded from above and below independently of $\mathbf{T}$ and $\mathbf{x}$. As we have noted in Assumption 2.1, we can assume that there exist some $(\mathbf{T}_i)_{i=0}^{\infty} \subset L^{\infty}(\Omega, \mathbb{R}^{n\times n})$ such that for every $\mathbf{T} \in \mathcal{T}$ there exist $(y_i(\mathbf{T}))_{i=1}^{\infty} \subset [-1,1]$ with $\mathbf{T} = \mathbf{T}_0 + \sum_{i=1}^{\infty} y_i(\mathbf{T})\mathbf{T}_i$. We restrict ourselves to the case of finitely supported sequences $(y_i)_{i=1}^{\infty}$. To be more precise, let $p \in \mathbb{N}$ be potentially very high, but fixed, let $\mathcal{Y} \coloneqq [-1,1]^p$ and consider for $y \in \mathcal{Y}$ and some fixed $f \in \mathcal{H}^*$ the parametric PDE

$$b_y(u_y, v) \coloneqq \int_{\Omega} \mathbf{T}_0 \nabla u_y \nabla v \; d\mathbf{x} + \sum_{i=1}^{p} y_i \int_{\Omega} \mathbf{T}_i \nabla u_y \nabla v \; d\mathbf{x} = f(v), \qquad \text{for all } v \in \mathcal{H}.$$

Then, the parameter-dependency of the bilinear forms is linear, hence analytic whereas the parameter-dependency of the right-hand side is constant, hence also analytic, implying that $\overline{W}(S(\mathcal{Y}))$ decays exponentially fast. This in turn implies existence of small RBs $(\psi_i)_{i=1}^{d(\tilde{\epsilon})}$ where $d(\tilde{\epsilon})$ depends at most polylogarithmically on $1/\tilde{\epsilon}$. In this case, Assumption 4.1 and Assumption 4.2 are trivially fulfilled: for $\tilde{\epsilon} > 0, \epsilon > 0$ we can construct one-layer NNs $\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon}$ with p-dimensional input and $d(\tilde{\epsilon})^2$-dimensional output as well as $\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon}$ with p-dimensional input and $d(\tilde{\epsilon})$-dimensional output the ReLU-realizations of which exactly implement the maps $y \mapsto \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}$ and $y \mapsto \mathbf{f}^{\mathrm{rb}}_{y,\tilde{\epsilon}}$, respectively.

In conclusion, in this example, we have, for $\tilde{\epsilon}, \epsilon > 0$,

$$B_L(\tilde{\epsilon}, \epsilon) = 1, \quad F_L(\tilde{\epsilon}, \epsilon) = 1, \quad B_M(\tilde{\epsilon}, \epsilon) \le p d(\tilde{\epsilon})^2, \quad F_M(\tilde{\epsilon}, \epsilon) \le d(\tilde{\epsilon}).$$

Theorem 4.3 hence implies the existence of a NN approximating $\mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$ up to error ϵ with a size that is linear in p, polylogarithmic in $1/\epsilon$, and, up to a log factor, cubic in $d(\tilde{\epsilon})$. Moreover, we have shown the existence of a NN approximating $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}$ with a size that is linear in p, polylogarithmic in $1/\epsilon$, linear in D and, up to a log factor, cubic in $d(\tilde{\epsilon})^3$.

4.2.3 Example II: Linear Elasticity Equation

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain, $\Gamma_D, \Gamma_{N_1}, \Gamma_{N_2}, \Gamma_{N_3} \subset \partial\Omega$, be disjoint such that $\Gamma_D \cup \Gamma_{N_1} \cup \Gamma_{N_2} \cup \Gamma_{N_3} = \partial\Omega$, $\mathcal{H} \coloneqq [H^1_{\Gamma_D}(\Omega)]^3$, where $H^1_{\Gamma_D}(\Omega) = \left\{ u \in H^1(\Omega) \colon u|_{\Gamma_D} = 0 \right\}$. In variational formulation, this problem can be formulated as an affinely decomposed problem dependent on five parameters, i.e. $p = 5$. Let $\mathcal{Y} \coloneqq [\tilde{y}^{1,1}, \tilde{y}^{2,1}] \times \ldots \times [\tilde{y}^{1,5}, \tilde{y}^{2,5}] \subset \mathbb{R}^5$ such that $[\tilde{y}^{1,2}, \tilde{y}^{2,2}] \subset]-1, 1/2[$ and for $y = (y_1, \ldots, y_5) \in \mathcal{Y}$ we consider the problem

$$b_y(u_y, v) = f_y(v), \qquad \text{for all } v \in \mathcal{H},$$

where

- $b_y(u_y, v) \coloneqq \frac{y_1}{1+y_2} \int_{\Omega} \operatorname{trace}\left(\left(\nabla u_y + (\nabla(u_y)^T \right) \cdot \left(\nabla v + (\nabla v)^T \right)^T \right) d\mathbf{x} + \frac{y_1 y_2}{1-2y_2} \int_{\Omega} \operatorname{div}(u_y) \operatorname{div}(v) \; d\mathbf{x}$,
- $f_y(v) \coloneqq y_3 \int_{\Gamma_1} \mathbf{n} \cdot v \, d\mathbf{x} + y_4 \int_{\Gamma_2} \mathbf{n} \cdot v \, d\mathbf{x} + y_5 \int_{\Gamma_3} \mathbf{n} \cdot v \, d\mathbf{x}$, and where $\mathbf{n}$ denotes the outward unit normal on $\partial\Omega$.

The parameter-dependency of the right-hand side is linear (hence analytic), whereas the parameter-dependency of the bilinear forms is rational, hence (due to the choice of $\tilde{y}^{1,2}, \tilde{y}^{2,2}$) also analytic and $\overline{W}_N(S(\mathcal{Y}))$ decays exponentially fast implying that we can choose $d(\tilde{\epsilon})$ to depend polylogarithmically on $\tilde{\epsilon}$. It is now easy to see that Assumption 4.1 and Assumption 4.2 are fulfilled with NNs the size of which is comparatively small: By [69], for every $\tilde{\epsilon}, \epsilon > 0$ we can find a NN with $\mathcal{O}(\log_2^2(1/\epsilon))$ layers and $\mathcal{O}(d(\tilde{\epsilon})^2 \log_2^3(1/\epsilon))$ non-zero weights the ReLU-realization of which approximates the map $y \mapsto \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}$ up to an error of ϵ. Moreover, there exists a one-layer NN $\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon}$ with p-dimensional input and $d(\tilde{\epsilon})$-dimensional output the ReLU-realization of which exactly implements the map $y \mapsto \mathbf{f}^{\mathrm{rb}}_{y,\tilde{\epsilon}}$. In other words, in these examples, for $\tilde{\epsilon}, \epsilon > 0$,

$$B_L(\tilde{\epsilon}, \epsilon) \in \mathcal{O}\left(\log_2^2(1/\epsilon)\right), \quad F_L(\tilde{\epsilon}, \epsilon) = 1, \quad B_M(\tilde{\epsilon}, \epsilon) \in \mathcal{O}\left(d(\tilde{\epsilon})^2 \log_2^3(1/\epsilon)\right), \quad F_M(\tilde{\epsilon}, \epsilon) \le 5d(\tilde{\epsilon}).$$

Thus, Theorem 4.3 implies the existence of NNs approximating $\mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$ up to error ϵ with a size that is polylogarithmic in $1/\epsilon$, and, up to a log factor, cubic in $d(\tilde{\epsilon})$. Moreover, there exist NNs approximating $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}$ up to error ϵ with a size that is linear in D, polylogarithmic in $1/\epsilon$, and, up to a log factor, cubic in $d(\tilde{\epsilon})$.

For a more thorough discussion of this example (a special case of the linear elasticity equation which describes the displacement of some elastic structure under physical stress on its boundaries), we refer to [59, Chapter 2.1.2, Chapter 2.3.2, Chapter 8.6].

4.3 Approximation of the Parametrized Solution

We observed above that the size of the NN emulation of $\mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$ does not depend on D and the emulation of $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}$ depends at worst linearly on D. Knowledge of $\mathbf{u}^{\mathrm{rb}}_{\cdot,\tilde{\epsilon}}$ is, however, only useful if the underlying RB is known. In contrast, we can always interpret $\tilde{\mathbf{u}}^{\mathrm{h}}_{\cdot,\tilde{\epsilon}}$ since we always know the high-fidelity basis.

In this section, we demonstrate how to approximate the parameter dependent solutions of the PDEs, without any dependence on D under the assumption that we do not know the RB, but instead assume that we can efficiently approximate it with ReLU-realizations of NNs.

We start by making the assumption on the efficient approximation of the RB precise:

Assumption 4.4. *Let $\mathcal{H} \subset \{h\colon \Omega \to \mathbb{R}^k\}$, $\Omega \subset \mathbb{R}^n$ be measurable and $n, k \in \mathbb{N}$. There exist constants $C^{\mathrm{rb}}_M, C^{\mathrm{rb}}_L, q_\infty, q_L, q_M \geq 1$ such that for RBs $(\psi_i)_{i=1}^d$, $d \in \mathbb{N}$, it holds that for each $i = \{1, \dots, d\}$ and each $\epsilon > 0$ there exists a NN $\Phi^{\mathrm{rb}}_{i,\epsilon}$ with n-dimensional input and k-dimensional output and the following properties:*

(i) $L(\Phi^{\mathrm{rb}}_{i,\epsilon}) \leq C^{\mathrm{rb}}_L \log_2^{q_L}(1/\epsilon) \log_2^{q_L}(i)$,

(ii) $M(\Phi^{\mathrm{rb}}_{i,\epsilon}) \leq C^{\mathrm{rb}}_M \log_2^{q_M}(1/\epsilon) \log_2^{q_M}(i)$,

(iii) $\left\| \psi_i - \mathrm{R}^{\Omega}_{\varrho}(\Phi^{\mathrm{rb}}_{i,\epsilon}) \right\|_{\mathcal{H}} \leq \epsilon$,

(iv) $\left\| \mathrm{R}^{\Omega}_{\varrho}(\Phi^{\mathrm{rb}}_{i,\epsilon}) \right\|_{L^\infty(\Omega, \mathbb{R}^k)} \leq C_\infty i^{q_\infty}$.

Assumption 4.4 is not unrealistic. Indeed, it has been observed numerous times in the literature, that NNs can approximate many classical basis functions very efficiently: By [73], Assumption 4.4 holds for polynomial bases. It was demonstrated in [71, 54] that Assumption 4.4 is satisfied if the reduced basis is made of finite elements. Moreover, [65] proves these approximation results for bases of wavelets, [9] extends these results to systems based on affine transforms, and [27] demonstrate the validity of Assumption 4.4 for bases of sinusoidal functions.

We will show below that under Assumptions 4.1, 4.2, and 4.4 the map $u(\cdot)\colon \mathcal{Y} \times \Omega \to \mathbb{R}^k$ can be approximated without a curse of dimension. To shorten the notion below, we define

$$H_L(\tilde{\epsilon}) := B_L\left(\frac{\tilde{\epsilon}}{4}\frac{C_{\mathrm{coer}}}{C_{\mathrm{cont}}}, \frac{3}{32}\frac{\tilde{\epsilon}\alpha C_{\mathrm{coer}}^2}{\max\{6, C_{\mathrm{rhs}}\}}\right) + F_L\left(\frac{\tilde{\epsilon}}{4}\frac{C_{\mathrm{coer}}}{C_{\mathrm{cont}}}, \frac{\tilde{\epsilon}}{12}C_{\mathrm{coer}}\right),$$
$$H_M(\tilde{\epsilon}) := 2B_M\left(\frac{\tilde{\epsilon}}{4}\frac{C_{\mathrm{coer}}}{C_{\mathrm{cont}}}, \frac{3}{32}\frac{\tilde{\epsilon}\alpha C_{\mathrm{coer}}^2}{\max\{6, C_{\mathrm{rhs}}\}}\right) + F_M\left(\frac{\tilde{\epsilon}}{4}\frac{C_{\mathrm{coer}}}{C_{\mathrm{cont}}}, \frac{\tilde{\epsilon}}{12}C_{\mathrm{coer}}\right).$$

Note that, if B_L, F_L, B_L, B_M are polylogarithmic in their inputs (as in both examples in Section 4.2), then the same holds for H_L and H_M.

Theorem 4.5. *Let $\tilde{\epsilon} \in [\hat{\epsilon}, \alpha/4 \cdot \min\{1, C_{\mathrm{coer}}\})$, $\tilde{\epsilon}' := \tilde{\epsilon}/4 \cdot C_{\mathrm{coer}}/C_{\mathrm{cont}}$. Under Assumptions 4.1, 4.2, and 4.4 there exists a NN $\Phi^u_{\tilde{\epsilon}}$ with $p + n$-dimensional input and k-dimensional output such that*

$$\sup_{y \in \mathcal{Y}} \left\| u_y - \mathrm{R}^{\mathcal{Y}\times\Omega}_{\varrho}(\Phi^u_{\tilde{\epsilon}})(y, \cdot) \right\|_{\mathcal{H}} \leq \tilde{\epsilon}. \tag{4.1}$$

Moreover, there exist $C_u, C'_u > 0$ independent of $\tilde{\epsilon}$ such that

$$L\left(\Phi^u_{\tilde\epsilon}\right) \leq C_u \cdot \left(\left(d(\tilde\epsilon') + \log_2(\log_2(1/\tilde\epsilon))\right)\log_2(1/\tilde\epsilon) + H_L(\tilde\epsilon) + \log_2^{2q_L}(d(\tilde\epsilon'))\log_2^{q_L}(1/\tilde\epsilon)\right), \tag{4.2}$$

$$M\left(\Phi^u_{\tilde\epsilon}\right) \leq C'_u \cdot \Big(d(\tilde\epsilon')^3\left(\log_2(d(\tilde\epsilon')) + \log_2^3(1/\tilde\epsilon)\right) + d(\tilde\epsilon')^2 H_L(\tilde\epsilon) + H_M(\tilde\epsilon)) + kd(\tilde\epsilon')\log_2^{2\max\{q_L,q_M\}}(d(\tilde\epsilon'))\log_2^{\max\{q_L,q_M\}}(1/\tilde\epsilon)\Big). \tag{4.3}$$

Remark 4.6. *In the two examples from Section 4.2 we had that* $d(\tilde\epsilon')$ *as well as* $H_L(\tilde\epsilon), H_M(\tilde\epsilon)$ *scale like* $\log_2^m(1/\tilde\epsilon)$ *for an* $m \in \mathbb{N}$ *and* $\tilde\epsilon \to 0$. *Hence, Theorem 4.5 yields an exponential approximation rate for the function* $u\colon \mathcal{Y} \times \Omega \to \mathbb{R}^k$, *where* $u_y \in \mathcal{H}$ *is the solution of the parametric problem with parameter* y.

A similar estimate would not be achievable by only analyzing the smoothness of $u\colon \mathcal{Y} \times \Omega \to \mathbb{R}^k$. *Indeed,* u_y *is not required to be smooth but only assumed to be in* $[H^1(\Omega)]^k$.

Acknowledgments

G. K. acknowledges partial support by the Bundesministerium für Bildung und Forschung (BMBF) through the Berliner Zentrum for Machine Learning (BZML), Project AP4, by the Deutsche Forschungsgemeinschaft (DFG) through grants CRC 1114 "Scaling Cascades in Complex Systems", Project B07, CRC/TR 109 "Discretization in Geometry and Dynamics", Projects C02 and C03, RTG DAEDALUS (RTG 2433), Projects P1 and P3, RTG BIOQIC (RTG 2260), Projects P4 and P9, and SPP 1798 "Compressed Sensing in Information Processing", Coordination Project and Project Massive MIMO-I/II, by the Berlin Mathematics Research Center MATH+ , Projects EF1-1 and EF1-4, and by the Einstein Foundation Berlin. P.P. is supported by a DFG Research Fellowship "Shearlet-based energy functionals for anisotropic phase-field methods". M.R. would like to thank Ingo Gühring for fruitful discussions on the topic. M.R. is supported by the Berlin Mathematical School and by the DFG Collaborative Research Center TRR 109 "Discretization in Geometry and Dynamics". R.S. acknowledges support by the DFG through grant RTG DAEDALUS (RTG 2433), Project P14.

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A Proofs of the Results from Section 3

A.1 Proof of Proposition 3.7

In this subsection, we will prove Proposition 3.7. As a preparation, we first prove the following special instance under which $M(\Phi^1 \bullet \Phi^2)$ can be estimated by $\max\left\{M(\Phi^1), M(\Phi^2)\right\}$.

Lemma A.1. *Let Φ be a NN with m-dimensional output and d-dimensional input. If $\mathbf{a} \in \mathbb{R}^{1\times m}$, then, for all $\ell = 1, \dots, L(\Phi)$,*

$$M_\ell(((\mathbf{a}, 0)) \bullet \Phi) \leq M_\ell(\Phi).$$

In particular, it holds that $M((\mathbf{a}, 0) \bullet \Phi) \leq M(\Phi)$. Moreover, if $\mathbf{D} \in \mathbb{R}^{d\times n}$ such that, for every $k \leq d$ there is at most one $l_k \leq n$ such that $\mathbf{D}_{k,l_k} \neq 0$, then, for all $\ell = 1, \dots, L(\Phi)$,

$$M_\ell(\Phi \bullet ((\mathbf{D}, \mathbf{0}_{\mathbb{R}^d}))) \leq M_\ell(\Phi).$$

In particular, it holds that $M(\Phi \bullet ((\mathbf{D}, \mathbf{0}_{\mathbb{R}^d}))) \leq M(\Phi)$.

Proof. Let $\Phi = \big((\mathbf{A}_1, \mathbf{b}_1), \ldots, (\mathbf{A}_L, \mathbf{b}_L)\big)$, and $\mathbf{a}, \mathbf{D}$ as in the statement of the lemma. Then the result follows if

$$\|\mathbf{a}\mathbf{A}_L\|_0 + \|\mathbf{a}\mathbf{b}_L\|_0 \leq \|\mathbf{A}_L\|_0 + \|\mathbf{b}_L\|_0 \tag{A.1}$$

and

$$\|\mathbf{A}_1\mathbf{D}\|_0 \leq \|\mathbf{A}_1\|_0.$$

It is clear that $\|\mathbf{a}\mathbf{A}_L\|_0$ is less than the number of nonzero columns of $\mathbf{A}_L$ which is certainly bounded by $\|\mathbf{A}_L\|_0$. The same argument shows that $\|\mathbf{a}\mathbf{b}_L\|_0 \leq \|\mathbf{b}_L\|_0$. This yields (A.1).

We have that for two vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^k$, $k \in \mathbb{N}$ and for all $\mu, \nu \in \mathbb{R}$

$$\|\mu\mathbf{p} + \nu\mathbf{q}\|_0 \leq I(\mu)\|\mathbf{p}\|_0 + I(\nu)\|\mathbf{q}\|_0,$$

where $I(\gamma) = 0$ if $\gamma = 0$ and $I(\gamma) = 1$ otherwise. Also,

$$\|\mathbf{A}_1\mathbf{D}\|_0 = \left\|\mathbf{D}^T\mathbf{A}_1^T\right\|_0 = \sum_{l=1}^{n} \left\|\left(\mathbf{D}^T\mathbf{A}_1^T\right)_{l,-}\right\|_0,$$

where, for a matrix $\mathbf{G}$, $\mathbf{G}_{l,-}$ denotes the l-th row of $\mathbf{G}$. Moreover, we have that for all $l \leq n$

$$\left(\mathbf{D}^T\mathbf{A}_1^T\right)_{l,-} = \sum_{k=1}^{d} \left(\mathbf{D}^T\right)_{l,k} \left(\mathbf{A}_1^T\right)_{k,-} = \sum_{k=1}^{d} \mathbf{D}_{k,l} \left(\mathbf{A}_1^T\right)_{k,-}.$$

As a consequence, we obtain

$$\begin{aligned}
\|\mathbf{A}_1\mathbf{D}\|_0 &\leq \sum_{l=1}^{n} \left\|\sum_{k=1}^{d} \mathbf{D}_{k,l} \left(\mathbf{A}_1^T\right)_{k,-}\right\|_0 \leq \sum_{l=1}^{n}\sum_{k=1}^{d} I\left(\mathbf{D}_{k,l}\right) \left\|\left(\mathbf{A}_1^T\right)_{k,-}\right\|_0 \\
&= \sum_{k=1}^{d} I\left(\mathbf{D}_{k,l_k}\right) \left\|\left(\mathbf{A}_1^T\right)_{k,-}\right\|_0 \leq \|\mathbf{A}_1\|_0.
\end{aligned}$$

□

Now we are ready to prove Proposition 3.7.

Proof of Proposition 3.7. Without loss of generality, assume that $Z \geq 1$. By [22, Lemma 6.2], there exists a NN $\times_\epsilon^Z$ with input dimension 2, output dimension 1 such that for $\Phi_\epsilon := \times_\epsilon^Z$

$$L\left(\Phi_\epsilon\right) \leq 0.5 \log_2\left(\frac{n\sqrt{dl}}{\epsilon}\right) + \log_2(Z) + 6, \tag{A.2}$$

$$M\left(\Phi_\epsilon\right) \leq 90 \cdot \left(\log_2\left(\frac{n\sqrt{dl}}{\epsilon}\right) + 2\log_2(Z) + 6\right), \tag{A.3}$$

$$M_1\left(\Phi_\epsilon\right) \leq 16, \text{ as well as } M_{L(\Phi_\epsilon)}\left(\Phi_\epsilon\right) \leq 3, \tag{A.4}$$

$$\sup_{|a|,|b|\leq Z} \left|ab - \mathrm{R}_\varrho^{\mathbb{R}^2}\left(\Phi_\epsilon\right)(a,b)\right| \leq \frac{\epsilon}{n\sqrt{dl}}. \tag{A.5}$$

Since $\|\mathbf{A}\|_2, \|\mathbf{B}\|_2 \leq Z$ we know that for every $i = 1, ..., d$, $k = 1, ..., n$, $j = 1, ..., l$ we have that $|\mathbf{A}_{i,k}|, |\mathbf{B}_{k,j}| \leq Z$. We define, for $i \in \{1, \ldots, d\}, k \in \{1, \ldots, n\}, j \in \{1, \ldots, l\}$, the matrix $\mathbf{D}_{i,k,j}$ such that, for all $\mathbf{A} \in \mathbb{R}^{d\times n}, \mathbf{B} \in \mathbb{R}^{n\times l}$

$$\mathbf{D}_{i,k,j}(\mathbf{vec}(\mathbf{A}), \mathbf{vec}(\mathbf{B})) = (\mathbf{A}_{i,k}, \mathbf{B}_{k,j}).$$

Moreover, let

$$\Phi^Z_{i,k,j;\epsilon} := \times^Z_\epsilon \bullet \left((\mathbf{D}_{i,k,j}, \mathbf{0}_{\mathbb{R}^2}) \right).$$

We have, for all $i \in \{1,\dots,d\}, k \in \{1,\dots,n\}, j \in \{1,\dots,l\}$, that $L\left(\Phi^Z_{i,k,j;\epsilon}\right) = L\left(\times^Z_\epsilon\right)$ and by Lemma A.1 that $\Phi^Z_{i,k,j;\epsilon}$ satisfies (A.2), (A.3), (A.4) with $\Phi_\epsilon := \Phi^Z_{i,k,j;\epsilon}$. Moreover, we have by (A.5)

$$\sup_{(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B}))\in K^Z_{d,n,l}} \left| \mathbf{A}_{i,k}\mathbf{B}_{k,j} - \mathrm{R}^{K^Z_{d,n,l}}_\varrho \left(\Phi^Z_{i,j,k;\epsilon}\right) \right| \leq \frac{\epsilon}{n\sqrt{dl}}. \tag{A.6}$$

As a next step, we set, for $\mathbf{1}_{\mathbb{R}^n} \in \mathbb{R}^n$ being a vector with each entry equal to 1,

$$\Phi^Z_{i,j;\epsilon} := \left((\mathbf{1}_{\mathbb{R}^n}, 0) \right) \bullet \mathrm{P}\left(\Phi^Z_{i,1,j;\epsilon}, \dots, \Phi^Z_{i,n,j;\epsilon}\right),$$

which by Lemma 3.6 is a NN with $n \cdot (d+l)$-dimensional input and 1-dimensional output such that (A.2) holds with $\Phi_\epsilon := \Phi^Z_{i,j;\epsilon}$. Moreover, by Lemmas A.1 and 3.6 and by (A.3) we have that

$$M\left(\Phi^Z_{i,j;\epsilon}\right) \leq M\left(\mathrm{P}\left(\Phi^Z_{i,1,j;\epsilon}, \dots, \Phi^Z_{i,n,j;\epsilon}\right)\right) \leq 90n \cdot \left(\log_2\left(\frac{n\sqrt{dl}}{\epsilon}\right) + 2\log_2(Z) + 6 \right). \tag{A.7}$$

Additionally, by Lemmas 3.6 and A.1 and (A.4), we obtain

$$M_1\left(\Phi^Z_{i,j;\epsilon}\right) \leq M_1\left(\mathrm{P}\left(\Phi^Z_{i,1,j;\epsilon}, \dots, \Phi^Z_{i,n,j;\epsilon}\right)\right) \leq 16n.$$

and

$$M_{L\left(\Phi^Z_{i,j;\epsilon}\right)}\left(\Phi^Z_{i,j;\epsilon}\right) = M_{L\left(\Phi^Z_{i,j;\epsilon}\right)}\left(\mathrm{P}\left(\Phi^Z_{i,1,j;\epsilon}, \dots, \Phi^Z_{i,n,j;\epsilon}\right)\right) \leq 2n. \tag{A.8}$$

By construction it follows that

$$\mathrm{R}^{K^Z_{d,n,l}}_\varrho \left(\Phi^Z_{i,j;\epsilon}\right)(\mathbf{vec}(\mathbf{A}), \mathbf{vec}(\mathbf{B})) = \sum_{k=1}^n \mathrm{R}^{K^Z_{d,n,l}}_\varrho \left(\Phi^Z_{i,k,j;\epsilon}\right)(\mathbf{vec}(\mathbf{A}), \mathbf{vec}(\mathbf{B}))$$

and hence we have, by (A.6),

$$\sup_{(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B}))\in K^Z_{d,n,l}} \left| \sum_{k=1}^n \mathbf{A}_{i,k}\mathbf{B}_{k,j} - \mathrm{R}^{K^Z_{d,n,l}}_\varrho \left(\Phi^Z_{i,j;\epsilon}\right)(\mathbf{vec}(\mathbf{A}), \mathbf{vec}(\mathbf{B})) \right| \leq \frac{\epsilon}{\sqrt{dl}}.$$

As a final step, we define $\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}} := \mathrm{P}\left(\Phi^Z_{1,1;\epsilon}, \dots, \Phi^Z_{d,1;\epsilon}, \dots, \Phi^Z_{1,l;\epsilon}, \dots, \Phi^Z_{d,l;\epsilon}\right)$. Then, by Lemma 3.6, we have that (A.2) is satisfied for $\Phi_\epsilon := \Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}$. This yields (i) of the asserted statement. Moreover, invoking Lemma 3.6 and (A.7) yields that

$$M\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right) \leq 90dln \cdot \left(\log_2\left(\frac{n\sqrt{dl}}{\epsilon}\right) + 2\log_2(Z) + 6 \right),$$

which yields (ii) of the result. Moreover, by Lemma 3.6 and (A.8) it follows that

$$M_1\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right) \leq 16dln \text{ and } M_{L\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right)}\left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}}\right) \leq 2dln,$$

completing the proof of (iii). By construction and using the fact that for any $\mathbf{N} \in \mathbb{R}^{d\times l}$ there holds

$$\|\mathbf{N}\|_2 \leq \sqrt{dl} \max_{i,j} |\mathbf{N}_{i,j}|,$$

we obtain that

$$\sup_{(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B}))\in K^Z_{d,n,l}} \left\| \mathbf{AB} - \mathbf{matr}\left(\mathrm{R}_\varrho^{K^Z_{d,n,l}} \left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}} \right) (\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B})) \right) \right\|_2$$

$$\leq \sqrt{dl} \sup_{(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B}))\in K^Z_{d,n,l}} \max_{i=1,\ldots,d,\ j=1,\ldots,l} \left| \sum_{k=1}^n \mathbf{A}_{i,k}\mathbf{B}_{k,j} - \mathrm{R}_\varrho^{K^Z_{d,n,l}} \left(\Phi^Z_{i,j;\epsilon} \right) (\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B})) \right| \leq \epsilon. \quad \text{(A.9)}$$

Equation (A.9) establishes (iv) of the asserted result. Finally, we have for any $(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B})) \in K^Z_{d,n,l}$ that

$$\begin{aligned}
&\left\| \mathbf{matr}\left(\mathrm{R}_\varrho^{K^Z_{d,n,l}} \left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}} \right) (\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B})) \right) \right\|_2 \\
&\leq \left\| \mathbf{matr}\left(\mathrm{R}_\varrho^{K^Z_{d,n,l}} \left(\Phi^{Z,d,n,l}_{\mathrm{mult};\tilde{\epsilon}} \right) (\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B})) \right) - \mathbf{AB} \right\|_2 + \|\mathbf{AB}\|_2 \\
&\leq \epsilon + \|\mathbf{A}\|_2\|\mathbf{B}\|_2 \leq \epsilon + Z^2 \leq 1 + Z^2.
\end{aligned}$$

This demonstrates that (v) holds and thereby finishes the proof. □

A.2 Proof of Theorem 3.8

The objective of this subsection is to prove of Theorem 3.8. Towards this goal, we construct NNs which emulate the map $\mathbf{A} \mapsto \mathbf{A}^k$ for $k \in \mathbb{N}$ and square matrices $\mathbf{A}$. This is done by heavily using Proposition 3.7. First of all, as a direct consequence of Proposition 3.7 we can estimate the sizes of the emulation of the multiplication of two squared matrices. Indeed, there exists a universal constant $C_1 > 0$ such that for all $d \in \mathbb{N}$, $Z > 0$, $\epsilon \in (0,1)$

(i) $L\left(\Phi^{Z,d,d,d}_{\mathrm{mult};\epsilon}\right) \leq C_1 \cdot (\log_2(1/\epsilon) + \log_2(d) + \log_2(\max\{1,Z\}))$,

(ii) $M\left(\Phi^{Z,d,d,d}_{\mathrm{mult};\epsilon}\right) \leq C_1 \cdot (\log_2(1/\epsilon) + \log_2(d) + \log_2(\max\{1,Z\}))\, d^3$,

(iii) $M_1\left(\Phi^{Z,d,d,d}_{\mathrm{mult};\epsilon}\right) \leq C_1 d^3$, as well as $M_{L\left(\Phi^{Z,d,d,d}_{\mathrm{mult};\epsilon}\right)}\left(\Phi^{Z,d,d,d}_{\mathrm{mult};\epsilon}\right) \leq C_1 d^3$,

(iv) $\sup_{(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B}))\in K^Z_{d,d,d}} \left\| \mathbf{AB} - \mathbf{matr}\left(\mathrm{R}_\varrho^{K^Z_{d,d,d}} \left(\Phi^{Z,d,d,d}_{\mathrm{mult};\epsilon} \right) (\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B})) \right) \right\|_2 \leq \epsilon$,

(v) for every $(\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B})) \in K^Z_{d,d,d}$ we have

$$\left\| \mathbf{matr}\left(\mathrm{R}_\varrho^{K^Z_{d,d,d}} \left(\Phi^{Z,d,d,d}_{\mathrm{mult};\epsilon} \right) (\mathbf{vec}(\mathbf{A}),\mathbf{vec}(\mathbf{B})) \right) \right\|_2 \leq \epsilon + \|\mathbf{A}\|_2\|\mathbf{B}\|_2 \leq \epsilon + Z^2 \leq 1 + Z^2.$$

One consequence of the ability to emulate the multiplication of matrices is that we can also emulate the squaring of matrices. We make this precise in the following definition.

Definition A.2. *For $d \in \mathbb{N}$, $Z > 0$, and $\epsilon \in (0,1)$ we define the NN*

$$\Phi^{Z,d}_{2;\epsilon} := \Phi^{Z,d,d,d}_{\mathrm{mult};\epsilon} \odot \left(\left(\begin{pmatrix} \mathbf{Id}_{\mathbb{R}^{d^2}} \\ \mathbf{Id}_{\mathbb{R}^{d^2}} \end{pmatrix}, \mathbf{0}_{\mathbb{R}^{2d^2}} \right) \right),$$

which has d^2-dimensional input and d^2-dimensional output. By Lemma 3.6 we have that there exists a constant $C_{\mathrm{sq}} > C_1$ such that for all $d \in \mathbb{N}$, $Z > 0$, $\epsilon \in (0,1)$

(i) $L\left(\Phi^{Z,d}_{2;\epsilon}\right) \leq C_{\mathrm{sq}} \cdot (\log_2(1/\epsilon) + \log_2(d) + \log_2(\max\{1,Z\}))$,

(ii) $M\left(\Phi^{Z,d}_{2;\epsilon}\right) \leq C_{\mathrm{sq}} d^3 \cdot (\log_2(1/\epsilon) + \log_2(d) + \log_2(\max\{1, Z\}))$,

(iii) $M_1\left(\Phi^{Z,d}_{2;\epsilon}\right) \leq C_{\mathrm{sq}} d^3$, *as well as* $M_{L\left(\Phi^{Z,d}_{2;\epsilon}\right)}\left(\Phi^{Z,d}_{2;\epsilon}\right) \leq C_{\mathrm{sq}} d^3$,

(iv) $\sup_{\mathbf{vec}(\mathbf{A}) \in K^Z_d} \left\| \mathbf{A}^2 - \mathbf{matr}\left(\mathrm{R}^{K^Z_d}_{\varrho}\left(\Phi^{Z,d}_{2;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon$,

(v) for all $\mathbf{vec}(\mathbf{A}) \in K^Z_d$ *we have*

$$\left\|\mathbf{matr}\left(\mathrm{R}^{K^Z_d}_{\varrho}\left(\Phi^{Z,d}_{2;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon + \|\mathbf{A}\|^2 \leq \epsilon + Z^2 \leq 1 + Z^2.$$

Our next goal is to approximate the map $\mathbf{A} \mapsto \mathbf{A}^k$ for an arbitrary $k \in \mathbb{N}_0$. We start with the case that k is a power of 2 and for the moment we only consider the set of all matrices the norm of which is bounded by 1/2.

Proposition A.3. *Let* $d \in \mathbb{N}$, $j \in \mathbb{N}$, *as well as* $\epsilon \in (0, 1/4)$. *Then there exists a NN* $\Phi^{1/2,d}_{2^j;\epsilon}$ *with* d^2*-dimensional input and* d^2*-dimensional output with the following properties:*

(i) $L\left(\Phi^{1/2,d}_{2^j;\epsilon}\right) \leq C_{\mathrm{sq}} j \cdot (\log_2(1/\epsilon) + \log_2(d)) + 2C_{\mathrm{sq}} \cdot (j-1)$,

(ii) $M\left(\Phi^{1/2,d}_{2^j;\epsilon}\right) \leq C_{\mathrm{sq}} j d^3 \cdot (\log_2(1/\epsilon) + \log_2(d)) + 4C_{\mathrm{sq}} \cdot (j-1) d^3$,

(iii) $M_1\left(\Phi^{1/2,d}_{2^j;\epsilon}\right) \leq C_{\mathrm{sq}} d^3$, *as well as* $M_{L\left(\Phi^{1/2,d}_{2^j;\epsilon}\right)}\left(\Phi^{1/2,d}_{2^j;\epsilon}\right) \leq C_{\mathrm{sq}} d^3$,

(iv) $\sup_{\mathbf{vec}(\mathbf{A}) \in K^{1/2}_d} \left\| \mathbf{A}^{2^j} - \mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{2^j;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon$,

(v) for every $\mathbf{vec}(\mathbf{A}) \in K^{1/2}_d$ *we have*

$$\left\|\mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{2^j;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon + \left\|\mathbf{A}^{2^j}\right\|_2 \leq \epsilon + \|\mathbf{A}\|_2^{2^j} \leq \frac{1}{4} + \left(\frac{1}{2}\right)^{2^j} \leq \frac{1}{2}.$$

Proof. We show the statement by induction over $j \in \mathbb{N}$. For $j = 1$, the statement follows by choosing $\Phi^{1/2,d}_{2;\epsilon}$ as in Definition A.2. Assume now, as induction hypothesis, that the claim holds for an arbitrary, but fixed $j \in \mathbb{N}$, i.e., there exists a NN $\Phi^{1/2,d}_{2^j;\epsilon}$ such that

$$\left\|\mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{2^j;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \mathbf{A}^{2^j}\right\|_2 \leq \epsilon, \qquad \left\|\mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{2^j;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon + \left(\frac{1}{2}\right)^{2^j} \tag{A.10}$$

and $\Phi^{1/2,d}_{2^j;\epsilon}$ satisfies (i),(ii),(iii). Now we define

$$\Phi^{1/2,d}_{2^{j+1};\epsilon} := \Phi^{1,d}_{2;\frac{\epsilon}{4}} \odot \Phi^{1/2,d}_{2^j;\epsilon}.$$

By the triangle inequality, we obtain for any $\mathbf{vec}(\mathbf{A}) \in K^{1/2}_d$

$$\begin{aligned}
&\left\|\mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \mathbf{A}^{2^{j+1}}\right\|_2 \\
&\leq \left\|\mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \mathbf{A}^{2^j}\mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{2^j;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \\
&\quad + \left\|\mathbf{A}^{2^j}\mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{2^j;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \left(\mathbf{A}^{2^j}\right)^2\right\|_2.
\end{aligned} \tag{A.11}$$

By construction of $\Phi^{1/2,d}_{2^{j+1};\epsilon}$, we know that

$$\left\| \mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \left(\mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right)^2 \right\|_2 \leq \frac{\epsilon}{4}.$$

Therefore, using the triangle inequality and the fact that $\|\cdot\|_2$ is a submultiplicative operator norm, we derive that

$$\begin{aligned}
&\left\| \mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \mathbf{A}^{2^j}\mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2 \\
&\leq \frac{\epsilon}{4} + \left\| \left(\mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right)^2 - \mathbf{A}^{2^j}\mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2 \\
&\leq \frac{\epsilon}{4} + \left\| \mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \mathbf{A}^{2^j} \right\|_2 \left\| \mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2 \\
&\leq \frac{\epsilon}{4} + \epsilon \cdot \left(\epsilon + \left(\frac{1}{2}\right)^{2^j}\right) \leq \frac{3}{4}\epsilon,
\end{aligned} \tag{A.12}$$

where the penultimate estimate follows by the induction hypothesis (A.10) and $\epsilon < 1/4$. Hence, since $\|\cdot\|_2$ is a submultiplicative operator norm, we obtain

$$\begin{aligned}
\left\| \mathbf{A}^{2^j}\mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \left(\mathbf{A}^{2^j}\right)^2 \right\|_2 &\leq \left\| \mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \mathbf{A}^{2^j} \right\|_2 \left\| \mathbf{A}^{2^j} \right\|_2 \\
&\leq \frac{\epsilon}{4},
\end{aligned} \tag{A.13}$$

where we used $\left\|\mathbf{A}^{2^j}\right\|_2 \leq 1/4$ and the induction hypothesis (A.10). Applying (A.13) and (A.12) to (A.11) yields

$$\left\| \mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) - \mathbf{A}^{2^{j+1}} \right\|_2 \leq \epsilon. \tag{A.14}$$

A direct consequence of (A.14) is that

$$\left\| \mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2 \leq \epsilon + \left\| \mathbf{A}^{2^{j+1}} \right\|_2 \leq \epsilon + \|\mathbf{A}\|_2^{2^{j+1}}. \tag{A.15}$$

The estimates (A.14) and (A.15) complete the proof of the assertions (iv) and (v) of the proposition statement. Now we estimate the size of $\Phi^{1/2,d}_{2^{j+1};\epsilon}$. By the induction hypothesis and Lemma 3.6(a)(i), we obtain

$$\begin{aligned}
L\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right) &= L\left(\Phi^{1,d}_{2;\frac{\epsilon}{4}}\right) + L\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right) \\
&\leq C_{\mathrm{sq}} \cdot (\log_2(1/\epsilon) + \log_2(d) + \log_2(4) + j\log_2(1/\epsilon) + 2\cdot(j-1) + j\log_2(d)) \\
&= C_{\mathrm{sq}} \cdot ((j+1)\log_2(1/\epsilon) + (j+1)\log_2(d) + 2j),
\end{aligned}$$

which implies (i). Moreover, by the induction hypothesis and Lemma 3.6(a)(ii), we conclude that

$$\begin{aligned}
M\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right) &\leq M\left(\Phi^{1,d}_{2;\frac{\epsilon}{4}}\right) + M\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right) + M_1\left(\Phi^{1,d}_{2;\frac{\epsilon}{4}}\right) + M_{L\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right)}\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right) \\
&\leq C_{\mathrm{sq}}d^3 \cdot (\log_2(1/\epsilon) + \log_2(d) + \log_2(4) + j\log_2(1/\epsilon) + j\log_2(d) + 4\cdot(j-1)) + 2C_{\mathrm{sq}}d^3 \\
&= C_{\mathrm{sq}}d^3 \cdot ((j+1)\log_2(1/\epsilon) + (j+1)\log_2(d) + 4j),
\end{aligned}$$

implying (ii). Finally, it follows from Lemma 3.6(a)(iii) in combination with the induction hypothesis as well Lemma 3.6(a)(iv) that

$$M_1\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right) = M_1\left(\Phi^{1/2,d}_{2^{j};\epsilon}\right) \leq C_{\mathrm{sq}} d^3,$$

as well as

$$M_{L\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right)}\left(\Phi^{1/2,d}_{2^{j+1};\epsilon}\right) = M_{L\left(\Phi^{1,d}_{2;\frac{\epsilon}{4}}\right)}\left(\Phi^{1,d}_{2;\frac{\epsilon}{4}}\right) \leq C_{\mathrm{sq}} d^3,$$

which finishes the proof. □

We proceed by demonstrating, how to build a NN that emulates the mapping $\mathbf{A} \mapsto \mathbf{A}^k$ for an arbitrary $k \in \mathbb{N}_0$. Again, for the moment we only consider the set of all matrices the norms of which are bounded by $1/2$. For the case of the set of all matrices the norms of which are bounded by an arbitrary $Z > 0$, we refer to Corollary A.5.

Proposition A.4. *Let $d \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $\epsilon \in (0, 1/4)$. Then, there exists a NN $\Phi^{1/2,d}_{k;\epsilon}$ with d^2- dimensional input and d^2-dimensional output satisfying the following properties:*

(i)

$$\begin{aligned} L\left(\Phi^{1/2,d}_{k;\epsilon}\right) &\leq \lfloor \log_2(\max\{k,2\}) \rfloor L\left(\Phi^{1,d}_{\mathrm{mult};\frac{\epsilon}{4}}\right) + L\left(\Phi^{1/2,d}_{2^{\lfloor \log_2(\max\{k,2\})\rfloor};\epsilon}\right) \\ &\leq 2C_{\mathrm{sq}} \lfloor \log_2(\max\{k,2\}) \rfloor \cdot (\log_2(1/\epsilon) + \log_2(d) + 2), \end{aligned}$$

(ii) $M\left(\Phi^{1/2,d}_{k;\epsilon}\right) \leq \frac{3}{2} C_{\mathrm{sq}} d^3 \cdot \lfloor \log_2(\max\{k,2\}) \rfloor \cdot (\lfloor \log_2(\max\{k,2\}) \rfloor + 1) \cdot (\log_2(1/\epsilon) + \log_2(d) + 4)$,

(iii) $M_1\left(\Phi^{1/2,d}_{k;\epsilon}\right) \leq C_{\mathrm{sq}} \cdot (\lfloor \log_2(\max\{k,2\}) \rfloor + 1)\, d^3$, *as well as* $M_{L\left(\Phi^{1/2,d}_{k;\epsilon}\right)}\left(\Phi^{1/2,d}_{k;\epsilon}\right) \leq C_{\mathrm{sq}} d^3$,

(iv) $\sup_{\mathbf{vec}(\mathbf{A}) \in K^{1/2}_d} \left\| \mathbf{A}^k - \mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{k;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon$,

(v) for any $\mathbf{vec}(\mathbf{A}) \in K^{1/2}_d$ *we have*

$$\left\|\mathbf{matr}\left(\mathrm{R}^{K^{1/2}_d}_{\varrho}\left(\Phi^{1/2,d}_{k;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon + \|\mathbf{A}^k\|_2 \leq \frac{1}{4} + \|\mathbf{A}\|_2^k.$$

Proof. We prove the result per induction over $k \in \mathbb{N}_0$. The cases $k = 0$ and $k = 1$ hold trivially by defining the NNs

$$\Phi^{1/2,d}_{0;\epsilon} := \left(\left(\mathbf{0}_{\mathbb{R}^{d^2} \times \mathbb{R}^{d^2}}, \mathbf{vec}(\mathbf{Id}_{\mathbb{R}^d})\right)\right), \qquad \Phi^{1/2,d}_{1;\epsilon} := \left(\left(\mathbf{Id}_{\mathbb{R}^{d^2}}, \mathbf{0}_{\mathbb{R}^{d^2}}\right)\right).$$

For the induction hypothesis, we claim that the result holds true for all $k' \leq k \in \mathbb{N}$. If k is a power of two, then the result holds per Proposition A.3, thus we can assume without loss of generality, that k is not a power of two. We define $j := \lfloor \log_2(k) \rfloor$ such that, for $t := k - 2^j$, we have that $0 < t < 2^j$. This implies that $A^k = A^{2^j} A^t$. Hence, by Proposition A.3 and by the induction hypothesis, respectively, there exist a NN $\Phi^{1/2,d}_{2^j;\epsilon}$ satisfying (i)-(v) of Proposition A.3 and a NN $\Phi^{1/2,d}_{t;\epsilon}$ satisfying (i)-(v) of the statement of this proposition. We now define the NN

$$\Phi^{1/2,d}_{k;\epsilon} := \Phi^{1,d,d,d}_{\mathrm{mult};\frac{\epsilon}{4}} \odot \mathrm{P}\left(\Phi^{1/2,d}_{2^j;\epsilon}, \Phi^{1/2,d}_{t;\epsilon}\right).$$

By construction and Lemma 3.6(a)(iv), we first observe that

$$M_{L\left(\Phi^{1/2,d}_{k;\epsilon}\right)}\left(\Phi^{1/2,d}_{k;\epsilon}\right) = M_{L\left(\Phi^{1,d,d,d}_{\mathrm{mult};\frac{\epsilon}{4}}\right)}\left(\Phi^{1,d,d,d}_{\mathrm{mult};\frac{\epsilon}{4}}\right) \leq C_{\mathrm{sq}} d^3.$$

Moreover, we obtain by the induction hypothesis as well as Lemma 3.6(a)(iii) in combination with Lemma 3.6(b)(iv) that

$$\begin{aligned} M_1\left(\Phi_{k;\epsilon}^{1/2,d}\right) &= M_1\left(\mathrm{P}\left(\Phi_{2^j;\epsilon}^{1/2,d},\Phi_{t;\epsilon}^{1/2,d}\right)\right) = M_1\left(\Phi_{2^j;\epsilon}^{1/2,d}\right) + M_1\left(\Phi_{t;\epsilon}^{1/2,d}\right) \\ &\leq C_{\mathrm{sq}}d^3 + (j+1)C_{\mathrm{sq}}d^3 = (j+2)C_{\mathrm{sq}}d^3. \end{aligned}$$

This shows (iii). To show (iv), we perform a similar estimate as the one following (A.11). By the triangle inequality,

$$\begin{aligned} &\left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{k;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right) - \mathbf{A}^k\right\|_2 \\ &\leq \left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{k;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right) - \mathbf{A}^{2^j}\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{t;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right)\right\|_2 \\ &\quad + \left\|\mathbf{A}^{2^j}\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{t;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right) - \mathbf{A}^{2^j}\mathbf{A}^t\right\|_2. \end{aligned} \tag{A.16}$$

By the construction of $\Phi_{k;\epsilon}^{1/2,d}$ and the Proposition 3.7, we conclude that

$$\begin{aligned} &\left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{k;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right)\right. \\ &\quad \left.- \mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{2^j;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right)\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{t;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right)\right\|_2 \\ &\quad \leq \frac{\epsilon}{4}. \end{aligned}$$

Hence, using (A.16), we can estimate

$$\begin{aligned} &\left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{k;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right) - \mathbf{A}^k\right\|_2 \\ &\leq \frac{\epsilon}{4} + \left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{2^j;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right)\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{t;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right)\right. \\ &\quad \left.- \mathbf{A}^{2^j}\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{t;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right)\right\|_2 \\ &\qquad + \left\|\mathbf{A}^{2^j}\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{t;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right) - \mathbf{A}^k\right\|_2 \\ &\leq \frac{\epsilon}{4} + \left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{t;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right)\right\|_2\left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{2^j;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right) - \mathbf{A}^{2^j}\right\|_2 \\ &\quad + \left\|\mathbf{A}^{2^j}\right\|_2\left\|\mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1/2}}\left(\Phi_{t;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right) - \mathbf{A}^t\right\|_2 =: \frac{\epsilon}{4} + \mathrm{I} + \mathrm{II}. \end{aligned}$$

We now consider two cases: If $t=1$, then we know by the construction of $\Phi_{1;\epsilon}^{1/2,d}$ that $\mathrm{II}=0$. Thus

$$\frac{\epsilon}{4} + \mathrm{I} + \mathrm{II} = \frac{\epsilon}{4} + \mathrm{I} \leq \frac{\epsilon}{4} + \|\mathbf{A}\|_{2}\epsilon \leq \frac{3\epsilon}{4} \leq \epsilon.$$

If $t \geq 2$, then

$$\frac{\epsilon}{4} + \mathrm{I} + \mathrm{II} \leq \frac{\epsilon}{4} + \left(\epsilon + \|\mathbf{A}\|^t + \|\mathbf{A}\|^{2^j}\right)\epsilon \leq \frac{\epsilon}{4} + \left(\frac{1}{4} + \left(\frac{1}{2}\right)^t + \left(\frac{1}{2}\right)^{2^j}\right)\epsilon \leq \frac{\epsilon}{4} + \frac{3\epsilon}{4} = \epsilon,$$

where we have used that $\left(\frac{1}{2}\right)^t \leq \frac{1}{4}$ for $t \geq 2$. This shows (iv). In addition, by an application of the triangle inequality, we have that

$$\left\|\mathbf{matr}\left(\mathrm{R}_{\varrho}^{K_d^{1/2}}\left(\Phi_{k;\epsilon}^{1/2,d}(\mathbf{vec}(\mathbf{A}))\right)\right)\right\|_2 \leq \epsilon + \left\|\mathbf{A}^k\right\|_2 \leq \epsilon + \|\mathbf{A}\|_2^k.$$

This shows (v). Now we analyze the size of $\Phi_{k;\epsilon}^{1/2,d}$. We have by Lemma 3.6(a)(i) in combination with Lemma 3.6(b)(i) and by the induction hypothesis that

$$\begin{aligned}
L\left(\Phi_{k;\epsilon}^{1/2,d}\right) &\leq L\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) + \max\left\{L\left(\Phi_{2^j;\epsilon}^{1/2,d}\right), L\left(\Phi_{t;\epsilon}^{1/2,d}\right)\right\} \\
&\leq L\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) + \max\left\{L\left(\Phi_{2^j;\epsilon}^{1/2,d}\right), (j-1)L\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) + L\left(\Phi_{2^{j-1};\epsilon}^{1/2,d}\right)\right\} \\
&\leq L\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) + \max\left\{(j-1)L\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) + L\left(\Phi_{2^j;\epsilon}^{1/2,d}\right), (j-1)L\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) + L\left(\Phi_{2^{j-1};\epsilon}^{1/2,d}\right)\right\} \\
&\leq jL\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) + L\left(\Phi_{2^j;\epsilon}^{1/2,d}\right) \\
&\leq C_{\mathrm{sq}}j \cdot (\log_2(1/\epsilon) + \log_2(d) + 2) + C_{\mathrm{sq}}j \cdot (\log_2(1/\epsilon) + \log_2(d)) + 2C_{\mathrm{sq}} \cdot (j-1) \\
&\leq 2C_{\mathrm{sq}}j \cdot (\log_2(1/\epsilon) + \log_2(d) + 2),
\end{aligned}$$

which implies (i). Finally, we address the number of non-zero weights of the resulting NN. We first observe that, by Lemma 3.6(a)(ii),

$$\begin{aligned}
M\left(\Phi_{k;\epsilon}^{1/2,d}\right) \leq& \left(M\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) + M_1\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right)\right) + M\left(\mathrm{P}\left(\Phi_{2^j;\epsilon}^{1/2,d}, \Phi_{t;\epsilon}^{1/2,d}\right)\right) \\
&+ M_{L\left(\mathrm{P}\left(\Phi_{2^j;\epsilon}^{1/2,d}, \Phi_{t;\epsilon}^{1/2,d}\right)\right)}\left(\mathrm{P}\left(\Phi_{2^j;\epsilon}^{1/2,d}, \Phi_{t;\epsilon}^{1/2,d}\right)\right) \\
=:& \mathrm{I}' + \mathrm{II}'(a) + \mathrm{II}'(b).
\end{aligned}$$

Then, by the properties of the NN $\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}$, we obtain

$$\begin{aligned}
\mathrm{I}' = M\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) + M_1\left(\Phi_{\mathrm{mult};\frac{\epsilon}{4}}^{1,d,d,d}\right) &\leq C_{\mathrm{sq}}d^3 \cdot (\log_2(1/\epsilon) + \log_2(d) + 2) + C_{\mathrm{sq}}d^3 \\
&= C_{\mathrm{sq}}d^3 \cdot (\log_2(1/\epsilon) + \log_2(d) + 3).
\end{aligned}$$

Next, we estimate

$$\mathrm{II}'(a) + \mathrm{II}'(b) = M\left(\mathrm{P}\left(\Phi_{2^j;\epsilon}^{1/2,d}, \Phi_{t;\epsilon}^{1/2,d}\right)\right) + M_{L\left(\mathrm{P}\left(\Phi_{2^j;\epsilon}^{1/2,d}, \Phi_{t;\epsilon}^{1/2,d}\right)\right)}\left(\mathrm{P}\left(\Phi_{2^j;\epsilon}^{1/2,d}, \Phi_{t;\epsilon}^{1/2,d}\right)\right).$$

Without loss of generality we assume that $L := L\left(\Phi_{t;\epsilon}^{1/2,d}\right) - L\left(\Phi_{2^j;\epsilon}^{1/2,d}\right) > 0$. The other cases follow similarly. We have that $L \leq 2C_{\mathrm{sq}}j \cdot (\log_2(1/\epsilon) + \log_2(d) + 2)$ and, by the definition of the parallelization of two NNs with a different number of layers that

$$\begin{aligned}
\mathrm{II}'(a) &= M\left(\mathrm{P}\left(\Phi_{2^j;\epsilon}^{1/2,d}, \Phi_{t;\epsilon}^{1/2,d}\right)\right) \\
&= M\left(\mathrm{P}\left(\Phi_{d^2,L}^{\mathbf{Id}} \odot \Phi_{2^j;\epsilon}^{1/2,d}, \Phi_{t;\epsilon}^{1/2,d}\right)\right) \\
&= M\left(\Phi_{d^2,L}^{\mathbf{Id}} \odot \Phi_{2^j;\epsilon}^{1/2,d}\right) + M\left(\Phi_{t;\epsilon}^{1/2,d}\right) \\
&\leq M\left(\Phi_{d^2,L}^{\mathbf{Id}}\right) + M_1\left(\Phi_{d^2,L}^{\mathbf{Id}}\right) + M_{L\left(\Phi_{2^j;\epsilon}^{1/2,d}\right)}\left(\Phi_{2^j;\epsilon}^{1/2,d}\right) + M\left(\Phi_{2^j;\epsilon}^{1/2,d}\right) + M\left(\Phi_{t;\epsilon}^{1/2,d}\right) \\
&\leq 2d^2(L+1) + C_{\mathrm{sq}}d^3 + M\left(\Phi_{t;\epsilon}^{1/2,d}\right) + M\left(\Phi_{2^j;\epsilon}^{1/2,d}\right),
\end{aligned}$$

where we have used the definition of the parallelization for the first two equalities, Lemma 3.6(b)(iii) for the third equality, Lemma 3.6(a)(ii) for the fourth inequality as well as the properties of $\Phi^{\mathbf{Id}}_{d^2,L}$ in combination with Proposition A.3(iii) for the last inequality. Moreover, by the definition of the parallelization of two NNs with different numbers of layers, we conclude that

$$\mathrm{II}'(b) = M_{L\left(\mathrm{P}\left(\Phi^{1/2,d}_{2^j;\epsilon},\Phi^{1/2,d}_{t;\epsilon}\right)\right)}\left(\mathrm{P}\left(\Phi^{1/2,d}_{2^j;\epsilon},\Phi^{1/2,d}_{d;\epsilon}\right)\right) \leq d^2 + C_{\mathrm{sq}}d^3.$$

Combining the estimates on I', $\mathrm{II}'(a)$, and $\mathrm{II}'(b)$, we obtain by using the induction hypothesis that

$$\begin{aligned} M\left(\Phi^{1/2,d}_{k;\epsilon}\right) &\leq C_{\mathrm{sq}}d^3\cdot(\log_2(1/\epsilon)+\log_2(d)+3) + 2d^2\cdot(L+1) + d^2 + C_{\mathrm{sq}}d^3 + M\left(\Phi^{1/2,d}_{t;\epsilon}\right) + M\left(\Phi^{1/2,d}_{2^j;\epsilon}\right) \\ &\leq C_{\mathrm{sq}}d^3\cdot(\log_2(1/\epsilon)+\log_2(d)+4) + 2d^2\cdot(L+2) + M\left(\Phi^{1/2,d}_{t;\epsilon}\right) + M\left(\Phi^{1/2,d}_{2^j;\epsilon}\right) \\ &\leq C_{\mathrm{sq}}\cdot(j+1)d^3\cdot(\log_2(1/\epsilon)+\log_2(d)+4) + 2d^2\cdot(L+2) + M\left(\Phi^{1/2,d}_{t;\epsilon}\right) \\ &\leq C_{\mathrm{sq}}\cdot(j+1)d^3\cdot(\log_2(1/\epsilon)+\log_2(d)+4) + 2C_{\mathrm{sq}}jd^2\cdot(\log_2(1/\epsilon)+\log_2(d)+2) \\ &\qquad + 4d^2 + M\left(\Phi^{1/2,d}_{t;\epsilon}\right) \\ &\leq 3C_{\mathrm{sq}}\cdot(j+1)d^3\cdot(\log_2(1/\epsilon)+\log_2(d)+4) + M\left(\Phi^{1/2,d}_{t;\epsilon}\right) \\ &\leq 3C_{\mathrm{sq}}d^3\cdot\left(j+1+\frac{j\cdot(j+1)}{2}\right)\cdot(\log_2(1/\epsilon)+\log_2(d)+4)) \\ &= \frac{3}{2}C_{\mathrm{sq}}\cdot(j+1)\cdot(j+2)d^3\cdot(\log_2(1/\epsilon)+\log_2(d)+4)\,. \end{aligned}$$

□

Proposition A.4 only provides a construction of a NN the ReLU-realization of which emulates a power of a matrix $\mathbf{A}$, under the assumption that $\|\mathbf{A}\|_2 \leq 1/2$. We remove this restriction in the following corollary by presenting a construction of a NN $\Phi^{Z,d}_{k;\epsilon}$ the ReLU-realization of which approximates the map $\mathbf{A} \mapsto \mathbf{A}^k$, on the set of all matrices $\mathbf{A}$ the norms of which are bounded by an arbitrary $Z > 0$.

Corollary A.5. *There exists a universal constant $C_{\mathrm{pow}} > C_{\mathrm{sq}}$ such that for all $Z > 0$, $d \in \mathbb{N}$ and $k \in \mathbb{N}_0$, there exists some NN $\Phi^{Z,d}_{k;\epsilon}$ with the following properties:*

(i) $L\left(\Phi^{Z,d}_{k;\epsilon}\right) \leq C_{\mathrm{pow}}\log_2(\max\{k,2\})\cdot(\log_2(1/\epsilon)+\log_2(d)+k\log_2(\max\{1,Z\}))$,

(ii) $M\left(\Phi^{Z,d}_{k;\epsilon}\right) \leq C_{\mathrm{pow}}\log_2^2(\max\{k,2\})\,d^3\cdot(\log_2(1/\epsilon)+\log_2(d)+k\log_2(\max\{1,Z\}))$,

(iii) $M_1\left(\Phi^{Z,d}_{k;\epsilon}\right) \leq C_{\mathrm{pow}}\log_2(\max\{k,2\})\,d^3$, *as well as* $M_{L\left(\Phi^{Z,d}_{k;\epsilon}\right)}\left(\Phi^{Z}_{k;\epsilon}\right) \leq C_{\mathrm{pow}}d^3$,

(iv) $\sup_{\mathbf{vec}(\mathbf{A})\in K^Z_d}\left\|\mathbf{A}^k - \mathbf{matr}\left(\mathrm{R}^{K^Z_d}_{\varrho}\left(\Phi^{Z,d}_{k;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon$,

(v) for any $\mathbf{vec}(\mathbf{A}) \in K^Z_d$ *we have*

$$\left\|\mathbf{matr}\left(\mathrm{R}^{K^Z_d}_{\varrho}\left(\Phi^{Z,d}_{k;\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \leq \epsilon + \|\mathbf{A}^k\|_2 \leq \epsilon + \|\mathbf{A}\|_2^k.$$

Proof. Let $((\mathbf{A}_1,\mathbf{b}_1),...,(\mathbf{A}_L,\mathbf{b}_L)) := \Phi^{1/2,d}_{k;\frac{\epsilon}{2\max\{1,Z^k\}}}$ according to Proposition A.4. Then the NN

$$\Phi^{Z,d}_{k;\epsilon} := \left(\left(\frac{1}{2Z}\mathbf{A}_1,\mathbf{b}_1\right),(\mathbf{A}_2,\mathbf{b}_2),...,(\mathbf{A}_{L-1},\mathbf{b}_{L-1}),\left(2Z^k\mathbf{A}_L,2Z^k\mathbf{b}_L\right)\right)$$

fulfills all of the desired properties. □

We have seen how to construct a NN that takes a matrix as an input and computes a power of this matrix. With this tool at hand, we are now ready to prove Theorem 3.8.

Proof of Theorem 3.8. By the properties of the partial sums of the Neumann series, for $m \in \mathbb{N}$ and every $\mathbf{vec}(\mathbf{A}) \in K_d^{1-\delta}$, we have that

$$\left\| (\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1} - \sum_{k=0}^{m} \mathbf{A}^k \right\|_2 = \left\| (\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1} \mathbf{A}^{m+1} \right\|_2 \leq \left\| (\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1} \right\|_2 \|\mathbf{A}\|_2^{m+1}$$
$$\leq \frac{1}{1-(1-\delta)} \cdot (1-\delta)^{m+1} = \frac{(1-\delta)^{m+1}}{\delta}.$$

Hence, for

$$m(\epsilon,\delta) = \left\lceil \log_{1-\delta}(2) \log_2\left(\frac{\epsilon\delta}{2}\right) \right\rceil - 1 = \left\lceil \frac{\log_2(\epsilon) + \log_2(\delta) - 1}{\log_2(1-\delta)} \right\rceil - 1 \leq \frac{\log_2(\epsilon) + \log_2(\delta) - 1}{\log_2(1-\delta)}$$

we obtain

$$\left\| (\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1} - \sum_{k=0}^{m(\epsilon,\delta)} \mathbf{A}^k \right\|_2 \leq \frac{\epsilon}{2}.$$

Let now

$$((\mathbf{A}_1,\mathbf{b}_1),...,(\mathbf{A}_L,\mathbf{b}_L)) \coloneqq \left(\left(\left(\mathbf{Id}_{\mathbb{R}^{d^2}}|...|\mathbf{Id}_{\mathbb{R}^{d^2}}\right), \mathbf{0}_{\mathbb{R}^{d^2}}\right)\right) \odot \mathrm{P}\left(\Phi^{1,d}_{1;\frac{\epsilon}{2(m(\epsilon,\delta)-1)}}, ..., \Phi^{1,d}_{m(\epsilon,\delta);\frac{\epsilon}{2(m(\epsilon,\delta)-1)}}\right),$$

where $\left(\mathbf{Id}_{\mathbb{R}^{d^2}}|...|\mathbf{Id}_{\mathbb{R}^{d^2}}\right) \in \mathbb{R}^{d^2 \times m(\epsilon,\delta)\cdot d^2}$. Then we set

$$\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon} \coloneqq ((\mathbf{A}_1,\mathbf{b}_1),...,(\mathbf{A}_L,\mathbf{b}_L + \mathbf{vec}(\mathbf{Id}_{\mathbb{R}^d}))).$$

We have for any $\mathbf{vec}(\mathbf{A}) \in K_d^{1-\delta}$

$$\left\| (\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1} - \mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1-\delta}}\left(\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2$$
$$\leq \left\| (\mathbf{Id}_{\mathbb{R}^d} - \mathbf{A})^{-1} - \sum_{k=0}^{m(\epsilon,\delta)} \mathbf{A}^k \right\|_2 + \left\| \sum_{k=0}^{m(\epsilon,\delta)} \mathbf{A}^k - \mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1-\delta}}\left(\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2$$
$$\leq \frac{\epsilon}{2} + \sum_{k=2}^{m(\epsilon,\delta)} \left\| \mathbf{A}^k - \mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1-\delta}}\left(\Phi^{1,d}_{k;\frac{\epsilon}{2(m(\epsilon,\delta)-1)}}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2$$
$$\leq \frac{\epsilon}{2} + (m(\epsilon,\delta)-1)\frac{\epsilon}{2(m(\epsilon,\delta)-1)} = \epsilon,$$

where we have used that

$$\left\| \mathbf{A} - \mathbf{matr}\left(\mathrm{R}_\varrho^{K_d^{1-\delta}}\left(\Phi^{1,d}_{1;\frac{\epsilon}{2(m(\epsilon,\delta)-1)}}\right)(\mathbf{vec}(\mathbf{A}))\right) \right\|_2 = 0.$$

This completes the proof of (iii). Moreover, (iv) is a direct consequence of (iii). Now we analyze the size of the resulting NN. First of all, we have by Lemma 3.6(b)(i) and Corollary A.5 that

$$L\left(\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}\right) = \max_{k=1,\dots,m(\epsilon,\delta)} L\left(\Phi^{1,d}_{k;\frac{\epsilon}{2(m(\epsilon,\delta)-1)}}\right)$$
$$\leq C_{\mathrm{pow}} \log_2(m(\epsilon,\delta)-1) \cdot (\log_2(1/\epsilon) + 1 + \log_2(m(\epsilon,\delta)-1) + \log_2(d))$$
$$\leq C_{\mathrm{pow}} \log_2\left(\frac{\log_2(0.5\epsilon\delta)}{\log_2(1-\delta)}\right) \cdot \left(\log_2(1/\epsilon) + 1 + \log_2\left(\frac{\log_2(0.5\epsilon\delta)}{\log_2(1-\delta)}\right) + \log_2(d)\right),$$

which implies (i). Moreover, by Lemma 3.6(b)(ii), Corollary A.5 and the monotonicity of the logarithm, we obtain

$$\begin{aligned}
M\left(\Phi^{1-\delta,d}_{\mathrm{inv};\epsilon}\right) &\leq 3\cdot\left(\sum_{k=1}^{m(\epsilon,\delta)} M\left(\Phi^{1,d}_{k;\frac{\epsilon}{2(m(\epsilon,\delta)-1)}}\right)\right)\\
&\quad + 4C_{\mathrm{pow}}m(\epsilon,\delta)d^2\log_2\left(m(\epsilon,\delta)\right)\cdot\left(\log_2\left(1/\epsilon\right)+1+\log_2\left(m(\epsilon,\delta)\right)+\log_2(d)\right)\\
&\leq 3C_{\mathrm{pow}}\cdot\left(\sum_{k=1}^{m(\epsilon,\delta)}\log_2^2(\max\{k,2\})\right)d^3\cdot\left(\log_2(1/\epsilon)+1+\log_2\left(m(\epsilon,\delta)\right)+\log_2(d)\right)\\
&\quad + 5m(\epsilon,\delta)d^2C_{\mathrm{pow}}\log_2\left(m(\epsilon,\delta)\right)\cdot\left(\log_2\left(1/\epsilon\right)+1+\log_2\left(m(\epsilon,\delta)\right)+\log_2(d)\right) =: \mathrm{I}.
\end{aligned}$$

Since $\sum_{k=1}^{m(\epsilon,\delta)}\log_2^2(\max\{k,2\}) \leq m(\epsilon,\delta)\log_2^2(m(\epsilon,\delta))$, we obtain for some constant $C_{\mathrm{inv}} > C_{\mathrm{pow}}$ that

$$\mathrm{I} \leq C_{\mathrm{inv}}m(\epsilon,\delta)\log_2^2(m(\epsilon,\delta))d^3\cdot\left(\log_2(1/\epsilon)+\log_2\left(m(\epsilon,\delta)\right)+\log_2(d)\right).$$

This completes the proof. □

B Proofs of the Results from Section 4

We start by establishing a bound on $\left\|\mathbf{Id}_{\mathbb{R}^{d(\tilde{\epsilon})}} - \alpha\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right\|_2$ and a preparatory result and then give the proofs of the results from Section 4 one after another.

Proposition B.1. *For any $\alpha \in (0, 1/C_{\mathrm{cont}})$ and $\delta := \alpha C_{\mathrm{cont}} \in (0,1)$ there holds*

$$\left\|\mathbf{Id}_{\mathbb{R}^{d(\tilde{\epsilon})}} - \alpha\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right\|_2 \leq 1-\delta < 1, \qquad \textit{for all } y\in\mathcal{Y},\ \tilde{\epsilon}>0.$$

Proof. Since $\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}$ is symmetric, there holds that

$$\left\|\mathbf{Id}_{\mathbb{R}^{d(\tilde{\epsilon})}} - \alpha\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right\|_2 = \max_{\mu\in\sigma\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)}|1-\alpha\mu| \leq \max_{\mu\in[C_{\mathrm{coer}},C_{\mathrm{cont}}]}|1-\alpha\mu| = 1-\alpha C_{\mathrm{cont}} = 1-\delta < 1,$$

for all $y\in\mathcal{Y},\ \tilde{\epsilon}>0$. □

With an approximation of the parameter-dependent map building the stiffness matrices with respect to a RB, due to Assumption 4.1, we can next state a construction of a NN the ReLU-realization of which approximates the map $y\mapsto\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}$. As a first step, we observe the following remark.

Remark B.2. *It is not hard to see that if $\left((\mathbf{A}^1_{\tilde{\epsilon},\epsilon},\mathbf{b}^1_{\tilde{\epsilon},\epsilon}),...,(\mathbf{A}^L_{\tilde{\epsilon},\epsilon},\mathbf{b}^L_{\tilde{\epsilon},\epsilon})\right) := \Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon}$ is the NN of Assumption 4.1, then for*

$$\Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon} := \left((\mathbf{A}^1_{\tilde{\epsilon},\epsilon},\mathbf{b}^1_{\tilde{\epsilon},\epsilon}),...,(-\mathbf{A}^L_{\tilde{\epsilon},\epsilon},-\mathbf{b}^L_{\tilde{\epsilon},\epsilon}+\mathbf{vec}\left(\mathbf{Id}_{\mathbb{R}^{d(\tilde{\epsilon})}}\right))\right)$$

we have that

$$\sup_{y\in\mathcal{Y}}\left\|\mathbf{Id}_{\mathbb{R}^{d(\tilde{\epsilon})}} - \alpha\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon}\right)(y)\right)\right\|_2 \leq \epsilon,$$

as well as $M\left(\Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon}\right) \leq B_M(\tilde{\epsilon},\epsilon)+d(\tilde{\epsilon})^2$ and $L\left(\Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon}\right) = B_L\left(\tilde{\epsilon},\epsilon\right)$.

Now we present the construction of the NN emulating $y\mapsto\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}$.

Proposition B.3. *Let $\tilde{\epsilon} \geq \hat{\epsilon}, \epsilon \in (0, \alpha/4 \cdot \min\{1, C_{\mathrm{coer}}\})$ and $\epsilon' := 3/8 \cdot \epsilon\alpha C_{\mathrm{coer}}^2 < \epsilon$. Assume that Assumption 4.1 holds. We define*

$$\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon} := ((\alpha \mathbf{Id}_{\mathbb{R}^{d(\tilde{\epsilon})}}, \mathbf{0}_{\mathbb{R}^{d(\tilde{\epsilon})}})) \bullet \Phi^{1-\delta/2, d(\tilde{\epsilon})}_{\mathrm{inv};\frac{\epsilon}{2\alpha}} \odot \Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon'},$$

which has p-dimensional input and $d(\tilde{\epsilon})^2$- dimensional output. Then, there exists a universal constant $C_B > 0$ such that

(i) $L\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right) \leq C_B \log_2(\log_2(1/\epsilon))\big(\log_2(1/\epsilon) + \log_2(\log_2(1/\epsilon)) + \log_2(d(\tilde{\epsilon}))\big) + B_L(\tilde{\epsilon}, \epsilon')$,

(ii) $M\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right) \leq C_B \log_2(1/\epsilon) \log_2^2(\log_2(1/\epsilon)) d(\tilde{\epsilon})^3 \cdot \big(\log_2(1/\epsilon) + \log_2(\log_2(1/\epsilon)) + \log_2(d(\tilde{\epsilon}))\big) + 2B_M(\tilde{\epsilon}, \epsilon')$,

(iii) $\sup_{y\in\mathcal{Y}} \left\| \left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right)(y)\right) \right\|_2 \leq \epsilon$,

(iv) $\sup_{y\in\mathcal{Y}} \left\| \mathbf{G}^{1/2} \mathbf{V}_{\tilde{\epsilon}} \cdot \left(\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right)(y)\right)\right) \right\|_2 \leq \epsilon$,

(v) $\sup_{y\in\mathcal{Y}} \left\| \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right)(y)\right) \right\|_2 \leq \epsilon + \frac{1}{C_{\mathrm{coer}}}$,

(vi) $\sup_{y\in\mathcal{Y}} \left\| \mathbf{G}^{1/2} \mathbf{V}_{\tilde{\epsilon}} \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right)(y)\right) \right\|_2 \leq \epsilon + \frac{1}{C_{\mathrm{coer}}}$.

Proof. First of all, for all $y \in \mathcal{Y}$ the matrix $\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)$ is invertible. This can be deduced from the fact that

$$\left\| \alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right) \right\|_2 \leq \epsilon' < \epsilon \leq \frac{\alpha \min\{1, C_{\mathrm{coer}}\}}{4} \leq \frac{\alpha C_{\mathrm{coer}}}{4}. \tag{B.1}$$

Indeed, we estimate

$$\begin{aligned}
&\min_{\mathbf{z}\in\mathbb{R}^{d(\tilde{\epsilon})}\setminus\{0\}} \frac{\left|\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\mathbf{z}\right|}{|\mathbf{z}|} \\
\text{[Reverse triangle inequality]}\quad &\geq \min_{\mathbf{z}\in\mathbb{R}^{d(\tilde{\epsilon})}\setminus\{0\}} \frac{\left|\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\mathbf{z}\right|}{|\mathbf{z}|} - \max_{\mathbf{z}\in\mathbb{R}^{d(\tilde{\epsilon})}\setminus\{0\}} \frac{\left|\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\mathbf{z} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\mathbf{z}\right|}{|\mathbf{z}|} \\
\text{[Definition of } \|.\|_2] \quad &\geq \left(\max_{\mathbf{z}\in\mathbb{R}^{d(\tilde{\epsilon})}\setminus\{0\}} \frac{|\mathbf{z}|}{\left|\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\mathbf{z}\right|}\right)^{-1} - \left\|\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right\|_2 \\
\text{[Set } \tilde{\mathbf{z}} := (\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}})\mathbf{z}] \quad &\geq \left(\max_{\tilde{\mathbf{z}}\in\mathbb{R}^{d(\tilde{\epsilon})}\setminus\{0\}} \frac{\left|(\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}})^{-1}\tilde{\mathbf{z}}\right|}{|\tilde{\mathbf{z}}|}\right)^{-1} - \left\|\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right\|_2 \\
\text{[Definition of } \|\cdot\|_2] \quad &\geq \left\|\left(\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}\right\|_2^{-1} - \left\|\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right\|_2 \\
\text{[By Equations (B.1) and (2.8)]} \quad &\geq \alpha C_{\mathrm{coer}} - \frac{\alpha C_{\mathrm{coer}}}{4} \geq \frac{3}{4}\alpha C_{\mathrm{coer}}.
\end{aligned}$$

Thus it follows that

$$\left\|\left(\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right)^{-1}\right\|_2 \leq \frac{4}{3}\frac{1}{C_{\mathrm{coer}}\alpha}. \tag{B.2}$$

Then

$$\begin{aligned}
&\left\| \frac{1}{\alpha}\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{1-\delta/2, d(\tilde{\epsilon})}_{\mathrm{inv};\frac{\epsilon}{2\alpha}} \odot \Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right\|_2 \\
&\quad\leq \left\| \frac{1}{\alpha}\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1} - \left(\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right)^{-1}\right\|_2 \\
&\qquad + \left\|\left(\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right)^{-1} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{1-\delta/2, d(\tilde{\epsilon})}_{\mathrm{inv};\frac{\epsilon}{2\alpha}} \odot \Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right\|_2 =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Due to the fact that for two invertible matrices $\mathbf{M}, \mathbf{N}$,

$$\left\|\mathbf{M}^{-1} - \mathbf{N}^{-1}\right\|_2 = \left\|\mathbf{M}^{-1}(\mathbf{N}-\mathbf{M})\mathbf{N}^{-1}\right\|_2 \leq \|\mathbf{M}-\mathbf{N}\|_2 \|\mathbf{M}^{-1}\|_2 \|\mathbf{N}^{-1}\|_2,$$

we obtain

$$\begin{aligned} \mathrm{I} &\leq \left\|\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right\|_2 \left\|\left(\alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}\right\|_2 \left\|\left(\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right)^{-1}\right\|_2 \\ &\leq \frac{3}{8}\epsilon\alpha C_{\mathrm{coer}}^2 \frac{1}{\alpha C_{\mathrm{coer}}} \frac{4}{3} \frac{1}{C_{\mathrm{coer}}\alpha} = \frac{\epsilon}{2\alpha}, \end{aligned}$$

where we have used Assumption 4.1, Equation (2.8) and Equation (B.2). Now we turn our attention to estimating II. First, observe that for every $y \in \mathcal{Y}$ by the triangle inequality and Remark B.2, that

$$\begin{aligned} \left\|\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right\|_2 &\leq \left\|\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right) - \left(\mathbf{Id}_{\mathbb{R}^{d(\tilde{\epsilon})}} - \alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)\right\|_2 + \left\|\mathbf{Id}_{\mathbb{R}^{d(\tilde{\epsilon})}} - \alpha \mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right\|_2 \\ &\leq \epsilon' + 1 - \delta \leq 1 - \delta + \frac{\alpha C_{\mathrm{coer}}}{4} \leq 1 - \delta + \frac{\alpha C_{\mathrm{cont}}}{4} \leq 1 - \delta + \frac{\delta}{2} = 1 - \frac{\delta}{2}. \end{aligned}$$

Moreover, have that $\epsilon/(2\alpha) \leq \alpha/(8\alpha) < 1/4$. Hence, by Theorem 3.8, we obtain that $\mathrm{II} \leq \epsilon/2\alpha$. Putting everything together yields

$$\sup_{y\in\mathcal{Y}} \left\|\frac{1}{\alpha}\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{1-\delta/2,d(\tilde{\epsilon})}_{\mathrm{inv};\frac{\epsilon}{2\alpha}} \odot \Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon'}\right)(y)\right)\right\|_2 \leq \mathrm{I} + \mathrm{II} \leq \frac{\epsilon}{\alpha}.$$

Finally, by construction we can conclude that

$$\sup_{y\in\mathcal{Y}} \left\|\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right)(y)\right)\right\|_2 \leq \epsilon.$$

This implies (iii) of the assertion. Now, by Equation (2.6) we obtain

$$\sup_{y\in\mathcal{Y}} \left\|\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1} - \mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right)(y)\right)\right\|_2 \leq \left\|\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}\right\|_2 \epsilon = \epsilon,$$

completing the proof of (iv). Finally, for all $y \in \mathcal{Y}$ we estimate

$$\begin{aligned} &\left\|\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right)(y)\right)\right\|_2 \\ &\leq \left\|\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}} \cdot \left(\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right)(y)\right)\right)\right\|_2 + \left\|\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}\right\|_2 \\ &\leq \epsilon + \frac{1}{C_{\mathrm{coer}}}. \end{aligned}$$

This yields (vi). A minor modification of the calculation above yields (v). At last, we show (i) and (ii). First of all, it is clear that $L\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right) = L\left(\Phi^{1-\delta/2,d(\tilde{\epsilon})}_{\mathrm{inv};\frac{\epsilon}{2\alpha}} \odot \Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon'}\right)$ and $M\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right) = M\left(\Phi^{1-\delta/2,d(\tilde{\epsilon})}_{\mathrm{inv};\frac{\epsilon}{2\alpha}} \odot \Phi^{\mathbf{B},\mathbf{Id}}_{\tilde{\epsilon},\epsilon'}\right)$. Moreover, by Lemma 3.6(a)(i) in combination with Theorem 3.8 (i) we have

$$L\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right) \leq C_{\mathrm{inv}} \log_2\left(m\left(\epsilon/(2\alpha), \delta/2\right)\right) \cdot \left(\log_2\left(2\alpha/\epsilon\right) + \log_2\left(m\left(\epsilon/(2\alpha), \delta/2\right)\right) + \log_2(d(\tilde{\epsilon}))\right) + B_L(\tilde{\epsilon}, \epsilon')$$

and, by Lemma 3.6(a)(ii) in combination with Theorem 3.8(ii), we obtain

$$\begin{aligned} &M\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon}\right) \\ &\quad \leq 2C_{\mathrm{inv}} m(\epsilon/(2\alpha), \delta/2) \log_2^2\left(m(\epsilon/(2\alpha), \delta/2)\right) d(\tilde{\epsilon})^3 \cdot \left(\log_2\left(2\alpha/\epsilon\right) + \log_2\left(m(\epsilon/(2\alpha), \delta/2)\right) + \log_2(d(\tilde{\epsilon}))\right) \\ &\quad + 2d(\tilde{\epsilon})^2 + 2B_M(\tilde{\epsilon}, \epsilon'). \end{aligned}$$

In addition, by definition of $m(\epsilon, \delta)$ in the statement of Theorem 3.8, for some constant $\tilde{C} > 0$ there holds $m\left(\epsilon/(2\alpha), \delta/2\right) \leq \tilde{C} \log_2(1/\epsilon)$. Hence, the claim follows for a suitably chosen constant $C_B > 0$.

□

B.1 Proof of Theorem 4.3

We start with proving (i) by deducing the estimate for $\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}$. The estimate for $\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon}$ follows in a similar, but simpler way. For $y \in \mathcal{Y}$, we have that

$$\begin{aligned}
&\left|\tilde{\mathbf{u}}^{\mathrm{h}}_{y,\tilde{\epsilon}} - \mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}\right)(y)\right|_{\mathbf{G}} \\
&= \left|\mathbf{G}^{1/2}\cdot\left(\mathbf{V}_{\tilde{\epsilon}}\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}\mathbf{f}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}\right)(y)\right)\right| \\
&\leq \left|\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}\cdot\left(\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}\mathbf{f}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)(y)\right)\right| \\
&\quad + \left|\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}\cdot\left(\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)(y) - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}\right)(y)\right)\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)(y)\right)\right| \\
&\quad + \left|\mathbf{G}^{1/2}\cdot\left(\mathbf{V}_{\tilde{\epsilon}}\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}\right)(y)\right)\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)(y) - \mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}\right)(y)\right)\right| =: \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

We now estimate $\mathrm{I},\mathrm{II},\mathrm{III}$ separately. By Equation (2.6), Equation (2.8), Assumption 4.2, and the definition of ϵ'' there holds for $y \in \mathcal{Y}$ that

$$\mathrm{I} \leq \left\|\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}\right\|_2 \left\|\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1}\right\|_2 \left|\mathbf{f}^{\mathrm{rb}}_{y,\tilde{\epsilon}} - \mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)(y)\right| \leq \frac{1}{C_{\mathrm{coer}}}\frac{\epsilon C_{\mathrm{coer}}}{3} = \frac{\epsilon}{3}.$$

We proceed with estimating II. It is not hard to see from Assumption 4.2 that

$$\sup_{y\in\mathcal{Y}}\left|\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon}\right)(y)\right| \leq \epsilon + C_{\mathrm{rhs}}. \tag{B.3}$$

By definition, $\epsilon' = \epsilon/\max\{6, C_{\mathrm{rhs}}\} \leq \epsilon$. Hence, by Assumption 4.1 and (B.3) in combination with Proposition B.3 (i), we obtain

$$\begin{aligned}
\mathrm{II} &\leq \left\|\mathbf{G}^{1/2}\mathbf{V}_{\tilde{\epsilon}}\cdot\left(\left(\mathbf{B}^{\mathrm{rb}}_{y,\tilde{\epsilon}}\right)^{-1} - \mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}\right)(y)\right)\right)\right\|_2 \left|\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)(y)\right| \leq \epsilon'\cdot\left(C_{\mathrm{rhs}} + \frac{\epsilon\cdot C_{\mathrm{coer}}}{3}\right) \\
&\leq \frac{\epsilon}{\max\{6,C_{\mathrm{rhs}}\}}C_{\mathrm{rhs}} + \frac{\epsilon C_{\mathrm{coer}}}{\max\{6,C_{\mathrm{rhs}}\}}\frac{\epsilon}{3} \leq \frac{2\epsilon}{6} = \frac{\epsilon}{3},
\end{aligned}$$

where we have used that $C_{\mathrm{coer}}\epsilon < C_{\mathrm{coer}}\alpha/4 < 1$. Finally, we estimate III. Per construction, we have that

$$\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}\right)(y) = \mathbf{V}_{\tilde{\epsilon}}\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\kappa,d(\tilde{\epsilon}),d(\tilde{\epsilon}),1}_{\mathrm{mult};\frac{\epsilon}{3}} \odot \mathrm{P}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}, \Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)\right)(y).$$

Moreover, we have by Proposition B.3(v)

$$\left\|\mathbf{matr}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}\right)(y)\right)\right\|_2 \leq \epsilon + \frac{1}{C_{\mathrm{coer}}} \leq 1 + \frac{1}{C_{\mathrm{coer}}} \leq \kappa$$

and by (B.3) that

$$\left|\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)(y)\right| \leq \epsilon'' + C_{\mathrm{rhs}} \leq \epsilon C_{\mathrm{coer}} + C_{\mathrm{rhs}} \leq 1 + C_{\mathrm{rhs}} \leq \kappa.$$

Hence, by the choice of κ and Proposition 3.7 we conclude that $\mathrm{III} \leq \epsilon/3$. Combining the estimates on I,II, and III yields (i) and using (i) implies (v). Now we estimate the size of the NNs. We start with proving (ii). First of all, we have by the definition of $\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon}$ and $\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}$ as well as Lemma 3.6(a)(i) in combination with Proposition 3.7 that

$$\begin{aligned}
L\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon}\right) < L\left(\Phi^{\mathbf{u},\mathrm{h}}_{\tilde{\epsilon},\epsilon}\right) &\leq 1 + L\left(\Phi^{\kappa,d(\tilde{\epsilon}),d(\tilde{\epsilon}),1}_{\mathrm{mult};\frac{\epsilon}{3}}\right) + L\left(\mathrm{P}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}, \Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)\right) \\
&\leq 1 + C_{\mathrm{mult}}\cdot\left(\log_2(3/\epsilon) + 3/2\log_2(d(\tilde{\epsilon})) + \log_2(\kappa)\right) + \max\left\{L\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}\right), F_L\left(\tilde{\epsilon},\epsilon''\right)\right\}.
\end{aligned}$$

As a result, we obtain (ii) after applying Proposition B.3(i) and choosing a suitable constant $C_L^{\mathbf{u}} > 0$. We now note that if we establish (iii), then (iv) follows immediately by Lemma 3.6(a)(ii). Thus, we proceed with proving (iii). First of all, by Lemma 3.6(a)(ii) in combination with Proposition 3.7 we have

$$\begin{aligned} M\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon},\epsilon}\right) &\leq 2M\left(\Phi^{\kappa,d(\tilde{\epsilon}),d(\tilde{\epsilon}),1}_{\mathrm{mult};\frac{\epsilon}{3}}\right) + 2M\left(\mathrm{P}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}, \Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)\right) \\ &\leq 2C_{\mathrm{mult}} d(\tilde{\epsilon})^2 \cdot \left(\log_2(3/\epsilon) + 3/2\log_2(d(\tilde{\epsilon})) + \log_2(\kappa)\right) + 2M\left(\mathrm{P}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}, \Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)\right). \end{aligned}$$

Next, by Lemma 3.6(b)(ii) in combination with Proposition B.3 as well as Assumption 4.1 and Assumption 4.2 we have that

$$\begin{aligned} &M\left(\mathrm{P}\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}, \Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right)\right) \\ &\leq M\left(\Phi^{\mathbf{B}}_{\mathrm{inv};\tilde{\epsilon},\epsilon'}\right) + M\left(\Phi^{\mathbf{f}}_{\tilde{\epsilon},\epsilon''}\right) \\ &\quad + 8d(\tilde{\epsilon})^2 \max\left\{C_L^{\mathbf{u}} \log_2(\log_2(1/\epsilon'))\left(\log_2(1/\epsilon') + \log_2(\log_2(1/\epsilon')) + \log_2(d(\tilde{\epsilon}))\right) + B_L(\tilde{\epsilon},\epsilon'''), F_L(\tilde{\epsilon},\epsilon'')\right\} \\ &\leq C_B \log_2(1/\epsilon')\log_2^2(\log_2(1/\epsilon'))d(\tilde{\epsilon})^3 \cdot \left(\log_2(1/\epsilon') + \log_2(\log_2(1/\epsilon')) + \log_2(d(\tilde{\epsilon}))\right) \\ &\quad + 8d(\tilde{\epsilon})^2 \max\left\{C_L^{\mathbf{u}} \log_2(\log_2(1/\epsilon'))\left(\log_2(1/\epsilon') + \log_2(\log_2(1/\epsilon')) + \log_2(d(\tilde{\epsilon}))\right) + B_L(\tilde{\epsilon},\epsilon'''), F_L(\tilde{\epsilon},\epsilon'')\right\} \\ &\quad + 2B_M(\tilde{\epsilon},\epsilon''') + F_M(\tilde{\epsilon},\epsilon'') \\ &\leq C_M^{\mathbf{u}} d(\tilde{\epsilon})^2 \cdot \Bigg(d(\tilde{\epsilon})\log_2(1/\epsilon)\log_2^2(\log_2(1/\epsilon))\big(\log_2(1/\epsilon) + \log_2(\log_2(1/\epsilon)) + \log_2(d(\tilde{\epsilon}))\big)... \\ &\qquad\qquad ... + B_L(\tilde{\epsilon},\epsilon''') + F_L(\tilde{\epsilon},\epsilon'')\Bigg) \quad + 2B_M(\tilde{\epsilon},\epsilon''') + F_M(\tilde{\epsilon},\epsilon''), \end{aligned}$$

for a suitably chosen constant $C_M^{\mathbf{u}} > 0$. This shows the claim.

B.2 Proof of Theorem 4.5

Defining $\tilde{\epsilon}'' := \tilde{\epsilon} \Big/ \left(8d(\tilde{\epsilon}') \cdot (2\max\{1, C_{\mathrm{rhs}}, 1/C_{\mathrm{coer}}\})^2\right)$, we set with Assumption 4.4

$$\Phi^{d(\tilde{\epsilon}')} := \mathrm{P}\left(\Phi^{\mathrm{rb}}_{1,\tilde{\epsilon}''}, \Phi^{\mathrm{rb}}_{2,\tilde{\epsilon}''}, \dots, \Phi^{\mathrm{rb}}_{d(\tilde{\epsilon}'),\tilde{\epsilon}''}\right).$$

We define $Z := \max\{(2\max\{1, C_{\mathrm{rhs}}, 1/C_{\mathrm{coer}}\})^2 + 1, C_\infty d(\tilde{\epsilon}')^{q_\infty+1}\}$ and set

$$\Phi^u_{\tilde{\epsilon}} := \Phi^{Z,1,d(\tilde{\epsilon}),1}_{\mathrm{mult};\frac{\tilde{\epsilon}}{4}} \odot \mathrm{P}\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}}, \Phi^{d(\tilde{\epsilon}')}\right),$$

which is a NN with n-dimensional input and $kd(\tilde{\epsilon}')$-dimensional output. Let $y \in \mathcal{Y}$. Then, by the triangle inequality, we derive

$$\begin{aligned} &\left\|u_y - \mathrm{R}^{\mathcal{Y}\times\Omega}_{\varrho}(\Phi^u)(y,\cdot)\right\|_{\mathcal{H}} \\ &\leq \left\|u_y - \sum_{i=1}^{d(\tilde{\epsilon}')}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}}\right)(y)\right)_i \psi_i\right\|_{\mathcal{H}} + \left\|\sum_{i=1}^{d(\tilde{\epsilon}')}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}}\right)(y)\right)_i \psi_i - \mathrm{R}^{\mathcal{Y}\times\Omega}_{\varrho}(\Phi^u_{\tilde{\epsilon}})(y,\cdot)\right\|_{\mathcal{H}} =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

We first estimate I. By Theorem 4.3, Equation (2.9), Equation (2.7), and Equation (2.4) we obtain

$$\mathrm{I} \leq \left\|u_y - u^{\mathrm{rb}}_{\tilde{\epsilon}'}(y)\right\|_{\mathcal{H}} + \left\|u^{\mathrm{rb}}_{\tilde{\epsilon}'}(y) - \sum_{i=1}^{d(\tilde{\epsilon}')}\left(\mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}}\right)(y)\right)_i \psi_i\right\|_{\mathcal{H}} \leq \frac{\tilde{\epsilon}}{4} + \left|\mathbf{u}^{\mathrm{rb}}_{y,\tilde{\epsilon}'} - \mathrm{R}^{\mathcal{Y}}_{\varrho}\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}}\right)(y)\right| \leq \frac{\tilde{\epsilon}}{2}.$$

Now we turn our attention to the estimation of II. We have that

$$\mathrm{II} \leq \left\| \sum_{i=1}^{d(\tilde{\epsilon}')} \left(\mathrm{R}_{\varrho}^{\mathcal{Y}} \left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}} \right)(y) \right)_i \psi_i - \sum_{i=1}^{d(\tilde{\epsilon}')} \mathrm{R}_{\varrho}^{\mathcal{Y}} \left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}}(y) \right)_i \mathrm{R}_{\varrho}^{\Omega} \left(\Phi^{\mathrm{rb}}_{i,\tilde{\epsilon}''} \right) \right\|_{\mathcal{H}}$$
$$+ \left\| \sum_{i=1}^{d(\tilde{\epsilon}')} \mathrm{R}_{\varrho}^{\mathcal{Y}} \left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}}(y) \right)_i \mathrm{R}_{\varrho}^{\Omega} \left(\Phi^{\mathrm{rb}}_{i,\tilde{\epsilon}''} \right) - \mathrm{R}_{\varrho}^{\mathcal{Y}\times\Omega}(\Phi^{u}_{\tilde{\epsilon}})(y,\cdot) \right\|_{\mathcal{H}} =: \mathrm{II}(a) + \mathrm{II}(b).$$

First of all, by Theorem 4.3(v) and Assumption 4.4 we have that

$$\mathrm{II}(a) \leq d(\tilde{\epsilon}') \left| \mathrm{R}_{\varrho}^{\mathcal{Y}} \left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}} \right)(y) \right| \max_{i=1,\dots,d(\tilde{\epsilon}')} \left\| \psi_i - \mathrm{R}_{\varrho}^{\Omega} \left(\Phi^{\mathrm{rb}}_{i,\tilde{\epsilon}''} \right) \right\|_{\mathcal{H}}$$
$$\leq d(\tilde{\epsilon}') \cdot \left((2\max\{1, C_{\mathrm{rhs}}, 1/C_{\mathrm{coer}}\})^2 + 1 \right) \frac{\tilde{\epsilon}}{8d(\tilde{\epsilon}')\,(2\max\{1, C_{\mathrm{rhs}}, 1/C_{\mathrm{coer}}\})^2} \leq \frac{\tilde{\epsilon}}{4}.$$

By Assumption 4.4(iv),

$$\sup_{\mathbf{x}\in\Omega} \left| \mathrm{R}_{\varrho}^{\Omega} \left(\Phi^{d(\tilde{\epsilon}')} \right)(\mathbf{x}) \right| \leq d(\tilde{\epsilon}') \max_{i=1,\dots,d(\tilde{\epsilon}')} \left\| \mathrm{R}_{\varrho}^{\Omega} \left(\Phi^{\mathrm{rb}}_{i,\epsilon''} \right) \right\|_{L^{\infty}(\Omega,\mathbb{R}^k)} \leq d(\tilde{\epsilon}') C_{\infty} d(\tilde{\epsilon}')^{q_{\infty}} \leq Z.$$

Moreover, by Theorem 4.3(v), it follows that

$$\sup_{y\in\mathcal{Y}} \left| \mathrm{R}_{\varrho}^{\mathcal{Y}} \left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}} \right)(y) \right| \leq (2\max\{1, C_{\mathrm{rhs}}, 1/C_{\mathrm{coer}}\})^2 + 1 \leq Z.$$

Hence, by Proposition 3.7, $\mathrm{II}(b) \leq \tilde{\epsilon}/4$. Combining the estimates on $\mathrm{I}, \mathrm{II}(a), \mathrm{II}(b)$ yields (4.1). We now estimate the size of $\Phi^{u}_{\tilde{\epsilon}}$. Proposition 3.7 implies that

$$L\left(\Phi^{Z,1,d(\tilde{\epsilon}),1}_{\mathrm{mult};\frac{\tilde{\epsilon}}{4}} \right) \leq C_{\mathrm{mult}} \cdot \left(\log_2(4/\tilde{\epsilon}) + \log_2(d(\tilde{\epsilon}')) + \log_2(Z) \right) \leq C_1(\log_2(1/\tilde{\epsilon}) + \log_2(d(\tilde{\epsilon}'))), \tag{B.4}$$

for a constant $C_1 > 0$, and

$$M\left(\Phi^{Z,1,d(\tilde{\epsilon}),1}_{\mathrm{mult};\frac{\tilde{\epsilon}}{4}} \right) \leq C_{\mathrm{mult}} \cdot \left(\log_2(4/\tilde{\epsilon}) + \log_2(d(\tilde{\epsilon}')) + \log_2(Z) \right) d(\tilde{\epsilon}')$$
$$\leq C_2 d(\tilde{\epsilon}') \cdot (\log_2(1/\tilde{\epsilon}) + \log_2(d(\tilde{\epsilon}'))), \tag{B.5}$$

for a constant $C_2 > 0$. Before we finish estimating the size of $\Phi^{u}_{\tilde{\epsilon}}$, we estimate the size of $\Phi^{d(\tilde{\epsilon}')}$. Towards this goal, note that for two real numbers $a, b \geq 2$ and $q \in \mathbb{N}$, we have

$$(a+b)^q \leq a^q b^q. \tag{B.6}$$

Thus, by the choice of $\tilde{\epsilon}''$ in combination with (B.6) and for some $C_3 > 0$,

$$L\left(\Phi^{d(\tilde{\epsilon}')} \right) \leq C_L^{\mathrm{rb}} \log_2^{q_L}(1/\tilde{\epsilon}'') \log_2^{q_L}(d(\tilde{\epsilon}')) \leq C_3 \log_2^{2q_L}(d(\tilde{\epsilon}')) \log_2^{q_L}(1/\tilde{\epsilon}) \tag{B.7}$$

and, for some $C_4 > 0$ by Lemma 3.6(b)(ii) in combination with (B.6),

$$M\left(\Phi^{d(\tilde{\epsilon}')} \right) \leq (2+4k) \max\{C_L^{\mathrm{rb}}, C_M^{\mathrm{rb}}\} \log_2^{\max\{q_L,q_M\}}(1/\tilde{\epsilon}'') d(\tilde{\epsilon}') \log_2^{\max\{q_L,q_M\}}(d(\tilde{\epsilon}'))$$
$$\leq C_4 k d(\tilde{\epsilon}') \log_2^{2\max\{q_L,q_M\}}(d(\tilde{\epsilon}')) \log_2^{\max\{q_L,q_M\}}(1/\tilde{\epsilon}). \tag{B.8}$$

Moreover, Theorem 4.3 yields

$$L\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde{\epsilon}',\frac{\tilde{\epsilon}}{4}} \right) \leq C_L^{\mathbf{u}} \log_2(\log_2(4/\tilde{\epsilon})) \cdot \left(\log_2(4/\tilde{\epsilon}) + \log_2(\log_2(4/\tilde{\epsilon})) + \log_2(d(\tilde{\epsilon}')) \right) + H_L(\tilde{\epsilon})$$
$$\leq C_5 \cdot (\log_2(1/\tilde{\epsilon}) \log_2(\log_2(1/\tilde{\epsilon})) + d(\tilde{\epsilon}') \log_2(1/\tilde{\epsilon}) + H_L(\tilde{\epsilon})), \tag{B.9}$$

for a constant $C_5 > 0$.

Similarly,

$$\begin{aligned} M\left(\Phi^{\mathbf{u},\mathrm{rb}}_{\tilde\epsilon',\frac{\tilde\epsilon}{4}}\right) &\leq C^{\mathbf{u}}_M d(\tilde\epsilon')^2 \cdot \bigg(d(\tilde\epsilon')\log_2(4/\tilde\epsilon)\log_2^2(\log_2(4/\tilde\epsilon))\big(\log_2(4/\tilde\epsilon) + \log_2(\log_2(4/\tilde\epsilon)) \\ &\quad + \log_2(d(\tilde\epsilon'))\big) + H_L(\tilde\epsilon)\bigg) + H_M\left(\tilde\epsilon\right), \\ &\leq C_6 d(\tilde\epsilon')^3 \cdot (\log_2^3(1/\tilde\epsilon) + \log_2(d(\tilde\epsilon'))) + d(\tilde\epsilon')^2 H_L(\tilde\epsilon) + H_M(\tilde\epsilon), \end{aligned} \tag{B.10}$$

for a constant $C_6 > 0$. Combining the estimates (B.4), (B.7) and (B.9) now yields (4.2) and combining (B.5), (B.8), and (B.10) implies (4.3).