A QUALITATIVE UNCERTAINTY PRINCIPLE FOR FUNCTIONS GENERATING A GABOR FRAME ON LCA GROUPS

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Abstract. Let G be a locally compact abelian group. In this paper we study in which way the qualitative uncertainty principle is modified when we consider only functions $f \in L^2(G)$ which generate a Gabor frame associated with a uniform lattice K in G. This provides us with sharp lower bounds for the measure of the support of such functions and their Plancherel transforms.

1. Introduction

Let G be a locally compact abelian (LCA) group equipped with a Haar measure m_G. The dual group is denoted by $\widehat{G}$. Let the Haar measure on $\widehat{G}$, μ_G, be normalized so that the Plancherel formula holds. The Fourier transform $\hat{f}$ of any function $f \in L^1(G)$ is defined by

$$\hat{f}(\omega) = \int_G f(t)\overline{\omega(t)}\, dm_G(t).$$

The transformation $f \mapsto \hat{f}$, $L^1(G) \to C_0(\widehat{G})$ extends to a Hilbert space isomorphism of $L^2(G)$ onto $L^2(\widehat{G})$, the so-called Plancherel isomorphism. The Plancherel transform shall also be denoted by $\hat{f}$. For $f \in L^2(G)$, let $\text{supp } f = \{x \in G : f(x) \neq 0\}$ and $\text{supp } \hat{f} = \{\omega \in \widehat{G} : \hat{f}(\omega) \neq 0\}$.

Uncertainty principles were studied extensively during the last fifty years. Although there exists an abundance of different types of them the common statement is that a nonzero function and its Fourier transform cannot both be sharply localized. The first qualitative uncertainty principle was derived in 1973 by Matolcsi and Szücs [MS73]. It states the following. Given a LCA group G, for $f \in L^2(G)$, we have

$$m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) \geq 1.$$

For L^1-functions this result was proven by Smith [Smi90]. Following Benedicks [Ben85] the appropriate formulation of the qualitative uncertainty principle which seems to be the right setting for LCA groups G and which is referred to as QUP is

$$m_G(\text{supp } f) < m_G(G) \quad \text{and} \quad \mu_G(\text{supp } \hat{f}) < \mu_G(\widehat{G}) \quad \Longrightarrow \quad f = 0.$$

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Hogan [Hog88] proved that the QUP holds for a non-compact non-discrete LCA group with connected component G_0 if and only if G_0 is non-compact. An infinite compact abelian group satisfies the QUP if and only if it is connected (see [Hog93]). There exists an abundance of extensions. For an excellent survey we refer to [FS97].

Frames were introduced in 1952 by Duffin and Schaeffer [DS52]. Since then, they have become a major tool in signal and image processing, data compression and sampling theory. Given a LCA group G, a sequence $\{g_i\}_{i\in I}$ in $L^2(G)$ is a *frame*, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, g_i\rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in L^2(G).$$

A and B are called the *frame bounds*. In this paper we will focus on *Gabor frames* associated with some uniform lattice K in G, which are frames of the form

$$S(f,K) := \{x \mapsto \gamma(x)f(xk) : (k,\gamma) \in K \times \widehat{G/K}\},$$

where $f \in L^2(G)$. Speaking of a *uniform lattice* we mean a discrete and cocompact subgroup. Further recall that provided H is a closed subgroup of G, we can identify $\widehat{G/H}$ with the annihilator $A(H,\widehat{G}) = \{\omega \in \widehat{G} : \omega(h) = 1 \text{ for all } h \in H\}$ of H in $\widehat{G}$ (compare [HR63/70, Theorem 23.25]).

In dealing with frames, it is especially interesting to know in which way uncertainty principles are modified when we consider only functions generating a frame. For example studying the QUP for such functions f establishes lower bounds for $m_G(\text{supp } f)$ and $\mu_G(\text{supp } \hat{f})$. This helps us to understand to which extent we can localize in time and frequency when constructing a frame. An important example for this approach is the Balian-Low Theorem, which was originally stated by Balian [Bal81] and independently by Low [Low85]. It shows that restriction to functions generating a frame maximizes the classical Heisenberg uncertainty principle. The first approach on other versions of uncertainty principles was made by Korn [Kor00]. He studied different types of uncertainty principles, e.g. the uncertainty principles of Donoho-Stark [DS89] and Landau-Pollak-Slepian [LP61, PS61], according to their modification when considering only functions in $L^2(\mathbb{R})$ which generate a Gabor frame.

In this paper we investigate in which way the QUP is modified when we consider only functions in $L^2(G)$, G a LCA group, which generate a Gabor frame associated with a uniform lattice in G.

In the second section of this paper we start with some basic results. Let G be a LCA group and let K be a uniform lattice in G. For $f \in L^2(G)$, we prove that provided $S(f,K)$ forms a frame the measure of the support of f and $\hat{f}$ is bounded from below by the measure of fundamental domains. In the case of a LCA group G with non-compact connected component, which in particular includes $G = \mathbb{R}^n$, one of those bounds always equals infinity, whereas the measure of a fundamental domain is a sharp bound for the measure of the other support. Hence the interesting case to look at are LCA groups with a compact connected component. Moreover, we deal with the question, whether one can classify all functions $f \in L^2(G)$ for which $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f})$ attains the infimum, i.e. $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) = 1$. We give a complete answer in the general situation as well as in the situation where we restrict to functions $f \in L^2(G)$ for which $S(f, K)$ forms a frame for $L^2(G)$.

To obtain more precise exact bounds in the case of LCA groups with compact connected component, we first deal with compact abelian groups G in Section 3. Provided that K is a finite subgroup of G which is contained in G_0 and $f \in L^2(G)$ we calculate lower bounds for the measure of the support of f and $\hat{f}$ if $S(f, K)$ is a frame and prove that they are sharp. In particular, we show that $m_G(\text{supp } f) = m_G(G)$ if $\mu_G(\text{supp } \hat{f}) < \infty$ and that $\mu_G(\text{supp } \hat{f}) \geq |K|\mu_G(1)$, where 1 denotes the neutral element in $\widehat{G}$. If the hypothesis is not fulfilled we give examples of functions which do not satisfy these bounds.

In Section 4 we use these results to obtain lower bounds for general LCA groups which have a compact connected component. Provided that there exist some compact open subgroup H of G such that $H \cap K$ is contained in G_0 and $f \in L^2(G)$ with supp $\hat{f}$ compact we have $m_G(\text{supp } f) \geq [G : HK] m_G(H)$ and a similar result for the dual side. These bounds are sharp. Again there exist examples of functions which do not satisfy the bounds if the hypotheses are not fulfilled. Moreover, we obtain a sharp bound for the product $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f})$ which does not depend on some compact open subgroup H. We finish with a result for the case that supp $\hat{f}$ is not compact.

2. Basic results

Let G be a LCA group, which we always assume to be second countable, and let K be a uniform lattice in G. A *fundamental domain* for K is a Borel subset S of G such that every $x \in G$ can be uniquely written in the form $x = sk$ where $s \in S$ and $k \in K$. The existence of a fundamental domain for K is always guaranteed by [KK98, Lemma 2]. A useful tool for studying frames is the so-called *Zak transform* associated with K of some function $f \in L^2(G)$, which is defined on $G \times \widehat{G}$ by

$$Zf(x, \omega) = \sum_{k \in K} f(xk)\omega(k).$$

By [KK98, Lemma 3], the Zak transform $Z : L^2(G) \to L^2(S \times \Omega)$ is an isometry, where S and Ω be fundamental domains for K and $\widehat{G/K}$ in G and $\widehat{G}$, respectively. The following property of this transform will be used very often throughout the paper, because it provides us with a condition for $S(f, K)$ being a frame for $L^2(G)$ which is easy to check. For a proof compare [Gro98, Corollary 6.4.4].

Theorem 2.1. *Let G be a LCA group, let K be a uniform lattice in G and let $f \in L^2(G)$. Then $S(f, K)$ is a frame for $L^2(G)$ with frame bounds A and B if and only if $A \leq |Zf|^2 \leq B$ a.e.. In this case $S(f, K)$ is an exact frame.*

Let us begin with a simple lemma which shows, provided $S(f, K)$ forms a frame, that the measure of the support of f and $\hat{f}$ is bounded from below by the measure of fundamental domains. In the sequel we will use the following notation. For $A, B \subseteq G$, $A = B$ almost everywhere always means $\chi_A = \chi_B$ almost everywhere or equivalently the measure of the symmetric difference of A and B equals zero, where $\chi_\cdot$ denotes the characteristic function.

Lemma 2.2. *Let G be a LCA group, let K be a uniform lattice in G and let $f \in L^2(G)$. Let S and Ω be fundamental domains for K and $\widehat{G/K}$ in G and $\widehat{G}$, respectively. If $S(f,K)$ is a frame for $L^2(G)$, we have*

$$m_G(\text{supp } f) \geq m_G(S) \quad \textit{and} \quad \mu_G(\text{supp } \hat{f}) \geq \mu_G(\Omega).$$

Proof. If the first claim is proven, the second follows immediately by just using the Plancherel isomorphism and the same arguments on the dual side. To prove $m_G(\text{supp } f) \geq m_G(S)$, let S be some arbitrarily fixed fundamental domain for K in G. On the one hand supp $(\gamma(\cdot)f(\cdot k)) = k^{-1} \cdot \text{supp } f$ for each $(k, \gamma) \in K \times \widehat{G/K}$. Therefore, since $S(f,K)$ is a frame for $L^2(G)$, we have $G = \bigcup_{k \in K} k \cdot \text{supp } f$ a.e.. This implies immediately that $m_G(xK \cap \text{supp } f) \neq 0$ for almost all $x \in G$. On the other hand we know that G is the disjoint union of the sets $k \cdot S$, where k runs through K. By normalizing the Haar measure on G/K, $m_{G/K}$, in an appropriate way, Weil's formula yields

$$m_G(S) = \int_G \chi_S(x) dm_G(x) = \int_{G/K} \sum_{k \in K} \chi_S(xk) dm_{G/K}(xK) = \int_{G/K} 1 dm_{G/K}(xK).$$

So we get

$$m_G(\text{supp } f) = \int_{G/K} \sum_{k \in K} \chi_{\text{supp } f}(xk) dm_{G/K}(xK) \geq \int_{G/K} 1 dm_{G/K}(xK) = m_G(S).$$

□

Concerning the measure of the support of f and $\hat{f}$ we start with the situation of a LCA group G with connected component G_0 being not compact. Notice that this case in particular includes $G = \mathbb{R}^n$ as a special case. We will see that the measure of at least one of supp f or supp $\hat{f}$ is infinite.

Proposition 2.3. *Let G be a LCA group such that G_0 is non-compact.*

(i) *For $f \in L^2(G)$, $f \neq 0$ we have $m_G(\text{supp } f) = \infty$ or $\mu_G(\text{supp } \hat{f}) = \infty$.*
(ii) *If K is a uniform lattice in G, then there exist functions $g, h \in L^2(G)$ such that $S(g,K)$ and $S(h,K)$ are frames for $L^2(G)$ and which satisfy $m_G(\text{supp } g) = m_G(S)$, $\mu_G(\text{supp } \hat{g}) = \infty$, $m_G(\text{supp } h) = \infty$ and $\mu_G(\text{supp } \hat{h}) = \mu_G(\Omega)$.*

Proof. The first claim follows immediately from [Hog88, Theorem 1]. For the second part let S and Ω be fixed fundamental domains for K and $\widehat{G/K}$ in G and $\widehat{G}$, respectively. We define g and h by $g := \chi_S$ and $h := \chi_\Omega$. Obviously, we have $m_G(\text{supp } g) = m_G(S)$. Moreover, [Hog88, Theorem 1] implies $\mu_G(\text{supp } \hat{g}) = \infty$. Finally, for each $(x, \omega) \in S \times \Omega$, we obtain $|Zg(x, \omega)| = |\omega(e)| = 1$. By Theorem 2.1, the set $S(g,K)$ is a frame for $L^2(G)$. This shows that g fulfills the assertion. Using the Plancherel isomorphism and the same arguments on the dual side, we obtain the claim for h. □

This result provides us with sharp bounds if G_0 is non-compact (see also Lemma 2.2). Therefore in the following we will focus on LCA groups G whose connected component is compact, for example $G = \mathbb{Z}^n$, $\mathbb{T}^n$ or G finite.

Let G be a LCA group. Since for each $f \in L^2(G)$, $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) \geq 1$ [MS73], it is an interesting question whether we can classify all functions $f \in L^2(G)$ for which $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f})$ attains the infimum. The following theorem gives a complete answer.

Theorem 2.4. *Let G be a LCA group and let $f \in L^2(G)$, $f \neq 0$. Then the following conditions are equivalent:*

(i) $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) = 1$.

(ii) *There exists $c > 0$ such that $|f(x)| = c$ and $|\hat{f}(\omega)| = m_G(\text{supp } f) \cdot c$ for almost all $(x,\omega) \in \text{supp } f \times \text{supp } \hat{f}$ and $m_G(\text{supp } f) < \infty$.*

(iii) *There exist a compact open subgroup H of G and some point $(x_0,\omega_0) \in G \times \widehat{G}$ such that $\text{supp } f = x_0H$ and $\text{supp } \hat{f} = \omega_0\widehat{G/H}$ a.e..*

Moreover, (i) *implies that G_0 is compact.*

Proof. The last claim follows from [Hog93, Corollary 2.5].

Provided that $m_G(\text{supp } f) < \infty$, we have

$$\|f\|_2^2 \overset{(1)}{\leq} \mu_G(\text{supp } \hat{f})\|\hat{f}\|_\infty^2 \overset{(2)}{\leq} \mu_G(\text{supp } \hat{f})\|f\|_1^2 \overset{(3)}{\leq} \mu_G(\text{supp } \hat{f})m_G(\text{supp } f)\|f\|_2^2.$$

Hence (i) holds if and only if we have equality in (1), (2) and (3) and moreover $m_G(\text{supp } f) < \infty$. In the following we will study the inequalities (1), (2) and (3) more detailed. For this, suppose that $m_G(\text{supp } f) < \infty$. We start examining inequality (1). Note that

$$\|f\|_2^2 = \|\hat{f}\|_2^2 = \int_{\text{supp } \hat{f}} |\hat{f}(\omega)|^2 d\mu_G(\omega)$$

and

$$\mu_G(\text{supp } \hat{f})\|\hat{f}\|_\infty^2 = \mu_G(\text{supp } \hat{f}) \max_{\omega \in \text{supp } \hat{f}} |\hat{f}(\omega)|^2.$$

Thus we have equality in (1) if and only if there exists some $d > 0$ such that $|\hat{f}(\omega)| = d$ for almost all $\omega \in \text{supp } \hat{f}$. Concerning inequality (3), it always hold

$$\|f\|_1^2 = \|f\chi_{\text{supp } f}\|_1^2 \text{ and } m_G(\text{supp } f)\|f\|_2^2 = \|f\|_2^2\|\chi_{\text{supp } f}\|_2^2.$$

So we get equality in (3) if and only if there exists $c > 0$ with $|f(x)| = c$ for almost all $x \in \text{supp } f$. Now suppose that we already have equality in (1) and (3). Then

$$\|\hat{f}\|_\infty^2 = d^2 \text{ and } \|f\|_1^2 = \left(\int_{\text{supp } f} |f(x)| dm_G(x)\right)^2 = m_G(\text{supp } f)^2 c^2.$$

This implies that provided we have equality in (1) and (3), we also have equality in (2) if and only if $d = m_G(\text{supp } f)c$. This proves (i) $\Leftrightarrow$ (ii).

Next suppose that (iii) holds. Since H is a compact open subgroup, we have $m_G(H)\mu_G(\widehat{G/H}) = 1$, which immediately implies (i).

Finally suppose there exists $c > 0$ such that $|f(x)| = c$ and $|\hat{f}(\omega)| = m_G(\text{supp } f) \cdot c$ for almost all $(x, \omega) \in \text{supp } f \times \text{supp } \hat{f}$ and $m_G(\text{supp } f) < \infty$. We will show that this implies (iii). Without loss of generality we can assume that $c = 1$. For almost all $\omega \in \text{supp } \hat{f}$, we have

$$\left| \int_{\text{supp } f} f(x)\overline{\omega(x)} dm_G(x) \right| = |\hat{f}(\omega)| = m_G(\text{supp } f) = \int_{\text{supp } f} |f(x)\overline{\omega(x)}| dm_G(x).$$

Now [HR63/70, Theorem 12.4] implies that there exists some constant λ_ω such that

$$f(x)\overline{\omega(x)} = \lambda_\omega \text{ for almost all } (x, \omega) \in \text{supp } f \times \text{supp } \hat{f}.$$

Hence, for almost all $(x, \omega) \in \text{supp } f \times \text{supp } \hat{f}$, we obtain

$$\hat{f}(\omega) = \int_{\text{supp } f} f(x)\overline{\omega(x)} dm_G(x) = \lambda_\omega m_G(\text{supp } f) = f(x)\overline{\omega(x)} m_G(\text{supp } f). \tag{4}$$

Without loss of generality we can assume that $1 \in \text{supp } \hat{f}$, since otherwise we may consider $g \in L^2(G)$ with $\hat{g} = \hat{f}(\omega_0 \cdot)$ for some $\omega_0 \in \text{supp } \hat{f}$. Then $1 \in \text{supp } \hat{g}$ and $g(x) = \overline{\omega_0(x)} f(x)$. Hence $\text{supp } g = \text{supp } f$ and $\text{supp } \hat{g} = \overline{\omega_0} \cdot \text{supp } \hat{f}$.

Therefore equation (4) implies $\hat{f}(1) = f(x) m_G(\text{supp } f)$ for almost all $x \in \text{supp } f$. Thus we can assume that f is constant on its support, i.e. $f(x) = c\chi_{\text{supp } f}(x)$ for some $|c| = 1$, and hence $\hat{f}(\omega) = c \int_{\text{supp } f} \overline{\omega(x)} dm_G(x)$. Now let $\omega \in \widehat{G}$. If $\omega \notin \text{supp } \hat{f}$, by the previous equation, there has to exist some $x \in \text{supp } f$ with $\omega(x) \neq 1$. If $\omega \in \text{supp } \hat{f}$, then

$$\left| c \int_{\text{supp } f} \overline{\omega(x)} dm_G(x) \right| = |\hat{f}(\omega)| = m_G(\text{supp } f),$$

which implies $\omega|_{\text{supp } f} \equiv 1$. This proves

$$\text{supp } \hat{f} = A(\text{supp } f, \widehat{G}),$$

which shows that $\text{supp } \hat{f}$ coincides almost everywhere with a subgroup of $\widehat{G}$. This subgroup has to be compact, since otherwise $\mu_G(\text{supp } \hat{f})$ would not be finite. Moreover, it has to be open, because the measure of the support of $\hat{f}$ has to be non-zero.

Now we turn our attention to the support of f. The smallest closed subgroup containing $\text{supp } f$, which we will denote by J, satisfies $A(J, \widehat{G}) = A(\text{supp } f, \widehat{G})$. Since $m_G(J)\mu_G(A(J, \widehat{G}))$ as well as $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f})$ equals 1, we have $\text{supp } f = J$ a.e.. □

Now we restrict ourselves to functions $f \in L^2(G)$ such that $S(f, K)$ forms a frame for $L^2(G)$ and ask the same question.

Theorem 2.5. *Let G be a LCA group and let K be a uniform lattice in G. Let $f \in L^2(G)$ be such that $S(f,K)$ is a frame for $L^2(G)$. Then the following conditions are equivalent.*

(i) $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) = 1$.
(ii) *There exist a closed subgroup S of G, which is a fundamental domain for K in G, and some $(x_0, \omega_0) \in G \times \widehat{G}$ such that* $\text{supp } f = x_0 S$ *and* $\text{supp } \hat{f} = \omega_0 \widehat{G/S}$ *a.e.*.

Proof. First let $f \in L^2(G)$ be such that $S(f,K)$ is a frame for $L^2(G)$ and such that it satisfies $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) = 1$. Theorem 2.4 implies that there exist a compact open subgroup H of G and some point $(x_0, \omega_0) \in G \times \widehat{G}$ such that $\text{supp } f = x_0 H$ and $\text{supp } \hat{f} = \omega_0 \widehat{G/H}$ a.e.. Since $S(f,K)$ is a frame for $L^2(G)$, there exists a fundamental domain S and Ω for K and $\widehat{G/K}$ in G and $\widehat{G}$, respectively, with $S \subseteq \text{supp } f$ and $\Omega \subseteq \text{supp } \hat{f}$. This implies $S \subseteq x_0 H$ and $\Omega \subseteq \omega_0 \widehat{G/H}$. Since it is well-known that $m_G(S)\mu_G(\Omega) = 1$ and $m_G(x_0 H)\mu_G(\omega_0 \widehat{G/H}) = 1$, we obtain $S = x_0 H$ and $\Omega = \omega_0 \widehat{G/H}$ a.e.. Hence also H is a fundamental domain for K in G.

Since S is compact and open, the implication (ii) $\Rightarrow$ (i) follows immediately from Theorem 2.4. □

Corollary 2.6. *Let G be a LCA group and let K be a uniform lattice in G. Then the following conditions are equivalent.*

(i) *There exists some $f \in L^2(G)$ such that $S(f,K)$ is a frame for $L^2(G)$ and* $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) = 1$.
(ii) *There exists a closed subgroup S of G such that $G = K \times S$.*
(iii) *There exists a closed subgroup Ω of $\widehat{G}$ such that $\widehat{G} = \widehat{G/K} \times \Omega$.*

Proof. This follows immediately from Theorem 2.5. □

3. Compact groups

Let G be a LCA group. An element $x \in G$ is said to be *compact*, if the smallest closed subgroup of G containing x is compact. Let G^c denote the set of compact elements in G, which is a closed subgroup of G [HR63/70, Theorem 9.10].

In this section we consider only compact abelian groups G. For such G, let m_G denote some Haar measure on G. In the next section we will prove generalizations to LCA groups of the results obtained here by reducing to the situation of compact groups.

We start with a basic property of functions on compact abelian groups, which gives rise to lower bounds for the measure of functions which generate a Gabor frame associated with a uniform lattice. Moreover, it will turn out to be useful in the next section.

Lemma 3.1. *Let G be a compact abelian group, and let $f \in L^2(G)$ such that* supp $\hat{f}$ *is finite. Then either $m_G(\text{supp } f) = m_G(G)$ or f vanishes on a coset of some open subgroup.*

Proof. By hypothesis, we have $\hat{f} = \sum_{j=1}^n \alpha_j \chi_{\omega_j}$ for certain $\omega_j \in \widehat{G}$ and $\alpha_j \in \mathbb{C}$. Then $f = \sum_{j=1}^n \alpha_j \omega_j$. Since G is a projective limit of Lie groups [HR63/70, (28.61) (c)], there exists a closed subgroup C of G such that G/C is a Lie group and $\omega_j \in \widehat{G/C}$ for all j. Let m_C be induced by the Haar measure on G and let the Haar measure on G/C be normalized so that Weil's formula holds. Define g on G/C by $g(xC) = \int_C f(xc) dm_C(c)$. An easy calculation shows that $\hat{g} = \hat{f}|_{\widehat{G/C}}$. Now, $(G/C)_0 = \mathbb{T}^m$ for some $m \in \mathbb{N}$, whence $G/C = F \cdot \mathbb{T}^m$ for some finite set F. For any $a \in F$, $z \mapsto g(az) = \sum_{j=1}^n \alpha_j \omega_j(a) \omega_j(z)$ is an analytic function on $\mathbb{T}^m$. However, a nonzero analytic function on $\mathbb{T}^m$ cannot vanish on a set of positive measure. Thus, for each $a \in F$, either $g|_{a\mathbb{T}^m} = 0$ or $g|_{a\mathbb{T}^m} \neq 0$ a.e.. Let $H := q^{-1}(\mathbb{T}^m)$, where $q : G \to G/C$ denotes the quotient map. Then $G = SH$, S finite, and, for each $s \in S$, either $g \circ q|_{sH} = 0$ or $g \circ q|_{sH} \neq 0$ a.e.. Then the same holds for f, since $f = g \circ q$. Indeed, $\hat{f} = \widehat{g \circ q}$ on $\widehat{G/C}$, $\hat{f} = 0$ on $\widehat{G} \backslash \widehat{G/C}$, and $\widehat{g \circ q} = 0$ on $\widehat{G} \backslash \widehat{G/C}$, because $\widehat{g \circ q}(\tau) = \int_{G/C} g(xC) \int_C \overline{\tau(xc)} dm_C(c) dm_{G/C}(xC)$ and $\int_C \overline{\tau(xc)} dm_C(c) = \overline{\tau(x)} \int_C \overline{\tau(c)} dm_C(c) = 0$ whenever $\tau|_C \neq 1$. □

Let $f \in L^2(G)$ be such that f generates a Gabor frame associated with a uniform lattice. The following proposition gives exact bounds for the measure of the support of f and its Fourier transform. Obviously, each uniform lattice K in a compact abelian group is finite. In the sequel the number of elements of K is denoted by $|K|$.

Proposition 3.2. *Let G be a compact abelian group, let K be a finite subgroup of G which is contained in G_0, and let $f \in L^2(G)$ with* supp $\hat{f}$ *finite be such that $S(f, K)$ is a frame for $L^2(G)$. Then the following hold.*

(i) $m_G(\text{supp } f) = m_G(G)$.
(ii) $\mu_G(\text{supp } \hat{f}) \geq |K| \mu_G(1)$.
(iii) $m_G(\text{supp } f) \mu_G(\text{supp } \hat{f}) \geq |K|$.

Proof. It suffices to prove (i), since (ii) is Lemma 2.2 and (iii) follows immediately from (i) and (ii). For this, let $f \in L^2(G)$ such that supp $\hat{f}$ is finite and $S(f, K)$ is a frame for $L^2(G)$. Assume that $m_G(\text{supp } f) < m_G(G)$. Then, by Lemma 3.1, f vanishes on a coset of some open subgroup H, i.e. there exists some $x_0 \in G$ with $f|_{x_0 H} = 0$. Now $G_0 \subseteq H$, hence K is a subgroup of H. Thus $Zf(x_0 y, 1) = \sum_{k \in K} f(k x_0 y) = 0$ for all $y \in H$. By Theorem 2.1, this is a contradiction. □

The next question we have to deal with is whether the bounds are sharp. This is confirmed by the next result.

Proposition 3.3. *Let G be a compact abelian group and let K be a finite subgroup of G which is contained in G_0. Then there exists a function $f \in L^2(G)$ such that*

(i) *$S(f, K)$ is a frame for $L^2(G)$,*

(ii) $m_G(\text{supp } f) = m_G(G)$, *and*
(iii) $\mu_G(\text{supp } \hat{f}) = |K|\mu_G(1)$.
In particular, we have

$$m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) = |K|.$$

Proof. Let Ω be a fixed fundamental domain for $\widehat{G/K}$ in $\widehat{G}$. Then we choose $f \in L^2(G)$ such that $\hat{f} = \chi_\Omega$, which clearly satisfies (iii). Obviously, $S(\hat{f}, \widehat{G/K})$ is a frame for $L^2(\widehat{G})$. By the Plancherel isomorphism, $S(f, K)$ is a frame for $L^2(G)$. This proves (i). Now we may apply Proposition 3.2 to obtain (ii). □

It remains to study the case when K is not contained in G_0.

Proposition 3.4. *Let G be a compact abelian group and let K be a finite subgroup of G which is not contained in G_0. Then there exists a function $f \in L^2(G)$ with*
(i) $S(f, K)$ *is a frame for* $L^2(G)$,
(ii) $m_G(\text{supp } f) < m_G(G)$, *and*
(iii) $\mu_G(\text{supp } \hat{f}) = |K|\mu_G(1)$.
In particular, we have

$$m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) < |K|.$$

Proof. We start constructing a special fundamental domain for $\widehat{G/K}$ in $\widehat{G}$. First recall that $[\widehat{G} : \widehat{G/K}] = |K|$. For the sake of brevity we denote $[\widehat{G} : (\widehat{G})^c\widehat{G/K}]$ by N. Since $G_0 = \widehat{\widehat{G}/(\widehat{G})^c}$ [HR63/70, Theorem 24.17], by hypothesis, $(\widehat{G})^c$ is not contained in $\widehat{G/K}$. Hence $\frac{|K|}{N} \geq 2$. Let $\{\omega_l : l = 1, \ldots, \frac{|K|}{N}\}$ be a fundamental domain for $(\widehat{G})^c \cap \widehat{G/K}$ in $(\widehat{G})^c$ and let $\pi_1, \ldots, \pi_N$ be a representative system for the $(\widehat{G})^c\widehat{G/K}$-cosets in $\widehat{G}$. Then we set

$$\tau_i := \pi_k\omega_l, \quad \text{if } i = (k-1)\frac{|K|}{N} + l,\ 1 \leq i \leq |K|.$$

Obviously, $\{\tau_i : i = 1, \ldots, |K|\}$ is a fundamental domain for $\widehat{G/K}$ in $\widehat{G}$. Now we define the function f by its Plancherel transform $\hat{f} = \sum_{i=1}^{|K|} \lambda_i \chi_{\tau_i}$, where $\lambda_i \neq 0$, $1 \leq i \leq |K|$, are chosen later on. Notice that (iii) is satisfied automatically.

To prove (ii), we first calculate the function f itself. We obtain

$$f(x) = \sum_{i=1}^{|K|} \lambda_i \tau_i(x) = \sum_{k=1}^{N} \left[\sum_{l=1}^{\frac{|K|}{N}} \lambda_{k_l} \omega_l(x) \right] \pi_k(x),$$

where the numbers λ_{k_l} have to be chosen in an appropriate way. Since $\omega_l \in (\widehat{G})^c$ for all $1 \leq l \leq \frac{|K|}{N}$, it follows that the order of each ω_l is finite and hence $\omega_l^{-1}(1)$ is an open subgroup of G for each l. Then we define G_1 by $G_1 := \bigcap_{l=1}^{\frac{|K|}{N}} \omega_l^{-1}(1)$, which is an open subgroup with $\omega_l(x) = 1$ for all $x \in G_1$, $1 \leq l \leq \frac{|K|}{N}$. We now fix the numbers λ_{k_l} in such a way that $\sum_{l=1}^{\frac{|K|}{N}} \lambda_{k_l} = 0$ for each k and $\lambda_{k_l} \neq 0$ for all k, l. Then $f(x) = 0$ for all $x \in G_1$. Thus supp $f \subseteq G \backslash G_1$. Since $m_G(G_1) > 0$, this proves (ii).

It remains to show (i). For this, let S be a fundamental domain for K in G. As a fundamental domain for $\widehat{G/K}$ in $\widehat{G}$ we choose $\Omega := \{\overline{\tau_i} : i = 1, \ldots, |K|\}$. Then, for $(x, \overline{\tau_j}) \in S \times \Omega$, we obtain

$$|Zf(x, \overline{\tau_j})| = \left| \sum_{k \in K} \overline{\overline{\tau_j}(k)} f(xk) \right| = \left| \sum_{k \in K} \overline{\overline{\tau_j}(k)} \sum_{i=1}^{|K|} \lambda_i \tau_i(xk) \right|$$

$$= \left| \sum_{i=1}^{|K|} \lambda_i \tau_i(x) \left[\sum_{k \in K} (\overline{\tau_j} \tau_i)(k) \right] \right| = \left| \sum_{i=1}^{|K|} \lambda_i \tau_i(x) \left[|K| \chi_{\widehat{G/K}}(\overline{\tau_j} \tau_i) \right] \right| = |K||\lambda_j|.$$

Now we can apply Theorem 2.1. $\square$

Notice that we cannot find a function $f \in L^2(G)$ generating a Gabor frame associated with a finite subgroup K of G, which does not satisfy the bound on the dual side $\mu_G(\text{supp } \hat{f}) \geq |K| \mu_G(1)$ because of Lemma 2.2.

4. General LCA groups with compact connected component

Throughout this section let G be a LCA group with compact connected component equipped with a Haar measure m_G and let K be a uniform lattice in G.

Let H be some open compact subgroup of G. The Haar measure m_H on such a subgroup shall always be induced by the Haar measure on G. We start by choosing a special fundamental domain S_H for K in G with respect to H, which will make the following calculations much easier. Since $H \cap K$ is a finite subgroup of H, there exists a fundamental domain $\tilde{S}_H$ for $H \cap K$ in H. Moreover, we have $[G : HK] < \infty$. Thus we can choose a finite representative system $\{y_H^{(i)} : 1 \leq i \leq [G : HK]\}$ for the HK-cosets in G, which shall be fixed during this section. Without loss of generality we may assume that $y_H^{(1)} = e$. Then we define the fundamental domain S_H by

$$S_H = \bigcup_{i=1}^{[G:HK]} y_H^{(i)} \tilde{S}_H.$$

Notice that this union is disjoint. It is straightforward to show that S_H is indeed a fundamental domain for K in G.

We choose a fundamental domain Ω_H for $\widehat{G/K}$ in $\widehat{G}$ in a similar way, i.e. by choosing $\tilde{\Omega}_H$ and $\gamma_H^{(j)}, 1 \leq j \leq [\widehat{G} : \widehat{G/H}\widehat{G/K}] = |H \cap K|$ in an analogous way and then following exactly the same steps on the dual side.

In the following we will consider functions $f \in L^2(G)$ with supp $\hat{f}$ being compact. Therefore we first extend Lemma 3.1 to LCA groups, since it explores the structure of such functions. For $f \in L^2(G)$ and $x_0 \in G$, let $L_{x_0} f$ be the left-translation of f, i.e. $L_{x_0} f(x) = f(x_0^{-1} x)$.

Lemma 4.1. *Let $f \in L^2(G)$ such that* supp $\hat{f}$ *is compact. Then, for each $x_0 \in G$ and for each compact open subgroup H of G, we have either $m_H(\text{supp }(L_{x_0}f)|_H) = m_G(H)$ or $(L_{x_0}f)|_H$ vanishes on a coset of some open subgroup of H.*

Proof. By hypothesis, $\hat{f} = \sum_{j=1}^n h_j \chi_{\tau_j \widehat{G/H}}$ for $\tau_j \in \widehat{G}$ and certain functions $h_j \in L^2(G)$ with supp $h_j \subseteq \tau_j \widehat{G/H}$. Let the Haar measure on $\widehat{G/H}$, $\mu_{G/H}$, be induced by μ_G. Then, for $x \in G$,

$$f(x) = \int_{\widehat{G}} \sum_{j=1}^n h_j(\omega)\chi_{\tau_j\widehat{G/H}}(\omega)\omega(x)d\mu_G(\omega) = \sum_{j=1}^n \int_{\widehat{G/H}} h_j(\tau_j\omega)(\tau_j\omega)(x)d\mu_{G/H}(\omega).$$

For each $x_0 \in G$ and $\alpha \in \widehat{H}$, this implies

$$\begin{aligned}\widehat{(L_{x_0}f)|}_H(\alpha) &= \int_H \left[\sum_{j=1}^n \int_{\widehat{G/H}} h_j(\tau_j\omega)(\tau_j\omega)(x_0^{-1}x)d\mu_{G/H}(\omega)\right]\overline{\alpha(x)}dm_H(x) \\ &= \sum_{j=1}^n \left[\int_{\widehat{G/H}} h_j(\tau_j\omega)\overline{(\tau_j\omega)(x_0)}d\mu_{G/H}(\omega)\right]\left[\int_H (\tau_j\overline{\alpha})(x)dm_H(x)\right].\end{aligned}$$

Since $\int_H (\tau_j\overline{\alpha})(x)dm_H(x) = 1$ if and only if $\alpha = \tau_j|_H$ and equal to zero otherwise, it follows that the support of $\widehat{(L_{x_0}f)|}_H$ is finite. Thus we may apply Lemma 3.1. This yields the claim. □

Let $f \in L^2(G)$ be a function which generates a Gabor frame associated with K. Then Lemma 2.2 tells us that $m_G(\text{supp } f) \geq m_G(S_H) = [G : HK]\frac{m_G(H)}{|H\cap K|}$. However, under weak conditions we can obtain an even better bound.

Theorem 4.2. *Suppose there exists some compact open subgroup H of G such that $H \cap K$ is contained in G_0. Further, let $f \in L^2(G)$ with* supp $\hat{f}$ *compact be such that $S(f, K)$ is a frame for $L^2(G)$. Then we have*

$$m_G(\text{supp } f) \geq [G : HK]m_G(H).$$

Proof. Let the fundamental domains S_H and Ω_H and the representative system $\{y_H^{(i)} : 1 \leq i \leq [G : HK]\}$ be chosen as in the beginning of this section. Let $i \in \{1, \dots, [G : HK]\}$ be arbitrarily fixed. Obviously, it suffices to show that $m_G(\text{supp } f|_{y_H^{(i)}HK}) \geq m_G(H)$.

Let $\tilde{x} \in \tilde{S}_H$ and $\tilde{\omega} \in \tilde{\Omega}_H$. Moreover, let F be a fixed representative system for the $H \cap K$-cosets in K. Then we have

$$|Zf(y_H^{(i)}\tilde{x}, \tilde{\omega})| = \left|\sum_{k\in K} \tilde{\omega}(k)f(y_H^{(i)}\tilde{x}k)\right| = \left|\sum_{l\in F}\sum_{k\in H\cap K} \tilde{\omega}(lk)f(y_H^{(i)}l\tilde{x}k)\right|$$

$$= \left|\sum_{k\in H\cap K}\sum_{l\in F} \tilde{\omega}(l)f(y_H^{(i)}l\tilde{x}k)\right| = |Z_{H\cap K}^H \left[\sum_{l\in F} \tilde{\omega}(l)f(y_H^{(i)}l\cdot)\right](\tilde{x}, 1)|,$$

where $Z^H_{H\cap K}$ denotes the Zak transform associated with $H \cap K$ in H. For the sake of brevity, we set $g_{\tilde{\omega}} := \sum_{l\in F} \tilde{\omega}(l) f(y_H^{(i)} l\cdot)$. Note that, since the Zak transform is a Hilbert space isomorphism (compare [Kut00, Theorem 3.1.7]), $g_{\tilde{\omega}} \in L^2(H)$. Since supp $\hat{f}$ is compact, we can write $\hat{f}$ in the form $\hat{f} = \sum_{k=1}^N h_k \chi_{\tau_k \widehat{G/H}}$, supp $h_k \subseteq \tau_k \widehat{G/H}$, where $\tau_k \in \widehat{G}$, $1 \le k \le N$ be such that $\tau_k \widehat{G/H}$ are pairwise different. With this notation we can write f as

$$f(x) = \int_{\widehat{G}} \sum_{k=1}^N h_k(\omega) \chi_{\tau_k \widehat{G/H}}(\omega) \omega(x) d\mu_G(\omega) = \sum_{k=1}^N \left[\int_{\widehat{G/H}} h_k(\tau_k \omega) \omega(x) d\mu_G(\omega) \right] \tau_k(x)$$

for all $x \in G$. For simplicity we set $f(x) =: \sum_{k=1}^N \lambda_k(x) \tau_k(x)$. Then, for $\alpha \in \widehat{H}$, we get

$$\begin{aligned} \widehat{g_{\tilde{\omega}}}(\alpha) &= \int_H \sum_{l\in F} \tilde{\omega}(l) f(y_H^{(i)} l x) \overline{\alpha(x)} dm_H(x) \\ &= \int_H \sum_{l\in F} \tilde{\omega}(l) \sum_{k=1}^N \lambda_k(y_H^{(i)} l) \tau_k(y_H^{(i)} l x) \overline{\alpha(x)} dm_H(x) \\ &= \sum_{k=1}^N \left[\sum_{l\in F} \tilde{\omega}(l) \lambda_k(y_H^{(i)} l) \tau_k(y_H^{(i)} l) \right] \left[\int_H (\tau_k \overline{\alpha})(x) dm_H(x) \right]. \end{aligned}$$

Since $\int_H (\tau_k \overline{\alpha})(x) dm_H(x) = 1$ if and only if $\alpha = \tau_k|_H$ and equal to zero otherwise, we know that $|\text{supp } \widehat{g_{\tilde{\omega}}}| < \infty$ for almost all $\tilde{\omega}$. We may now apply Lemma 3.1 to H and $g_{\tilde{\omega}}$. Hence we have $m_H(\text{supp } g_{\tilde{\omega}}) = m_G(H)$ or there exist an open subgroup $L_{\tilde{\omega}}$ of H and some $x_0 \in H$ such that $g_{\tilde{\omega}}|_{x_0 L_{\tilde{\omega}}} = 0$ a.e.. By hypothesis, $H \cap K$ is a subgroup of $G_0 = H_0$. Thus either $m_H(\text{supp } g_{\tilde{\omega}}) = m_G(H)$ or $|Z^H_{H\cap K}(g_{\tilde{\omega}})(x_0 y, 1)| = 0$ for all $y \in L_{\tilde{\omega}}$ (compare the proof of Proposition 3.2). The latter case implies $|Zf(y_H^{(i)} x_0 y, \tilde{\omega})| = 0$ for all $y \in L_{\tilde{\omega}}$, which contradicts $S(f, K)$ being a frame for $L^2(G)$. Hence the first case holds true. This implies $\sum_{l\in F} m_H(\text{supp } f(y_H^{(i)} l\cdot)) \ge m_G(H)$. We conclude $m_G(\text{supp } f|_{y_H^{(i)} HK}) \ge m_G(H)$. This finishes the proof. □

We also obtain a lower bound on the dual side. In particular, this gives rise to a lower bound for the product of the measures of the support of f and $\hat{f}$.

Corollary 4.3. *Suppose there exists some compact open subgroup H of G such that $\widehat{G/H} \cap \widehat{G/K}$ is contained in $(\widehat{G})_0$. Further, let $f \in L^2(G)$ with* supp f *compact be such that $S(f, K)$ is a frame for $L^2(G)$. Then*

$$\mu_G(\text{supp } \hat{f}) \ge |H \cap K| \mu_G(\widehat{G/H}).$$

If, in addition, $H \cap K$ is contained in G_0 and supp $\hat{f}$ *is compact, we have*

$$m_G(\text{supp } f) \mu_G(\text{supp } \hat{f}) \ge [G : G^c K][\widehat{G} : (\widehat{G})^c \widehat{G/K}].$$

Proof. Using similar arguments as in the proof of Theorem 4.2, we obtain the lower bound for $\mu_G(\text{supp } \hat{f})$.

It remains to prove the lower bound for the product $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f})$. Suppose $\widehat{G/H} \cap \widehat{G/K} \subseteq (\widehat{G})_0$. Since $\widehat{G}/(\widehat{G/H} \cap \widehat{G/K}) = HK$ and $\widehat{G}/(\widehat{G})_0 = G^c$ [HR63/70, Theorem 24.17], this implies $G^c \subseteq HK$. It always holds $H \subseteq G^c$. Thus $HK = G^c K$, which gives $[G : HK] = [G : G^c K]$. Using the same arguments on the dual side yields $|H \cap K| = [\widehat{G} : \widehat{G/H}\widehat{G/K}] = [\widehat{G} : (\widehat{G})^c \widehat{G/K}]$. Applying Theorem 4.2 and the first part of Corollary 4.3 finishes the proof. □

Note that the lower bound for $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f})$ does not depend on the fixed subgroup H anymore. This is not astonishing, since also the product of the Haar measures of G and $\widehat{G}$ does not depend on the normalization.

The next step is to check whether the bounds are sharp.

Theorem 4.4. *Suppose there exists some compact open subgroup H of G such that $H \cap K$ is contained in G_0 or $\widehat{G/H} \cap \widehat{G/K}$ is contained in $(\widehat{G})_0$. Then there exists a function $f \in L^2(G)$ with the following properties.*

(i) *$S(f, K)$ is a frame for $L^2(G)$.*
(ii) supp f *and* supp $\hat{f}$ *are compact.*
(iii) *$m_G(\text{supp } f) = [G : HK] m_G(H)$.*
(iv) *$\mu_G(\text{supp } \hat{f}) = |H \cap K| \mu_G(\widehat{G/H})$.*

In particular, we have

$$m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) = [G : G^c K][\widehat{G} : (\widehat{G})^c \widehat{G/K}].$$

Proof. It suffices to deal with the case that $H \cap K$ is contained in G_0. If $\widehat{G/H} \cap \widehat{G/K}$ is contained in $(\widehat{G})_0$, we can construct f in a similar way.

Since H is compact, we may apply Proposition 3.3. Hence there exists a function $g \in L^2(H)$ such that

(i') $S(g, H \cap K)$ is a frame for $L^2(H)$,
(ii') $m_H(\text{supp } g) = m_H(H)$, and
(iii') $\mu_H(\text{supp } \hat{g}) = |H \cap K| \mu_H(1)$.

Consider S_H, Ω_H, $\{y_H^{(i)} : 1 \leq i \leq [G : HK]\}$, and $\{\gamma_H^{(j)} : 1 \leq j \leq |H \cap K|\}$ as fixed in the beginning of this section. We define the function $f \in L^2(G)$ by

$$f(x) = \begin{cases} 0 & : \ x \notin y_H^{(i)} H \quad \text{for all } 1 \leq i \leq [G : HK], \\ g(h) & : \ x = y_H^{(i)} h \quad \text{for some } i \in \{1, \ldots, [G : HK]\}, h \in H. \end{cases}$$

Obviously, the support of f is compact. Moreover, by (ii') we have

$$m_G(\text{supp } f) = [G : HK] m_H(\text{supp } g) = [G : HK] m_H(H) = [G : HK] m_G(H),$$

which proves (iii). Then the Fourier transform of f is given by

$$\hat{f}(\omega) = \sum_{i=1}^{[G:HK]} \int_H g(h)\overline{\omega(h)}dm_H(h)\overline{\omega(y_H^{(i)})} = \hat{g}(\omega|_H) \sum_{i=1}^{[G:HK]} \overline{\omega(y_H^{(i)})}.$$

Thus, (iii') implies $\mu_G(\text{supp } \hat{f}) = |H \cap K|\mu_G(\widehat{G/H})$. Also supp $\hat{f}$ is compact.

It remains to prove (i). For this, let $(x, \omega) \in S_H \times \Omega_H$. By choice of S_H and Ω_H, we can write $x = y_H^{(i)}\tilde{x}$ and $\omega = \gamma_H^{(j)}\tilde{\omega}$ with $\tilde{x} \in \tilde{S}_H$ and $\tilde{\omega} \in \tilde{\Omega}_H$. Then

$$|Zf(x,\omega)| = \left|\sum_{k\in K} \omega(k) f(y_H^{(i)}\tilde{x}k)\right| = \left|\sum_{k\in H\cap K} \gamma_H^{(j)}(k) g(\tilde{x}k)\right| = |Z^H_{H\cap K} g(\tilde{x}, \gamma_H^{(j)}|_H)|.$$

Now (i') implies the existence of $0 < A \le B < \infty$ with $A \le |Z^H_{H\cap K} g(\tilde{x}, \gamma_H^{(j)}|_H)| \le B$ for almost all $\tilde{x} \in \tilde{S}_H$ and for all $1 \le j \le |H \cap K|$. Hence $A \le |Zf(x, \omega)| \le B$ for almost all $(x, \omega) \in S_H \times \Omega_H$. By Theorem 2.1, it follows that $S(f, K)$ is a frame for $L^2(G)$. □

We still have to deal with the case when there exists some compact open subgroup H such that $H \cap K$ is not contained in G_0.

Theorem 4.5. *Suppose there exists some compact open subgroup H of G such that $H \cap K$ is not contained in G_0. Then there exists a function $f \in L^2(G)$ with the following properties.*

(i) *$S(f, K)$ is a frame for $L^2(G)$.*
(ii) supp f *and* supp $\hat{f}$ *are compact.*
(iii) *$m_G(\text{supp } f) < [G : HK]m_G(H)$.*
(iv) *$\mu_G(\text{supp } \hat{f}) = |H \cap K|\mu_G(\widehat{G/H})$.*

In particular, we have

$$m_G(\text{supp } f)\mu_G(\text{supp } \hat{f}) < [G : G^c K][\widehat{G} : (\widehat{G})^c \widehat{G/K}].$$

Proof. Since H is compact, we may apply Proposition 3.4. This shows that there exists a function $g \in L^2(H)$ with

(i') $S(g, H \cap K)$ is a frame for $L^2(H)$,
(ii') $m_H(\text{supp } g) < m_H(H)$, and
(iii') $\mu_H(\text{supp } \hat{g}) = |H \cap K|\mu_H(1)$.

We choose $f \in L^2(G)$ by

$$f(x) = \begin{cases} 0 & : \; x \notin y_H^{(i)}H \quad \text{for all } 1 \le i \le [G : HK], \\ g(h) & : \; x = y_H^{(i)}h \quad \text{for some } i \in \{1, \ldots, [G : HK]\}, h \in H, \end{cases}$$

where $\{y_H^{(i)} : 1 \le i \le [G : HK]\}$ is the representative system chosen in the beginning of this section. Following exactly the same steps as in the proof of Theorem 4.4 yields the claim. □

Let us mention that we may transfer this result to the dual side to obtain a function which does not satisfy the bound for the measure of the support of the Plancherel transform.

We further have to ask, what happens in case that supp $\hat{f}$ is not compact. Unfortunately, in some of these cases we loose our lower bound.

Theorem 4.6. *Suppose there exists some compact open subgroup H of G such that $H \cap K$ is contained in G_0. Then the following conditions are equivalent.*

(i) *For each $f \in L^2(G)$ with* supp $\hat{f}$ *not compact such that $S(f, K)$ is a frame for $L^2(G)$, we have $m_G(\text{supp } f) \geq [G : HK] m_G(H)$.*

(ii) $|H \cap K| = 1$.

Proof. Let S_H be the fixed fundamental domain. Suppose that $|H \cap K| > 1$. Then define $f \in L^2(G)$ by $f = \chi_{S_H}$. Obviously, $S(f, K)$ is a frame for $L^2(G)$. But $m_G(\text{supp } f) = m_G(S_H) = [G : HK] \frac{m_G(H)}{|H \cap K|} < [G : HK] m_G(H)$. Moreover, by Theorem 4.2, supp $\hat{f}$ is not compact.

To prove the opposite direction let $|H \cap K| = 1$. Then $m_G(S_H) = [G : HK] \frac{m_G(H)}{|H \cap K|}$ $= [G : HK] m_G(H)$. Now we can apply Lemma 2.2. □

Let us remark that this theorem gives rise to similar results concerning the measure of the support of $\hat{f}$ and concerning the product $m_G(\text{supp } f)\mu_G(\text{supp } \hat{f})$.

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