A QUANTITATIVE NOTION OF REDUNDANCY AND ITS APPLICATIONS

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ABSTRACT

Although redundancy is at the heart of frame theory, just very recently an intuitively accessible, quantitative notion of redundancy for finite frames was established in [3]. This notion is based on a redundancy function for points on the sphere which measures the concentration of frame vectors around this point. The upper and lower redundancy are then defined as the maximum and the minimum of this function, respectively. In this paper, after presenting this novel quantitative notion of redundancy and its main properties, we analyze its contribution to erasure suppression as well as to sparse approximation and sparse recovery.

***Keywords*—** Frames, Erasures, Linearly Independent Sets, Redundancy, Spanning Sets, Sparse Approximation

1. INTRODUCTION

Over the last decades, frame theory has matured into a subject with applications to a wide variety of areas in mathematics, computer science and engineering. Frames are the key notion when redundant, yet stable expansions are required; examples include resilience against noise, quantization errors, and erasures in signal transmissions [8], or provision of sparse representations and approximations [4].

Let us be more precise: A *frame* for $\mathbb{R}^n$ is a sequence of vectors $\Phi = (\varphi_i)_{i=1}^N$, such that there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|x\|^2 \leq \sum_{i=1}^N |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathbb{R}^n$$

with A and B called the *lower* and *upper frame bound*, respectively. Indeed, in the finite-dimensional situation, a frame is equivalent to being a complete system. When using frames for transmission of some signal $x \in \mathbb{R}^n$, the *frame coefficients* $(\langle x, \varphi_i \rangle)_{i=1}^N$ are transmitted with reconstruction performed by

$$x = \sum_{i=1}^N \langle x, \varphi_i \rangle S_\Phi^{-1} \varphi_i,$$

where $S_\Phi = \sum_{i=1}^N \langle \cdot\, , \varphi_i \rangle \varphi_i$ is the so-called *frame operator* associated with Φ. Redundancy of Φ now allows erasures of some coefficients, while the remaining frame coefficients still preserve the complete information of a transmitted vector x. From a computational point of view, particularly convenient frames are so-called *A-tight frames*, which satisfy $A = B$, or *Parseval frames*, which allow $A = B = 1$. In these cases S_Φ just equals $A \cdot Id$ or even Id, respectively, thereby reducing the complexity of a reconstruction algorithm significantly.

A different viewpoint is taken when asking for an expansion of a signal $x \in \mathbb{R}^n$ into a frame $\Phi = (\varphi_i)_{i=1}^N$. In this case, frame theory provides the formula

$$x = \sum_{i=1}^N \langle x, S_\Phi^{-1} \varphi_i \rangle \varphi_i.$$

If the frame is not a basis, there exist uncountably many other coefficient sequences $(c_i)_{i=1}^N$ also satisfying $x = \sum_{i=1}^N c_i \varphi_i$. The sequence $(\langle x, S_\Phi^{-1} \varphi_i \rangle)_{i=1}^N$ is distinguished by the fact that this is the smallest in ℓ_2 norm. However, in order to enable usage of Compressed Sensing or, in general, sparse recovery techniques, one might want to search for the coefficient sequence with the least number of non-vanishing coefficients; and redundancy gives the freedom to do so.

It is therefore quite surprising to notice that, although redundancy is the key concept behind frame theory, there did not exist an appropriate quantitative measure on the 'degree' of redundancy of a given frame. The widely spread answer of scientists, if asked about their view of redundancy, is to use the quotient of the number of frame vectors divided by the dimension of the space as a measure for such. From a scholarly point of view, this is a very unsatisfactory definition. It does not distinguish between the two toy frames $\{e_1, e_1, e_1, e_2\}$ and $\{e_1, e_1, e_2, e_2\}$ in $\mathbb{R}^2$ as it intuitively should (e_1, e_2 being the unit vectors), nor does it provide any insight into properties of the frame. Hence, although the idea of redundancy is the crucial property in various applications and thus the foundation of frame theory, a mathematically precise, meaningful definition was missing.

In [3], a first quantitative notion of redundancy for a finite frame Φ, in fact, an upper redundancy $\mathcal{R}_\Phi^+$ and a lower redundancy $\mathcal{R}_\Phi^-$ were introduced. The upper redundancy measures the maximally achievable local redundancy, and it was indeed

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proven that Φ can be partitioned into $\lceil \mathcal{R}_\Phi^+ \rceil$ linearly independent sets. The lower redundancy measures the degree of least locally achievable redundancy in the sense that any set of $\lfloor \mathcal{R}_\Phi^- \rfloor - 1$ vectors can be deleted yet leave a frame. It was a surprise that the correct definition for those redundancies were the upper and the lower frame bound of the normalized frame.

Let us briefly mention that for infinite-dimensional Hilbert spaces, notions of redundancy have already been introduced three years ago in [2] (see also [1]), which are however not applicable to the finite-dimensional case. We would also like to draw the reader's attention to [5], in which the notion from [3] was recently generalized to infinite dimensions with however less satisfactory results.

In this paper, we will first recall the definition of lower and upper redundancy and state their main properties. Then we will analyze how these notions enable precise estimates for reconstruction error under erasures and for the error of sparse approximations.

2. UPPER AND LOWER REDUNDANCY

2.1. Definitions

We now take a viewpoint as in [3] and first introduce a notion of local redundancy, which encodes the concentration of frame vectors around one point. This local viewpoint will provide us with the correct intuition about the meaning of upper and lower redundancy.

Since the norms of the frame vectors do not matter for concentration, the given frame is first normalized and also only points on the unit sphere $\mathbb{S} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ in $\mathbb{R}^n$ are considered. We then define a so-called redundancy function, which might be also viewed as some sort of density function on the sphere or as a redundancy pattern on the sphere, which measures redundancy at each single point. For this and throughout the paper, we let $\langle y \rangle$ denote the span of some $y \in \mathbb{R}^n$ and $P_{\langle y \rangle}$ the orthogonal projection onto $\langle y \rangle$.

Definition 2.1. *Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for $\mathbb{R}^n$. For each $x \in \mathbb{S}$, the* redundancy function $\mathcal{R}_\Phi : \mathbb{S} \to \mathbb{R}^+$ *is defined by*

$$\mathcal{R}_\Phi(x) = \sum_{i=1}^N \|P_{\langle \varphi_i \rangle}(x)\|^2 = \|\varphi_i\|^{-2} \sum_{\{i:\varphi_i \neq 0\}} |\langle x, \varphi_i \rangle|^2.$$

We wish to notice that this notion is reminiscent of the fusion frame condition [6], here for rank-one projections.

Since the sphere is compact, we can now define the maximal and minimal value the redundancy function attains as upper and lower redundancy.

Definition 2.2. *Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for $\mathbb{R}^n$. Then the* upper redundancy of Φ *is defined by*

$$\mathcal{R}_\Phi^+ = \max_{x \in \mathbb{S}} \mathcal{R}_\Phi(x)$$

and the lower redundancy of Φ *by*

$$\mathcal{R}_\Phi^- = \min_{x \in \mathbb{S}} \mathcal{R}_\Phi(x).$$

Moreover, Φ has a uniform redundancy, *if*

$$\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+.$$

The reader might already have noticed that this is indeed the upper and lower frame bound of the normalized frame. The different viewpoint taken gives though an intuition why the main theorem of [3] stated in the next subsection holds true.

Before we state this theorem, let us briefly consider the special case of an *equal-norm frame*, i.e., a frame $\Phi = (\varphi_i)_{i=1}^N$ for which there exists some $c > 0$ such that $\|\varphi_i\| = c$ for all $i = 1, \ldots, N$. Let us at this point also recall the notion of a *unit-norm frame*, for which $c = 1$. In the case of an equal-norm frame, the upper and lower redundancy is immediately computed from the frame bounds as the follow simple result shows.

Lemma 2.3. *Let $\Phi = (\varphi_i)_{i=1}^N$ be an equal-norm frame for $\mathbb{R}^n$, having frame bounds A and B. Set $c = \|\varphi_i\|^2$ for all $i = 1, \ldots, N$. Then*

$$\mathcal{R}_\Phi^- = \frac{A}{c} \quad \textit{and} \quad \mathcal{R}_\Phi^+ = \frac{B}{c}.$$

2.2. Main Properties

The main result from [3] now states various properties these two redundancy measures have. We would like to draw the attention of the reader to [D6] and [D7], which show that the viewpoint of upper and lower redundancy measuring the maximally and minimally achievable local redundancy, respectively, is indeed valid.

Theorem 2.4 ([3]). *Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for $\mathbb{R}^n$.*

[D1] Generalization. *If Φ is an equal-norm Parseval frame, then*

$$\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = \frac{N}{n}.$$

[D2] Nyquist Property. *The following conditions are equivalent:*

(i) *We have $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+$.*
(ii) *The normalized version of Φ is tight.*

Also the following conditions are equivalent.

(i') *We have $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = 1$.*
(ii') *Φ is orthogonal.*

[D3] Upper and Lower Redundancy. *We have*

$$0 < \mathcal{R}_\Phi^- \le \mathcal{R}_\Phi^+ < \infty.$$

[D4] Additivity. *For each orthonormal basis $(e_i)_{i=1}^n$,*

$$\mathcal{R}^\pm_{\Phi \cup (e_i)_{i=1}^n} = \mathcal{R}_\Phi^\pm + 1.$$

Moreover, for each frame Φ' in $\mathbb{R}^n$,

$$\mathcal{R}^-_{\Phi \cup \Phi'} \ge \mathcal{R}_\Phi^- + \mathcal{R}_{\Phi'}^- \quad \textit{and} \quad \mathcal{R}^+_{\Phi \cup \Phi'} \le \mathcal{R}_\Phi^+ + \mathcal{R}_{\Phi'}^+.$$

In particular, if Φ and Φ' have uniform redundancy, then

$$\mathcal{R}^-_{\Phi \cup \Phi'} = \mathcal{R}_\Phi + \mathcal{R}_{\Phi'} = \mathcal{R}^+_{\Phi \cup \Phi'}.$$

[D5] Invariance. *Redundancy is invariant under application of a unitary operator U on $\mathbb{R}^n$, i.e.,*

$$\mathcal{R}^{\pm}_{U(\Phi)} = \mathcal{R}^{\pm}_{\Phi},$$

under scaling of the frame vectors, i.e.,

$$\mathcal{R}^{\pm}_{(c_i\varphi_i)_{i=1}^N} = \mathcal{R}^{\pm}_{\Phi}, \quad c_i \text{ scalars},$$

and under permutations, i.e.,

$$\mathcal{R}^{\pm}_{(\varphi_{\pi(i)})_{i=1}^N} = \mathcal{R}^{\pm}_{\Phi}, \quad \pi \in S_{\{1,\dots,N\}},$$

[D6] Spanning Sets. *Φ contains at least $\lfloor \mathcal{R}^-_{\Phi} \rfloor$ disjoint spanning sets. In particular, any set of $\lfloor \mathcal{R}^-_{\Phi} \rfloor - 1$ vectors can be deleted yet leave a frame.*

[D7] Linearly Independent Sets. *If Φ does not contain any zero vectors, then it can be partitioned into $\lceil \mathcal{R}^+_{\Phi} \rceil$ linearly independent sets.*

2.3. Examples

Let us now illustrate the introduced notions and, in particular, Theorem 2.4 [D6] and [D7] by analyzing three different frames. For this, let $\{e_j\}_{j=1}^n$ denote the unit basis of $\mathbb{R}^n$.

Example 2.5. Let $\Phi_{1,s}$ be defined by

$$\Phi_{1,s} = \{e_1, \dots, e_1, e_2, e_3, \dots, e_n\}, \text{ where } e_1 \text{ occurs } s \text{ times.}$$

It is easy to see that

$$\mathcal{R}^-_{\Phi_{1,s}} = 1 \quad \text{and} \quad \mathcal{R}^+_{\Phi_{1,s}} = s.$$

By Theorem 2.4 [D6] $\Phi_{1,s}$ can be split into 1 spanning set, which is indeed the maximal number, since, for instance, e_2 occurs one time and is orthogonal to all other elements from the frame. Concluding from Theorem 2.4 [D7], $\Phi_{1,s}$ can be partitioned into s linearly independent sets, which can be chosen as $\{e_1\}$ $s-1$ times and $\{e_1, \dots, e_n\}$. It is also evident that this is the minimal possible number, since the s vectors e_1 need to be placed into separate linearly independent sets.

Example 2.6. Let Φ_2 be defined by

$$\Phi_2 = \{e_1, e_1, e_2, e_2, e_3, e_3, \dots, e_n, e_n\}.$$

Then Φ_2 possesses a uniform redundancy. More precisely,

$$\mathcal{R}^-_{\Phi_2} = 2 \quad \text{and} \quad \mathcal{R}^+_{\Phi_2} = 2.$$

By Theorem 2.4 [D6] and [D7], Φ_2 can be partitioned into 2 spanning sets and also into 2 linearly independent sets. Those partitions can here in fact be chosen to be the same, more precisely, can be chosen to be the two orthonormal bases of which Φ_2 is composed.

Example 2.7. We add a third example in which the frame is not merely composed of vectors from $\{e_1, \dots, e_n\}$. Letting $0 < \varepsilon < 1$, we choose $\Phi_3 = (\varphi_i)_{i=1}^N$ as

$$\varphi_i = \begin{cases} e_1 & : \quad i = 1, \\ \sqrt{1-\varepsilon^2}e_1 + \varepsilon e_i & : \quad i \neq 1. \end{cases}$$

This frame is strongly concentrated around the vector e_1. We can prove that

$$0 < \mathcal{R}^-_{\Phi_3} \le \varepsilon^2.$$

The frame Φ_3 shows that Theorem 2.4 [D6] and [D7] in this case only provide an estimate near the extreme cases: $\mathcal{R}^- \approx 0$ and $\mathcal{R}^+ \approx N$, which however can be shown to become increasingly more accurate as $\mathcal{R}^-$ and $\mathcal{R}^+$ become closer to one another. For Φ_3, Theorem 2.4 [D6] gives $\lfloor \mathcal{R}^-_{\Phi_3} \rfloor = 0$, although there does exist a partition into one spanning set. Also Theorem 2.4 [D7] implies that this frame can be partitioned into N linearly independent sets. Again, we see that we can do better than this by merely taking the whole frame which happens to be linearly independent.

3. APPLICATIONS

We mentioned before that redundancy of a frame enables, in particular, resilience with respect to noise and erasures as well as the possibility to derive sparse expansions. Using the notion of redundancy certainly does not improve the various derived results, but phrasing results in terms of the redundancy function and upper and lower redundancy provides insight in how and where redundancy plays a role for these applications. For illustrative purposes, we focus on resilience against erasures and sparse expansions.

3.1. Erasures

Here we study the resilience of a frame $\Phi = \{\varphi_i\}_{i=1}^N$ against deletion of one single vector. For the statement of the result, we denote the worst-case (blind) reconstruction error under deletion of the vector φ_{i_0} by

$$E(\Phi, \varphi_{i_0}) = \max_{x \in \mathbb{S}^n} \left\| x - \sum_{i=1, i\neq i_0}^{N} \langle x, \varphi_i \rangle S^{-\frac{1}{2}} \varphi_i \right\|^2.$$

The following result now states estimates for the worst-case (blind) reconstruction error using redundancy notions.

Proposition 3.1. *Let $\Phi = (\varphi_i)_{i=1}^N$ be a unit-norm frame for $\mathbb{R}^n$. Then, setting $C = \max_{i=1,\dots,N} \|\varphi_i\|^2$,*

$$\max_{i=1,\dots,N} E(\Phi, \varphi_i) \le C \|S^{-\frac{1}{2}}\|^2 \mathcal{R}^+_{\Phi}. \tag{3.1}$$

Also, there exists a frame vector φ_{i_0} such that

$$E(\Phi, \varphi_{i_0}) \le \frac{1}{N} C^2 \|S^{-\frac{1}{2}}\|^2 \mathcal{R}^-_{\Phi}. \tag{3.2}$$

Proof. To prove the first claim, observe that, for $i_0 \in \{1,\ldots,N\}$,

$$\begin{aligned} E(\Phi,\varphi_{i_0}) &= \max_{x\in\mathbb{S}^n}\left\|\langle x,\varphi_{i_0}\rangle S^{-\frac{1}{2}}\varphi_{i_0}\right\|^2 \\ &\le C\|S^{-\frac{1}{2}}\|^2\max_{x\in\mathbb{S}^n}|\langle x,f_{i_0}\rangle|^2. \end{aligned} \tag{3.3}$$

By definition of $\mathcal{R}_\Phi^+$,

$$\begin{aligned} \max_{i=1,\ldots,N} E(\Phi,\varphi_i) &\le C\|S^{-\frac{1}{2}}\|^2\max_{i=1,\ldots,N}\max_{x\in\mathbb{S}^n}|\langle x,f_i\rangle|^2 \\ &\le C\|S^{-\frac{1}{2}}\|^2\mathcal{R}_\Phi^+, \end{aligned} \tag{3.4}$$

which proves (3.1).

Now, let $x_0 \in \mathbb{S}^n$ be such that $\mathcal{R}_\Phi(x_0) = \mathcal{R}_\Phi^-$, and choose φ_{i_0} to satisfy

$$\frac{|\langle x_0,\varphi_{i_0}\rangle|^2}{\|\varphi_{i_0}\|^2} \le \frac{1}{N}\mathcal{R}_\Phi(x_0) \quad \text{for all } x\in\mathbb{S}^n.$$

By (3.3),

$$E(\Phi,\varphi_{i_0}) \le C\|S^{-\frac{1}{2}}\|^2\max_{x\in\mathbb{S}^n}|\langle x,f_{i_0}\rangle|^2 \le C^2\|S^{-\frac{1}{2}}\|^2\frac{1}{N}\mathcal{R}_\Phi^-.$$

This proves (3.2). □

We observe that the estimate in (3.1) exploiting $\mathcal{R}_\Phi^+$ is weaker than a more tailored analysis due to the averaging over the distances of a vector to frame elements (see, for instance, [7]). Also the estimate in (3.2) is weaker again caused by the averaging. But despite the slightly weaker estimates, this result supports the intuition that upper redundancy is more like a local property, whereas lower redundancy can be regarded as a property of all frame vectors (note that here we are driving a ‘worst case’ analysis!). Also it shows that redundancy plays the role in erasure analysis it is anticipated to play.

3.2. Sparse Approximation

Finally, we would like to see redundancy in action when considering sparse approximations. For this, we first need to introduce some notation. Given a vector $x \in \mathbb{R}^n$, and a subspace H in $\mathbb{R}^n$, we denote the best approximation error of x through vectors in H by

$$\sigma(x,H) := \inf_{y\in H}\|x-y\|_2^2.$$

As before, the following theorem shall be viewed as a proof of concept; it certainly does not compete with much more adapted methodologies to this particular problem (see [4]).

Proposition 3.2. *Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for $\mathbb{R}^n$. Then, for each $x \in \mathbb{S}$, there exists a frame vector φ_{i_0} such that*

$$\sigma(x,\langle\varphi_{i_0}\rangle) \le 1 - \frac{\mathcal{R}_\Phi(x)}{N}. \tag{3.5}$$

In particular, for each $x \in \mathbb{S}^n$,

$$\min_{i=1,\ldots,N}\sigma(x,\langle\varphi_i\rangle) \le 1 - \frac{\mathcal{R}_\Phi^+}{N}. \tag{3.6}$$

Both bounds are sharp.

Proof. For a given $x \in \mathbb{S}^n$, $i_0 \in \{1,\ldots,N\}$ can be chosen such that

$$\frac{|\langle x,\varphi_{i_0}\rangle|^2}{\|\varphi_{i_0}\|^2} \ge \frac{1}{N}\sum_{i=1}^N\frac{|\langle x,\varphi_i\rangle|^2}{\|\varphi_i\|^2} = \frac{1}{N}\mathcal{R}_\Phi(x). \tag{3.7}$$

By definition,

$$\sigma(x,\langle\varphi_{i_0}\rangle) = \inf_c\|x-c\varphi_{i_0}\|_2^2.$$

But this infimum is attained for $c\varphi_{i_0}$ being the orthogonal projection of x onto $\langle\varphi_{i_0}\rangle$, hence $c = \|\varphi_{i_0}\|^{-2}\langle x,\varphi_{i_0}\rangle$. Now, using (3.7),

$$\sigma(x,\langle\varphi_{i_0}\rangle) = \|x\|^2 - \frac{|\langle x,\varphi_{i_0}\rangle|^2}{\|\varphi_{i_0}\|^2} \le 1 - \frac{\mathcal{R}_\Phi(x)}{N}.$$

The ‘in particular’-part follows immediately by taking the supremum over $x \in \mathbb{S}^n$. □

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4. REFERENCES

[1] R. Balan, P. G. Casazza, and Z. Landau, *Redundancy for localized frames*, preprint.

[2] R. Balan and Z. Landau, *Measure functions for frames*, J. Funct. Anal. **252** (2007), 630–676.

[3] B. G. Bodmann, P. G. Casazza, and G. Kutyniok, *A Quantitative Notion of Redundancy for Finite Frames*, Appl. Comput. Harmon. Anal., to appear.

[4] A.M. Bruckstein, D.L. Donoho, and M. Elad, *From Sparse Solutions of Systems of Equations to Sparse Modeling of Signals and Images*, SIAM Review **51** (2009), 34–81.

[5] J. Cahill, P. G. Casazza, and A. Heinecke, *A quantitative notion of redundancy for infinite frames*, preprint.

[6] P. G. Casazza, G. Kutyniok, and S. Li, *Fusion Frames and Distributed Processing*, Appl. Comput. Harmon. Anal. **25** (2008), 114–132.

[7] R. B. Holmes und V. I. Paulsen, *Optimal frames for erasures*, Linear Algebra Appl. **377** (2004), 31–51.

[8] J. Kovačević and A. Chebira, *An Introduction to Frames*, Found. Trends Signal Proc. **2** (2008), 1–94.