# Fusion Frames and Wireless Sensor Networks

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***Abstract*—Fusion frames provide a novel mathematical framework to design and analyze applications under distributed processing requirements. They are particularly advantageous for applications in which stability through redundancy plays a vital role. Frames – redundant yet stable systems – have already been frequently utilized in imaging science, information theory, and signal processing, to name a few. Fusion frames extend this notion, and can be regarded as weighted sets of redundant subspaces. In this paper, we will discuss the connection of fusion frames with wireless sensor networks and use this connection to analyze robustness under noise and erasures.**

## I. Introduction

Frames are nowadays a standard notion for modeling applications in which redundancy plays a vital role such as filter bank theory, sigma-delta quantization, signal and image processing, and wireless communications, see [8]. However, this framework obscures and neglects the often present substructure of a system such as, for instance, the subarrays present in sensor networks. Fusion frame theory extends frame theory by providing an extensive, flexible framework to incorporate such substructures.

### A. Fusion Frames

Fusion frames were introduced in [10] (see also the survey [9]) with various ideas already lurking in the work [6]. In short, fusion frames are weighted sets of redundant subspaces each of which contains a spanning set of local frame vectors with the subspaces having to satisfy special overlapping properties. Thus they also provide the possibility to model hierarchical structures which appear, for instance, in wireless sensor networks. Their main difference to frames is the fact that *scalar* measurements of a frame are now substituted by *vector* measurements. To date, several theoretical aspects of fusion frames have already been studied such as the sparsity of both the fusion frame measurement vectors [3], [5] as well as the signal itself [2] and its connection to compressed sensing (cf. [15]), constructions of fusion frames [4], and robustness against noise and erasures [7], [16]. On the application side, fusion frames were, for instance, utilized for the modeling of the visual cortex [17], the analysis of packet encoding [1], and the design of filter banks [13]. Finally, some first ideas on modeling of sensor networks are already contained in [11].

### B. Connection with Wireless Sensor Networks

Wireless sensor networks aim to measure environmental characteristics such as temperature, sound, vibration, or pollutants by resource-constrained sensor nodes which are spread over a large area. Typically, the data sensed by neighboring sensors has a high degree of overlap, hence wireless sensor networks are highly redundant. Thus a common model for the set of sensors is a frame such that the sensed data of one single sensor is one frame measurement. To reduce the cost, often small and inexpensive sensors, which do not have the capability to communicate over a long distance to one single centralized location, are utilized for processing. A common strategy to circumvent this problem consists in dividing the network into (possibly overlapping) subgroups where each sensor communicates its data to a single node within the cluster, which has considerably more power or is selected by a rotating assignment. These so-called head nodes then communicate the collected data to a central location for final processing. This process can be modeled by assuming that the frame vectors (the sensors) are divided into groups (the clusters) that span individual subspaces. The signal of interest is now first measured by the single frame vectors, then reconstructed from those within each subspace, followed by a transmission of the locally reconstructed parts of the signal to a central location for final reconstruction.

### C. Outline of the Paper

After having introduced fusion frames in Subsection II-B, we discuss its potential for modeling applications under distributed processing requirements with a particular emphasis on wireless sensor networks (Section III). In Section IV, we then focus on the analysis of robustness under noise and erasures.

## II. Fusion Frames

### A. Frame Theory revisited

Let us first recall the notion of a frame. A set of vectors $(\varphi_k)_{k=1}^K$ in $\mathbb{R}^N$ forms a *frame* for $\mathbb{R}^N$, if there exist constants $0 < A \le B < \infty$ such that

$$A\|x\|_2^2 \le \sum_{k=1}^{K} |\langle x, \varphi_k \rangle|_2^2 \le B\|x\|_2^2 \quad \text{for all } x \in \mathbb{R}^N.$$

The constants are referred to as *(lower and upper) frame bounds*. A frame is called *tight* if $A = B$ is possible. In this case, sometimes the expression $A$*-tightness* is used. Further, if $A = B = 1$ can be chosen, the frame is a *Parseval frame*.

The properties of a signal $x \in \mathbb{R}^N$ are analyzed by a frame $\Phi = (\varphi_k)_{k=1}^K$ through the *analysis operator* $T_\Phi : \mathbb{R}^N \to \mathbb{R}^K$, defined by

$$T_\Phi(x) = (\langle x, \varphi_k \rangle)_{k=1}^K.$$

Coining the adjoint operator $T_\Phi^*: \mathbb{R}^K \to \mathbb{R}^N$, i.e.

$$T_\Phi^*(c) = \sum_{k=1}^K c_k \varphi_k,$$

the *synthesis operator*, the key operator associated with a frame can be defined, which is the *frame operator*

$$S_\Phi : \mathbb{R}^N \to \mathbb{R}^N, \quad S_\Phi(x) = T_\Phi^* T_\Phi(x) = \sum_{k=1}^K \langle x, \varphi_k \rangle \varphi_k.$$

This operator allows the reconstruction of each signal $x \in \mathbb{R}^N$ from its *frame coefficients* $(\langle x, \varphi_k \rangle)_{k=1}^K$ by the reconstruction formula

$$x = \sum_{k=1}^K \langle x, \varphi_k \rangle S_\Phi^{-1} \varphi_k.$$

We remark that $(S_\Phi^{-1}\varphi_k)_{k=1}^K$ forms a frame for $\mathbb{R}^N$, the so-called *(canonical) dual frame*.

### B. Definitions

Generalizing the scalar measurements $(\langle x, \varphi_k \rangle)_{k=1}^K$ by vector measurements $(P_i(x))_{i=1}^M$, $P_i$ being orthogonal projections, a fusion frame was defined in [10] as follows. We emphasize that in the following definition, the subspaces are not necessarily orthogonal, in fact, for instance, robustness against erasures can only be achieved if the subspaces are *not* orthogonal.

*Definition 2.1:* Let $(\mathcal{W}_i)_{i=1}^M$ be a set of subspaces in $\mathbb{R}^N$, and let $(w_i)_{i=1}^M \subseteq \mathbb{R}^+$ be associated weights. Then $((\mathcal{W}_i, w_i))_{i=1}^M$ is a *fusion frame* for $\mathbb{R}^N$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|_2^2 \leq \sum_{i=1}^M w_i^2 \|P_i(x)\|_2^2 \leq B\|x\|_2^2 \quad \text{for all } x \in \mathbb{R}^N,$$

where $P_i$ denotes the orthogonal projection onto $\mathcal{W}_i$. The fusion frame $((\mathcal{W}_i, w_i))_{i=1}^M$ is *tight*, if $A = B$ is possible, and *Parseval*, if $A = B = 1$. A fusion frame is *equi-dimensional*, if all subspaces have the same dimension, and *equi-distant*, if the chordal distance between each pair $(\mathcal{W}_i, \mathcal{W}_j)$, $i \neq j$ coincides.

Recall that, as introduced by Conway, Hardin, and Sloane in [14], the *chordal distance* between two subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ with $m := \dim \mathcal{W}_1 = \dim \mathcal{W}_2$ is given by

$$d_c^2(\mathcal{W}_1, \mathcal{W}_2) = m - Tr[P_1 P_2],$$

where $P_i$ denotes the orthogonal projection onto $\mathcal{W}_i$, $i = 1, 2$.

### C. Signal Processing by a Fusion Frame

Given a fusion frame $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M$ for $\mathbb{R}^N$, the *fusion frame measurements* are the images of a signal of interest $x \in \mathbb{R}^N$ under the *analysis operator* $T_\mathcal{W}$ defined by

$$T_\mathcal{W} : \mathbb{R}^N \to \mathbb{R}^{MN}, \quad x \mapsto (w_i P_i(x))_{i=1}^M.$$

Its adjoint can be easily computed to be the operator

$$T_\mathcal{W}^* : \mathbb{R}^{MN} \to \mathbb{R}^N, \quad (c_i)_{i=1}^M \mapsto \sum_{i=1}^M w_i c_i,$$

called the *synthesis operator*. Then the so-called *fusion frame operator* $S_\mathcal{W}$ is given by

$$S_\mathcal{W} = T_\mathcal{W}^* T_\mathcal{W} : \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto \sum_{i=1}^M w_i^2 P_i(x).$$

Using this operator, a signal $x \in \mathbb{R}^N$ can be reconstructed from fusion fame measurements $(w_i P_i(x))_{i=1}^M$ as follows:

*Theorem 2.1 ([6]):* Let $\mathcal{W} = ((\mathcal{W}_i, w_i))_{i=1}^M$ be a fusion frame for $\mathbb{R}^N$ with associated fusion frame operator $S_\mathcal{W}$. Then $S_\mathcal{W}$ is a positive, self-adjoint, invertible operator on $\mathbb{R}^N$ and

$$x = \sum_{i=1}^M w_i^2 S_\mathcal{W}^{-1}(P_i(x)) \quad \text{for all } x \in \mathbb{R}^N. \tag{1}$$

### D. Frames versus Fusion Frames

Regarding the question whether the notion of fusion frames generalizes the notion of frames, the following lemma shows that this is indeed true. Since the proof is short and elegant, we decided to include it.

*Lemma 2.1 ([6]):* Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathbb{R}^N$ with frame bounds $A$ and $B$. Then $(\text{span}\{\varphi_i\}, \|\varphi_i\|_2)_{i=1}^M$ constitutes a fusion frame for $\mathbb{R}^N$ with fusion frame bounds $A$ and $B$.

*Proof:* For all $x \in \mathbb{R}^N$, we have

$$\begin{aligned} \sum_{i=1}^M \|\varphi_i\|_2^2 \|P_i(x)\|_2^2 &= \sum_{i=1}^M \|\varphi_i\|_2^2 \|\langle x, \tfrac{\varphi_i}{\|\varphi_i\|_2} \rangle \tfrac{\varphi_i}{\|\varphi_i\|_2}\|_2^2 \\ &= \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2. \end{aligned}$$

Applying the definitions of frames and fusion frames finishes the proof. ∎

However, also the converse is true. A fusion frame can be regarded as a frame with a particular substructure which will follow from Theorem 3.1 presented in the next section in the form of Corollary 3.2.

## III. Fusion Frames and Distributed Processing

### A. Local and Global Properties

Fusion frames already bear a hierarchical structure, namely first the layer of subspaces and second the local frames spanning those. This raises the question of frame theoretic properties of the set of all local frames. Interestingly, loosely speaking, a set of subspaces forming a fusion frame with 'good' fusion frame bounds is equivalent to the set of all local frames constituting a large frame with 'good' frame bounds. This is made precise in the following result.

*Theorem 3.1 ([6]):* Let $(\mathcal{W}_i)_{i=1}^M$ be a set of subspaces in $\mathbb{R}^N$, and let $(w_i)_{i=1}^M \subseteq \mathbb{R}^+$ be associated weights. Further, let $(\varphi_{ij})_{j=1}^{J_i}$ be a (local) frame for $\mathcal{W}_i$ with frame bounds $A_i$ and $B_i$ for each $i$, and set $A := \min_i A_i$ and $B := \max_i B_i$. Then the following conditions are equivalent.

(i) $((\mathcal{W}_i, w_i))_{i=1}^M$ is a fusion frame for $\mathbb{R}^N$.
(ii) $(w_i \varphi_{ij})_{i=1,j=1}^{M,J_i}$ is a frame for $\mathbb{R}^N$.

In particular, if $((\mathcal{W}_i, w_i))_{i=1}^M$ is a fusion frame with fusion frame bounds $C$ and $D$, then $(w_i\varphi_{ij})_{i=1,j=1}^{M,J_i}$ is a frame with bounds $AC$ and $BD$. On the other hand, if $(w_i\varphi_{ij})_{i=1,j=1}^{M,J_i}$ is a frame with bounds $C$ and $D$, then $((\mathcal{W}_i, w_i))_{i=1}^M$ is a fusion frame with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.

We wish to emphasize that this result also immediately provides constructions of fusion frames by taking a large frame and partitioning its frame vectors, each partition then spanning one of the subspaces. However, to perform the splitting aiming to optimize the frame bounds of the local frames is still intractable and equivalent to a problem in mathematics being unsolved since 1959, the Kadison-Singer Problem [12].

Let us next draw the following corollary from Theorem 3.1, which discusses the case of tight (fusion) frames.

*Corollary 3.1:* Let $(\mathcal{W}_i)_{i=1}^M$ be a set of subspaces in $\mathbb{R}^N$, and let $(w_i)_{i=1}^M \subseteq \mathbb{R}^+$ be associated weights. Further, let $(\varphi_{ij})_{j=1}^{J_i}$ be an A-tight frame for $\mathcal{W}_i$ for each $i$. Then the following conditions are equivalent.

(i) $((\mathcal{W}_i, w_i))_{i=1}^M$ is a $C$-tight fusion frame for $\mathbb{R}^N$.
(ii) $(w_i\varphi_{ij})_{i=1,j=1}^{M,J_i}$ is an $AC$-tight frame for $\mathbb{R}^N$.

The final corollary shows that indeed, by utilizing Theorem 3.1, a fusion frame can be also regarded as a frame with a particular substructure (cf. Subsection II-D).

*Corollary 3.2:* Let $(\mathcal{W}_i)_{i=1}^M$ be a fusion frame for $\mathbb{R}^N$, and let $(w_i)_{i=1}^M \subseteq \mathbb{R}^+$ be associated weights. Further, for each $i = 1, \ldots, M$, let $(e_{ij})_{j=1}^{J_i}$ be an orthonormal basis for $\mathcal{W}_i$. Then $(e_{ij})_{i=1,j=1}^{M,J_i}$ is a frame for $\mathbb{R}^N$.

### B. Modeling Wireless Sensor Networks by Fusion Frames

Relating to Subsection I-B, we now describe how fusion frames can serve as a model for wireless sensor networks. In this model, each single sensor is modeled by a frame vector $\varphi_k \in \mathbb{R}^N$, which captures the characteristics of the sensor, with $(\varphi_k)_{k=1}^K$ forming a frame for $\mathbb{R}^N$. Then the clusters of sensors are modeled by the partition $\{1, \ldots, K\} = \bigcup_{i=1}^M \{1, \ldots, J_i\}$. Finally, the induced sets of vectors $\Phi_i = \{\varphi_{ij} : j = 1, \ldots, J_i\}$ for $i = 1, \ldots, M$, and the associated subspaces

$$\mathcal{W}_i := \operatorname{span}\{\varphi_{ij} : j = 1, \ldots, J_i\}$$

model the clusters. Notice that in this situation, for each $i$, $\Phi_i$ forms a frame for $\mathcal{W}_i$.

In the basic model situation, the measurement each sensor – modeled by a frame vector $\varphi_{ij} \in \mathbb{R}^N$ – produces of a signal of interest $x \in \mathbb{R}^N$ is

$$\langle x, \varphi_{ij} \rangle.$$

Once the measurements of sensors belonging to the same cluster $i \in \{1, \ldots, M\}$ are transmitted to the associated head node, it performs the (frame) reconstruction – within the respective subspace $\mathcal{W}_i$ – resulting in the local reconstructions

$$\left( \sum_{j=1}^{J_i} \langle x, \varphi_{ij} \rangle S_{\Phi_i}^{-1} \varphi_{ij} \right)_{i=1}^M = (P_i(x))_{i=1}^M,$$

where $S_{\Phi_i}$ is the frame operator associated with the local frame $\Phi_i$. For this, the head nodes do only require knowledge about the frame of their respective cluster, not of the complete fusion frame. But observe that the collection $(P_i(x))_{i=1}^M$ now coincides with the fusion frame measurements of the fusion frame $(\mathcal{W}_i)_{i=1}^M$. Thus fusion frames are the correct model for the final fusion process, and the reconstruction at the central location is computed by (1).

This basic mathematical model is amenable to various generalizations and extensions. We mention a few examples:

- *Weights.* The fusion frame model can be extended to also include weights, which can be interpreted to model the fact that clusters might be of different importance or that clusters might be able to transmit with different signal strength.
- *Noise and Erasures.* The model can be extended to model noisy measurements and transmissions as well as erasures of both sensor measurements and measurements of head nodes. The situation of noisy measurements and erasures of measurements of head nodes will be analyzed in Section IV.
- *Quantization.* The impact of quantization can be modeled and analyzed by scalar quantization at the sensor level and vector quantization at the level of the head nodes. This will be one topic for future research.
- *Transmission.* The cost of transmission can be incorporated by, for example, weighting the frame elements; again a topic which is still under investigation.

## IV. Noise and Erasure Analysis

### A. Signal Model

Let $(\mathcal{W}_i)_{i=1}^M$ be a fusion frame for $\mathbb{R}^N$ with fusion frame bounds $A$ and $B$, and for $i = 1, \ldots, M$, let $m_i$ be the dimension of $\mathcal{W}_i$ and $U_i$ be an $N \times m_i$-matrix whose columns form an orthonormal basis of $\mathcal{W}_i$ for $i = 1, \ldots, M$. A signal $x \in \mathbb{R}^N$ is then modeled as a zero-mean random vector with covariance matrix $E[xx^T] = R_{xx} = \sigma_x^2 Id$.

### B. Analysis of Noise

The model for noisy fusion frame measurements is

$$z_i = U_i^T x + n_i, \quad i = 1, \ldots, M,$$

where $n_i \in \mathbb{R}^{m_i}$ is an additive white noise vector with zero mean and covariance matrix $E[n_i n_i^T] = \sigma_n^2 Id$, $i = 1, \ldots, M$. We also assume that the noise vectors $n_i \in \mathbb{R}^{m_i}$ are mutually uncorrelated as well as that the signal $x$ and the noise vectors $n_i$, $i = 1, \ldots, N$, are uncorrelated.

We now set

$$z = (z_1^T \quad z_2^T \quad \cdots \quad z_M^T)^T \quad \text{and} \quad U = (U_1 \quad U_2 \quad \cdots \quad U_M).$$

Then the composite covariance matrix between $x$ and $z$ can be written as

$$E\left[\begin{pmatrix} x \\ z \end{pmatrix} \begin{pmatrix} x^T & z^T \end{pmatrix}\right] = \begin{pmatrix} R_{xx} & R_{xz} \\ R_{zx} & R_{zz} \end{pmatrix}.$$

Here

$$R_{xz} = E[xz^T] = R_{xx} U$$

is the $M \times L$ ($L = \sum_{i=1}^{M} m_i$) cross-covariance matrix between $x$ and $z$ with $R_{zx} = R_{xz}^T$, and

$$R_{zz} = E[zz^T] = U^T R_{xx} U + \sigma_n^2 Id_L$$

is the $L \times L$ composite measurement covariance matrix.

For the reconstruction of a signal $x$ from the measurements $z$ the Wiener filter or LMMSE (linear minimum mean-squared error) filter $F = R_{xz} R_{zz}^{-1}$ is utilized. Notice that this is indeed the linear MSE minimizer. The actual reconstruction/estimation of $x$ is performed by computing

$$\hat{x} = Fz.$$

The associated error covariance matrix $R_{ee}$ is then given by

$$R_{ee} = E[(x-\hat{x})(x-\hat{x})^T] = \Big(R_{xx}^{-1} + \frac{1}{\sigma_n^2}\sum_{i=1}^{M} P_i\Big)^{-1},$$

and the MSE is obtained by taking the trace of $R_{ee}$.

The following result shows that in this case a fusion frame is optimally resilient against noise if it is tight.

*Theorem 4.1 ([16]):* Assuming the model previously introduced, the following conditions are equivalent.

(i) The MSE is minimized.
(ii) The fusion frame is tight.

### C. Analysis of Erasures

We now restrict ourselves to the class of $A$-tight fusion frames, which we just proved to be optimally robust against noise. To model the erasures, we let $K \subset \{1, 2, \ldots, M\}$ be the set of indices corresponding to the erased subspaces – in this analysis we regard all subspaces as equal. Then, the measurements are of the form

$$\tilde{z} = (Id - E)z,$$

where $E$ is an $L \times L$ block-diagonal erasure matrix whose $i$th diagonal block is an $m_i \times m_i$ zero matrix, if $i \notin K$, or an $m_i \times m_i$ identity matrix, if $i \in K$.

The estimate of $x$ is now given – using again the LMMSE filter $F$ – by

$$\tilde{x} = F\tilde{z},$$

with associated error covariance matrix

$$\begin{aligned} \widetilde{R}_{ee} &= E\left[(x-\tilde{x})(x-\tilde{x})^T\right] \\ &= E\left[(x - F(Id-E)z)(x - F(Id-E)z)^T\right]. \end{aligned}$$

The MSE for this estimate can then be written as

$$MSE = Tr[\widetilde{R}_{ee}] = Tr[R_{ee}] + \overline{MSE},$$

where $\overline{MSE}$ is the extra MSE due to erasures given by

$$\overline{MSE} = \frac{\sigma_x^4}{(A\sigma_x^2 + \sigma_n^2)^2} Tr\left[\sigma_x^2\left(\sum_{i\in\mathbb{S}} P_i\right)^2 + \sigma_n^2\left(\sum_{i\in\mathbb{S}} P_i\right)\right].$$

The following result then shows that optimal robustness against one subspace erasure is equivalent to equi-dimensionality of the tight fusion frame.

*Theorem 4.2 ([16]):* Assuming the model previously introduced and letting $(\mathcal{W}_i)_{i=1}^M$ be a tight fusion frame, the following conditions are equivalent.

(i) The MSE due to the erasure of one subspace is minimized.
(ii) All subspaces $\mathcal{W}_i$ have the same dimension, i.e. $(\mathcal{W}_i)_{i=1}^M$ is an *equi-dimensional* fusion frame.

We next state the result for two and more erasures. Similar as before, we restrict to the class of fusion frames, already shown to behave optimally under noise and one erasure.

*Theorem 4.3 ([16]):* Assuming the model previously introduced and letting $(\mathcal{W}_i)_{i=1}^M$ be a tight equi-dimensional fusion frame, the following conditions are equivalent.

(i) The MSE due to the erasure of two subspaces is minimized.
(ii) The chordal distance between each pair of subspaces is the same and maximal, i.e. $(\mathcal{W}_i)_{i=1}^M$ is a *maximal equi-distant* fusion frame.

Finally, let $(\mathcal{W}_i)_{i=1}^M$ be an equi-dimensional, maximally equi-distant tight fusion frame. Then the MSE due to $k$ subspace erasures, $3 \le k < N$, is constant.

### D. Optimal Fusion Frames and Grassmannian Packings

There exists an intriguing connection between optimally robust fusion frames and Grassmannian packings. To state this, we first formally introduce the classical packing problem (see, e.g., [14]).

*Classical Packing Problem:* For given $m, M, N$, find a set of $m$-dimensional subspaces $(\mathcal{W}_i)_{i=1}^M$ in $\mathcal{H}^N$ such that $\min_{i\neq j} d_c(i,j)$ is as large as possible. In this case we call $(\mathcal{W}_i)_{i=1}^M$ an *optimal packing*.

A lower bound is given by the so-called *simplex bound* determined by the next result.

*Theorem 4.4 ([14]):* Each packing of $m$-dimensional subspaces $(\mathcal{W}_i)_{i=1}^M$ in $\mathcal{H}^N$ satisfies

$$d_c^2(i,j) \le \frac{m(N-m)}{N}\frac{M}{M-1}, \quad i,j = 1,\ldots,M.$$

The following theorem shows that in fact equi-distant tight fusion frames are optimal Grassmannian packings.

*Theorem 4.5 ([16]):* Let $(\mathcal{W}_i)_{i=1}^M$ be a fusion frame of equi-dimensional subspaces with pairwise equal chordal distances $d_c$. Then, the fusion frame is tight if and only if $d_c^2$ equals the simplex bound.

In this sense, optimal Grassmannian packings as fusion frames lead to provably optimally robust wireless sensor networks.

## V. Conclusion

We discussed the novel mathematical notion of fusion frames as a means to model wireless sensor networks. We presented results on their robustness against noise and subspace erasures. We further discussed the connection to Grassmannian packings, thereby highlighting their importance for the modeling of wireless sensor networks. Open questions in this general direction are, for instance, a comprehensive analysis of robustness against single sensor in combination with subspace erasures and the analysis of the impact of quantization both of the sensor and the head node measurements.

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## REFERENCES

[1] B. G. Bodmann, *Optimal Linear Transmission by Loss-Insensitive Packet Encoding*, Appl. Comput. Harmon. Anal. **22** (2007), 274–285.

[2] B. Boufounos, G. Kutyniok, and H. Rauhut, *Sparse Recovery from Combined Fusion Frame Measurements*, IEEE Trans. Inform. Theory **57** (2011), 3864–3876.

[3] R. Calderbank, P. G. Casazza, A Heinecke, G. Kutyniok, and A. Pezeshki, *Sparse fusion frames: existence and construction*, Adv. Comput. Math. **35** (2011), 1–31.

[4] P. G. Casazza, M. Fickus, D. Mixon, Y. Wang, and Z. Zhou, *Constructing tight fusion frames*, Appl. Comput. Harmon. Anal. **30** (2011), 175–187.

[5] P. G. Casazza, A. Heinecke, and G. Kutyniok, *Optimally Sparse Fusion Frames: Existence and Construction*, SampTA'11 (Singapore, 2011), Proc., 2011.

[6] P. G. Casazza and G. Kutyniok, *Frames of subspaces*, in: Wavelets, Frames and Operator Theory (College Park, MD, 2003), Contemp. Math. **345**, Amer. Math. Soc., Providence, RI, 2004, 87–113.

[7] P. G. Casazza and G. Kutyniok, *Robustness of Fusion Frames under Erasures of Subspaces and of Local Frame Vectors*, in: Radon transforms, geometry, and wavelets (New Orleans, LA, 2006), 149–160, Contemp. Math. **464**, Amer. Math. Soc., Providence, RI, 2008.

[8] P. G. Casazza and G. Kutyniok, eds., *Finite Frames: Theory and Applications*, Birkhäuser, Boston, 2012.

[9] P. G. Casazza and G. Kutyniok, *Fusion Frames*, in: Finite Frames: Theory and Applications, Birkhäuser, Boston, 2012.

[10] P. G. Casazza, G. Kutyniok, and S. Li, *Fusion Frames and Distributed Processing*, Appl. Comput. Harmon. Anal. **25** (2008), 114–132.

[11] P. G. Casazza, G. Kutyniok, S. Li, and C. J. Rozell, *Modeling Sensor Networks with Fusion Frames*, In: Wavelets XII: San Diego 2007, SPIE Proc. **6701**, 67011M-1–67011M-11. SPIE, Bellingham, WA (2007).

[12] P. G. Casazza and J. C. Tremain, *The Kadison-Singer Problem in Mathematics and Engineering*, Proc. Nat. Acad. Sci. **103** (2006), 2032–2039.

[13] A. Chebira, M. Fickus, D.G. Mixon, *Filter bank fusion frames*, IEEE Trans. Signal Proc. **59** (2011), 953–963.

[14] J. H. Conway, R. H. Hardin, and N. J. A. Sloane, *Packing lines, planes, etc.: packings in Grassmannian spaces,* Experiment. Math, **5** (1996), 139–159.

[15] Y. C. Eldar and G. Kutyniok, eds., *Compressed Sensing: Theory and Applications*, Cambridge University Press, 2012.

[16] G. Kutyniok, A. Pezeshki, A. R. Calderbank, and T. Liu, *Robust Dimension Reduction, Fusion Frames, and Grassmannian Packings*, Appl. Comput. Harmon. Anal. **26** (2009), 64–76.

[17] C. J. Rozell and D. H. Johnson, *Analyzing the Robustness of Redundant Population Codes in Sensory and Feature Extraction Systems*, Neurocomputing **69** (2006), 1215–1218.