Computation of the density of weighted wavelet systems

Gitta Kutyniok

Department of Mathematics, University of Paderborn, 33095 Paderborn, Germany

ABSTRACT

Density conditions have turned out to be a powerful tool for deriving necessary conditions for weighted wavelet systems to possess an upper or lower frame bound. In this paper we study different definitions of density and compare them with respect to their appropriateness and practicality.

Keywords: Affine group, affine system, density, frame, Gabor system, wavelets

1. INTRODUCTION

Frames were introduced in 1952 by Duffin and Schaeffer.[1] Since then, they have become a major tool in signal and image processing, data compression, and sampling theory. We say that a family $(f_n)_{n\in\mathbb{N}}$ is a *frame* for $L^2(\mathbb{R})$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|_2^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|_2^2, \quad f \in L^2(\mathbb{R}).$$

The constants A and B are called *lower* and *upper frame bounds*, respectively. Frames can be viewed as generalizations of orthonormal bases which allow redundancy. For more information on the theory of frames we refer to the book by Daubechies[2] or the research-tutorial by Casazza.[3] Recently, the theory is beginning to grow even more rapidly, since several new applications have been developed. For example, frames are now used to mitigate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission (compare Casazza and Kovačević[4]) and to design high-rate constellations with full diversity in multiple-antenna code design (see Hassibi, Hochwald, Shokrollahi, and Sweldens[5]).

Weighted wavelet systems (or *weighted affine systems*), which are systems of the form

$$\{w(a,b)^{\frac{1}{2}} a^{-\frac{1}{2}} \psi(\tfrac{x}{a} - b) : (a,b) \in \Lambda\}, \quad \text{where } \psi \in L^2(\mathbb{R}),\ \Lambda \subset \mathbb{R}^+ \times \mathbb{R},\ \text{and } w : \Lambda \to \mathbb{R}^+,$$

play an important role in many of these applications. Hence it is essential to know when such a system is a frame. Especially interesting are necessary conditions, which the subset Λ and the weight function w have to satisfy, when they can lead to a wavelet system. This would help us to exclude many sets Λ and functions w in advance, when we would like to construct a wavelet system suitable for a particular application. This problem has already been independently studied by Heil and Kutyniok[6,7] and Sun and Zhou.[8,9] They gave definitions of density with respect to the geometry of the affine group, which is the appropriate group for doing wavelet analysis, and studied necessary conditions with respect to these densities. These definitions were inspired by the definition of Beurling density for Gabor systems (compare Christensen, Deng, and Heil[10]). But the two research groups used different ways to define the affine group, which led to different definitions of density. In this paper we will compare both definitions and also examine a third possibility to define density with respect to their appropriateness and practicality. Let us remark that this is the first comparison between the results of these two groups.

In Section 2 we first give two definitions of the affine group together with their main properties. We further define weighted wavelet systems with respect to both definitions and determine the connections between them. In Subsection 2.2 we then define density with respect to the geometry of the two versions of the affine group and with respect to either left- or right-multiplication. This leads to four different definitions of affine density.

E-mail: gittak@uni-paderborn.de

We conclude this section by stating the main results from Heil and Kutyniok[6] and Sun and Zhou,[9] which give necessary conditions on the density of the associated set for a weighted wavelet system to possess frame bounds.

In the next section we first study the definition of density with respect to left-multiplication (Subsection 3.1). After calculating the densities of oversampled affine systems, which are a generalization of the well-known classical affine systems, we show that both densities are equivalent in a certain sense. The next subsection deals with the definitions of density with respect to right-multiplication. Here we will see that these definitions are not appropriate. In Subsection 3.3 we finally investigate how the definitions depend on the choice of a sequence of sets, which we need to exhaust the affine group.

In Section 4 we finish the paper with a conclusion. We especially emphasize, which density is most appropriate for a special type of application.

2. DEFINITIONS AND BACKGROUND

In this section we will define our terminology and provide some background of the connection between the existence of frame bounds and the weighted density of the associated set.

2.1. The affine group and weighted wavelet systems

The affine group plays a vital role in the theory of wavelets. There are two main ways to define this group. However, although the resulting groups are isomorphic, concrete calculations can be quite different to carry out in practice. For the connection of the different versions of the affine group with wavelet systems in higher dimensions we refer to Weiss and Wilson.[11]

2.1.1. The group $\mathbb{A}_1$

Let $\mathbb{A}_1 = \mathbb{R}^+ \times \mathbb{R}$ denote the group endowed with the multiplication

$$(a,b) \bullet (x,y) = (ax, \tfrac{b}{x} + y).$$

The identity element of $\mathbb{A}_1$ is $e = (1,0)$, and inverses are given by

$$(a,b)^{-1} = (\tfrac{1}{a}, -ab).$$

A locally compact group is always equipped with a left-invariant Haar measure, which is unique up to a constant multiple. For $\mathbb{A}_1$ such a measure is given by $\mu_1 = \frac{dx}{x}\,dy$. There also exists a right-invariant Haar measure on a locally compact group, which for $\mathbb{A}_1$ we can choose to be $\nu_1 = dx\,dy$.

Wavelet systems can be constructed using unitary representations of $\mathbb{A}_1$. For this, let σ_1 be the unitary representation of $\mathbb{A}_1$ on $L^2(\mathbb{R})$ defined by

$$(\sigma_1(a,b)f)(x) = a^{-1/2} f(\tfrac{x}{a} - b).$$

Given a function $\psi \in L^2(\mathbb{R})$, a subset $\Lambda \subset \mathbb{A}_1$, and a weight function $w\colon \Lambda \to \mathbb{R}^+$, the *weighted wavelet system* with respect to $\mathbb{A}_1$ and σ_1 generated by ψ, Λ, and w is the weighted collection of time-scale shifts of ψ given by

$$\mathcal{W}_1(\psi,\Lambda,w) = \{w(a,b)^{1/2}\,\sigma_1(a,b)\psi\}_{(a,b)\in\Lambda} = \{w(a,b)^{1/2}\,a^{-1/2}\psi(\tfrac{x}{a} - b)\}_{(a,b)\in\Lambda}.$$

2.1.2. The group $\mathbb{A}_2$

Let $\mathbb{A}_2 = \mathbb{R}^+ \times \mathbb{R}$ be equipped with multiplication given by

$$(a,b) \star (x,y) = (ax, b + ay).$$

The identity element of $\mathbb{A}_1$ is $e = (1,0)$, and the inverse of an element $(a,b) \in \mathbb{A}_2$ is given by

$$(a,b)^{-1} = (\tfrac{1}{a}, -\tfrac{b}{a}).$$

A left-invariant Haar measure for the group $\mathbb{A}_2$ is given by $\mu_2 = \frac{dx}{x^2}\,dy$, and we can choose a right-invariant Haar measure to be $\nu_2 = \frac{dx}{x}\,dy$.

To construct a wavelet system, let σ_2 be the unitary representation of $\mathbb{A}_2$ on $L^2(\mathbb{R})$ defined by

$$(\sigma_2(a,b)f)(x) = a^{-1/2} f(\tfrac{x-b}{a}).$$

Given a function $\psi \in L^2(\mathbb{R})$, a subset $\Lambda \subset \mathbb{A}_2$, and a weight function $w\colon \Lambda \to \mathbb{R}^+$, the *weighted wavelet system* with respect to $\mathbb{A}_2$ and σ_2 generated by ψ, Λ, and w is defined by

$$\mathcal{W}_2(\psi,\Lambda,w) = \{w(a,b)^{1/2}\,\sigma_2(a,b)\psi\}_{(a,b)\in\Lambda} = \{w(a,b)^{1/2}\,a^{-1/2}\psi(\tfrac{x-b}{a})\}_{(a,b)\in\Lambda}.$$

2.1.3. Connection between $\mathbb{A}_1$ and $\mathbb{A}_2$ and between $\mathcal{W}_1$ and $\mathcal{W}_2$

Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ be defined by

$$\Phi(x,y) = (x, xy).$$

It is easy to check that we have

$$\Phi((a,b)\bullet(x,y)) = (ax, ab + axy) = \Phi(a,b) \star \Phi(x,y).$$

Since Φ is also bijective, $\Phi : \mathbb{A}_1 \to \mathbb{A}_2$ is a group isomorphism. Thus $\mathbb{A}_1$ and $\mathbb{A}_2$ are indeed isomorphic.

We obtain the following connection between the weighted wavelet systems $\mathcal{W}_1(\psi,\Lambda,w)$ and $\mathcal{W}_2(\psi,\Lambda,w)$. We will use the notation $(w\circ\Phi^{-1})(a,b) = w(\Phi^{-1}(a,b))$. The proof is just a straightforward calculation, therefore we omit it.

LEMMA 2.1. *For each $\psi \in L^2(\mathbb{R})$, $\Lambda \subset \mathbb{A}_1$, and $w\colon \Lambda \to \mathbb{R}^+$, we have*

$$\mathcal{W}_1(\psi,\Lambda,w) = \mathcal{W}_2(\psi,\Phi(\Lambda),w\circ\Phi^{-1}).$$

2.2. Definition of density

Now we define a Beurling density that is suited to the geometry of the affine group. We allow different elements of Λ to be given different weights in determining the density of Λ.

2.2.1. Density of subsets of $\mathbb{A}_1$

Given $h > 1$, we let Q_h denote a fixed family of neighborhoods of the identity element $e = (1,0)$ in $\mathbb{A}_1$. For simplicity of computation, we will take

$$Q_h = [\tfrac{1}{h}, h) \times [-h, h).$$

In Subsection 3.3 we will see that the affine Beurling density, in contrast to the classical Beurling density (compare Landau[12]), indeed depends on the choice of the sets Q_h, $h > 1$. For $(x,y) \in \mathbb{A}_1$, we let $L_h^1(x,y)$ be the set Q_h left-translated via the group action, so that it is "centered" at the point (x,y), i.e.,

$$L_h^1(x,y) = (x,y)\bullet Q_h = \{(xa, \tfrac{y}{a} + b) : a \in [\tfrac{1}{h}, h),\ b \in [-h,h)\}.$$

Note that while Q_h is a rectangle, in general the sets $L_h^1(x,y)$ are not rectangles, and hence, strictly speaking, while the point (x,y) is in the interior of $L_h^1(x,y)$, it is not the center of $L_h^1(x,y)$. However, for simplicity we will still refer to the sets $L_h^1(x,y)$ as "boxes" and the points (x,y) as their "centers".

We could also use right-translations to define the boxes. That is, for $(x,y) \in \mathbb{A}_1$, we define $R_h^1(x,y)$ by

$$R_h^1(x,y) = Q_h \bullet (x,y) = \{(ax, \tfrac{b}{x} + y) : a \in [\tfrac{1}{h}, h),\ b \in [-h,h)\}.$$

Note that in contrast to the sets $L_h^1(x,y)$, the sets $R_h^1(x,y)$ are true rectangles for each $(x,y) \in \mathbb{A}_1$.

We next define upper and lower affine density analogously to the classical Beurling density, by taking the maximum and minimum weighted number of elements of Λ in all possible boxes, dividing by the volume of each box, and then letting h go to infinity. When taking the boxes $L_h^1(x,y)$, we choose the left-invariant Haar measure μ_1 to define the volume. We see that then the volume of each box $L_h^1(x,y)$ does not depend on the point $(x,y) \in \mathbb{A}_1$. In fact,

$$\mu_1(L_h^1(x,y)) = \mu_1(Q_h) = \int_{-h}^{h} \int_{1/h}^{h} \frac{dx}{x}\, dy = 4h \ln h.$$

Analogously, when choosing the boxes $R_h^1(x,y)$, we apply the right-invariant Haar measure ν_1, so that again all boxes have the same volume. We obtain

$$\nu_1(R_h^1(x,y)) = \nu_1(Q_h) = \int_{-h}^{h} \int_{1/h}^{h} dx\, dy = 2(h^2-1).$$

In order to allow different points to be weighted differently in determining the density of a subset Λ of $\mathbb{A}_1$, let $w\colon \Lambda \to \mathbb{R}^+$ be a given weight function. Then we define the weighted number of elements of Λ lying in a subset K of $\mathbb{A}_1$ to be the value

$$\#_w(K) = \sum_{(a,b)\in K} w(a,b).$$

Now we can give a definition of the density of a weighted subset of $\mathbb{A}_1$. The definition of weighted affine Beurling density with respect to left-translation that we give below follows that given in Heil and Kutyniok.[6]

DEFINITION 2.2. *Let Λ be a subset of $\mathbb{A}_1$ and let $w\colon \Lambda \to \mathbb{R}^+$ be a weight function. Then the* upper weighted affine Beurling density *of Λ with respect to left-translation is*

$$\mathcal{D}^+_{1,L}(\Lambda, w) = \limsup_{h\to\infty} \frac{\sup_{(x,y)\in\mathbb{A}_1} \#_w(\Lambda \cap L_h^1(x,y))}{4h \ln h},$$

and the lower weighted affine Beurling density *of Λ with respect to left-translation is*

$$\mathcal{D}^-_{1,L}(\Lambda, w) = \liminf_{h\to\infty} \frac{\inf_{(x,y)\in\mathbb{A}_1} \#_w(\Lambda \cap L_h^1(x,y))}{4h \ln h}.$$

If $\mathcal{D}^-_{1,L}(\Lambda,w) = \mathcal{D}^+_{1,L}(\Lambda,w)$, then we say that Λ has uniform weighted affine Beurling density *with respect to left-translation and denote this density by $\mathcal{D}_{1,L}(\Lambda,w)$.*

Moreover, the upper weighted affine Beurling density *of Λ with respect to right-translation is*

$$\mathcal{D}^+_{1,R}(\Lambda, w) = \limsup_{h\to\infty} \frac{\sup_{(x,y)\in\mathbb{A}_1} \#_w(\Lambda \cap R_h^1(x,y))}{2(h^2-1)},$$

and the lower weighted affine Beurling density *of Λ with respect to right-translation is*

$$\mathcal{D}^-_{1,R}(\Lambda, w) = \liminf_{h\to\infty} \frac{\inf_{(x,y)\in\mathbb{A}_1} \#_w(\Lambda \cap R_h^1(x,y))}{2(h^2-1)}.$$

If $\mathcal{D}^-_{1,R}(\Lambda,w) = \mathcal{D}^+_{1,R}(\Lambda,w)$, we again say that Λ has uniform weighted affine Beurling density *with respect to right-translation and denote this density by $\mathcal{D}_{1,R}(\Lambda,w)$.*

If $w = 1$, we always omit writing it.

2.2.2. Density of subsets of $\mathbb{A}_2$

We can define density with respect to the geometry of $\mathbb{A}_2$ in a similar way as in the previous subsection. For this, for each $h > 1$ and $(x,y) \in \mathbb{A}_2$, we define the boxes $L_h^2(x,y)$ and $R_h^2(x,y)$ by

$$L_h^2(x,y) = (x,y) \star Q_h = \{(xa, y + xb) : a \in [\tfrac{1}{h}, h),\ b \in [-h,h)\}$$

and

$$R_h^2(x,y) = Q_h \star (x,y) = \{(ax, b + ay) : a \in [\tfrac{1}{h}, h),\ b \in [-h,h)\}.$$

Note that in this situation the sets $L_h^2(x,y)$ are always rectangles, whereas the sets $R_h^2(x,y)$ become non-rectangular as (x,y) varies.

By choosing the appropriate Haar measure, we can compute the volumes of these sets to be

$$\mu_2(L_h^2(x,y)) = \mu_2(Q_h) = \int_{-h}^{h}\int_{1/h}^{h} \frac{dx}{x^2}\,dy = 2(h^2-1)$$

and

$$\nu_2(R_h^2(x,y)) = \nu_2(Q_h) = \int_{-h}^{h}\int_{1/h}^{h} \frac{dx}{x}\,dy = 4h\ln h.$$

Now we define density of a weighted subset of $\mathbb{A}_2$. The definition of weighted affine Beurling density with respect to left-translation we give below follows that given in Sun and Zhou.[9]

Definition 2.3. *Let Λ be a subset of $\mathbb{A}_2$ and let $w\colon \Lambda \to \mathbb{R}^+$ be a weight function. Then the* upper weighted affine Beurling density *of Λ with respect to left-translation is*

$$\mathcal{D}_{2,L}^+(\Lambda,w) = \limsup_{h\to\infty} \frac{\sup_{(x,y)\in\mathbb{A}_2} \#_w(\Lambda \cap L_h^2(x,y))}{2(h^2-1)},$$

and the lower weighted affine Beurling density *of Λ with respect to left-translation is*

$$\mathcal{D}_{2,L}^-(\Lambda,w) = \liminf_{h\to\infty} \frac{\inf_{(x,y)\in\mathbb{A}_2} \#_w(\Lambda \cap L_h^2(x,y))}{2(h^2-1)}.$$

If $\mathcal{D}_{2,L}^-(\Lambda,w) = \mathcal{D}_{2,L}^+(\Lambda,w)$, then we say that Λ has uniform weighted affine Beurling density *with respect to left-translation and denote this density by $\mathcal{D}_{2,L}(\Lambda,w)$.*

Moreover, the upper weighted affine Beurling density *of Λ with respect to right-translation is*

$$\mathcal{D}_{2,R}^+(\Lambda,w) = \limsup_{h\to\infty} \frac{\sup_{(x,y)\in\mathbb{A}_2} \#_w(\Lambda \cap R_h^2(x,y))}{4h\ln h},$$

and the lower weighted affine Beurling density *of Λ with respect to right-translation is*

$$\mathcal{D}_{2,R}^-(\Lambda,w) = \liminf_{h\to\infty} \frac{\inf_{(x,y)\in\mathbb{A}_2} \#_w(\Lambda \cap R_h^2(x,y))}{4h\ln h}.$$

If $\mathcal{D}_{2,R}^-(\Lambda,w) = \mathcal{D}_{2,R}^+(\Lambda,w)$, then we again say that Λ has uniform weighted affine Beurling density *with respect to right-translation and denote this density by $\mathcal{D}_{2,R}(\Lambda,w)$.*

If $w=1$, we always omit writing it.

2.3. Necessary conditions for a weighted wavelet system to possess frame bounds

It was shown in Heil and Kutyniok[6] that the following necessary condition for a weighted wavelet system to possess an upper frame bound holds.

Theorem 2.4. *Let a nonzero $\psi \in L^2(\mathbb{R})$, a subset Λ of $\mathbb{A}_1$, and a weight function $w\colon \Lambda \to \mathbb{R}^+$ be given. If $\mathcal{W}_1(\psi,\Lambda,w)$ possesses an upper frame bound for $L^2(\mathbb{R})$, then $\mathcal{D}_{1,L}^+(\Lambda,w) < \infty$.*

Using this result we can exclude an abundance of subsets Λ of $\mathbb{A}_1$ at once, when we want to construct irregular wavelet frames.

An analogous result was proven in Sun and Zhou,[9] but without weight functions.

Theorem 2.5. *Let a nonzero $\psi \in L^2(\mathbb{R})$ and a subset Λ of $\mathbb{A}_2$ be given. If $\mathcal{W}_2(\psi,\Lambda)$ possesses an upper frame bound for $L^2(\mathbb{R})$, then $\mathcal{D}_{2,L}^+(\Lambda) < \infty$.*

Concerning the lower frame bound it was conjectured that the lower weighted affine density has to be strictly positive if a weighted wavelet system is a frame. Partial results can be found in Heil and Kutyniok[6] and Sun and Zhou.[9]

3. COMPARISON OF THE DIFFERENT DEFINITIONS OF DENSITY

In this section we will study the four different definitions of density and compare them with respect to their usefulness for determining necessary conditions for a weighted wavelet system to possess lower or upper frame bounds and with respect to their ease of computation. Moreover, we will show that the density indeed depends on the chosen sets Q_h, $h > 1$.

3.1. Left-multiplication

It will turn out that both definitions of density with respect to left-multiplication are equivalent in a special sense.

3.1.1. Examples

The best-known class of wavelet systems are the classical affine systems $\mathcal{W}_1(\psi, \Lambda, w)$ defined by

$$\Lambda = \{(a^j, bk) : j, k \in \mathbb{Z}\} \quad \text{with } a > 1,\ b > 0,\ w = 1 \text{ and } \psi \in L^2(\mathbb{R}).$$

Recently, a general definition of oversampled affine systems has been introduced in Hernandez, Labate, Weiss, and Wilson.[13] These systems include the classical affine systems, the quasi-affine systems of Ron and Shen,[14] and the more general quasi-affine systems of Bownik[15] as special cases.

DEFINITION 3.1. *Given $a > 1$, $b > 0$, and $r_j > 0$, an* oversampled affine system *is a weighted wavelet system of the form $\mathcal{W}_1(\psi, \Lambda, w)$ with*

$$\Lambda = \{(a^j, \tfrac{bk}{r_j})\}_{j,k\in\mathbb{Z}} \quad \textit{and} \quad w(a^j, \tfrac{bk}{r_j}) = \frac{1}{r_j}.$$

All oversampled affine systems possess exactly the same weighted affine Beurling density with respect to the geometry of $\mathbb{A}_1$. The proof of the following result is given in Heil and Kutyniok.[6]

PROPOSITION 3.2. *If $\mathcal{W}_1(\psi, \Lambda, w)$ is an oversampled affine system, then Λ has the uniform weighted affine Beurling density*

$$\mathcal{D}_{1,L}(\Lambda, w) = \frac{1}{b \ln a}.$$

We remark that it was already suggested in Daubechies[2] that $\frac{1}{b\ln a}$ might play the role of a density for affine systems, since this is an ubiquitous constant in a variety of formulas in wavelet theory.

Next we will consider the oversampled affine systems with respect to the affine group $\mathbb{A}_2$. Let $a > 1$, $b > 0$, and $r_j > 0$. By Lemma 2.1, the oversampled affine systems can be regarded as systems $\mathcal{W}_2(\psi, \Lambda, w)$ with

$$\Lambda = \{\Phi((a^j, \tfrac{bk}{r_j}))\}_{j,k\in\mathbb{Z}} = \{(a^j, \tfrac{a^j bk}{r_j})\}_{j,k\in\mathbb{Z}} \quad \text{and} \quad w(a^j, \tfrac{a^j bk}{r_j}) = \frac{1}{r_j}.$$

Interestingly, no oversampled system possesses a uniform weighted affine Beurling density. Moreover, both the upper and lower affine densities differ from the value $\frac{1}{b\ln a}$.

PROPOSITION 3.3. *If $\mathcal{W}_2(\psi, \Lambda, w)$ is an oversampled affine system, then*

$$\mathcal{D}^-_{2,L}(\Lambda, w) = \frac{1}{b(a-1)} \quad \textit{and} \quad \mathcal{D}^+_{2,L}(\Lambda, w) = \frac{a}{b(a-1)}.$$

Proof. Fix $(x, y) \in \mathbb{A}$. If $(a^j, \frac{a^j bk}{r_j}) \in L_h^2(x, y)$, then

$$\left(\tfrac{a^j}{x}, \tfrac{a^j bk}{r_j x} - \tfrac{y}{x}\right) = (x, y)^{-1} \star (a^j, \tfrac{a^j bk}{r_j}) \in Q_h.$$

This requires

$$\log_a x - \log_a h \le j < \log_a x + \log_a h \quad \text{and} \quad \frac{r_j(y-xh)}{a^j b} \le k < \frac{r_j(y+xh)}{a^j b}.$$

We compute

$$\begin{aligned}
\#_w(\Lambda \cap L_h^2(x,y)) &\le \sum_{j=\lceil \log_a x - \log_a h \rceil}^{\lfloor \log_a x + \log_a h \rfloor} \left[\frac{1}{r_j} \frac{2r_j xh}{a^j b} + 1 \right] \\
&\le \frac{2xh}{b} \left[\frac{a - a^{-\lfloor \log_a x + \log_a h \rfloor}}{a-1} - \frac{a - a^{-(\lceil \log_a x - \log_a h \rceil - 1)}}{a-1} \right] + 2\log_a h + 1.
\end{aligned}$$

Now there exists some $c \in [0,1)$ such that

$$\begin{aligned}
\#_w(\Lambda \cap L_h^2(x,y)) &\le \frac{2xh}{b} \left[\frac{a - a^{-(\log_a x + \log_a h - (2\log_a h - \lfloor 2\log_a h \rfloor - c))}}{a-1} - \frac{a - a^{-(\log_a x - \log_a h + c - 1)}}{a-1} \right] + 2\log_a h + 1 \\
&= \frac{2a^{-c}(ah^2 - h^{2-\lfloor 2\log_a h \rfloor})}{b(a-1)} + 2\log_a h + 1.
\end{aligned}$$

Similarly, with the additional observation that in the case $\log_a x + \log_a h \in \mathbb{Z}$ we have to subtract $\frac{1}{r_j}\frac{2r_j xh}{a^j b} - 1$ for $j = \log_a x + \log_a h$ to obtain a lower estimate, we obtain

$$\#_w(\Lambda \cap L_h^2(x,y)) \ge \frac{2a^{-c}(ah^2 - h^{2-\lfloor 2\log_a h \rfloor})}{b(a-1)} - \left(\frac{2}{b} - 1\right) - (2\log_a h + 1).$$

By changing x, the variable c can attain any value in the interval $[0,1)$. Thus

$$\mathcal{D}_{2,L}^{+}(\Lambda, w) \le \limsup_{h\to\infty} \left[\frac{2(ah^2 - h^{2-\lfloor 2\log_a h \rfloor})}{b(a-1)2(h^2-1)} + \frac{2\log_a h + 1}{2(h^2-1)} \right] = \frac{a}{b(a-1)}$$

and

$$\mathcal{D}_{2,L}^{+}(\Lambda, w) \ge \limsup_{h\to\infty} \left[\frac{2a^{-c}(ah^2 - h^{2-\lfloor 2\log_a h \rfloor})}{b(a-1)2(h^2-1)} - \frac{2-b}{2b(h^2-1)} - \frac{2\log_a h + 1}{2(h^2-1)} \right] = \frac{a}{b(a-1)}.$$

A similar argument shows $\mathcal{D}_{2,L}^{-}(\Lambda, w) = \frac{1}{b(a-1)}$. □

Note that we can choose different weight functions for oversampled affine systems such that we obtain a uniform affine density, which equals $\frac{1}{b \ln a}$. In the special case of classical affine systems this was done in Sun and Zhou.[9]

3.1.2. General results

We will prove that the density with respect to $\mathbb{A}_1$ and the density with respect to $\mathbb{A}_2$ are equivalent in the sense that the one upper density is finite if and only if the other upper density is finite. Also the one lower density is strictly positive if and only if the other lower density is strictly positive. This shows that concerning determining necessary conditions for wavelet frames both densities yield the same results. Thus, depending on which lattice we are considering, we may choose the one that is the easiest to compute with.

First we study in which way we can obtain a covering of the affine group with the defined boxes. Considering the group $\mathbb{A}_1$ we can obtain a covering, but it is not a disjoint one. However, the number of sets of the covering, which every set $L_{rh}^1(x,y)$ intersects at most, is independent of (x,y). (We include the additional parameter $r \ge 1$ for later use.) For the proof compare Heil and Kutyniok.[6]

Lemma 3.4. *Let $h > 1$ and $r \ge 1$ be given.*

1. *$\{L_h^1(h^{2j}, 2k) : j,k \in \mathbb{Z}\}$ covers $\mathbb{A}_1$.*

2. *Any set $L^1_{rh}(x,y)$ intersects at most $(\log_h r+3)(rh^2+h^2+1)$ sets of the form $L^1_h(h^{2j},2k)$.*

For the group $\mathbb{A}_2$ there exists a much nicer covering consisting of sets of the form $L^2_h(x,y)$. In fact we even obtain a disjoint covering.

LEMMA 3.5. *Let $h>1$ and $r\geq 1$ be given.*

1. *$\{L^2_h(h^{2j},2kh^{2j+1}) : j,k\in\mathbb{Z}\}$ is a disjoint covering of $\mathbb{A}_2$.*

2. *Any set $L^2_{rh}(x,y)$ intersects at most $\frac{r^2h^6-h^{-4}}{h^2-1}+2(\log_h r+4)$ sets of the form $L^2_h(h^{2j},2kh^{2j+1})$.*

Proof. Fix any $(x,y)\in\mathbb{A}_2$. Then $[\log_h x-1,\log_h x+1)$ contains a unique integer of the form $2j$, and there exists a unique $a\in[\frac{1}{h},h)$ such that $\log_h x=2j+\log_h a$. Further there exists a unique integer k and a number $b\in[-h,h)$ such that $y=h^{2j}(2kh+b)$. Hence

$$(x,y)=(h^{2j}a,2kh^{2j+1}+h^{2j}b)=(h^{2j},2kh^{2j+1})\star(a,b)\in L^2_h(h^{2j},2kh^{2j+1}).$$

To prove part 2, fix $(x,y)\in\mathbb{A}_2$, and suppose that $(u,v)\in L^2_{rh}(x,y)\cap L^2_h(h^{2j},2kh^{2j+1})$. Then there exist points $(a,b)\in Q_{rh}$, $(c,d)\in Q_h$ such that

$$(u,v)=(x,y)\star(a,b)=(ax,y+xb)\in L^2_{rh}(x,y)$$

and

$$(u,v)=(h^{2j},2kh^{2j+1})\star(c,d)=(h^{2j}c,2kh^{2j+1}+h^{2j}d)\in L^2_h(h^{2j},2kh^{2j+1}).$$

In particular, $\frac{ax}{c}=h^{2j}$ with $\frac{1}{rh}\leq a<rh$ and $\frac{1}{h}\leq c<h$, so $\frac{x}{rh^2}\leq h^{2j}\leq rh^2x$. Therefore

$$\frac{\log_h x}{2}-\frac{\log_h r}{2}-1\leq j\leq\frac{\log_h x}{2}+\frac{\log_h r}{2}+1.$$

Further, $2k=\frac{y}{h^{2j+1}}+\frac{xb}{h^{2j+1}}-\frac{d}{h}$, so

$$\frac{y}{2h^{2j+1}}-\frac{rx}{2h^{2j}}-\frac{1}{2}\leq k\leq\frac{y}{2h^{2j+1}}+\frac{rx}{2h^{2j}}+\frac{1}{2}.$$

For a given value of j, this is satisfied for at most $\frac{rx}{h^{2j}}+2$ values of k. Thus $L^2_{rh}(x,y)$ can intersect at most

$$\begin{aligned}\sum_{j=\lfloor\frac{\log_h x}{2}-\frac{\log_h r}{2}-1\rfloor}^{\lceil\frac{\log_h x}{2}+\frac{\log_h r}{2}+1\rceil}\frac{rx}{h^{2j}}+2 &\leq rx\left[\frac{h^2-h^{-2(\frac{\log_h x}{2}+\frac{\log_h r}{2}+2)}}{h^2-1}-\frac{h^2-h^{-2(\frac{\log_h x}{2}-\frac{\log_h r}{2}-3)}}{h^2-1}\right]+2(\log_h r+4)\\ &= \frac{r^2h^6-h^{-4}}{h^2-1}+2(\log_h r+4)\end{aligned}$$

sets of the form $L^2_h(h^{2j},2kh^{2j+1})$. □

The next proposition provides some information on the structure of subsets Λ of the affine group $\mathbb{A}_1$ which have finite upper or strictly positive lower weighted affine density. The following result is given in Heil and Kutyniok.[6]

PROPOSITION 3.6. *If $\Lambda\subset\mathbb{A}_1$ and $w\colon\Lambda\to\mathbb{R}^+$, then the following conditions are equivalent.*

1. *$\mathcal{D}^+_{1,L}(\Lambda,w)<\infty$.*

2. *There exists $h>1$ such that $\sup_{(x,y)\in\mathbb{A}_1}\#_w(\Lambda\cap L^1_h(x,y))<\infty$.*

Also the following conditions are equivalent.

1. $\mathcal{D}^-_{1,L}(\Lambda, w) > 0$.

2. *There exists* $h > 1$ *such that* $\inf_{(x,y)\in\mathbb{A}_1} \#_w(\Lambda \cap L^1_h(x,y)) > 0$.

The same result also holds for subsets Λ of the affine group $\mathbb{A}_2$.

PROPOSITION 3.7. *If* $\Lambda \subset \mathbb{A}_2$ *and* $w\colon \Lambda \to \mathbb{R}^+$, *then the following conditions are equivalent.*

1. $\mathcal{D}^+_{2,L}(\Lambda, w) < \infty$.

2. *There exists* $h > 1$ *such that* $\sup_{(x,y)\in\mathbb{A}_2} \#_w(\Lambda \cap L^2_h(x,y)) < \infty$.

Also the following conditions are equivalent.

1. $\mathcal{D}^-_{2,L}(\Lambda, w) > 0$.

2. *There exists* $h > 1$ *such that* $\inf_{(x,y)\in\mathbb{A}_2} \#_w(\Lambda \cap L^2_h(x,y)) > 0$.

Proof. We prove only the first equivalence, concerning the upper density. The proof of the result for the lower density is similar.

1. $\Rightarrow$ 2. is trivial. To prove the converse implication, suppose there exists some $h > 1$ such that $R = \sup_{(x,y)\in\mathbb{A}_2} \#_w(\Lambda \cap L^2_h(x,y)) < \infty$. If we let $N_r = \frac{r^2h^6 - h^{-4}}{h^2-1} + 2(\log_h r + 4)$ be as given in part 2 of Lemma 3.5, then each set $L^2_{rh}(x,y)$ is covered by a union of at most N_r sets of the form $L^2_h(h^{2j}, 2kh^{2j+1})$. This implies

$$\sup_{(x,y)\in\mathbb{A}_2} \#_w(\Lambda \cap L^2_{rh}(x,y)) \le N_r \cdot \sup_{j,k\in\mathbb{Z}} \#_w(\Lambda \cap L^2_h(h^{2j}, 2kh^{2j+1})) \le N_r R.$$

Thus

$$\mathcal{D}^+_{2,L}(\Lambda, w) \le \limsup_{r\to\infty} \frac{N_r R}{2(r^2h^2-1)} = \limsup_{r\to\infty} \frac{R(r^2h^6 - h^{-4})}{(h^2-1)2(r^2h^2-1)} + \frac{2R(\log_h r + 4)}{2(r^2h^2-1)} = \frac{Rh^4}{2(h^2-1)} < \infty.$$

□

We will now derive the equivalence of the upper and lower weighted affine densities associated with the affine groups $\mathbb{A}_1$ and $\mathbb{A}_2$. Note that we cannot expect both densities to coincide, since this is not true even for the class of oversampled affine systems.

THEOREM 3.8. *If* $\Lambda \subset \mathbb{A}_1$ *and* $w\colon \Lambda \to \mathbb{R}^+$, *then the following conditions are equivalent.*

1. $\mathcal{D}^+_{1,L}(\Lambda, w) < \infty$.

2. $\mathcal{D}^+_{2,L}(\Phi(\Lambda), w \circ \Phi^{-1}) < \infty$.

Also the following conditions are equivalent.

1. $\mathcal{D}^-_{1,L}(\Lambda, w) > 0$.

2. $\mathcal{D}^-_{2,L}(\Phi(\Lambda), w \circ \Phi^{-1}) > 0$.

Proof. We will only prove the first part. The results for the lower density can be obtained by the same arguments.

By Proposition 3.6, Statement 1. holds if and only if there exists $h > 1$ such that $\sup_{(x,y)\in\mathbb{A}_1} \#_w(\Lambda \cap L^1_h(x,y)) < \infty$. We have $\#_w(\Lambda \cap L^1_h(x,y)) = \#_{w\circ\Phi^{-1}}(\Phi(\Lambda) \cap \Phi(L^1_h(x,y)))$ and $\Phi(L^1_h(x,y)) = \Phi(x,y) \star \Phi(Q_h)$. This shows that 1. is equivalent to the existence of some $h > 1$ such that

$$\sup_{(x,y)\in\mathbb{A}_2} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda) \cap (x,y) \star \Phi(Q_h)) < \infty.$$

On the other hand, using Proposition 3.7, we get that Statement 2. holds if and only if there exists $h > 1$ such that $\sup_{(x,y)\in\mathbb{A}_2} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda) \cap (x,y) \star Q_h) < \infty$. It is easy to check that we have $Q_{\sqrt{h}} \subseteq \Phi(Q_h) \subseteq Q_{h^2}$ for all $h > 1$. This implies

$$\sup_{(x,y)\in\mathbb{A}_2} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star Q_{\sqrt{h}}) \leq \sup_{(x,y)\in\mathbb{A}_2} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star\Phi(Q_h)) \leq \sup_{(x,y)\in\mathbb{A}_2} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star Q_{h^2}),$$

which shows that there exists $h > 1$ such that $\sup_{(x,y)\in\mathbb{A}_2} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda) \cap (x,y) \star \Phi(Q_h)) < \infty$ if and only if there exists $h > 1$ such that $\sup_{(x,y)\in\mathbb{A}_2} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda) \cap (x,y) \star Q_h) < \infty$. This finishes the proof. □

The following corollary shows that it does not matter which definition of density we apply when testing whether a discrete set gives rise to a wavelet frame.

COROLLARY 3.9. *Let $\psi \in L^2(\mathbb{R})$, $\Lambda_1 \subset \mathbb{A}_1$, $\Lambda_2 \subset \mathbb{A}_2$ and $w\colon \Lambda \to \mathbb{R}^+$ be such that $\mathcal{W}_1(\psi, \Lambda_1, w) = \mathcal{W}_2(\psi, \Lambda_2, w \circ \Phi^{-1})$. Then the following conditions are equivalent.*

1. $\mathcal{D}^+_{1,L}(\Lambda_1, w) < \infty$.
2. $\mathcal{D}^+_{2,L}(\Lambda_2, w \circ \Phi^{-1}) < \infty$.

Also the following conditions are equivalent.

1. $\mathcal{D}^-_{1,L}(\Lambda_1, w) > 0$.
2. $\mathcal{D}^-_{2,L}(\Lambda_2, w \circ \Phi^{-1}) > 0$.

Proof. This follows immediately from $\Lambda_2 = \Phi(\Lambda_1)$ and Theorem 3.8. □

This shows in particular that Theorem 2.5 also holds for weighted wavelet systems. Moreover, it indicates that as far as necessary conditions for a weighted wavelet system to be a frame are concerned, we can choose the density that is the easiest to apply.

3.2. Right-multiplication

We will show now that right-multiplication is not the correct way to define density. We start by studying oversampled affine systems, which were introduced in Definition 3.1. If we define density with respect to right-multiplication, we obtain the following result.

LEMMA 3.10.

1. *If $\mathcal{W}_1(\psi, \Lambda, w)$ is an oversampled affine system, then*
$$\mathcal{D}^-_{1,R}(\Lambda, w) = 0 \quad \text{and} \quad \mathcal{D}^+_{1,R}(\Lambda, w) = \infty.$$
2. *If $\mathcal{W}_2(\psi, \Lambda, w)$ is an oversampled affine system, then*
$$\mathcal{D}^-_{2,R}(\Lambda, w) = 0 \quad \text{and} \quad \mathcal{D}^+_{2,R}(\Lambda, w) = \infty.$$

Proof. We prove only the first part. Part 2 can be shown in a similar way.

Fix $(x, y) \in \mathbb{A}$. If $(a^j, \frac{bk}{r_j}) \in R^1_h(x, y)$, then

$$\left(\tfrac{a^j}{x}, x(\tfrac{bk}{r_j} - y)\right) = (a^j, \tfrac{bk}{r_j}) \bullet (x, y)^{-1} \in Q_h.$$

In particular, $\frac{a^j}{x} \in [\frac{1}{h}, h)$. Since terms ± 1 are not significant in the limit, it suffices to observe that there are approximately $2\log_a h$ integers j satisfying this condition. Additionally, we have $\frac{r_j(xy-h)}{bx} \leq k < \frac{r_j(xy+h)}{bx}$. For a given j, there are approximately $\frac{2hr_j}{bx}$ integers k satisfying this condition. Taking the weight into account, we conclude that

$$\#_w(\Lambda \cap R_h^1(x,y)) \approx 2\log_a h \cdot \frac{1}{r_j} \cdot \frac{2hr_j}{bx} = \frac{4h\ln h}{xb\ln a}.$$

By changing x, we can make this quantity arbitrarily large or small, which yields the conclusion $\mathcal{D}^-_{1,R}(\Lambda, w) = 0$ and $\mathcal{D}^+_{1,R}(\Lambda, w) = \infty$. $\square$

This shows that we cannot expect to derive useful necessary conditions for a wavelet system to have frame bounds using this definition of density. A useful notion of density should at least satisfy the following conditions. First, an upper density being infinite should imply that the associated wavelet system possesses "too many" elements, and hence does not possess an upper frame bound. Second, the lower density being zero should imply that the associated wavelet system does not possess "enough" elements, and hence does not possess a lower frame bound. Since even the classical affine systems have zero lower density and infinite upper density, this notion of density is inappropriate.

3.3. Dependency on the sets Q_h, $h > 1$

We will present an example, which shows that the density indeed depends on the choice of the sets Q_h, $h > 1$. For this we compare the density $\mathcal{D}^+_{1,L}$ with respect to our choice of sets $Q_h = [\frac{1}{h}, h) \times [-h, h)$, $h > 1$, which is

$$\mathcal{D}^+_{1,L}(\Lambda, w) = \limsup_{h\to\infty} \frac{\sup_{(x,y)\in\mathbb{A}_1} \#_w(\Lambda \cap (x,y) \bullet Q_h)}{4h\ln h},$$

with the density $\widetilde{\mathcal{D}}^+_{1,L}$ with respect to the sets $\Phi^{-1}(Q_h) = \{(a, \frac{b}{a}) : a \in [\frac{1}{h}, h), b \in [-h, h)\}$, $h > 1$, which is given by

$$\widetilde{\mathcal{D}}^+_{1,L}(\Lambda, w) = \limsup_{h\to\infty} \frac{\sup_{(x,y)\in\mathbb{A}_1} \#_w(\Lambda \cap (x,y) \bullet \Phi^{-1}(Q_h))}{2(h^2-1)},$$

since $\mu_1(\Phi^{-1}(Q_h)) = \int_{\frac{1}{h}}^{h} \int_{-\frac{h}{x}}^{\frac{h}{x}} dy\, \frac{dx}{x} = 2(h^2-1)$. We define the lower and uniform affine Beurling densities in the same way. Then we obtain the following result.

Lemma 3.11. *Let $\Lambda = \{(a^j, bk) : j,k \in \mathbb{Z}\} \subset \mathbb{A}_1$ and $w = 1$, i.e., Λ is the set associated with the classical affine system. Then, for all $a > 1$, $b > 0$, we have*

$$\mathcal{D}_{1,L}(\Lambda) = \frac{1}{b\ln a} \neq \frac{1}{b(a-1)} = \widetilde{\mathcal{D}}^-_{1,L}(\Lambda) \quad \textit{and} \quad \mathcal{D}_{1,L}(\Lambda) = \frac{1}{b\ln a} \neq \frac{a}{b(a-1)} = \widetilde{\mathcal{D}}^+_{1,L}(\Lambda).$$

Proof. The first equality follows from Proposition 3.2. For the second equality we compute

$$\begin{aligned}
\widetilde{\mathcal{D}}^-_{1,L}(\Lambda) &= \liminf_{h\to\infty} \frac{\inf_{(x,y)\in\mathbb{A}_1} \#(\Lambda \cap (x,y) \bullet \Phi^{-1}(Q_h))}{2(h^2-1)} \\
&= \liminf_{h\to\infty} \frac{\inf_{(x,y)\in\mathbb{A}_2} \#(\Phi^{-1}[\Phi(\Lambda) \cap (x,y) \star Q_h])}{2(h^2-1)} \\
&= \liminf_{h\to\infty} \frac{\inf_{(x,y)\in\mathbb{A}_2} \#(\Phi(\Lambda) \cap (x,y) \star Q_h)}{2(h^2-1)} \\
&= \mathcal{D}^-_{2,L}(\Phi(\Lambda)).
\end{aligned}$$

Now $\mathcal{D}^-_{2,L}(\Phi(\Lambda)) = \frac{1}{b(a-1)}$ follows from Lemma 2.1 and Proposition 3.3. Moreover, it is an easy calculation to show that $\frac{1}{b\ln a} \neq \frac{1}{b(a-1)}$ for all $a > 1$, $b > 0$. The other claim can be proven in a similar way. $\square$

Thus different choices of sets Q_h, $h > 1$ may lead to different values of the associated densities and even may change a uniform affine density into a non-uniform one.

4. CONCLUSION

First of all we have seen that $\mathcal{D}^{\pm}_{1,R}$ as well as $\mathcal{D}^{\pm}_{2,R}$ are both inappropriate as densities, since they do not provide suitable necessary conditions for a weighted wavelet system to possess frame bounds.

The definition of density with respect to left-multiplication is appropriate, since we get very useful necessary conditions (see Subsection 2.3) for a weighted wavelet system to possess an upper or lower frame bound. If we use $\mathcal{D}^{\pm}_{1,L}$ the classical affine system has a uniform density, which attains exactly the ubiquitous constant $\frac{1}{b \ln a}$. Hence it can be viewed as the more natural definition. The problem is that the boxes $L^1_h(x,y)$ become non-rectangular as (x,y) varies. This makes calculations more difficult. Using $\mathcal{D}^{\pm}_{2,L}$ does not lead to this magical constant for the classical affine systems and it does not even lead to a uniform affine density. However, the boxes $L^2_h(x,y)$ remain rectangles as they are moved around by the group action. This makes calculations much easier. Moreover, we obtain a disjoint covering of the affine group using these boxes. But we have seen that both definitions are equivalent in the sense that the one upper density is finite if and only if the other upper density is finite. Also the one lower density is strictly positive if and only if the other lower density is strictly positive. Hence we can use whatever definition suits our application best, if we are only interested in necessary conditions for a weighted wavelet system to possess frame bounds.

We finally showed that the value of the density indeed depends on the choice of the exhausting sequence $(Q_h)_{h>1}$ we choose.

ACKNOWLEDGMENTS

The author is indebted to Chris Heil for valuable comments and suggestions which improved the presentation.

REFERENCES

1. R. J. Duffin and A. C. Schaeffer, "A class of nonharmonic Fourier series," *Trans. Amer. Math. Soc.* **72**, pp. 341–366, 1952.
2. I. Daubechies, *Ten lectures on wavelets*, vol. 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
3. P. G. Casazza, "The art of frame theory," *Taiwanese J. Math.* **4**(2), pp. 129–201, 2000.
4. P. G. Casazza and J. Kovačević, "Equal-norm tight frames with erasures," *Adv. Comput. Math.* **18**(2-4), pp. 387–430, 2003.
5. B. Hassibi, B. M. Hochwald, A. Shokrollahi, and W. Sweldens, "Representation theory for high-rate multiple-antenna code design," *IEEE Trans. Inform. Theory* **47**(6), pp. 2335–2367, 2001.
6. C. Heil and G. Kutyniok, "Density of weighted wavelet frames," *J. Geom. Anal.* **13**, pp. 479–493, 2003.
7. C. Heil and G. Kutyniok, "The homogeneous approximation property for wavelet frames and schauder bases," preprint.
8. W. Sun and X. Zhou, "Density of irregular wavelet frames," *Proc. Amer. Math. Soc.* , to appear.
9. W. Sun and X. Zhou, "Density and stability of wavelet frames," *Appl. Comput. Harmon. Anal.* , to appear.
10. O. Christensen, B. Deng, and C. Heil, "Density of Gabor frames," *Appl. Comput. Harmon. Anal.* **7**(3), pp. 292–304, 1999.
11. G. Weiss and E. N. Wilson, "The mathematical theory of wavelets," in *Twentieth century harmonic analysis—a celebration (Il Ciocco, 2000)*, *NATO Sci. Ser. II Math. Phys. Chem.* **33**, pp. 329–366, Kluwer Acad. Publ., Dordrecht, 2001.
12. H. J. Landau, "Necessary density conditions for sampling and interpolation of certain entire functions," *Acta Math.* **117**, pp. 37–52, 1967.
13. E. Hernández, D. Labate, G. Weiss, and E. Wilson, "Oversampling, quasi affine frames and wave packets," preprint.
14. A. Ron and Z. Shen, "Affine systems in $L_2(\mathbf{R}^d)$: the analysis of the analysis operator," *J. Funct. Anal.* **148**(2), pp. 408–447, 1997.
15. M. Bownik, "Quasi-affine systems and the calderón condition," in *Proc. Conf. Harmonic Analysis (Mt. Holyoke, MA, 2001)*, *Contemp. Math.*, American Mathematical Society, (Providence, RI), to appear.