Reconstruction of Finite-Valued Sparse Signals

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ABSTRACT

The need of reconstructing discrete-valued sparse signals from few measurements, that is solving an undetermined system of linear equations, appears frequently in science and engineering. Those signals appear, for example, in error correcting codes as well as massive Multiple-Input Multiple-Output (MIMO) channel and wideband spectrum sensing. A particular example is given by wireless communications, where the transmitted signals are sequences of bits, i.e., with entries in $\{0, 1\}$. Whereas classical compressed sensing algorithms do not incorporate the additional knowledge of the discrete nature of the signal, classical lattice decoding approaches do not utilize sparsity constraints. In this talk, we present an approach that incorporates a discrete values prior into basis pursuit. In particular, we address finite-valued sparse signals, i.e., sparse signals with entries in a finite alphabet. We will introduce an equivalent null space characterization and show that phase transition takes place earlier than when using the classical basis pursuit approach. We will further discuss robustness of the algorithm and show that the nonnegative case is very different from the bipolar one. One of our findings is that the positioning of the zero in the alphabet - i.e., whether it is a boundary element or not - is crucial.

Keywords: Compressed Sensing, Sparse Recovery, Null Space Property, Finite Alphabet, Phase Transition, Box Constraints

1. INTRODUCTION

Compressed sensing was introduced as an effective tool to acquire signals from an underdetermined system of linear equations under some realistic additional constraint. More precisely, in compressed sensing we aim to solve the underdetermined system

$$Ax_0 = b,$$

with $A \in \mathbb{R}^{m\times N}$ $(m < N)$ and $b \in \mathbb{R}^m$ by using the a priori information that most of the entries of x_0 are zero. In this situation, necessary and sufficient conditions, for instance, null space and incoherence properties of the measurement matrix A for the exact recovery of the signal x_0, even when b is contaminated with noise, are known.

However, there are many applications, where the signal fulfills a secondary structure constraint besides sparsity. That is, the nonzero entries of x_0 stem from a finite or discrete alphabet. Those signals appear, for example, in error correcting codes[1] as well as massive Multiple-Input Multiple-Output (MIMO) channel[2] and wideband spectrum sensing.[3] There also exist several examples of applications, where the transmitted data originate from a general finite set $\mathcal{A} \subset \mathbb{R}$ such as in source decoding[4] or radar.[5]

In the following we will focus on signals with entries from a bounded lattice and show that compressed sensing recovery guarantees for those signals can be improved significantly in some cases. More precisely, we will focus on the following two structural assumptions. First the signal $x_0 \in \mathbb{R}^N$ is assumed to be k-sparse, for some $k \in [N] = \{1, \dots, N\}$, i.e., $|\{i : (x_0)_i \neq 0\}| \leq k$. And, secondly, we will assume that the entries of the signal $x_0 \in \mathbb{R}^N$ stem from a finite alphabet $\mathcal{A}$, i.e., $\mathcal{A} \subset \mathbb{R}$ is a finite set of real numbers.

We first consider the general case $\mathcal{A} = \{-L_1, \dots, L_2\}$, $L_1, L_2 \in \mathbb{N}$, and then $\mathcal{A} = \{0, \dots, L\}$, $L \in \mathbb{N}$. Surprisingly, it will turn out that those alphabets exhibit quite different phenomena due to the positioning of

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the zero within the set. Note, that the cases $\mathcal{A} = \{0, 1\}$ and $\mathcal{A} = \{-1, 0, 1\}$ are particularly included. We will see that the proposed recovery algorithm will exploit the structure of those signals exceptionally well.

So far sparsity and finiteness have been considered mostly separately. Compressed sensing focusses almost entirely on sparsity without considering finiteness,[6,7] whereas approaches such as lattice decoding[8,9] utilize the finite nature of a signal without taking sparsity into account. There are only few examples, which deal with signals having entries from a finite alphabet. Drapper & Malekpour,[10] and Tian, Leus & Lottici[11] assumed that $\mathcal{A} = \{1, \dots, p\}$ is a field, i.e., p is prime. More closely related is the work,[12] in which signals with entries in $\mathcal{A} = \{-1, 1\}$ have been considered as so-called *saturated vectors*. In [Lange, Pfetsch, Seib, Tillmann][13] conditions for the unique recoverability of integer-valued signals have been studied. The therein considered problems are in general NP-hard. However, for binary variables, medium-sized problems are shown to be solvable in reasonable time.

There already exist some few cases in which compressed sensing algorithms were adapted to the finite or rather discrete situation. One case is orthogonal matching pursuit (OMP), which has been considered in connection with quantization, soft feedback[14] and the sphere decoder.[15] Additionally, in [Flinth, Kutyniok][16] the knowledge of the discrete nature has been used to initialize the support set for the OMP algorithm. This approach is able to slightly beat conventional compressed sensing algorithms. The first mentioned approaches show improvements of the symbol error rates by incorporating the finite or discrete structure in OMP, however, they do not consider the reduction of the number of measurements.

In this work we will consider versions of the so-called basis pursuit instead of the OMP. *Basis pursuit* is a popular and by now well-understood approach to recover sparse signals from an underdetermined linear system and given by:[17]

$$\min \|x\|_1 \quad \text{subject to} \quad Ax = b. \tag{P_1}$$

As this problem is convex, it can be solved elementary with the help of convex optimization methods. A necessary and sufficient condition under which x_0 is uniquely recovered by basis pursuit is given as follows: The set of all feasible solutions, $x_0 + \ker A$, intersects with the set $\{x : \|x\|_1 \leq \|x_0\|_1\}$ exactly at x_0 (cf. Figure **??**). This condition provides a useful geometric intuition about properties of measurement matrices to ensure uniqueness of the solution. One of those properties is the so-called *null space property (NSP)* given by

$$\ker A \cap \{w \in \mathbb{R}^N : \|w_K\|_1 \geq \|w_{K^C}\|_1\} = \{0\}, \tag{NSP}$$

where $K = \operatorname{supp}(x_0)$ and $K^C = [N] \setminus K$, where $[N] = \{1, \dots, N\}$. Notice, that for a set $B \subset [N]$ and a vector $x \in \mathbb{R}^N$, we denote by x_B the vector in $\mathbb{R}^N$ which coincides with x on the indices in B and is zero outside of B.

It is well-known that, if A fulfills the NSP with respect to some subset $K \subset [N]$, every signal x_0 supported on K is the unique minimizer of (P_1) with $b = Ax_0$.[7,18–21]

By using random matrices A such as a matrix with i.i.d. Gaussian entries it is possible to achieve a very high probability of A having the NSP and therefore of (P_1) to succeed, given that the number of measurements satisfies $m \geq Ck \log(N)$, where k is the sparsity of the signal x_0 and C some positive constant not depending on k and N.[22] In the following, we aim to decrease the number of measurements m further using additional structural assumptions.

Note that Flinth & Kutyniok[16] showed that running basis pursuit followed by post-projection to the alphabet does not help to recover the exact solution; one intuition behind this result being that the finite nature of the signal is not incorporated in the reconstruction algorithm.

This implies that a better performance for finite-valued signals can only be achieved, if the finite nature of the signals is incorporated into basis pursuit. One first idea could be to solve the problem given by

$$\min \|x\|_1 \quad \text{subject to} \quad Ax = b \quad \text{and} \quad x \in \mathcal{A}^N.$$

Unfortunately, this is a very hard problem due to the nonconvexity of $\mathcal{A}$. To resolve the problem of nonconvexity of the set $\mathcal{A}^N$, one can instead consider the following minimization problem

$$\min \|x\|_1 \quad \text{subject to} \quad Ax = b \quad \text{and} \quad x \in \operatorname{conv} \mathcal{A}^N,$$

to which we usually refer to as *basis pursuit with box constraints*, which was already considered by Stojnic[23] for $\mathcal{A} = \{0, 1\}$.

In this talk we give an overview of the results in [Keiper, Kutyniok, Lee, Pfander].[24] In particular we will present analytical results for the recovery of finite-valued k-sparse signals using basis pursuit with box constraints in full generality. The related alphabets belong to either the unipolar ($\mathcal{A} = \{0, \dots, L\}, L \in \mathbb{Z}$) or the bipolar ($\mathcal{A} = \{-L_1, \dots, L_2\}$, $L_1, L_2 \in \mathbb{N}$) situation.

We will introduce null space properties for the recovery of finite-valued k-sparse signals, which allow equivalent conditions for unique recoverability of such signals (cf. Theorems 2.2, 3.2). We will then state that all versions of basis pursuit with box constraints – adapted to the specific alphabet considered – are stable under noisy measurements with precise error bounds. Finally, one can analytically compute the phase transitions of all versions of basis pursuit with box constraints (see Theorems 2.3, 3.3) in the case of a *Gaussian measurement matrix* A, i.e.,

$$A = m^{-1/2} \, [a_{i,j}]_{i,j=1}^{m,N}, \quad \text{with i.i.d.} \quad a_{i,j} \sim \mathcal{N}(0, 1). \tag{1}$$

The Results[24] will surprisingly show that the unipolar situation is very different from the bipolar one. One of the important findings is that the positioning of the zero – i.e., whether it is a boundary element or not – is crucial. A second key observation is the fact that mainly the boundary elements play a role in the sense of $-L_1$ and L_2 in the case of bipolar finite-valued signals.

2. BIPOLAR FINITE-VALUED SIGNALS

We first consider the most general type of finite-valued sparse signals, namely signals $x_0 \in \mathcal{A}^N$, where $\mathcal{A} = \{-L_1, \dots, L_2\}$. In the sequel it will turn out that performance guarantees can be significantly improved in the case of an alphabets $\mathcal{A} = \{0, \dots, L\}$. We, thus, will call a signal *bipolar finite-valued signal*, if we allow for both, nonnegative and nonpositive entries to emphasize this case.

In the subsequent we will frequently use the following notation for a finite-valued signal $x \in [-L_1, L_2]^N$:

$$K := K(x) := \{j : x_j \neq 0\}, \quad K_- := K_-(x) = \{j : x_j < 0\} \quad \text{and} \quad K_+ := K_+(x) = \{j : x_j > 0\}. \tag{2}$$

We further introduce a notation for the sets of indices, where x either takes the value $-L_1$ or L_2, i.e., the values with largest amplitude:

$$K_{-L_1} {:=} K_{-L_1}(x) {:=} \{j : x_j = -L_1\} \quad \text{and} \quad K_{L_2} {:=} K_{L_2}(x) {:=} \{j : x_j = L_2\}. \tag{3}$$

The key objective now is to solve the underdetermined system of linear equations

$$Ax_0 = b,$$

with $A \in \mathbb{R}^{m \times N}$ and $b \in \mathbb{R}^m$, $m < N$, under the additional assumption $x_0 \in \mathcal{A}^N$, $\mathcal{A} = \{-L_1, \dots, L_2\}$, $L_1, L_2 \in \mathbb{N}$, and $\|x_0\|_0 \leq k$. A natural approach would be to exploit basis pursuit under the additional constraint that $x_0 \in \mathcal{A}^N$. However, this problem would be very hard to solve, wherefore we 'convexify' the additional constraint. This yields *basis pursuit with box constraints*

$$\min \|x\|_1 \quad \text{subject to} \quad Ax = b \quad \text{and} \quad x \in [-L_1, L_2]^N. \tag{$P_{\mathcal{F}}$}$$

2.1 Finite NSP

One can now introduce the following weakend NSP as a necessary and sufficient condition on the measurement matrix A, such that ($P_{\mathcal{F}}$) uniquely recovers a given finite-valued sparse signal.

Definition 2.1. *Let $B_1 \subset B_-, B_2 \subset B_+ \subset [N]$ with $B_- \cap B_+ = \emptyset$. A matrix $A \in \mathbb{R}^{m \times N}$ is said to satisfy the* finite NSP *with respect to B_1, B_2, B_- and B_+, if*

$$\ker A \cap N_{B_-, B_+} \cap H_{B_1, B_2} = \{0\}, \tag{$\mathcal{F}$-NSP}$$

where

$$N_{B_-,B_+} := \left\{ w \in \mathbb{R}^N : \sum_{i \in B_-} w_i - \sum_{i \in B_+} w_i \geq \sum_{i \in B^C} \|w_i\| \right\},$$

with $B = B_+ \cup B_-$ *and*

$$H_{B_1,B_2} = \left\{ w \in \mathbb{R}^N : w_i \geq 0 \text{ for } i \in B_1 \text{ and } w_i \leq 0 \text{ for } i \in B_2 \right\}.$$

Observe that the (classical) NSP can be written as $\ker A \cap N_B = \{0\}$, with $N_B = \{w \in \mathbb{R}^N : \sum_{i \in B_-} \|w_i\| + \sum_{i \in B_+} \|w_i\| \geq \sum_{i \in B^C} \|w_i\|\}$. Because of $N_{B_-,B_+} \subset N_B$, the $\mathcal{F}$-NSP is indeed weaker than the NSP.

One can indeed verify that the $\mathcal{F}$-NSP is necessary and sufficient to recover a bipolar finite-valued signal via ($P_{\mathcal{F}}$). This fact is stated by the following theorem.

Theorem 2.2. *Let $x_0 \in \mathcal{A}^N$ be a bipolar finite-valued signal and let $K_{-L_1} \subset K_-, K_{L_2} \subset K_+$ be defined as in (2) and (3) for x_0, and let $A \in \mathbb{R}^{m \times N}$. Then the following conditions are equivalent:*

(i) The vector x_0 is the unique solution of ($P_{\mathcal{F}}$) with $b = Ax_0$.

(ii) The matrix A fulfills the $\mathcal{F}$-NSP with respect to the sets K_{-L_1}, K_{L_2}, K_-, and K_+.

2.2 Phase Transition in Basis Pursuit with Box Constraints

The next aim is to show that for m large, a Gaussian matrix fulfills the $\mathcal{F}$-NSP with respect to some fixed but unknown support sets K_{-L_1}, K_{L_2}, K_-, and K_+ with high probability, i.e., that the kernel of a Gaussian matrix trivially intersects the convex cone $N_{K_-,K_+} \cap H_{K_{-L_1},K_{L_2}}$ with high probability.

Amelunxen, Lotz, McCoy & Tropp[25] have shown that the probability of $\ker(A) \cap B = \{0\}$, for some convex set $B \subset \mathbb{R}^N$, can be estimated in terms of the statistical dimension of B. This becomes particularly computable, if B is a descent cone. We, therefore rely on the fact that $N_{K_-,K_+} \cap H_{K_{-L_1},K_{L_2}}$ can be recast into the form of a descent cone of the function $f : \mathbb{R}^N \to \mathbb{R}_+$ defined by

$$f(x) = \begin{cases} \sum_{i \in K^C} \|x_i\| - \sum_{i \in K_-} x_i + \sum_{i \in K_+} x_i, & \text{if } x \in [-L_1, L_2]^N, \\ \infty, & \text{otherwise.} \end{cases}$$

Consequently, one obtains the number of measurements necessary to recover finite-valued signals with high probability.

Theorem 2.3. *Fix a tolerance $\varepsilon > 0$. Let x_0 be a bipolar finite-valued signal and let $K_{-L_1} \subset K_-, K_{L_2} \subset K_+ \subset [N]$ be defined as in (2) and (3) for x_0. Further, let $A \in \mathbb{R}^{m \times N}$ be a Gaussian matrix according to (1) as well as $b = Ax_0$. Set $k_{bnd} = |K_{-L_1} \cup K_{L_2}|$ and $k_0 = |[N] \setminus (K_- \cup K_+)|$. Define*

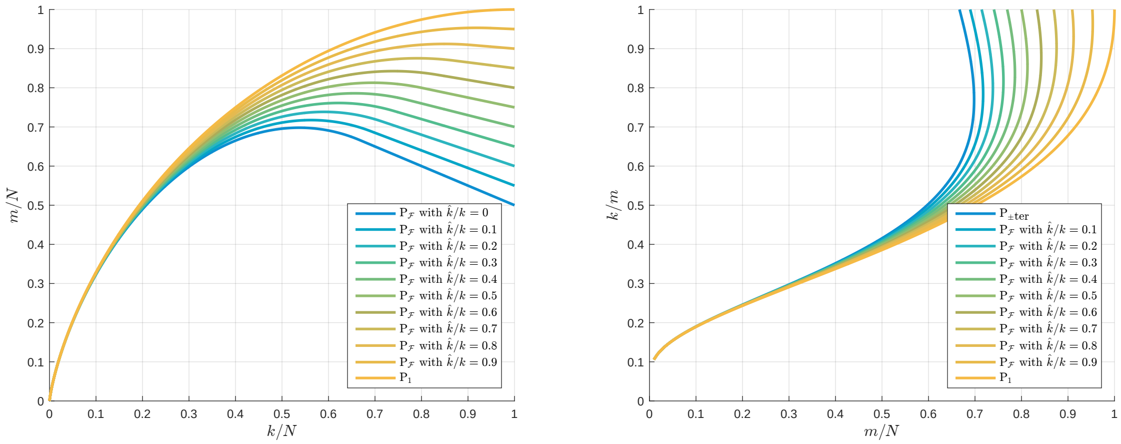
$$\Delta_{\mathcal{F}}(k_0, k_{bnd}) := \inf_{\tau \geq 0} \left\{ 2k_0 \int_{\tau}^{\infty} (u-\tau)^2 \phi(u) du + (N - k_0 - k_{bnd})(1+\tau^2) + k_{bnd} \int_{-\infty}^{\tau} (u-\tau)^2 \phi(u) du \right\},$$

where $\phi(u) = (2\pi)^{-1/2} e^{-u^2/2}$ is the probability density of the standard Gaussian distribution. If

$$m \geq \Delta_{\mathcal{F}}(k_0, k_{bnd}) + \sqrt{8 \log(4/\varepsilon) N},$$

then ($P_{\mathcal{F}}$) will succeed to recover x_0 uniquely with probability larger than $1 - \varepsilon$.

The behavior of the function $\Delta_{\mathcal{F}}$ is sketched in Figure 1 for different pairs of k_0 and k_{bnd}. One can observe, that $\Delta_{\mathcal{F}}$ does not only depend on the number of nonzero entries, but it also depends on the number of entries which do not have largest amplitude. We observe, that in the worst case, we cannot improve performance guarantees compared to classical basis pursuit. However, if the ratio of k_{bnd} and $|K| = |K_- \cup K_+|$ is close to one, performance guarantees improve significantly (see Figure 1). This phenomenon in particular influences recovery results for the alphabets $\mathcal{A} = \{-1, 0, 1\}$ and $\mathcal{A} = \{0, 1\}$, where we specifically have $|K| = k_{\text{bnd}}$ and thus receive the lowest of the curves in Figure 1 as phase transition.



(a) (b)

Figure 1. Phase transition of the convex program ($P_{\mathcal{F}}$) according to the ratio of $\tilde{k} = k - k_{\text{bnd}}$ to k, where k is the size of the whole support of a bipolar finite-valued signal and k_{bnd} the number of entries having largest amplitude. For the convenience of the reader, the following two illustrations are provided: Successful recovery related to the area above the curves in (a), and below the curves in (b).

2.3 Robust Finite NSP

In most applications we cannot measure signals with infinite precision. We therefore need to answer the question of the stability of a recovery algorithm with respect to noisy measurements, i.e., in case the measurement vector b is only an approximation of the vector Ax_0 with $\|Ax_0 - b\|_2 \leq \eta$ for some $\eta \geq 0$. One way to tackle this problem numerically is to replace the constraint $Ax = b$ in ($P_{\mathcal{F}}$) by $\|Ax - b\| \leq \eta$.[7] In our situation, this yields the following algorithm, which we refer to as *basis pursuit denoising with box constraints*:

$$\min \|z\|_1 \quad \text{subject to} \quad \|Az - b\|_2 \leq \eta \quad \text{and} \quad z \in [-L_1, L_2]^N. \tag{$\mathrm{P}_{\mathcal{F}}^{\eta}$}$$

After introducing a sufficient null space property of the measurement matrix A, Theorem 2.5 will state that ($\mathrm{P}_{\mathcal{F}}^{\eta}$) is indeed robust to noise under this condition.

Definition 2.4. *A matrix $A \in \mathbb{R}^{m \times N}$ is said to satisfy the* robust finite null space property *with constants $0 < \rho < 1$ and $\tau > 0$ relative to the disjoint sets $B_1, B_2, \hat{B} \subset [N]$, if*

$$\sum_{i \in B_1} v_i - \sum_{i \in B_2} v_i + \sum_{i \in \hat{B}} \|v_i\| \leq \rho \sum_{i \in B^C} \|v_i\| + \tau \|Av\|_2 \quad \text{for any } v \in H_{B_1, B_2}, \tag{R$\mathcal{F}$-NSP}$$

where $B = B_1 \cup B_2 \cup \hat{B}$.

Theorem 2.5. *Let x_0 be a bipolar finite-valued signal and $K_-, K_+, K_{-L_1}, K_{L_2}$ be defined as in (2), (3). Further let $A \in \mathbb{R}^{m \times N}$ satisfy the R$\mathcal{F}$-NSP with constants $0 < \rho < 1$ and $\tau > 0$ relative to the sets $K_{-L_1}, K_{L_2}, \hat{K} = (K_- \cup K_+) \setminus (K_{-L_1} \cup K_{L_2})$. Further, assume that the measurements satisfy $b = Ax_0 + e$, where the noise $e \in \mathbb{R}^N$ satisfies $\|e\|_2 \leq \eta$, for some $\eta > 0$. Then a solution $\hat{z}$ of ($\mathrm{P}_{\mathcal{F}}^{\eta}$) approximates x_0 with ℓ_1-error*

$$\|\hat{z} - x_0\|_1 \leq \frac{4\tau}{1 - \rho} \eta.$$

2.4 Phase Transition under Noisy Measurements

We have seen that ($P^{\eta}_{\mathcal{F}}$) is robust provided that the sensing matrix A satisfies the R$\mathcal{F}$-NSP. However, it is hard to verify in general, if a measurement matrix fulfills this property. However, one can show that Gaussian matrices are in terms of the NSP well-suited for robust recovery of unipolar binary signals provided that m is sufficiently large.

Since the outcome of the adapted basis pursuit ($P^{\eta}_{\mathcal{F}}$) is not necessarily integral, we utilize the finite nature of the signal and post-project the outcome to the integers in the spirit of.[16] Hence, we consider the following algorithm:

$$\tilde{z} = \mathrm{round}(\hat{z}) \quad \text{with} \quad \hat{z} = \operatorname{argmin} \|x\|_1 \quad \text{subject to} \quad \|Ax - b\|_2 \le \eta \quad \text{and} \quad x \in [-L_1, L_2]^N. \qquad (P^{\eta,\mathrm{r}}_{\mathcal{F}})$$

Note, that even though post-projection will not help to improve performance guarantees in the noiseless case,[16] we can use it to round an approximate solution of the noisy case to the exact and unique solution. This fact is included in the next theorem.

THEOREM 2.6. *Let Let x_0 be a bipolar finite-valued signal and $K_-, K_+, K_{-L_1}, K_{L_2}$ be defined as in (2), (3). Further let $A \in \mathbb{R}^{m\times N}$ be a Gaussian measurement matrix as in (1). Further, assume that the measurements satisfy $b = A\hat{x} + e$ with $\|e\|_2 \le \eta$ for some $\eta > 0$, and let $0 < \varepsilon < 1$, $\tau > 0$. If m fulfills*

$$m - 1 \ge \left(\sqrt{\ln(2\varepsilon^{-1})} + \sqrt{\Delta_{\mathcal{F}}(k_0, k_{bnd})} + \tau\right)^2,$$

where $k_{bnd} = |K_{-L_1} \cup K_{L_2}|$ and $k_0 = |[N] \setminus (K_- \cup K_+)|$, then, with probability at least $1-\varepsilon$, x_0 is the unique solution of ($P^{\eta,\mathrm{r}}_{\mathcal{F}}$).

3. UNIPOLAR FINITE-VALUED SIGNALS

We already mentioned that the positioning of the zero within the alphabet plays a crucial role in the performance of basis pursuit with box constraints. This phenomenon already appeared in the classical basis pursuit approach, as Stojnic showed that basis pursuit restricted to the nonnegative orthant provides an improved performance in recovering nonnegative-valued sparse signals.[26] The same effect appears for basis pursuit with box constraints.

In the sequel we will call a signal $x_0 \in \{0, \dots, L\}$ *unipolar finite-valued* and use the following notation for specific support sets of x_0:

$$K := K(x) := \{j : x_j > 0\}, \quad K_L := K_L(x) := \{j : x_j = L\}. \qquad (4)$$

To recover a unipolar finite-valued signal, we solve the underdetermined system of linear equations

$$Ax_0 = b,$$

with $A \in \mathbb{R}^{m\times N}$ and $b \in \mathbb{R}^m$, $m < N$, under the additional assumption that $x_0 \in \mathcal{A}_{\mathcal{U}}^N$ with $\mathcal{A}_{\mathcal{U}} = \{0, \dots, L\}$, and $\|x_0\|_0 \le k$. We will again use basis pursuit with box constraint, which yields for unipolar finite-valued signals the program given by

$$\min \|x\|_1 \quad \text{subject to} \quad Ax = b \quad \text{and} \quad x \in [0, L]^N. \qquad (P_{\mathcal{UF}})$$

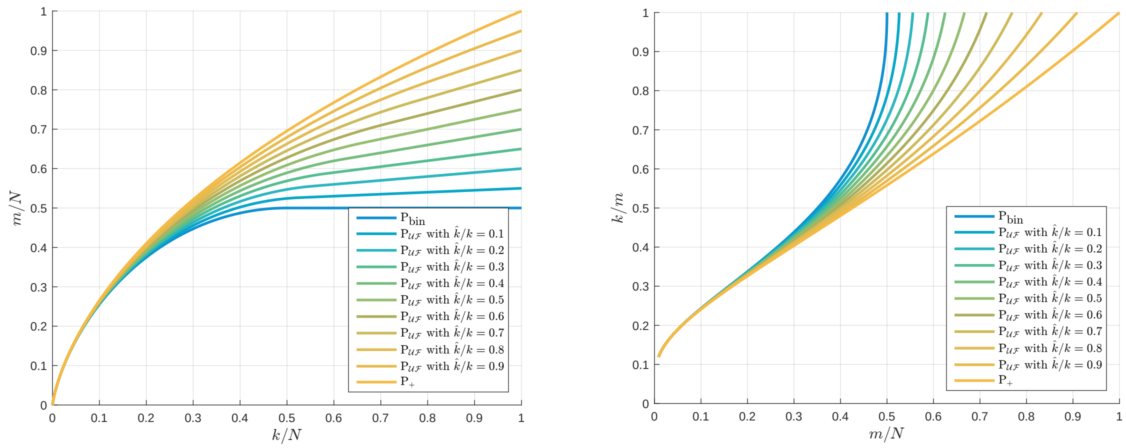
The following null space property can be shown to be necessary and sufficient for ($P_{\mathcal{UF}}$) to succeed in recovering a unipolar finite-valued signal.

DEFINITION 3.1. *Let $B_1 \subset B \subset [N]$. A matrix $A \in \mathbb{R}^{m\times N}$ is said to satisfy the* unipolar finite NSP *with respect to B_1 and B, if*

$$\ker A \cap N^+ \cap H_{B^C, B_1} = \{0\}, \qquad (\mathcal{UF}\text{-NSP})$$

where $H_{B^C,B_1} = \{w \in \mathbb{R}^N : w_i \le 0 \text{ for } i \in B_1 \text{ and } w_i \ge 0 \text{ for } i \in B^C\}$ and $N^+ = \{w \in \mathbb{R}^N : \sum_{i=1}^N w_i \le 0\}$.

THEOREM 3.2. *Let $x_0 \in \mathcal{A}_{\mathcal{U}}^N$ be a unipolar finite-valued signal and K_L, K be defined as in (4) for x_0. Further, let $A \in \mathbb{R}^{m\times N}$. Then the following conditions are equivalent:*



(a) (b)

Figure 2. Phase transition of the convex program ($P_{\mathcal{UF}}$) according to the ratio of $\hat{k}$ to k, where k is the size of the entire support of a unipolar finite-valued signal and $\hat{k} = K \setminus K_L$ the number of entries in the signal not equal to zero or to the largest value of the given alphabet. For the convenience of the reader, the following two illustrations are provided: Successful recovery related to the area above the curves in (a), and below the curves in (b).

(i) The measurement matrix A satisfies the $\mathcal{UF}$-NSP with respect to sets K_L and K.

(ii) The vector x_0 is the unique solution of ($P_{\mathcal{UF}}$) with $b = Ax_0$.

Using this result, one can again compute the number of sufficient measurements for ($P_{\mathcal{UF}}$) to succeed, given that $A \in \mathbb{R}^{m\times N}$ is a Gaussian matrix. The following theorem gives a sharp bound.

THEOREM 3.3. *Fix a tolerance $\varepsilon > 0$. Let x_0 be a unipolar finite-valued signal and $K_L \subset K \subset [N]$ defined as in (4). Further, let $A \in \mathbb{R}^{m\times N}$ be a Gaussian matrix as defined in (1), $b = Ax_0$, and let $\phi(u) = (2\pi)^{-1/2}e^{-u^2/2}$ the probability density of the Gaussian distribution. Setting $k_L := |K_L|$, and $k_0 := |[N] - K|$, provided that*

$$m \geq \Delta_{\mathcal{UF}}(k_0, k_L) + \sqrt{8\log(4/\varepsilon)N},$$

where

$$\Delta_{\mathcal{UF}}(k_0, \hat{k}) = \inf_{\tau\geq 0}\left\{(N - k_0 - k_L)(1+\tau^2) + k_0\int_{\tau}^{\infty}(u-\tau)^2\phi(u)du + (k_L)\int_{-\infty}^{\tau}(u-\tau)^2\phi(u)du\right\},$$

the program ($P_{\mathcal{UF}}$) will succeed to recover x_0 uniquely with probability larger than $1-\varepsilon$.

Note, that Theorem 3.3 is true for all signals of the form $\hat{x} = \breve{x}_{\hat{K}} + L\mathbb{1}_{K_L}$, where $\hat{K} = K \setminus K_L$ and $\breve{x} \in (0, L)^N$. In fact, the phase transition highly depends on the size of K_L relative to K. In the worst case, namely when $K_L = \emptyset$, the phase transition coincides with the one of nonnegative basis pursuit;[26] in the best case, namely when the ratio of k_L to $k = |k|$ is close to one, it appears much earlier. For an illustration we refer to Figure 2.

The question of robustness in the unipolar finite-valued case is of course also crucial. We will again use the a priori knowledge that the original signal is discrete-valued. We therefore consider the following variant of quadratically constrained basis pursuit:

$$\tilde{z} = \text{round}(\hat{z}) \quad \text{with} \quad \hat{z} = \text{argmin}\,\|x\|_1 \quad \text{subject to} \quad \|Ax - b\|_2 \leq \eta \quad \text{and} \quad x \in [0, L]^N. \qquad (P_{\mathcal{UF}}^{\eta,\text{r}})$$

[Keiper, Kutyniok, Lee, Pfander][24] have shown that ($P_{\mathcal{UF}}^{\eta,\mathrm{r}}$) is indeed robust provided that the sensing matrix $A \in \mathbb{R}^{m\times N}$ fulfills the an adapted $\mathcal{UF}$-NSP condition. Using this sufficient condition one can prove the following result for ($P_{\mathcal{UF}}^{\eta,\mathrm{r}}$).

THEOREM 3.4. *Let x_0 be a unipolar finite-valued signal and K, K_L as in (4), and let $A \in \mathbb{R}^{m\times N}$ be a Gaussian measurement matrix. Further, assume that the measurements satisfy $b = Ax_0 + e$, where $\|e\|_2 \le \eta$, for some $\eta > 0$, and let $0 < \varepsilon < 1$, and m fulfill*

$$m - 1 \ge \left(\sqrt{\ln(2\varepsilon^{-1})} + \sqrt{\Delta_{\mathcal{UF}}(k_0, k_L)} + 4\eta\right)^2,$$

where $k_0 = |K^C|$, $k_L = |K_L|$. Then, with probability at least $1-\varepsilon$, x_0 is the unique solution of ($P_{\mathcal{UF}}^{\eta,\mathrm{r}}$).

4. NONUNIFORM AND UNIFORM RECOVERY

In compressed sensing one usually distinguishes between uniform and nonuniform recovery guarantees. A nonuniform guarantee means that we can recover a fixed sparse signal using a random drawn measurement matrix with high probability. A uniform recovery result on the other hand states that we can recover all sparse signals using the same random matrix with high probability. Uniform guarantees are clearly stronger than nonuniform. The authors[24] have presented nonuniform recovery result for finite-valued signals. Thus the question remains whether one can also show uniform recovery results. In fact, this is not the case, which is implied by the next theorem.

THEOREM 4.1. *For $A \in \mathbb{R}^{m\times N}$ and $K \subset [N]$, the following statements are equivalent:*

(i) Every unipolar binary vector x_0 with $\operatorname{supp} x_0 \subset K$ is the unique solution of ($P_{\mathcal{UF}}$) with $b = Ax_0$.

(ii) Every vector $x_0 \in [0,1]^N$ with $\operatorname{supp} x_0 \subset K$ is the unique solution of ($P_{\mathcal{UF}}$) with $b = Ax_0$.

(iii) The measurement matrix A satisfies the NSP$_+$ with respect to K, i.e., $\ker A \cap N^+ \cap H_K^+ = \{0\}$, where

$$H_K^+ = \{w \in \mathbb{R}^N : w_i \ge 0 \text{ for } i \in K^C\} \quad \text{and} \quad N^+ = \{w \in \mathbb{R}^N : 0 \ge \sum_{i=1}^N w_i\}.$$

Thus, even if we only wish to recover every unipolar binary signal x_0 with $\operatorname{supp} x_0 \subset K$, the measurement matrix A needs to fulfill a much stronger property. This property is actually sufficient to uniquely recover *every* nonnegative signal supported on K via nonnegative basis pursuit. Thus, if we wish to show uniform recovery results, additional assumptions on the signal to be unipolar binary are not beneficial.

REFERENCES

[1] Candes, E. J., Rudelson, M., Tao, T., and Vershynin, R., "Error correction via linear programming," in [*46th Annual IEEE Symposium on Foundations of Computer Science (FOCS'05)*], 668–681 (2005).

[2] Rossi, M., Haimovich, A., and Eldar, Y., "Spatial compressive sensing for mimo radar," *IEEE Trans. Signal Process.* **62**(2), 419–430 (2013).

[3] Axell, E., Leus, G., Larsson, E. G., and Poor, H. V., "Spectrum sensing for cognitive radio: State-of-the-art and recent advances," *IEEE Signal Process. Mag.* **29**(3), 101–116 (2012).

[4] Dymarski, P. and Romaniuk, R., "Sparse signal modeling in a scalable celp coder," in [*Proc. EUSIPCO, Marrakech, Morocco*], 201–205 (2013).

[5] Ens, A., Yousaf, A., Ostertag, T., and Reindl, L., "Optimized sinus wave generation with compressed sensing for radar applications," in [*Proc. CoSeRa, Bonn, Germany*], (2013).

[6] Donoho, D. L., "Compressed sensing," *IEEE Trans. Inform. Theory* **52**, 1289–1306 (2006).

[7] Foucart, S. and Rauhut, H., [*A Mathematical Introduction to Compressive Sensing*], Applied and Numerical Harmonic Analysis. Birkhäuser Basel (2013).

[8] Agrell, E., Eriksson, T., Vardy, A., and Zeger, K., "Closest point search in lattices," *IEEE Trans. Inform. Theory* **48**(8), 2201–2214 (2002).

[9] Yao, H. and Wornell, G. W., "Lattice-reduction-aided detectors for mimo communication systems," in [*IEEE Global Telecommunications Conference (GLOBECOM)*], **1**, 424–428 (2002).

[10] Draper, S. C. and Malekpour, S., "Compressed sensing over finite fields," in [*IEEE International Symposium on Information Theory (ISIT)*], 669–673 (2009).

[11] Tian, Z., Leus, G., and Lottici, V., "Detection of sparse signals under finite-alphabet constraints," in [*IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*], 2349–2352 (2009).

[12] Mangasarian, O. L. and Recht, B., "Probability of unique integer solution to a system of linear equations," *EJOR* **214**(1), 27–30 (2011).

[13] Lange, H., Pfetsch, M., Seib, B., and Tillmann, A., "Sparse recovery with integrality constraints," *preprint, arXiv:1608.08678* (2016).

[14] Sparrer, S. and Fischer, R. F. H., "Adapting compressed sensing algorithms to discrete sparse signals," in [*Proceedings of International ITG/IEEE Workshop on Smart Antennas*], (2014).

[15] Sparrer, S. and Fischer, R. F. H., "Soft-feedback omp for the recovery of discrete-valued sparse signals," in [*23. European Signal Processing Conference (EUSIPCO)*], 1461–1465 (2015).

[16] Flinth, A. and Kutyniok, G., "Promp: A sparse recovery approach to lattice-valued signals," *Appl. Comput. Harm. Anal., in press.* (2017).

[17] Chen, S., Donoho, D., and Saunders, M., "Atomic decomposition by basis pursuit," *SIAM J. Sci. Comput.* **20**(1), 33–61 (1998).

[18] Cohen, A., Dahmen, W., and DeVore, R. A., "Compressed sensing and best k-term approximation," *J. Amer. Math. Soc.* **22**(1), 211–231 (2009).

[19] Donoho, D. L. and Elad, M., "Optimally sparse representations in general (non-orthogonal) dictionaries via ℓ_1 minimization," *Proc. Nat. Acad. Sci.* **100**(5), 2197–2202 (2003).

[20] Donoho, D. L. and Huo, X., "Uncertainty principles and ideal atomic decompositions," *IEEE Trans. Inform. Theor.* **47**(7), 2845–2862 (2001).

[21] Elad, M. and Bruckstein, A. M., "A generalized uncertainty principle and sparse representation in pairs of bases," *IEEE Trans. Inform. Theor.* **48**(9), 2558–2567 (2002).

[22] Candes, E. J., Romberg, J., and Tao, T., "Stable signal recovery from incomplete and inaccurate measurements," *Commun. Pure Appl. Math.* **59**(8), 1207–1223 (2006).

[23] Stojnic, M., "Recovery thresholds for ℓ_1 optimization in binary compressed sensing," in [*IEEE International Symposium on Information Theory (ISIT)*], 1593–1597 (2010).

[24] Keiper, S., Kutyniok, G., Lee, D., and Pfander, G., "Compressed sensing for finite-valued signals," *Linear Algebra Appl., in press.* (2017).

[25] Amelunxen, D., Lotz, M., McCoy, M. B., and Tropp, J. A., "Living on the edge: Phase transitions in convex programs with random data," *Inf. Inference* **3**, 224–294 (2014).

[26] Stojnic, M., "A simple performance analysis of ℓ_1 optimization in compressed sensing," in [*IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP)*], 3021–3024 (2009).