Perturbations of Fusion Frames and the Effect on Their Canonical Dual

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ABSTRACT

Measurements in real world applications are often corrupted by noise due to exterior influences. It is therefore convenient to investigate perturbations of the utilized measurement systems. In the present paper, we consider small perturbations of fusion frames in Hilbert spaces and study the effect on the canonical duals of orignal and perturbed fusion frame. It turns out that the duals are stable under these perturbations.

Keywords: Fusion frame, perturbation, canonical dual

1. INTRODUCTION

Motivated by the task of piecing frames together which for example becomes necessary when dealing with a huge amount of data, P.G. Casazza and G. Kutyniok introduced fusion frames as *frames of subspaces* in [2]. However, it soon became evident (see [4]) that these objects are perfectly suited for the needs of novel applications requiring distributed processing since they allow to effectively process data in the particular subspaces.

Fusion frames can be employed, similarly to standard frames, for sensing data but also for its reconstruction. In the sensing process, the data is stored in the fusion frame coefficients. On the other hand, for the reconstruction one typically uses so-called duals which synthesize the fusion frame coefficients. Although a widely accepted theory of duality for fusion frames is still not available (for approaches in this direction, see [5, 7, 8]), in analogy to the frame case there is always one special dual for performing the recovery task: the so-called canonical dual.

Since in real world application scenarios we often face corruption of sensored or processed data due to interferences and noise etc., it is natural to assume that the fusion frame at hand – and thereby the coefficients – might undergo small perturbations. However, for recovering the data, the dual of the original fusion frame is applied instead of the perturbed dual which would provide exact recovery. Therefore, it seems desirable to obtain results on how perturbation effects the canonical dual. This is our main objective in the present paper. In fact, the analogue situation for frames has been investigated already in [6]. However, although proofs of frame properties can often be carried over easily to the fusion frame setting, this is not the case here and new methods have to be developed. Our main result is Theorem 3.7 below which states that the canonical dual is stable under the kind of perturbations we consider. We also provide constants for measuring the degree of stability.

The paper is organized as follows. First, we provide some notation at the end of this introduction. In Section 2, we give a brief introduction on frames, fusion frames, and their canonical duals. The main theorem is stated and proved in Section 3 with the help of two auxiliary lemmas.

We close this introduction by fixing the notation we will use. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. The set of all bounded and everywhere defined linear operators between $\mathcal{H}$ and $\mathcal{K}$ will be denoted by $\mathcal{B}(\mathcal{H},\mathcal{K})$. As ususal, we set $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. The norm on $\mathcal{B}(\mathcal{H},\mathcal{K})$ will be the usual operator norm, i.e.

$$\|T\| := \sup\left\{\|Tx\| : x \in \mathcal{H},\, \|x\| = 1\right\}.$$

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The restriction of an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ to a subspace $V \subset \mathcal{H}$ will be denoted by $T|V$. If V is closed, by P_V we denote the orthogonal projection onto V in $\mathcal{H}$. Recall that the space of $\mathcal{H}$-valued ℓ^2-sequences over I, defined by

$$\ell^2(I,\mathcal{H}) := \left\{ (x_i)_{i\in I} : x_i \in \mathcal{H} \ \forall\ i \in I,\ \sum_{i\in I} \|x_i\|^2 < \infty \right\},$$

is a Hilbert space with scalar product

$$\big\langle (x_i)_{i\in I}, (y_i)_{i\in I} \big\rangle = \sum_{i\in I} \langle x_i, y_i \rangle, \quad (x_i)_{i\in I}, (y_i)_{i\in I} \in \ell^2(I,\mathcal{H}).$$

We shall often denote

$$\mathfrak{H} := \ell^2(I,\mathcal{H}).$$

2. FUSION FRAMES AND THEIR CANONICAL DUALS

For the rest of this paper, $\mathcal{H}$ is a fixed Hilbert space and $I \subset \mathbb{N}$ stands for a finite or countable index set. Recall that (see [1]) a *frame* for $\mathcal{H}$ is a sequence $\Phi = (\varphi_i)_{i\in I}$ of vectors in $\mathcal{H}$ for which there exist numbers $A, B > 0$ such that

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, \varphi_i \rangle|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

The numbers A and B are called *frame bounds* of Φ. It is immediately seen that the inequality above implies that the *analysis operator* $T_\Phi : \mathcal{H} \to \ell^2(I)$ of Φ, defined by

$$T_\Phi x := \left(\langle x, \varphi_i \rangle \right)_{i\in I}, \quad x \in \mathcal{H},$$

is bounded. It is well known that its adjoint – the *synthesis operator* of Φ – is given by

$$T_\Phi^*(c_i)_{i\in I} = \sum_{i\in I} c_i \varphi_i, \quad (c_i)_{i\in I} \in \ell^2(I).$$

The *frame operator* of Φ is then defined by $S_\Phi := T_\Phi^* T_\Phi$ and has the form

$$S_\Phi x = \sum_{i\in I} \langle x, \varphi_i \rangle \varphi_i, \quad x \in \mathcal{H}.$$

It is a positive selfadjoint operator in $\mathcal{H}$ which is bounded and boundedly invertible. The *canonical dual* of the frame Φ is defined by $\widetilde{\Phi} = (S_\Phi^{-1}\varphi_i)_{i\in I}$.

The concept of fusion frames is a generalization of frames. Let us recall that a *fusion frame* (see, e.g., [3]) for $\mathcal{H}$ is a sequence $\mathcal{W} = ((W_i, c_i))_{i\in I}$ of closed subspaces W_i of $\mathcal{H}$ and weights $c_i > 0$ for which there exist numbers $A, B > 0$ such that

$$A\|x\|^2 \le \sum_{i\in I} c_i^2 \|P_{W_i} x\|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

The numbers A and B are called the *fusion frame bounds* of $\mathcal{W}$. If only the right hand side inequality holds, $\mathcal{W}$ is called a *Bessel fusion sequence* and B its *Bessel bound*. The *analysis operator* $T_\mathcal{W} : \mathcal{H} \to \mathfrak{H} = \ell^2(I,\mathcal{H})$ of a Bessel fusion sequence $\mathcal{W}$ is defined by

$$T_\mathcal{W} x := \left(c_i P_{W_i} x \right)_{i\in I}, \quad x \in \mathcal{H}.$$

Again, it is clear that $T_\mathcal{W}$ is bounded. The *synthesis operator* of $\mathcal{W}$ is given by

$$T_\mathcal{W}^*(x_i)_{i\in I} = \sum_{i\in I} c_i P_{W_i} x_i, \quad (x_i)_{i\in I} \in \mathfrak{H},$$

and the *fusion frame operator* of $\mathcal{W}$, defined by $S_{\mathcal{W}} := T_{\mathcal{W}}^* T_{\mathcal{W}}$, reads as

$$S_{\mathcal{W}} x = \sum_{i\in I} c_i^2 P_{W_i} x, \quad x \in \mathcal{H}.$$

It is a bounded non-negative selfadjoint operator. It is boundedly invertible if and only if $\mathcal{W}$ is a fusion frame.

REMARK 2.1. *In previous works on fusion frames (see, e.g., [2,3,4]), the analysis operator was not defined to map into $\mathfrak{H} = \ell^2(I, \mathcal{H})$, but into $\bigoplus_{i\in I} W_i$ (which can be regarded as a closed subspace of $\mathfrak{H}$). We have thus slightly modified the definition. The reason is that in this paper we compare fusion frames with each other by considering the difference of their analysis operators.*

Let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ be a fusion frame for $\mathcal{H}$. We now define the *canonical dual* of $\mathcal{W}$ by

$$\widetilde{\mathcal{W}} := \left(\left(S_{\mathcal{W}}^{-1} W_i, \|S_{\mathcal{W}}^{-1}|W_i\| c_i\right)\right)_{i\in I}.$$

We remark that other authors (see, e.g., [5]) choose c_i as the weights instead of $\|S_{\mathcal{W}}^{-1}|W_i\| c_i$, $i \in I$. Therefore, we justify our choice: Consider the canonical dual $\widetilde{\Phi} = (S_{\Phi}^{-1}\varphi_i)_{i\in I}$ of a frame $\Phi = (\varphi_i)_{i\in I}$ for $\mathcal{H}$. Translated to the fusion frame setting, we have

$$W_i = \operatorname{span}\{\varphi_i\} \text{ and } c_i = \|\varphi_i\| \qquad \text{as well as} \qquad \widetilde{W}_i = S_{\Phi}^{-1} W_i \text{ and } \widetilde{c}_i = \|S_{\Phi}^{-1}\varphi_i\|.$$

Thus, if $c_i \neq 0$ (i.e., $\varphi_i \neq 0$), the weights of the canonical dual are $\widetilde{c}_i = c_i\|S_{\Phi}^{-1}(\varphi_i/\|\varphi_i\|)\| = c_i\|S_{\Phi}^{-1}|W_i\|$. The same trivially holds for $c_i = 0$.

LEMMA 2.2. *Let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ be a fusion frame for $\mathcal{H}$. Then also the canonical dual $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ is a fusion frame for $\mathcal{H}$.*

Proof. In [5] it was proved that $\widehat{\mathcal{W}} := ((S_{\mathcal{W}}^{-1} W_i, c_i))_{i\in I}$ is a fusion frame for $\mathcal{H}$ (see also Corollary 3.6 below). Let $A \le B$ be fusion frame bounds of $\mathcal{W}$ and $\widehat{A} \le \widehat{B}$ fusion frame bounds of $\widehat{\mathcal{W}}$. Since $B^{-1} \le \|S_{\mathcal{W}}^{-1}|W_i\| \le A^{-1}$ if $W_i \neq \{0\}$, we have for $x \in \mathcal{H}$

$$\widehat{A} B^{-2}\|x\|^2 \;\le\; \sum_{i\in I'} c_i^2 \|S_{\mathcal{W}}^{-1}|W_i\|^2 \|P_{S_{\mathcal{W}}^{-1} W_i} x\|^2 \;\le\; \widehat{B} A^{-2}\|x\|^2,$$

where $I' := \{i \in I : W_i \neq \{0\}\}$. But the above sum remains the same if I' is replaced by I. $\square$

3. PERTURBATIONS OF FUSION FRAMES

This section is devoted to studying the behavior of the canonical dual of a fusion frame under perturbations. We consider perturbations of fusion frames in analogy to the perturbations in [6] for frames.

DEFINITION 3.1. *Let $\mu > 0$, and let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ and $\mathcal{V} = ((V_i, d_i))_{i\in I}$ be two Bessel fusion sequencces in $\mathcal{H}$. We say that $\mathcal{V}$ is a* μ-perturbation *of $\mathcal{W}$ (and vice versa) if $\|T_{\mathcal{W}} - T_{\mathcal{V}}\| \le \mu$.*

REMARK 3.2. *If $\mathcal{V} = ((V_i, d_i))_{i\in I}$ is a μ-perturbation of $\mathcal{W} = ((W_i, c_i))_{i\in I}$, then for each $i \in I$ and each $x \in \mathcal{H}$ we have*

$$\|c_i P_{W_i} x - d_i P_{V_i} x\| = \|(T_{\mathcal{W}}^* - T_{\mathcal{V}}^*)(\delta_{ij} x)_{j\in I}\| \le \|T_{\mathcal{W}}^* - T_{\mathcal{V}}^*\|\|x\| \le \mu\|x\|,$$

and therefore

$$\|c_i P_{W_i} - d_i P_{V_i}\| \le \mu. \tag{1}$$

Since $c_i = \|c_i P_{W_i}\| \le \|c_i P_{W_i} - d_i P_{V_i}\| + d_i$ and $d_i \le \|c_i P_{W_i} - d_i P_{V_i}\| + c_i$, relation (1) implies

$$|c_i - d_i| \;\le\; \|c_i P_{W_i} - d_i P_{V_i}\| \;\le\; \mu.$$

LEMMA 3.3. *Let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds $A \le B$ and $\mathcal{V} = ((V_i, d_i))_{i\in I}$ a Bessel fusion sequence in $\mathcal{H}$ which is a μ-perturbation of $\mathcal{W}$, $\mu > 0$. If $\mu < \sqrt{A}$ then $\mathcal{V}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds $(\sqrt{A} - \mu)^2$ and $(\sqrt{B} + \mu)^2$.*

Proof. The claim directly follows from

$$(\sqrt{A}-\mu)\|x\| \le \|T_{\mathcal{W}}x\| - \|(T_{\mathcal{W}}-T_{\mathcal{V}})x\| \le \|T_{\mathcal{V}}x\| \le \|(T_{\mathcal{V}}-T_{\mathcal{W}})x\| + \|T_{\mathcal{W}}x\| \le (\mu+\sqrt{B})\|x\|,$$

where $x \in \mathcal{H}$. $\square$

In the following, we shall show that the canonical dual of a μ-perturbation of a fusion frame $\mathcal{W}$ will be a $C\mu$-perturbation of the canonical dual of $\mathcal{W}$, where $C > 0$ depends on μ and $\mathcal{W}$. In the proof, we will have to estimate expressions of the type

$$\left\|cP_{S_1^{-1}W} - dP_{S_2^{-1}V}\right\|,$$

where $c, d > 0$, S_1, S_2 are positive definite operators and V and W closed subspaces. The next two lemmas will be helpful for tackling this problem.

LEMMA 3.4. *Let P and Q be orthogonal projections in $\mathcal{H}$ and $c, d > 0$. Then*

$$\|P-Q\| \le \sqrt{\frac{1}{c^2}+\frac{1}{d^2}}\,\|cP-dQ\|.$$

Proof. Let $x \in \mathcal{H}$, $\|x\| = 1$. Then we have

$$\|cPx-dQx\|^2 = \|cQPx + c(I-Q)Px - dQx\|^2 = \|Q(cPx-dx)\|^2 + c^2\|(I-Q)Px\|^2 \ge c^2\|(I-Q)Px\|^2.$$

Analogously, one obtains $\|cPx-dQx\|^2 \ge d^2\|(I-P)Qx\|^2$. Thus,

$$\|(I-Q)P\| \le \frac{1}{c}\|cP-dQ\| \qquad \text{and} \qquad \|(I-P)Q\| \le \frac{1}{d}\|cP-dQ\|.$$

Hence, also $\|Q(I-P)\| = \|((I-P)Q)^*\| = \|(I-P)Q\| \le \frac{1}{d}\|cP-dQ\|$. Since, for $x \in \mathcal{H}$,

$$\|(P-Q)x\|^2 = \|QPx + (I-Q)Px - Qx\|^2 = \|Q(I-P)x\|^2 + \|(I-Q)Px\|^2,$$

the claim follows from the above inequalities. $\square$

LEMMA 3.5. *Let $W \subset \mathcal{H}$ be a closed subspace and $A \in \mathcal{B}(\mathcal{H})$ boundedly invertible. Then, for every $\lambda > 0$, the operator*

$$R(\lambda) := AP_W + \lambda A^{-*}P_{W^\perp},$$

where $A^{-*} = (A^*)^{-1}$, is boundedly invertible and we have

$$P_{AW} = R(\lambda)^{-*}P_W A^*. \tag{2}$$

Moreover, if $c, d > 0$ are such that $c\|x\| \le \|Ax\| \le d\|x\|$ for $x \in \mathcal{H}$ then

$$d^{-1}\min\{1, \lambda^{-1}cd\}\|x\| \le \|R(\lambda)^{-1}x\| \le c^{-1}\max\{1, \lambda^{-1}cd\}\|x\|. \tag{3}$$

As a consequence, we obtain

$$d^{-1}\|P_W A^*x\| \le \|P_{AW}x\| \le c^{-1}\|P_W A^*x\|. \tag{4}$$

Proof. First of all, we note that $(AW)^\perp = A^{-*}W^\perp$. From this, it immediately follows that $R(\lambda)$ is boundedly invertible and that $P_{AW}R(\lambda) = AP_W$. The latter implies $P_{AW} = AP_W R(\lambda)^{-1}$. Adjoining this gives (2). For the proof of (3) let $x \in \mathcal{H}$. Then we obtain

$$\begin{aligned}\|R(\lambda)x\|^2 &= \|AP_Wx\|^2 + \lambda^2\|A^{-*}P_{W^\perp}x\|^2 \ge c^2\|P_Wx\|^2 + \lambda^2 d^{-2}\|P_{W^\perp}x\|^2 \\ &\ge \min\{c^2, \lambda^2 d^{-2}\}\|x\|^2,\end{aligned}$$

as well as

$$\begin{aligned}\|R(\lambda)x\|^2 &= \|AP_W x\|^2 + \lambda^2\|A^{-*}P_{W^\perp}x\|^2 \le d^2\|P_W x\|^2 + \lambda^2 c^{-2}\|P_{W^\perp}x\|^2 \\ &\le \max\{d^2, \lambda^2 c^{-2}\}\|x\|^2.\end{aligned}$$

This implies (3). Setting $\lambda = cd$ and using (2) yields (4). $\square$

We briefly remark that Lemma 3.5 immediately implies the following corollary which was already proved by P. Găvruta in [5, Theorem 2.4].

COROLLARY 3.6. *Let $((W_i, c_i))_{i\in I}$ be a fusion frame for $\mathcal{H}$ with bounds $A \le B$ and let $T \in \mathcal{B}(\mathcal{H})$ be boundedly invertible. Then also $((TW_i, c_i))_{i\in I}$ is a fusion frame for $\mathcal{H}$ with bounds $A\gamma^{-2} \le B\gamma^2$, where $\gamma = \|T\|\|T^{-1}\|$.*

We are now ready to formulate and prove our main theorem. Recall that the canonical dual of a fusion frame $\mathcal{W}$ is denoted by $\widetilde{\mathcal{W}}$.

THEOREM 3.7. *Let $\mathcal{W} = ((W_i, c_i))_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds $A \le B$ and let $\mathcal{V} = ((V_i, d_i))_{i\in I}$ be a μ-perturbation of $\mathcal{W}$, where $0 < \mu < \sqrt{A}$. If the sequences $(c_i)_{i\in I}$ and $(d_i)_{i\in I}$ are bounded from below by some $\tau > 0$ and $\sigma > 0$, respectively, then $\widetilde{\mathcal{V}}$ is a $C\mu$-perturbation of $\widetilde{\mathcal{W}}$, where*

$$C = \frac{c^2+d^2}{A}\left[\frac{1+(A^{-1}+B)^2}{\sqrt{A}}\left(\sqrt{\frac{1}{\tau^2}+\frac{1}{\sigma^2}} + cd^2\right) + d^2\left(1+c^2d^2\right)\right]$$

with $c := 2\sqrt{B} + \mu$ and $d := (\sqrt{A} - \mu)^{-1}$.

Proof. For $i \in I$ we define the operators

$$R_{W_i} := S_{\mathcal{W}}^{-1}P_{W_i} + S_{\mathcal{W}}P_{W_i^\perp} \qquad \text{and} \qquad R_{V_i} := S_{\mathcal{V}}^{-1}P_{V_i} + S_{\mathcal{V}}P_{V_i^\perp}.$$

By Lemma 3.5, these are boundedly invertible and

$$P_{S_{\mathcal{W}}^{-1}W_i} = R_{W_i}^{-*}P_{W_i}S_{\mathcal{W}}^{-1} \qquad \text{and} \qquad P_{S_{\mathcal{V}}^{-1}V_i} = R_{V_i}^{-*}P_{V_i}S_{\mathcal{V}}^{-1}.$$

Put $\hat{c}_i := \|S_{\mathcal{W}}^{-1}|W_i\|c_i$ and $\hat{d}_i := \|S_{\mathcal{V}}^{-1}|V_i\|d_i$. Then $\hat{c}_i \le A^{-1}c_i$ and $\hat{d}_i \le (\sqrt{A}-\mu)^{-2}d_i = d^2 d_i$. For $x \in \mathcal{H}$ define

$$\Delta_i(x) := \left\|\hat{c}_i P_{S_{\mathcal{W}}^{-1}W_i}x - \hat{d}_i P_{S_{\mathcal{V}}^{-1}V_i}x\right\| = \left\|\hat{c}_i R_{W_i}^{-*}P_{W_i}S_{\mathcal{W}}^{-1}x - \hat{d}_i R_{V_i}^{-*}P_{V_i}S_{\mathcal{V}}^{-1}x\right\|.$$

Since $\|T_{\widetilde{\mathcal{W}}}x - T_{\widetilde{\mathcal{V}}}x\|^2 = \sum_{i\in I}\Delta_i^2(x)$, it is our aim to estimate $\Delta_i(x)$. We have

$$\begin{aligned}\Delta_i(x) &\le \left\|\hat{c}_i(R_{W_i}^{-*} - R_{V_i}^{-*})P_{W_i}S_{\mathcal{W}}^{-1}x\right\| + \left\|R_{V_i}^{-*}\left(\hat{c}_iP_{W_i}S_{\mathcal{W}}^{-1}x - \hat{d}_iP_{V_i}S_{\mathcal{V}}^{-1}x\right)\right\| \\ &\le \left\|R_{W_i}^{-1} - R_{V_i}^{-1}\right\|A^{-1}c_i\|P_{W_i}S_{\mathcal{W}}^{-1}x\| + \left\|R_{V_i}^{-1}\right\|\left\|\hat{c}_iP_{W_i}S_{\mathcal{W}}^{-1}x - \hat{d}_iP_{V_i}S_{\mathcal{W}}^{-1}x\right\| \\ &\qquad + \left\|R_{V_i}^{-1}\right\|d^2d_i\left\|P_{V_i}\left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1}\right)x\right\|.\end{aligned}$$

Since $R_{W_i}^{-1} - R_{V_i}^{-1} = R_{W_i}^{-1}(R_{V_i} - R_{W_i})R_{V_i}^{-1}$ and $\|R_{V_i}^{-1}\| \le (\sqrt{A}-\mu)^{-2} + (\sqrt{B}+\mu)^2 \le c^2 + d^2$ by Lemma 3.5, with

$$\Delta_i^{(1)} := A^{-1}\left\|R_{W_i}^{-1}\right\|\|R_{W_i} - R_{V_i}\| \qquad \text{and} \qquad \Delta_i^{(2)} := \left|\left\|S_{\mathcal{W}}^{-1}|W_i\right\| - \left\|S_{\mathcal{V}}^{-1}|V_i\right\|\right|$$

we obtain

$$\begin{aligned}\frac{\Delta_i(x)}{c^2+d^2} &\le \Delta_i^{(1)}c_i\left\|P_{W_i}S_{\mathcal{W}}^{-1}x\right\| + d^2d_i\left\|P_{V_i}\left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1}\right)x\right\| + \Delta_i^{(2)}c_i\left\|P_{W_i}S_{\mathcal{W}}^{-1}x\right\| \\ &\qquad + \left\|S_{\mathcal{V}}^{-1}|V_i\right\|\left\|\left(c_iP_{W_i} - d_iP_{V_i}\right)S_{\mathcal{W}}^{-1}x\right\| \\ &\le \left(\Delta_i^{(1)} + \Delta_i^{(2)}\right)c_i\left\|P_{W_i}S_{\mathcal{W}}^{-1}x\right\| + d^2d_i\left\|P_{V_i}\left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1}\right)x\right\| + d^2\left\|\left(c_iP_{W_i} - d_iP_{V_i}\right)S_{\mathcal{W}}^{-1}x\right\|.\end{aligned}$$

Let us start with estimating $\Delta_i^{(2)}$. First of all, we note that

$$\|S_{\mathcal{W}} - S_{\mathcal{V}}\| \leq \|T_{\mathcal{W}}^*(T_{\mathcal{W}} - T_{\mathcal{V}})\| + \|(T_{\mathcal{W}}^* - T_{\mathcal{V}}^*)T_{\mathcal{V}}\| \leq (\|T_{\mathcal{W}}\| + \|T_{\mathcal{V}}\|)\,\mu \leq (2\sqrt{B} + \mu)\mu. \tag{5}$$

Now, from $\|S_{\mathcal{W}}^{-1}|W_i\| = \|S_{\mathcal{W}}^{-1}P_{W_i}\|$, Lemma 3.4, and Remark 3.2 it follows that

$$\begin{aligned}
\Delta_i^{(2)} &= \left|\left\|S_{\mathcal{W}}^{-1}P_{W_i}\right\| - \left\|S_{\mathcal{V}}^{-1}P_{V_i}\right\|\right| \;\leq\; \left\|S_{\mathcal{W}}^{-1}P_{W_i} - S_{\mathcal{V}}^{-1}P_{V_i}\right\| \leq \left\|S_{\mathcal{W}}^{-1}\left(P_{W_i} - P_{V_i}\right)\right\| + \left\|S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1}\right\| \\
&\leq A^{-1}\sqrt{\frac{1}{c_i^2} + \frac{1}{d_i^2}}\ \|c_iP_{W_i} - d_iP_{V_i}\| + \|S_{\mathcal{W}}^{-1}(S_{\mathcal{V}} - S_{\mathcal{W}})S_{\mathcal{V}}^{-1}\| \\
&\leq \frac{\mu}{A}\sqrt{\frac{1}{\tau^2} + \frac{1}{\sigma^2}} + \frac{(2\sqrt{B} + \mu)\mu}{A(\sqrt{A} - \mu)^2} = M\mu,
\end{aligned}$$

where

$$M = \frac{1}{A}\left(\sqrt{\frac{1}{\tau^2} + \frac{1}{\sigma^2}} + \frac{2\sqrt{B} + \mu}{(\sqrt{A} - \mu)^2}\right) = \frac{1}{A}\left(\sqrt{\frac{1}{\tau^2} + \frac{1}{\sigma^2}} + cd^2\right).$$

In order to estimate $\Delta_i^{(1)}$, we observe that (see (5))

$$\begin{aligned}
\|R_{W_i} - R_{V_i}\| &= \left\|S_{\mathcal{W}}^{-1}P_{W_i} + S_{\mathcal{W}}P_{W_i^\perp} - S_{\mathcal{V}}^{-1}P_{V_i} - S_{\mathcal{V}}P_{V_i^\perp}\right\| \leq M\mu + \left\|S_{\mathcal{W}}P_{W_i^\perp} - S_{\mathcal{V}}P_{V_i^\perp}\right\| \\
&\leq M\mu + \left\|S_{\mathcal{W}}\left(P_{W_i^\perp} - P_{V_i^\perp}\right)\right\| + \left\|\left(S_{\mathcal{W}} - S_{\mathcal{V}}\right)P_{V_i^\perp}\right\| \\
&\leq \left(\frac{1}{A}\left(\sqrt{\frac{1}{\tau^2} + \frac{1}{\sigma^2}} + cd^2\right) + \sqrt{\frac{1}{\tau^2} + \frac{1}{\sigma^2}}B + c\right)\mu \\
&= \left(A^{-1} + B\right)\left(\sqrt{\frac{1}{\tau^2} + \frac{1}{\sigma^2}} + cd^2 \cdot \frac{1 + Ad^{-2}}{1 + AB}\right)\mu \\
&\leq \left(A^{-1} + B\right)\left(\sqrt{\frac{1}{\tau^2} + \frac{1}{\sigma^2}} + cd^2\right)\mu = A\left(A^{-1} + B\right)M\mu,
\end{aligned}$$

where in the last inequality we have used that $d^{-2} \leq B$. Thus, we have

$$\Delta_i^{(1)} + \Delta_i^{(2)} \;\leq\; \left(A^{-1} + B\right)^2 M\mu + M\mu = \left(1 + \left(A^{-1} + B\right)^2\right)M\mu.$$

Now, define the functionals

$$\begin{aligned}
r_i(x) &:= \left(1 + \left(A^{-1} + B\right)^2\right)M\mu c_i\left\|P_{W_i}S_{\mathcal{W}}^{-1}x\right\|, \\
s_i(x) &:= d^2 d_i\left\|P_{V_i}\left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1}\right)x\right\|, \\
t_i(x) &:= d^2\left\|\left(c_iP_{W_i} - d_iP_{V_i}\right)S_{\mathcal{W}}^{-1}x\right\|.
\end{aligned}$$

as well as

$$R(x) := \sqrt{\sum_{i\in I} r_i^2(x)}, \qquad S(x) := \sqrt{\sum_{i\in I} s_i^2(x)}, \qquad \text{and} \qquad T(x) := \sqrt{\sum_{i\in I} t_i^2(x)}.$$

Then $(c^2 + d^2)^{-1}\Delta_i(x) \leq r_i(x) + s_i(x) + t_i(x)$ and, by applying the Cauchy-Schwarz inequality, one obtains

$$\frac{1}{(c^2 + d^2)^2}\sum_{i\in I}\Delta_i^2(x) \leq \sum_{i\in I}\left(r_i(x) + s_i(x) + t_i(x)\right)^2 \;\leq\; \left(R(x) + S(x) + T(x)\right)^2.$$

Since $\|T_{\mathcal{W}} S_{\mathcal{W}}^{-1}\|^2 = \|(T_{\mathcal{W}} S_{\mathcal{W}}^{-1})^* T_{\mathcal{W}} S_{\mathcal{W}}^{-1}\| = \|S_{\mathcal{W}}^{-1}\| \leq A^{-1}$, we have

$$\begin{aligned}
R^2(x) &= \left(1 + \left(A^{-1} + B\right)^2\right)^2 M^2\mu^2 \left\|T_{\mathcal{W}} S_{\mathcal{W}}^{-1} x\right\|^2 \leq \left(\frac{1 + \left(A^{-1} + B\right)^2}{\sqrt{A}}\right)^2 M^2\mu^2\|x\|^2, \\
S^2(x) &= d^4 \left\|T_{\mathcal{V}}\left(S_{\mathcal{W}}^{-1} - S_{\mathcal{V}}^{-1}\right) x\right\|^2 \leq A^{-2} c^4 d^8 \mu^2 \|x\|^2, \\
T^2(x) &= d^4 \left\|\left(T_{\mathcal{W}} - T_{\mathcal{V}}\right) S_{\mathcal{W}}^{-1} x\right\|^2 \leq A^{-2} d^4 \mu^2 \|x\|^2.
\end{aligned}$$

That is,

$$\frac{1}{c^2 + d^2}\sqrt{\sum_{i \in I} \Delta_i^2(x)} \leq \left[\frac{1 + \left(A^{-1} + B\right)^2}{\sqrt{A}} M + A^{-1} c^2 d^4 + A^{-1} d^2\right] \mu\|x\|.$$

This shows that $\|T_{\widetilde{\mathcal{W}}} - T_{\widetilde{\mathcal{V}}}\| \leq C\mu$. □

The following corollary is an immediate consequence of Remark 3.2 and Theorem 3.7.

Corollary 3.8. *Let $\mathcal{W} = ((W_i, c_i))_{i \in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds $A \leq B$ such that $\tau := \inf c_i > 0$ and let $\mathcal{V} = ((V_i, d_i))_{i \in I}$ be a μ-perturbation of $\mathcal{W}$, where $0 < \mu < \min\{\sqrt{A}, \tau\}$. Then $\widetilde{\mathcal{V}}$ is a $C\mu$-perturbation of $\widetilde{\mathcal{W}}$, where*

$$C = \frac{c^2 + d^2}{A}\left[\frac{1 + (A^{-1} + B)^2}{\sqrt{A}}\left(\sqrt{\frac{1}{\tau} + \frac{1}{\tau - \mu}} + cd^2\right) + d^2\left(1 + c^2 d^2\right)\right]$$

with $c := 2\sqrt{B} + \mu$ and $d := (\sqrt{A} - \mu)^{-1}$.

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