# Modeling Sensor Networks with Fusion Frames

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## ABSTRACT

The new notion of fusion frames will be presented in this article. Fusion frames provide an extensive framework not only to model sensor networks, but also to serve as a means to improve robustness or develop efficient and feasible reconstruction algorithms. Fusion frames can be regarded as sets of redundant subspaces each of which contains a spanning set of local frame vectors, where the subspaces have to satisfy special overlapping properties. Main aspects of the theory of fusion frames will be presented with a particular focus on the design of sensor networks. New results on the construction of Parseval fusion frames will also be discussed.

**Keywords:** Data Fusion, Distributed Processing, Frames, Fusion Frames, Parallel Processing, Sensor Networks, Signal Reconstruction

## 1. INTRODUCTION

Frames, which are systems that provide robust, stable and usually non-unique representations of vectors, have been a focus of study in the last two decades in applications where redundancy plays a vital role (e.g., filter bank theory, sigma-delta quantization, signal and image processing, and wireless communications). However, a large number of new applications have emerged where the set-up can hardly be modeled naturally by one single frame system. Many of these applications share a common structure: redundant information sources must be managed through distributed processing structures that cannot aggregate data in a single central location. One prominent example that has garnered recent attention is the notion of a sensor network [18]. Fusion frames are an extension to frames that provide a framework for both modeling these applications and providing efficient and robust information processing algorithms.

### 1.1. Sensor Networks

Although fusion frames can be used to model general distributed processing applications, in this paper we intend to focus on the modeling of sensor networks. In (wireless) sensor networks, resource-constrained sensor nodes are spread over a (potentially large) area to measure environmental characteristics such as temperature, sound, vibration, pressure, motion, and/or pollutants. Such a sensor system is typically redundant in the sense that there is a high degree of overlap among the environmental conditions being sensed by neighboring sensors. In other words, the point spread functions denoting the sensor receptive fields often have a high degree of overlap and are not orthogonal. Therefore each sensor functions as a frame element in the system [23].

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Most sensors employed in such applications are power constrained due to their small onboard batteries. This practical consideration often translates into severe constraints on sensor's individual and collective wireless communication. In particular, communication constraints prohibit every sensor from communicating raw sensed data directly to a centralized location for processing. Consequently, a common sensor network strategy is to divide the network into (possibly overlapping) subgroups where each sensor communicates its data to a single node within the cluster. These head nodes (which may have more power or be a rotating assignment) then communicate the summarized data from their clusters to a more central location for collective processing. In frame theory terminology, the frame elements are divided into groups that span individual subspaces. The signal of interest is reconstructed within each subspace and communicated to a central location for final reconstruction.

Due to both the unpredictable nature of low-cost devices and hostile geographical factors, certain local sensor systems can be less reliable than others. While facing the task of combining local subspace information coherently, one has also to consider weighting the more reliable sets of substation information more than suspected less reliable ones. Consequently, the coherent combination mechanism we just saw as necessary often requires a weighted structure as well. We will show that fusion frames fit such weighted and coherent fusion needs.

### 1.2. Development of Fusion Frames

In [7], two of the authors studied redundant subspaces for the purpose of easing the construction of frames by building them locally in (redundant) subspaces and then piecing the local frames together by employing a special structure of the set of subspaces. Related approaches were undertaken by Fornasier [16] and by Sun [27, 28]. In [10], it was observed that this idea can be far more reaching than that of building large frames from smaller local ones. The weighted and coherent subspace combination in such a mechanism is exactly what is needed in distributed and parallel processing for many fusion applications as motivated above. Therefore this theory was developed further already with a particular focus on the modeling of distributed processing applications by studying stability aspects of fusion frames and deriving different reconstruction strategies.

Additional progress on the theory of fusion frames has been made by Casazza and Kutyniok [9], who studied optimal fusion frames under erasures of subspaces and erasures of local vectors. Further, in [4], Calderbank, Kutyniok, and Pezeshki have studied fusion frames which are optimally resilient against noise and erasures for random signals by using a Wiener filter for the reconstruction. They also showed a close connection of this problem with Grassmannian packings. Moreover, Asgari and Khosravi [1] studied atomic resolutions of the identity in Hilbert spaces, which relates to Parseval fusion frames, and in [25], Ruiz and Stojanoff obtained results on the behavior of fusion frames under application of operators and on the excess of fusion frames.

Some aspects of the theory of fusion frames have already been applied. Rozell, Goodman, and Johnson [20–22] used fusion frames to study noise reduction in sensor networks and to study overlapping feature spaces of neurons in visual and hearing systems. Also, Bodmann, Kribs, and Paulsen [3] and Bodmann [2] employed Parseval fusion frames under the term *weighted projective resolution of the identity* for optimal transmission of quantum states and for packet encoding.

### 1.3. Outline of the Paper

The organization of this article is as follows. After having briefly recalled the basic notation and definition of frames in Section 2, Section 3 provides an elaborate introduction into the theory of fusion frames. We first state the main definitions and discuss the notion of redundancy for sets of subspaces. Then we illustrate how sensor networks can be modeled using this notion, while also providing reconstruction strategies. In Section 4 we focus on Parseval fusion frames, and derive new results concerning their existence and construction. Section 5 provide various examples and constructions of fusion frames, and concluding remarks appear in the last section.

## 2. REVIEW OF FRAMES

Frames are systems that provide robust, stable and usually non-unique representations of vectors in a Hilbert space $\mathcal{H}$. In this sense they are a natural generalization of orthonormal bases which are now capable of providing various different representations of the same signal, hence resilience against erasures and noise. This immediately raises questions concerning detection of the "best" representation for a particular signal. Frame theory focusses on the representation with coefficients having the smallest possible $\ell_2$-norm, while providing the possibility of a linear reconstruction strategy. Other approaches such as basis pursuit study representations with coefficients having the smallest possible $\ell_1$-norm, which is closely related to sparse representations [12].

Let us now state the main definitions and introduce the notation we will require throughout this paper. A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ in $\mathcal{H}$ is a *frame* for $\mathcal{H}$, if there exist $0 < A \le B < \infty$ (*lower* and *upper frame bounds*) such that

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 = \|\{\langle f, f_i\rangle\}_{i\in I}\|_2^2 \le B\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{1}$$

The representation space associated with a frame is $\ell_2(I)$. In order to analyze a signal $f \in \mathcal{H}$, i.e., to map it into the representation space, the *analysis operator* $T_\mathcal{F} : \mathcal{H} \to \ell_2(I)$ given by $T_\mathcal{F} f = \{\langle f, f_i\rangle\}_{i\in I}$ is applied. The associated *synthesis operator*, which provides a mapping from the representation space to $\mathcal{H}$, is defined to be the adjoint operator $T_\mathcal{F}^* : \ell_2(I) \to \mathcal{H}$ which can be computed to be $T_\mathcal{F}^*(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$. By composing $T_\mathcal{F}$ and $T_\mathcal{F}^*$ we obtain the *frame operator*

$$S_\mathcal{F} : \mathcal{H} \to \mathcal{H}, \quad S_\mathcal{F} f = T_\mathcal{F}^* T_\mathcal{F} f = \sum_{i\in I} \langle f, f_i\rangle f_i,$$

which is a bounded, invertible, and positive operator. This provides the reconstruction formula

$$f = S_\mathcal{F}^{-1} S_\mathcal{F}(f) = \sum_{i\in I} \langle f, f_i\rangle \tilde{f}_i = \sum_{i\in I} \langle f, \tilde{f}_i\rangle f_i, \tag{2}$$

where $\tilde{f}_i = S_\mathcal{F}^{-1} f_i$. The family $\{\tilde{f}_i\}_{i\in I}$ is also a frame for $\mathcal{H}$, called the *canonical dual frame* of $\{f_i\}_{i\in I}$. Indeed, it can be proven that, given $f \in \mathcal{H}$,

$$\|\{\langle f, \tilde{f}_i\rangle\}_{i\in I}\|_2^2 \le \|\{c_i\}_{i\in I}\|_2^2 \quad \text{for all } \{c_i\}_{i\in I} \in \ell_2(I) \text{ satisfying } f = \sum_{i\in I} c_i f_i.$$

Notice further that $S_\mathcal{F}^{-1} S_\mathcal{F} = (S_\mathcal{F}^{-1} T_\mathcal{F}^*) T_\mathcal{F}$, where $S_\mathcal{F}^{-1} T_\mathcal{F}^*$ is the pseudo-inverse to the analysis operator $T_\mathcal{F}$, thus providing optimal resilience against additive zero-mean white noise.

Of particular interest are *A-tight frames*, i.e., when the frame bounds can be chosen as $A = B$ in the frame definition (1). Provided (1) holds with $A = B = 1$, we call $\mathcal{F}$ a *Parseval frame*. The advantage of working with these frames can be clearly seen by considering the reconstruction formula (2). In these cases the canonical dual frame equals $\{\frac{1}{A} f_i\}_{i\in I}$, and hence we obtain $f = \frac{1}{A} T_\mathcal{F}^* T_\mathcal{F} f$ for each $f \in \mathcal{H}$, i.e., we can employ the frame elements for both the analysis and the synthesis. There exist many procedures to construct tight or Parseval frames (cf. [5, 8]). However, Parseval frames with special properties are usually particularly difficult to construct, see, e.g., [26].

For more details about the theory and applications of frames we refer the reader to the books by Christensen [13], Daubechies [15], and Mallat [19], and the research-tutorial by Heil and Walnut [17].

## 3. FUSION FRAMES

### 3.1. Introduction to Fusion Frames

A fusion frame can be regarded as a frame of subspaces in the sense that it is a notion for a set of closed subspaces with a special overlapping property, thereby interpreting the overlapping as redundancy. Intuitively, we can speak of redundancy of a set of closed subspaces $\{W_i\}_{i\in I}$ in $\mathcal{H}$ if the decomposition of some $f \in \mathcal{H}$ in $f = \sum_{i\in I} f_i$ ($f_i \in W_i$ for each $i \in I$) is not unique. Hence a set of orthogonal subspaces would not provide redundancy, whereas a set consisting of equal subspaces would have "maximal" redundancy. The notion of a fusion frame now provides the means to make this redundancy more precise and also to study decompositions of the above mentioned type.

We will start by stating the definition of a fusion frame, where we allow the flexibility to endow the subspaces with additional weights.

DEFINITION 3.1. *Let $I$ be a countable index set, let $\{W_i\}_{i\in I}$ be a family of closed subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a* fusion frame*, if there exist constants $0 < C \leq D < \infty$ such that*

$$C\|f\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2 \quad \text{for all } f \in \mathcal{H}, \tag{3}$$

*where $\pi_{W_i}$ is the orthogonal projection onto the subspace $W_i$. We call $C$ and $D$ the* fusion frame bounds*. The family $\{(W_i, v_i)\}_{i\in I}$ is called a $C$-*tight fusion frame*, if in (3) the constants $C$ and $D$ can be chosen so that $C = D$. We call it a* Parseval fusion frame *provided that $C = D = 1$.*

It is easy to derive that frames are a special case of this definition. Indeed, if $\mathcal{F} = \{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with frame bounds $A, B$ and frame operator $S_{\mathcal{F}}$, then $\{(\text{span}\{f_i\}, \|f_i\|)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds $A, B$ and fusion frame operator $S_{\mathcal{F}}$. This result immediately follows from the observation that for any $f \in \mathcal{H}$,

$$\sum_{i\in I} \|f_i\|^2 \pi_{\text{span}\{f_i\}}(f) = \sum_{i\in I} \|f_i\|^2 \langle f, \tfrac{f_i}{\|f_i\|}\rangle \tfrac{f_i}{\|f_i\|} = \sum_{i\in I} \langle f, f_i\rangle f_i = S_{\mathcal{F}} f.$$

Now, heading towards modeling of sensor networks using fusion frames, we need to provide a fusion frame with a second processing layer, which we then call a fusion frame system. For this, we consider a fusion frame together with sets of vectors which span each of the subspaces belonging to the fusion frame. The precise definition is as follows.

DEFINITION 3.2. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i}$ be a frame for $W_i$ for each $i \in I$. Then we call $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ a* fusion frame system *for $\mathcal{H}$. $C$ and $D$ are the associated* fusion frame bounds *if they are the fusion frame bounds for $\{(W_i, v_i)\}_{i\in I}$, and $A$ and $B$ are the* local frame bounds *if these are the common frame bounds for the* local frames *$\{f_{ij}\}_{j\in J_i}$ for each $i \in I$. A collection of dual frames $\{\tilde{f}_{ij}\}_{j\in J_i}$ for each $i \in I$ associated with the local frames will be called* local dual frames.

One motivation for the notion of a fusion frame was the study of redundancy of sets of subspaces. The following result from [Prop. 2.4, 10] gives some insight into a possible interpretation of redundancy in this case. In fact, it states that the redundancy of a $C$-tight fusion frame can be made precise in terms of the fusion frame bound.

PROPOSITION 3.3. *Let $\{(W_i, v_i)\}_{i=1}^n$ be a $C$-tight fusion frame for $\mathcal{H}$ with $\dim \mathcal{H} < \infty$. Then we have*

$$C = \frac{\sum_{i=1}^n v_i^2 \dim W_i}{\dim \mathcal{H}}.$$

In this sense, we can interpret the frame bound $C$ as the *redundancy* of the tight fusion frame $\{(W_i, v_i)\}_{i=1}^n$.

Before we delve into the precise way fusion frame systems can be employed as a model for sensor networks, we will provide a quick inside-look at a fundamental relation between fusion frame properties and frame properties of the local frames. For the proof of this result we refer to [Thm. 3.2, 7].

THEOREM 3.4. *For each $i \in I$, let $v_i > 0$, let $W_i$ be a closed subspace of $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i}$ be a frame for $W_i$ with frame bounds $A_i$ and $B_i$. Suppose that*

$$0 < A = \inf_{i\in I} A_i \leq \sup_{i\in I} B_i = B < \infty.$$

*Then the following conditions are equivalent.*

(i) *$\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$.*

(ii) *$\{v_i f_{ij}\}_{j\in J_i, i\in I}$ is a frame for $\mathcal{H}$.*

*In particular, if $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ is a fusion frame system for $\mathcal{H}$ with fusion frame bounds $C$ and $D$, then $\{v_i f_{ij}\}_{j\in J_i, i\in I}$ is a frame for $\mathcal{H}$ with frame bounds $AC$ and $BD$. Also if $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ with frame bounds $C$ and $D$, then $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ is a fusion frame system for $\mathcal{H}$ with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.*

### 3.2. Modeling Sensor Networks

To model a sensor network, we associate each sensor with a vector $f_{ij}$ ($j \in J_i, i \in I$) in $\mathbb{R}^M$ that quantifies how it measures the environment. This vector models the point-spread function of the sensor [24] (possibly corresponding to the Green's function [14] of the propagation medium). The sensor measurement of a signal $f$ is given by the inner product $\langle f, f_{ij} \rangle$. If all of these measurements were available at a single central location, reconstruction of the environmental signal $f$ would be a straightforward application of frame theory.

Because of the prohibitive communication constraints of the typical sensor nodes under consideration, we focus on a common wireless sensor network paradigm where sensors are grouped into clusters. Specifically, the sensors are grouped so that for each group $J_i$ with $i \in I$ the sensors $\{f_{ij}\}_{j \in J_i}$ belong to that cluster. Each sensor is able to communicate its information to a collection point assigned to the relevant cluster, which may simply be a fixed or rotating assignment of a sensor node within the group. Remembering that the vectors $\{f_{ij}\}_{j \in J_i}$ form a frame for their closed linear span (i.e., for $W_i$), we can obtain a partial reconstruction of the signal by the "normal" frame reconstruction inside $W_i$. More precisely, we obtain $\sum_{j \in J_i} \langle f, f_{ij} \rangle S_i^{-1} f_{ij} = \pi_{W_i}(f)$, where $S_i$ is the local frame operator for $\{f_{ij}\}_{j \in J_i}$. Therefore, each collection point can use a frame theoretic approach to reconstruct the orthogonal projection of the (environmental) signal onto the subspace spanned by the sensors in the cluster.

After collecting data from the individual sensors in the cluster and performing a local subspace reconstruction, the collection point transmits this information to a master collection point (central station) for the whole sensor network. The central station is tasked with reconstructing the signal $f$ from the processed data it receives from each cluster collection point. This reconstruction now depends entirely on the set of subspaces, i.e., only on the fusion frame itself. The choice of the local frame vectors does not interfere anymore. The next section studies how this reconstruction can be preformed by making extensive use of the notion of a fusion frame.

### 3.3. Reconstruction Strategies

Let $\mathcal{W} = \{(W_i, v_i)\}_{i \in I}$ be a fusion frame for $\mathcal{H}$. In order to derive a reconstruction strategy we proceed in the spirit of frame reconstruction. In frame theory an input signal is represented by a collection of scalar coefficients that measure the projection of that signal onto each frame vector. The representation space employed in this theory is $\ell^2(I)$. However, in fusion frame theory an input signal is represented by a collection of *vector* coefficients that represent the projection (not just the projection energy) onto each subspace. Therefore the representation space employed in this setting is

$$\left( \sum_{i \in I} \oplus W_i \right)_{\ell^2} = \{\{f_i\}_{i \in I} | f_i \in W_i \text{ and } \{\|f_i\|\}_{i \in I} \in \ell^2(I)\}.$$

In order to map a signal to the representation space, i.e., to analyze it, the *analysis operator* $T_{\mathcal{W}}$ is employed, which is defined by

$$T_{\mathcal{W}} : \mathcal{H} \to \left( \sum_{i \in I} \oplus W_i \right)_{\ell_2} \quad \text{with} \quad T_{\mathcal{W}}(f) = \{v_i \pi_{W_i}(f)\}_{i \in I}.$$

It can easily be shown that the *synthesis operator* $T_{\mathcal{W}}^*$, which is defined to be the adjoint operator, is given by

$$T_{\mathcal{W}}^* : \left( \sum_{i \in I} \oplus W_i \right)_{\ell_2} \to \mathcal{H} \text{ with } T_{\mathcal{W}}^*(f) = \sum_{i \in I} v_i f_i, \; f = \{f_i\}_{i \in I} \in \left( \sum_{i \in I} \oplus W_i \right)_{\ell_2}.$$

The *fusion frame operator* $S_{\mathcal{W}}$ for $\mathcal{W}$ is defined by

$$S_{\mathcal{W}}(f) = T_{\mathcal{W}}^* T_{\mathcal{W}}(f) = \sum_{i \in I} v_i^2 \pi_{W_i}(f).$$

Interestingly, a fusion frame operator exhibits properties similar to a frame operator concerning invertibility. In fact, if $\{(W_i, v_i)\}_{i \in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds $C$ and $D$, then the associated fusion frame operator $S_{\mathcal{W}}$ is positive and invertible on $\mathcal{H}$, and

$$C\, Id \leq S_{\mathcal{W}} \leq D\, Id. \tag{4}$$

We refer the reader to [Prop. 3.16, 7] for details.

Thus we can obtain the following reconstruction from the collected and preprocessed data $\pi_{W_i}(f)$, $i \in I$, which is [Prop. 4.1, 10].

Proposition 3.5. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with fusion frame operator $S_{\mathcal{W}}$ and fusion frame bounds $C$ and $D$. Then we have the reconstruction formula*

$$f = \sum_{i\in I} v_i^2 S_{\mathcal{W}}^{-1} \pi_{W_i}(f) \quad \text{for all } f \in \mathcal{H}.$$

This reconstruction needs to be performed in the master collection point to recover the initial signal $f$. Thus, fusion frames serve as a complete model for signal processing in a sensor network. We wish to remark that this way of modeling sensor networks was already successfully applied to study noise reduction properties of distributed versus centralized reconstruction by Goodman, Johnson, and Rozell [20–22].

## 4. PARSEVAL FUSION FRAMES

Parseval fusion frames are of particular importance due to their advantageous reconstruction properties, as already indicated before. Thus we would like to raise three questions, whose answers will be particularly useful for the design of efficient sensor networks:

- Which geometry of sensor placement and grouping is possible which still provide the possibility for an easy reconstruction strategy?
  In terms of fusion frames, we would ask: how can we characterize the existence of Parseval fusion frames?
- If a collection of sensors needs to be grouped, how can this be accomplished while simultaneously leading to an efficient structure?
  In terms of fusion frames, we would ask: when and how can a collection of vectors be grouped to yield a Parseval fusion frame?
- A sensor network is often given a priori. For such a given network, we might ask whether the groups of sensors can be weighted to give rise to a more efficient reconstruction strategy?
  In terms of fusion frames, we would ask: when does there exist a combination of weights for a given fusion frame which yield a Parseval fusion frame, and how can we construct them?

In this section we would like to discuss the first question. The remaining questions will be studied in forthcoming papers.

First we would like to state a derivation of Theorem 3.4, which focusses on the special case of choosing Parseval frames as local frames.

Corollary 4.1. *Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i}$ be a Parseval frame for $W_i$ for all $i \in I$. Then the following conditions are equivalent.*

(i) *$\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ with frame bounds $C$ and $D$.*

(ii) *$\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with frame bounds $C$ and $D$.*

*In particular, $\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{H}$ if and only if $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a Parseval frame for $\mathcal{H}$.*

*Proof.* This follows from the proof of [Thm. 3.2, 7]. The second part is [Cor. 3.21, 7]. □

The following theorem now gives a characterization of Parseval fusion frames in terms of projections from larger spaces. This result also indicates how Parseval fusion frames can be constructed. For a characterization of general fusion frames we refer to [Thm. 3.1, 10].

THEOREM 4.2. *The following conditions are equivalent.*

(i) $\{(W_i, v_i)\}_{i\in I}$ *is a Parseval fusion frame for* $\mathcal{H}$.

(ii) *There exists a larger Hilbert space* $\mathcal{K}$ *and a projection* $P : \mathcal{K} \to \mathcal{H}$ *satisfying* $\mathcal{K} = (\sum_{i\in I} \oplus \widetilde{W}_i)_{\ell_2}$ *for some closed subspaces* $\{\widetilde{W}_i\}_{i\in I}$ *of* $\mathcal{H}$, *and* $P|_{\widetilde{W}_i}$ *is a* $v_i$*-isometry of* $\widetilde{W}_i$ *onto* $W_i$ *for each* $i \in I$.

*Proof.* (i) $\Rightarrow$ (ii): Let $T_{\mathcal{W}}$ denote the analysis operator of $\{(W_i, v_i)\}_{i\in I}$. Since $\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame, for every $f, g \in \mathcal{H}$ we have

$$\langle T_{\mathcal{W}} f, T_{\mathcal{W}} g\rangle = \langle T_{\mathcal{W}}^* T_{\mathcal{W}} f, g\rangle = \langle f, g\rangle.$$

So we may associate $\mathcal{H}$ with the space $T_{\mathcal{W}}(\mathcal{H})$, and let $\widetilde{W}_i = W_i$ for all $i \in I$. Let $P = T_{\mathcal{W}} T_{\mathcal{W}}^*$. First we notice that $P$ is a projection, since by using the fact that $\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame we obtain

$$P^2 = T_{\mathcal{W}} T_{\mathcal{W}}^* T_{\mathcal{W}} T_{\mathcal{W}}^* = T_{\mathcal{W}} T_{\mathcal{W}}^* = P.$$

Moreover, $P$ is onto $T_{\mathcal{W}}(\mathcal{H})$, since for each $f \in T_{\mathcal{W}}(\mathcal{H})$, hence $f = T_{\mathcal{W}}(g)$ for some $g \in \mathcal{H}$, we have

$$Pf = T_{\mathcal{W}} T_{\mathcal{W}}^* f = T_{\mathcal{W}} T_{\mathcal{W}}^* T_{\mathcal{W}}(g) = T_{\mathcal{W}}(g) = f.$$

Now fix $j \in I$ and let $f \in W_j$. Then $T_{\mathcal{W}}^* f = v_j f$, and hence

$$Pf = T_{\mathcal{W}} T_{\mathcal{W}}^* f = \{v_i \pi_{W_i}(v_j f)\}_{i\in I} = \{v_i v_j \pi_{W_i}(f)\}_{i\in I}.$$

Again using the fact that $\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame, we obtain

$$\|Pf\|^2 = \sum_{i\in I} \|v_i v_j \pi_{W_i}(f)\|^2 = v_j^2 \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = v_j^2 \|f\|^2.$$

(ii) $\Rightarrow$ (i): Let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\widetilde{W}_i$ for each $i \in I$. Since $P|_{\widetilde{W}_i}$ is a $v_i$-isometry, the sequence $\{v_i^{-1} P e_{ij}\}_{j\in J_i}$ is an orthonormal basis, hence in particular a Parseval frame, for $W_i = P(\widetilde{W}_i) = \text{span}_{j\in J_i}\{Pe_{ij}\}$. Furthermore, we know that $\{v_i(v_i^{-1} P e_{ij})\}_{j\in J_i} = \{Pe_{ij}\}_{j\in J_i}$ is a Parseval frame, since it is a projection of an orthonormal basis [6]. Employing Corollary 4.1, the last two observations together imply that $\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{H}$. □

This result being very general in its formulation calls for the consideration of special cases to illustrate its application. Therefore in the following we restrict our attention to the situation of having just two closed subspaces in $\mathcal{H}$.

For a Parseval fusion frame consisting of two subspaces we have an explicit characterization concerning its existence.

LEMMA 4.3. *Let* $W_1, W_2$ *be closed non-trivial subspaces of* $\mathcal{H}$, *and let* $v_1, v_2 > 0$. *The following conditions are equivalent.*

(i) $\{(W_1, v_1), (W_2, v_2)\}$ *is a Parseval fusion frame for* $\mathcal{H}$.

(ii) *Either we have* $W_1 \perp W_2$ *and* $v_1 = v_2 = 1$ *or we have* $W_1 = W_2 = \mathcal{H}$ *and* $v_1^2 + v_2^2 = 1$.

*Proof.* (i) $\Rightarrow$ (ii): First we assume that $W_2 \neq \mathcal{H}$. Fix some $g \perp W_2$. Then, by (i),

$$\|g\|^2 = v_1^2 \|\pi_{W_1}(g)\|^2 + v_2^2 \|\pi_{W_2}(g)\|^2 = v_1^2 \|\pi_{W_1}(g)\|^2 \le v_1^2 \|g\|^2,$$

hence $v_1^2 \ge 1$. But for all $f \in W_1$, we have

$$\|f\|^2 = v_1^2 \|f\|^2 + v_2^2 \|\pi_{W_2}(f)\|^2 \ge v_1^2 \|f\|^2.$$

This implies $v_1^2 \leq 1$, and therefore $v_1 = 1$. Now for all $f \in W_1$,

$$\|f\|^2 = \|f\|^2 + v_2^2 \|\pi_{W_2}(f)\|^2 .$$

This shows that $W_2 \perp W_1$, and $v_2 = 1$ follows immediately.

If $W_2 = \mathcal{H}$, towards a contradiction assume that $W_1 \neq \mathcal{H}$. Fix $g \perp W_1$. Then

$$\|g\|^2 = v_1^2 \|\pi_{W_1}(g)\|^2 + v_2^2 \|\pi_{W_2}(g)\|^2 = v_2^2 \|g\|^2 ,$$

hence $v_2^2 = 1$. Now for $g \in W_1$, we obtain

$$\|g\|^2 = v_1^2 \|\pi_{W_1}(g)\|^2 + v_2^2 \|\pi_{W_2}(g)\|^2 = (v_1^2 + 1) \|g\|^2 .$$

But this can only be true if $v_1 = 0$, a contradiction. Thus $W_1 = \mathcal{H}$. Now $v_1^2 + v_2^2 = 1$ follows from

$$\|f\|^2 = v_1^2 \|f\|^2 + v_2^2 \|\pi_{W_2}(f)\|^2 = (v_1^2 + v_2^2) \|f\|^2 \quad \text{for all } f \in \mathcal{H}.$$

(ii) $\Rightarrow$ (i): This is obvious. □

Now we apply Theorem 4.2 to this result yielding the following characterization of Parseval fusion frames in the situation of two (weighted) subspaces.

PROPOSITION 4.4. *Let $W_1$, $W_2$ be two closed subspaces of $\mathcal{H}$. Then the following conditions are equivalent.*

(i) *$\{(W_1, v_1), (W_2, v_2)\}$ is a Parseval fusion frame for $\mathcal{H}$.*

(ii) *There exist $v_1, v_2 > 0$ and a projection $P : \widetilde{W_1} \oplus \widetilde{W_2} \to \mathcal{H}$, where $\widetilde{W_1}$ and $\widetilde{W_2}$ are closed subspaces, such that $P|_{\widetilde{W_i}}$ is a $v_i$-isometry of $\widetilde{W_i}$ onto $W_i$for each $i = 1, 2$.*

(iii) *Either we have $W_1 \perp W_2$ and $v_1 = v_2 = 1$ or we have $W_1 = W_2 = \mathcal{H}$ and $v_1^2 + v_2^2 = 1$.*

*Proof.* This follows immediately from Theorem 4.2 and Proposition 4.3. □

Although Theorem 4.2 provides a complete characterization, we see the need to study property (ii) more closely, for instance, by using operator theoretic tools.

## 5. EXAMPLES OF FUSION FRAMES

Since almost all applications – in particular the modeling of sensor networks – require a finite-dimensional model, we restrict ourselves to finite-dimensional Hilbert spaces $\mathcal{H}$. We intend to discuss several examples, some of which indicate the mathematical depth of the construction of "good" fusion frames in the sense of Parseval fusion frames.

First we present a very general construction.

EXAMPLE 5.1 (GENERAL CONSTRUCTION). *Let $\{f_j\}_{j=1}^N$ be a frame for $\mathcal{H}$ with frame bounds $A$ and $B$. Given a partition $\{J_i\}_{i=1}^K$ of the indexing set $\{1, \ldots, N\}$, the set of subspaces $W_i := \text{span}_{j \in J_i}\{f_j\}$, $i = 1, \ldots, K$ forms a fusion frame with bounds $\frac{A}{B}$ and $K$. This was proven in [Prop. 5.2 and 5.3, 7]. We immediately see that although the bound is sharp, if we, for instance, choose $K = 1$, for "most" integers $K$ it is however a very rough estimate. However, finer estimates have to be adapted to the particular partition and to the frame bounds of the partition vectors. Exactly the same does hold for the lower estimate. Also in most cases we are interested in the lower and the upper bound being close, which in this case would mean $A \approx BK$.*

The next example will address the raised concerns.

EXAMPLE 5.2 (FUSION FRAMES AND THE KADISON-SINGER PROBLEM). *Suppose $\{f_j\}_{j=1}^N$ is a frame for $\mathcal{H} = \mathbb{R}^M$ say, with frame bounds $A$, $B$. As in the previous example, we split $\{1, \ldots, N\}$ into $K$ sets $J_1, \ldots, J_K$, and define $W_i = \text{span}\{f_j\}_{j \in J_i}$, $1 \leq i \leq K$. In the finite-dimensional situation each finite set of vectors forms* *a frame, in particular* $\{f_j\}_{j\in J_i}$ *is a frame for* $W_i$ *for each* $1 \le i \le K$. *Let* $C$ *and* $D$ *be a common lower and upper frame bound, respectively. Theorem 3.4 now implies that* $\{(W_i, 1, \{f_j\}_{j\in J_i})\}_{i=1}^K$ *is a fusion frame system with fusion frame bounds* $\frac{C}{B}$, $\frac{D}{A}$.

*In order for this process to work effectively, the* local frames *have to possess (uniformly) good lower frame bounds, since these control the computational complexity of reconstruction. However, it is known [11] that the problem of dividing a frame into a finite number of subsets each of which has good lower frame bounds is equivalent to one of the deepest and most intractable unsolved problems in mathematics:* the 1959 Kadison-Singer Problem. *Therefore, constructing fusion frames in the setting we need will require much more sophisticated methods than trying to merely divide a frame into fusion frame parts.*

Although we explained that general constructions of Parseval fusion frames by way of partitioning a frame are out of reach at the moment, we still have plenty of examples of Parseval fusion frames to present.

Example 5.3 (Examples of Parseval fusion frames). *Parseval fusion frames can be easily constructed by subspaces generated by vectors of a given orthonormal basis* $\{e_1, \dots, e_M\}$ *of* $\mathbb{R}^M$. *For this, set* $W_i = \mathrm{span}\{e_j\}_{j\in J_i}$, *where* $J_i \subset \{1, \dots, M\}$ *for* $1 \le i \le K$. *It is easy to see that the set of subspaces* $\{(W_i, v_i)\}_{i\in I}$ *is a Parseval fusion frame, if*

$$v_i = \frac{1}{\#\{j \in I : i \in J_j\}} \quad \text{for all } i \in \{1, \dots, K\},$$

*i.e., if, in particular, each element* $i \in \{1, \dots, K\}$ *is contained in* $v_i^{-1}$ *sets* $J_i$.

A special class of Parseval fusion frames will be illustrated in the last example.

Example 5.4 (Harmonic fusion frames). *Here we restrict our attention to finite fusion frames with weights being equal to* 1. *In analogy to a harmonic frame, we call a fusion frame* $\{W_i\}_{i=1}^n$ *a* harmonic fusion frame, *if there exists a unitary operator* $U$ *on* $\mathcal{H}$ *so that*

$$UW_n = W_1 \quad \text{and} \quad UW_i = W_{i+1} \quad \text{for all } 1 \le i \le n-1$$

*(compare [Sect. 6, 7]).*

*Then we can construct a Parseval fusion frame as follows. Let* $\varphi \in \mathcal{H}$ *and let* $V$ *be a unitary operator on* $\mathcal{H}$ *such that* $\{V^j\varphi\}_{j=1}^m$ *is an unit-norm Parseval frame sequence. Define the subspace* $W_1$ *by* $W_1 = \mathrm{span}\{V^j\varphi\}_{j=1}^m$. *Further let* $U$ *be a unitary operator on* $\mathcal{H}$. *Then the following conditions are equivalent.*

(i) $\{U^iW_1\}_{i=1}^l$ *is a Parseval fusion frame for* $\mathcal{H}$.

(ii) $\{U^iV^j\varphi\}_{i=1,\, j=1}^{l,\ \ m}$ *is a Parseval frame for* $\mathcal{H}$.

*This construction even leads to a Parseval fusion frame, for which all subspaces are generated from one single subspace by repeated application of a unitary operator.*

Additional construction can also be found in [4].

## 6. CONCLUSIONS

An introduction to the theory of fusion frames is elaborated in this article. We demonstrated that fusion frames are a well adapted tool for modeling data processing in a sensor network similar to the modeling of signal processing by frames. The theory of fusion frames does contain the theory of frames as a special case. In this sense, it goes beyond the frame theory. Fusion frames are just at the beginning of their exploration, and the theory is much more involved than frame theory due to the fact that a "frame of subspaces" is the object of study, instead of a "frame of vectors". We expect this theory to become an extensive mathematical tool to study problems not only derived in sensor networks but in even more general distributed processing applications.

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