OPTIMALLY SPARSE FUSION FRAMES: EXISTENCE AND CONSTRUCTION

Peter G. Casazza, Andreas Heinecke[*]

University of Missouri
Department of Mathematics
Columbia, MO 65211, USA

Gitta Kutyniok[†]

University of Osnabrück
Institute of Mathematics
49069 Osnabrück, Germany

ABSTRACT

Fusion frame theory is an emerging mathematical theory that provides a natural framework for performing hierarchical data processing. In this paper, we introduce the notion of a *sparse fusion frame*, that is, a fusion frame whose subspaces are generated by orthonormal basis vectors that are sparse in a 'uniform basis' over all subspaces, thereby enabling low-complexity fusion frame decompositions. We then provide an algorithmic construction to compute fusion frames with desired fusion frame operators, including tight fusion frames. Surprisingly, we can even prove that our algorithm constructs *optimally* sparse fusion frames.

Keywords— Computational complexity, frame decompositions, frame operator, frames, redundancy, sparse approximations, sparse matrices, tight frames.

1. INTRODUCTION

Recent advances in hardware technology have enabled the economic production and deployment of sensing and computing networks consisting of a large number of low-cost components, which through collaboration enable reliable and efficient operation. Across different disciplines there is a fundamental shift from centralized information processing to distributed or network-wide information processing. *Fusion frame theory* [1] is an emerging mathematical theory that provides a natural framework for two-stage (or, more generally, hierarchical) data processing. A *fusion frame* can be regarded as a frame-like collection of subspaces in a Hilbert space, and thereby generalizes the concept of a frame for signal representation.

Over the past few years, sparsity has become a key concept in various areas of applied mathematics, computer science, and electrical engineering. Sparse signal processing methodologies explore the fundamental fact that many types of signals can be represented by only a few non-zero coefficients when choosing a suitable basis or, more generally, a frame. A signal representable by only k, say, basis or frame elements is called k-*sparse*. If signals possess such a sparse representation, they can in general be recovered from few measurements using ℓ_1 minimization techniques (see, e.g., [2] and the references therein).

In this paper, we however pose a different question concerning sparsity, by viewing sparsity from a very different standpoint. Typically, data processing applications face low on-board computing power and/or small bandwidth budget. When the signal dimension is large, the decomposition of the signal into its fusion frame measurements requires a large number of additions and multiplications, which may be infeasible for on-board data processing. It would hence be a significant improvement, if the vectors of each orthonormal basis for the subspaces would contain very few non-zero entries, hence - phrasing it differently – be sparse in the standard unit vector basis, thereby ensuring low-complexity processing. Thus, our objective will be an algorithmic construction of optimally sparse fusion frames for which we only assume a given fusion frame operator, thereby including tight fusion frames as a special case while allowing additional flexibility.

In Section 2, we first review the basic notations and definitions for fusion frames, followed in Section 3 by the introduction of the novel concept of sparsity in fusion frame theory. We then present an algorithm constructing fusion frames with a prescribed fusion frame operator in Section 6. Surprisingly, in Section 5, in the case of frames, we can even prove that this algorithm leads to optimally sparse frames. We finally finish with some conclusions in Section 6.

2. FRAMES AND FUSION FRAMES

The notion of *fusion frames* (or *frames of subspaces*) was introduced in [1].

In contrast to frame theory, where a signal is represented by a collection of *scalars*, which measure the amplitudes of the projections of the signal onto the frame vectors, in fusion frame theory the signal is represented by a collection of *vectors*, more precisely, the projections of the signal onto the fusion frame subspaces. In a two-stage data processing setup, these projections serve as locally processed data, which can be combined to reconstruct the signal of interest.

Let $(\mathcal{W}_i)_{i=1}^N$ be a family of subspaces in $\mathbb{R}^n$ with associated

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positive weights v_i, $i = 1, \ldots, N$. Then $(\mathcal{W}_i, v_i)_{i=1}^N$ forms a *fusion frame* for $\mathbb{R}^n$ if there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|x\|^2 \leq \sum_{i=1}^N v_i^2 \|P_i x\|^2 \leq B\|x\|^2 \qquad \text{for any } x \in \mathbb{R}^n,$$

where P_i is the orthogonal projection onto $\mathcal{W}_i$. The constants A and B are called *fusion frame bounds*. We refer to a fusion frame as being A-*tight*, if A and B can be chosen as $A = B$, and *Parseval*, if $A = B = 1$. If $v_i = 1$ for all $i \in I$,we sometimes write $(\mathcal{W}_i)_{i=1}^N$ instead of $(\mathcal{W}_i, 1)_{i=1}^N$.

The decomposition of any signal $x \in \mathbb{R}^n$ according to a fusion frame $(\mathcal{W}_i, v_i)_{i=1}^N$ is given by the *fusion frame measurements* $(v_i P_i x)_{i=1}^N$, which are generated by the *synthesis operator* $\mathbb{R}^n \ni x \mapsto (v_i P_i x)_{i=1}^N \in \mathbb{R}^{Nn}$. These completely characterize the signal x, which can be reconstructed from

$$x = \sum_{i=1}^N v_i S^{-1}(v_i P_i x),$$

where $S = \sum_{i=1}^N v_i^2 P_i f$ is the *fusion frame operator* known to be positive and self-adjoint. We refer the interested reader to [1] for more details.

By [1, Thm. 2.3], a fusion frame is associated with a frame by letting $(f_{i,\ell})_{\ell=1}^{m_i}$ be an orthonormal basis for each $\mathcal{W}_i$ and considering $(f_{i,\ell})_{i=1,\ell=1}^{N,m_i}$, which we will refer to as an *associated frame* to $(\mathcal{W}_i)_{i=1}^N$. This allows us to write the synthesis operator in the matrix form $[f_{1,1}|\ldots|f_{N,m_N}]$ with the associated frame vectors as columns, and we refer to this matrix as the *synthesis matrix associated with* $(f_{i,\ell})_{i=1,\ell=1}^{N,m_i}$.

A frame can be regarded as a special fusion frame in the following way: Let $(\varphi_i)_{i=1}^N \subseteq \mathbb{R}^n$ and define subspaces $(\mathcal{W}_i)_{i=1}^N$ by $\mathcal{W}_i := \text{span}\{\varphi_i\}$, $i = 1, \ldots, N$. Then $(\varphi_i)_{i=1}^N$ is a *frame* for $\mathbb{R}^n$ with *frame bounds* A and B if and only if $(\mathcal{W}_i, \|\varphi_i\|_{i=1}^N)$ is a fusion frame with fusion frame bounds A and B. This can be directly deduced from the relation

$$\sum_{i=1}^N \|\varphi_i\|^2 \|P_i x\|^2 = \sum_{i=1}^N |\langle x, \varphi_i \rangle|^2.$$

3. NEW PARADIGM FOR FUSION FRAME CONSTRUCTIONS: SPARSITY

3.1. Sparseness Measure

As already elaborated upon before, we aim for sparsity in the standard unit vector basis of an orthonormal basis for the subspaces, which ensures low-complexity processing. Since we are interested in the performance of the whole fusion frame, the total number of non-zero entries seems to be a suitable sparsity measure. This viewpoint can also be slightly generalized by assuming that there exists a unitary transformation mapping the fusion frame into one having this 'sparsity' property. Taking these considerations into account, we are led to proclaim the following definition for a sparse fusion frame, which then reduces to the notion of a sparse frame, which we also state for the convenience of the reader.

Definition 3.1 *Let $(e_j)_{j=1}^n$ be an orthonormal basis for $\mathbb{R}^n$. Then a fusion frame $(\mathcal{W}_i)_{i=1}^N$ for $\mathbb{R}^n$ with $\dim \mathcal{W}_i = m_i$ for all $i = 1, \ldots, N$ is called k-*sparse *with respect to $(e_j)_{j=1}^n$, if, for each $i \in \{1, \ldots, N\}$, there exists an orthonormal basis $(f_{i,\ell})_{\ell=1}^{m_i}$ for $\mathcal{W}_i$ and, for each $\ell = 1, \ldots, m_i$ an $(J_{i,\ell}) \subset \{1, \ldots, n\}$ such that*

$$f_{i,\ell} \in \text{span}\{e_j : j \in J_{i,\ell}\} \text{ and } \sum_{i=1}^n \sum_{\ell=1}^{m_i} |J_{i,\ell}| = k.$$

We refer to $(f_{i,\ell})_{i=1,\ell=1}^{N,m_i}$ as an associated k-sparse frame.

*In particular, a frame $(\varphi_i)_{i=1}^N$ for $\mathbb{R}^n$ is called k-*sparse *with respect to $(e_j)_{j=1}^n$, if, for each $i \in \{1, \ldots, N\}$, there exists $J_i \subseteq \{1, \ldots, n\}$ such that*

$$\varphi_i \in \text{span}\{e_j : j \in J_i\} \text{ and } \sum_{i=1}^n |J_i| = k.$$

3.2. Notion of Optimality

We now have the necessary machinery at hand to introduce a notion of an *optimally* sparse fusion frame. Optimality will typically – as also in this paper - be considered within a particular class of fusion frames, e.g., in the class of tight ones.

Definition 3.2 *Let $\mathcal{FF}$ be a class of fusion frames for $\mathbb{R}^n$, let $(\mathcal{W}_i)_{i=1}^N \in \mathcal{FF}$, and let $(e_j)_{j=1}^n$ be an orthonormal basis for $\mathbb{R}^n$. Then $(\mathcal{W}_i)_{i=1}^N$ is called* optimally sparse in $\mathcal{FF}$ with respect to $(e_j)_{j=1}^n$, *if $(\mathcal{W}_i)_{i=1}^N$ is k_1-sparse with respect to $(e_j)_{j=1}^n$ and there does not exist a fusion frame $(\mathcal{V}_i)_{i=1}^N \in \mathcal{FF}$ which is k_2-sparse with respect to $(e_j)_{j=1}^n$ with $k_2 < k_1$.*

In particular, letting $\mathcal{F}$ be a class of frames for $\mathbb{R}^n$, $(\varphi_i)_{i=1}^N \in \mathcal{F}$, and $(e_j)_{j=1}^n$ an orthonormal basis for $\mathbb{R}^n$. Then $(\varphi_i)_{i=1}^N$ is called optimally sparse in $\mathcal{F}$ with respect to $(e_j)_{j=1}^n$, *if $(\varphi_i)_{i=1}^N$ is k_1-sparse with respect to $(e_j)_{j=1}^n$ and there does not exist a frame $(\psi_i)_{i=1}^N \in \mathcal{F}$ which is k_2-sparse with respect to $(e_j)_{j=1}^n$ with $k_2 < k_1$.*

4. CONSTRUCTION OF SPARSE FUSION FRAMES WITH A DESIRED FUSION FRAME OPERATOR

4.1. Spectral Tetris for Fusion Frames

Given a desired fusion frame operator associated with eigenvalues $\lambda_1, \ldots, \lambda_n \geq 2$ satisfying $\sum_{j=1}^n \lambda_j = mN$, we are interested in the construction of associated fusion frames $(\mathcal{W}_i)_{i=1}^N$ with $\dim \mathcal{W}_i = m$. Our algorithm will contain the original form of Spectral Tetris from [3] which constructed unit norm tight frames as a special case. Figure 1 states the steps of our algorithm, which we coin *Spectral Tetris for Fusion Frames*; in short, STFF.

STFF: SPECTRAL TETRIS FOR FUSION FRAMES

Parameters:

- Dimension $n \in \mathbb{N}$.
- Number of fusion frame subspaces $N \in \mathbb{N}$.
- Dimension of the subspaces $m \in \mathbb{N}$.
- Sequence of eigenvalues $\lambda_1, \ldots, \lambda_n \geq 2$ satisfying $\sum_{j=1}^{n} \lambda_j = mN$.

Algorithm:

Set $i := 1$.
For $j = 1, \ldots, n$ do
    Repeat
        If $\lambda_j < 1$ then
            $\varphi_i := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
            $\varphi_{i+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
            $i := i + 2$.
            $\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
            $\lambda_j := 0$.
        else
            $\varphi_i := e_j$.
            $i := i + 1$.
            $\lambda_j := \lambda_j - 1$.
        end.
    until $\lambda_j = 0$.
end.

Output:

- Fusion frame $\mathrm{STFF}(N, m; \lambda_1, \ldots, \lambda_n) := (\mathcal{W}_i := \mathrm{span}\{\varphi_{i+kN} : k = 0, \ldots, m-1\})_{i=1}^{N}$.

Fig. 1. The Spectral Tetris algorithm for constructing a fusion frame $\mathrm{STFF}(N, m; \lambda_1, \ldots, \lambda_n)$ for $\mathbb{R}^n$ with N subspaces of dimension m and associated fusion frame operator having eigenvalues $\lambda_1, \ldots, \lambda_n$.

4.2. Analysis of STFF

The following result shows that this algorithm indeed succeeds in its task.

Theorem 4.1 ([4]) *Suppose the eigenvalues $\lambda_1, \ldots, \lambda_n \geq 2$ satisfy $\sum_{j=1}^{n} \lambda_j = mN$ and, for all $j = 1, \ldots, n$, we have $\lfloor \lambda_j \rfloor \leq N - 3$. Then the fusion frame $(\mathcal{W}_i)_{i=1}^{N}$ constructed by STFF satisfies $\dim \mathcal{W}_i = m$ for all $i = 1 \ldots, N$ and the associated fusion frame operator has $\{\lambda_j\}_{j=1}^{n}$ as its eigenvalues.*

4.3. Special Case: Spectral Tetris for Frames

In the special case of frame constructions with a prescribed frame operator, only the dimension $n \in \mathbb{N}$ of the ambient space, the number of frame vectors $N \in \mathbb{N}$, and the sequence of eigenvalues $\lambda_1, \ldots, \lambda_n \geq 2$ satisfying $\sum_{j=1}^{n} \lambda_j = N$ need to be given. The body of the algorithm in Figure 1 remains unchanged. Only the output differs in that now the frame $\mathrm{STF}(N; \lambda_1, \ldots, \lambda_n) := (\varphi_i)_{i=1}^{N}$ is the chosen one.

5. SPECTRAL TETRIS GENERATES OPTIMALLY SPARSE FUSION FRAMES

We now prove that the fusion frames constructed by STFF (see Figure 1) are indeed optimally sparse in the sense introduced in Section 3.

5.1. Class for Optimality

We start by defining the class of fusion frames generated by STFF, which will be the class optimality will be considered in.

Let $n, N > 0$ and let the real values $\lambda_1, \ldots, \lambda_n \geq 2$ satisfy $\sum_{j=1}^{n} \lambda_j = N$. Then the class of fusion frames $(\mathcal{W}_i)_{i=1}^{N}$ in $\mathbb{R}^n$ with $\dim \mathcal{W}_i = m$ for all $i = 1, \ldots, N$ whose fusion frame operator has eigenvalues $\lambda_1, \ldots, \lambda_n$ will be denoted by $\mathcal{FF}(N, m, \{\lambda_i\}_{i=1}^{n})$. It is important to mention that by writing $\{\lambda_i\}_{i=1}^{n}$, we wish to indicate that the ordering does not play a role here, however, multiplicities are counted.

5.2. Novel Structural Property of Synthesis Matrices

Aiming for determining the maximally achievable sparsity for a class $\mathcal{FF}(N, m, \{\lambda_i\}_{i=1}^{n})$, we first need to introduce a particular measure associated with the set of eigenvalues $\{\lambda_i\}_{i=1}^{n}$. This measure indicates the maximal number of partial sums which are an integer when maximizing over the ordering of the eigenvalues.

Definition 5.1 *A finite sequence of real values $\lambda_1, \ldots, \lambda_n$ is* ordered blockwise*, if for any permutation π of $\{1, \ldots, n\}$ the set of partial sums $\{\sum_{j=1}^{s} \lambda_j\}_{s=1}^{n}$ contains at least as many integers as the set $\{\sum_{j=1}^{s} \lambda_{\pi(j)}\}_{s=1}^{n}$. The* maximal block number *of a finite sequence of real values $\lambda_1, \ldots, \lambda_n$, denoted by $\mu(\lambda_1, \ldots, \lambda_n)$, is the number of integers in $\{\sum_{j=1}^{s} \lambda_{\sigma(j)}\}_{s=1}^{n}$, where σ is a permutation of $\{1, \ldots, n\}$ such that $\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}$ is ordered blockwise.*

Surprisingly, the notion of maximal block number can illuminatingly be transferred to a particular decomposition property of the synthesis matrix of a fusion frame. Let us first define the decomposition property we are interested in:

Definition 5.2 *Let $n, N > 0$, and let $(\mathcal{W}_i)_{i=1}^{N}$ be a fusion frame for $\mathbb{R}^n$ with associated frame $(f_{i,\ell})_{\ell=1}^{m_i}$. Then we say that the synthesis matrix of $(\mathcal{W}_i)_{i=1}^{N}$ associated with $(f_{i,\ell})_{\ell=1}^{m_i}$ has* block decomposition of order M*, if there exists a partition $\{1, \ldots, N\} = I_1 \cup \ldots \cup I_M$ such that, for any $k_1 \in I_{i_1}$ and $k_2 \in I_{i_2}$ with $i_1 \neq i_2$, we have $\mathrm{supp}\, \varphi_{k_1} \cap \mathrm{supp}\, \varphi_{k_2} = \emptyset$ and M is maximal.*

The following result now connects the maximal block number of the sequence of eigenvalues of a fusion frame operator with the block decomposition order of an associated fusion frame. It follows directly from [1, Thm. 2.3] and [5, Prop. 4.3].

Proposition 5.3 *Let $n, m, N > 0$ and let the real values $\lambda_1, \ldots, \lambda_n \geq 2$ satisfy $\sum_{j=1}^{n} \lambda_j = mN$. Then the synthesis matrix of any fusion frame in the class $\mathcal{FF}(N, m, \{\lambda_i\}_{i=1}^{n})$ with any associated frame has block decomposition of order at most $\mu(\lambda_1, \ldots, \lambda_n)$.*

5.3. Maximally Achievable Sparsity

Having introduced the required new notions, we are now in the position to state the exact value for the maximally achievable sparsity for a class $\mathcal{FF}(N, m, \{\lambda_i\}_{i=1}^{n})$.

Theorem 5.4 *Let $n, m, N > 0$, and let $\lambda_1, \ldots, \lambda_n \geq 2$ satisfy $\sum_{j=1}^{n} \lambda_j = mN$. Then any fusion frame in $\mathcal{FF}(N, m, \{\lambda_i\}_{i=1}^{n})$ is at least $mN + 2(n - \mu(\lambda_1, \ldots, \lambda_n))$-sparse with respect to any orthonormal basis.*

This result is a consequence of [1, Thm. 2.3] and [5, Thm. 4.4]. It should be mentioned that an optimally sparse frame from $\mathcal{FF}(N, m, \{\lambda_i\}_{i=1}^{n})$ is in general not uniquely determined.

5.4. Main Result

Having set the benchmark, we now prove that fusion frames constructed by STFF in fact achieve the optimal sparsity rate. The result can be proven by using [1, Thm. 2.3] and [5, Thm. 4.5].

Theorem 5.5 *Let $n, m, N > 0$, and let $\lambda_1, \ldots, \lambda_n \geq 2$ be ordered blockwise and satisfy $\sum_{j=1}^{n} \lambda_j = mN$. Then the fusion frame STFF$(N, m; \lambda_1, \ldots, \lambda_n)$ is optimally sparse in the class $\mathcal{FF}(N, m, \{\lambda_i\}_{i=1}^{n})$ with respect to the standard unit vector basis.*

In particular, this fusion frame is $mN + 2(n - \mu(\lambda_1, \ldots, \lambda_n))$-sparse with respect to the standard unit vector basis, and the vectors generated by STFF are an associated $mN + 2(n - \mu(\lambda_1, \ldots, \lambda_n))$-sparse frame.

The reader will have realized that Spectral Tetris generates frames which are 'only' optimally sparse with respect to the standard unit vector basis. This seems at first sight like a drawback. However, if sparsity with respect to a different orthonormal basis is required, Spectral Tetris can easily be modified to accommodate this request by using vectors of this orthonormal basis instead of the standard unit vector basis.

5.5. Special Case: Optimally Sparse Tight Frames

In the special case of frames and of equal eigenvalues, i.e., of tight frames, with N elements in $\mathbb{R}^n$, all eigenvalues need to equal $\frac{N}{n}$ for the equality $\sum_{j=1}^{n} \lambda_j = N$ to be satisfied. The maximal block number can be easily computed to be $\gcd(N, n)$. Theorem 5.5 then takes the following form:

Corollary 5.6 *For $n, N > 0$, the frame STF$(N; \frac{N}{n}, \ldots, \frac{N}{n})$ is optimally sparse in the class of all N-element tight frames for $\mathbb{R}^n$ with respect to the standard unit vector basis.*

In particular, this frame is $N + 2(n - \gcd(N, n))$-sparse with respect to the standard unit vector basis.

6. CONCLUSIONS AND DISCUSSION

In this paper we considered the design of fusion frames associated with a desired fusion frame operator while simultaneously enabling efficient computations of the associated fusion frame measurements. This led to the introduction of the notion of sparsity measure for fusion frames, thereby introducing optimal sparsity as a new paradigm into the construction of fusion frames. We then introduced an extended version of Spectral Tetris for fusion frames capable of constructing fusion frames with a desired fusion frame operator. We could then even prove that the fusion frames constructed by this algorithm are indeed optimally sparse.

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