Spectral properties of dual frames

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***Abstract*—We study spectral properties of dual frames of a given finite frame. We give a complete characterization for which spectral patterns of dual frames are possible for a fixed frame. For many cases, we provide simple explicit constructions for dual frames with a given spectrum, in particular, if the constraint on the dual is that it be tight.**

I. Introduction

In signal processing, one of the primary objectives is to obtain suitable representations of the signals of interest. Finite frames are redundant systems in a finite dimensional Hilbert space, which give redundant representations of finite dimensional signals. The representation process can be split into two steps: the decomposition and the reconstruction. For each frame decomposition method, there is one canonical reconstruction using a least-squares approach. However, due to the redundancy of frames, there many alternative reconstruction methods. Each of these alternative reconstruction methods is associated to a so-called dual frame.

It is therefore natural to ask which dual frame for the reconstruction step is the best to choose in case the decomposition frame is given by the application at hand, e.g., by the way of measuring the data. The precise answer to this question is, of course, dependent on the application, but universal desirable properties of the dual can, nonetheless, be recognized. Among such desirable properties are fast and stable reconstruction. It turns out that the computational properties of the dual frames such as the *stability* of the reconstructions are directly linked to spectral properties of the frame. In particular, the Frobenius norm and the spectral norm of the so-called dual frame matrix play an important role in this context. In Subsection I-B below, we will illustrate the importance of these matrix norms in a situation, where we want to minimize the effect of noise from a noisy decomposition. Before we embark on this, we will need some basic definition from frame theory.

A. Setup and basic observations

Let us recall some basic definitions and facts from frame theory. For an extensive exposition on frames and their applications, we refer the reader to the books [1], [2]. We let $\mathbb{K}$ denote either $\mathbb{C}$ or $\mathbb{R}$ and define frames in $\mathbb{K}^n$ as follows.

Definition I.1. A collection of vectors $\Phi = (\phi_i)_{i=1}^m \subset \mathbb{K}^n$ is called a *frame* for $\mathbb{K}^n$ if there are two constants $0 < A \le B$ such that

$$A \left\| x \right\|_2^2 \le \sum_{i=1}^m \left| \langle x, \phi_i \rangle \right|^2 \le B \left\| x \right\|_2^2, \quad \text{for all } x \in \mathbb{K}^n. \tag{1}$$

If the frame bounds A and B are equal, the frame $(\phi_i)_{i=1}^m$ is called a *tight frame* for $\mathbb{K}^n$.

In this paper, we are interested in the case $m > n$, where the frame $(\phi_i)_{i=1}^m$ is *redundant*, i.e., it consists of more vectors than necessary for the spanning property. For these frames there exist infinitely many dual frames. The precise definition of dual frames is the following:

Definition I.2. Given a frame Φ, another frame $\Psi = (\psi_i)_{i=1}^m \subset \mathbb{K}^n$ is said to be a *dual frame* of Φ if the following reproducing formula holds:

$$x = \sum_{i=1}^m \langle x, \phi_i \rangle \psi_i \quad \text{for all } x \in \mathbb{K}^n. \tag{2}$$

In matrix notation this definition reads

$$\Psi \Phi^* = I_n, \tag{3}$$

where the maps induced by Φ^* and Ψ correspond to the decomposition and reconstruction procedure, respectively, and where I_n is the $n \times n$ identity matrix. Hence, the set of all duals of Φ is the set of all left-inverses Ψ to Φ^*. The particular choice of Ψ as the Moore-Penrose pseudoinverse of Φ^* is the *canonical dual frame* of Φ.

From (3) it is immediate that the set of all duals Ψ to a frame Φ is an $n(m-n)$-dimensional affine subspace of $\mathrm{Mat}(\mathbb{K}, n \times m)$. A natural parametrization of this space is obtained using the singular value decomposition. Let $\Phi = U \Sigma_\Phi V^*$ be a full SVD of Φ, i.e., $U \in \mathbb{K}^{n \times n}$ and $V \in \mathbb{K}^{m \times m}$ are unitary and $\Sigma_\Phi \in \mathbb{R}^{n \times m}$ is a diagonal matrix whose entries, namely $\sqrt{B} = \sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n = \sqrt{A} > 0$, are non-negative and arranged in a non-increasing order. We will sometimes write the ith singular value of Φ as σ_i^Φ. Let Ψ be a frame and define $M_\Psi := U^* \Psi V \in \mathbb{K}^{n \times m}$, where U and V are the right and left singular vectors of Φ. Then Ψ factors as $\Psi = U M_\Psi V^*$. By $\Phi \Psi^* = I_n$, we then see that

$$I_n = U^* I_n U = U^* \Phi \Psi^* U = \Sigma_\Phi M_\Psi^*.$$

Therefore, Ψ is a dual frame of Φ precisely when

$$\Sigma_\Phi M_\Psi^* = I_n, \tag{4}$$

where $\Psi = U M_\Psi V^*$. The solutions to (4) are given by

$$M_\Psi = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 & s_{1,1} & s_{1,2} & \cdots & s_{1,r} \\ 0 & \frac{1}{\sigma_2} & & 0 & s_{2,1} & s_{2,2} & \cdots & s_{2,r} \\ & & \ddots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \frac{1}{\sigma_n} & s_{n,1} & s_{n,2} & \cdots & s_{n,r} \end{bmatrix}, \tag{5}$$

where $s_{i,k} \in \mathbb{K}$ for $i = 1, \ldots, n$ and $k = 1, \ldots, r = m - n$. Note that the canonical dual frame is obtained by taking $s_{i,k} = 0$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, m-n$. More importantly, since U and V are unitaries, the possible spectrum of duals Ψ is completely described by the matrices M_Ψ in (5).

B. *Measures of the goodness of dual frames*

In this subsection we consider the important scenario when the frame coefficients $c = \Phi^* x$ of the signal $x \in \mathbb{K}^n$ are corrupted by noise e. We will assume that the noise components e_i corresponding to the different frame coefficients are centered, uncorrelated, and of the same variance. This is a standard setup used e.g., in [3] for unit-norm frames, and in [7] for the case of Gaussian white noise. We will here follow the above general setup from [5]. We remark that it is possible to study an alternative scenario of corruptions through erasures, see [8]–[10].

The reconstruction error is given by

$$\|\Psi \tilde{c} - x\|_2 = \|\Psi(\Phi^* x + e) - x\|_2 = \|\Psi e\|_2, \tag{6}$$

where the corrupted frame coefficients are $\tilde{c} = c + e$. Hence, we see that different duals Ψ yield different reconstruction accuracy. It can be shown, see e.g., [5], that the expected error is controlled by the Frobenius norm of the matrix Ψ. To be precise, one has the expected reconstruction accuracy

$$\mathbb{E}\|\Psi \tilde{c} - x\|_2 \le \sqrt{\frac{\delta B}{m}} \|\Psi\|_F, \tag{7}$$

where the variance satisfies $\sigma^2 \le \frac{\delta B}{m}$ with B being the upper frame bound of Φ and $\delta < 1$. This shows that the Frobenius norm of the dual frame matrix Ψ is crucial in the average case scenario.

For the worst case scenario the spectral norm of Ψ is the correct measure. This is seen as follows. Recall that the condition number of an $n \times n$ invertible matrix T is given by $\operatorname{cond}(T) = \max\left(\frac{\text{relative output error}}{\text{relative input error}}\right) = \sigma_1^T / \sigma_n^T$. For a pair of dual frames similar considerations give

$$\begin{aligned} \operatorname{cond}(\Phi, \Psi) &:= \max\left(\frac{\|\Psi e\|_2 \,/\, \|x\|_2}{\|e\|_2 \,/\, \|\Phi^* x\|_2}\right) \\ &= \max\left(\frac{\|\Psi e\|_2}{\|e\|_2} \frac{\|\Phi^* x\|_2}{\|x\|_2}\right) \\ &= \|\Phi\|_{2\to 2} \|\Psi\|_{2\to 2} = \sigma_1^\Phi \, \sigma_1^\Psi . \end{aligned}$$

Note that if Φ is an invertible matrix, we recover the usual definition: $\operatorname{cond}(\Phi, \Psi) = \sigma_1^\Phi / \sigma_n^\Phi$. We see that only the largest singular value of Ψ plays a role in the measure of goodness of dual frames for the worst case scenario.

In this subsection we have set up two important measures for the goodness of a dual frame. Since both of these measures are determined by the singular values of the dual frame, we are interested in understanding the possible spectra in the set of all duals of a given frame. This is the theme of the second part of this paper, Section II, where we characterize the possible spectral patterns of dual frames.

II. Spectral properties of duals

In this section we characterize the possible spectra in the set of all dual frames of a given frame. However, we begin with the special case of characterizing frames that admits tight duals, which is exactly the situation when the spectrum of the dual frame is a one point spectrum. The characterization was obtained in [11] and extended in [6].

It turns out that a frame always has a tight dual if the redundancy is two or larger. If the redundancy is less than two, it will only be possible under certain assumptions on the singular values of Φ.

Theorem II.1 ([6], [11]). *Let $n, m \in \mathbb{N}$. Suppose Φ is a frame for $\mathbb{K}^n$ with m frame vectors and lower frame bound A. Then the following assertions hold:*

(i) If $m \ge 2n$, then for every $c \ge \frac{1}{A}$, there exists a tight dual frame Ψ with frame bound c.

(ii) If $m = 2n - 1$, then there exists a tight dual frame Ψ; the only possible frame bound is $\frac{1}{A}$.

(iii) Suppose $m < 2n - 1$. Then there exists a tight dual frame Ψ if and only if the smallest $2n - m \in \{2, \ldots, n\}$ singular values of Φ are equal. In the positive case, the only possible frame bound is $\frac{1}{A}$.

Before we turn to a proof of Theorem II.1, let us give a simple dimension counting argument to explain why $m = 2n-1$ is the borderline case. Any dual frame Ψ will be row bi-orthogonal to Φ. Hence, for each $j_0 = 1, \ldots, n$, the j_0th row vector ψ^{j_0} of Ψ needs to be orthogonal to the jth row vector ϕ^j of Φ for $j \ne j_0$. For the dual frame Ψ to be tight, the matrix Ψ furthermore needs to be row orthogonal, hence ψ^{j_0} needs to be orthogonal to ψ^j for each $j \ne j_0$. In total, the vector $\psi^{j_0} \in \mathbb{K}^m$ needs to be orthogonal to $2(n-1)$ other vectors. If $m \ge 2n - 1$, it is possible to find $2(n-1) + 1 = 2n - 1$ orthogonal vectors in $\mathbb{K}^m$, which shows that we can find n orthogonal vectors $(\psi^j)_{j=1}^n$ being bi-orthogonal to $(\phi^j)_{j=1}^n$. As a final step to make Ψ a tight dual, we need to scale the vectors ψ^j, $j = 1, \ldots, n$, to have equal norm.

The above argument is almost a proof of Theorem II.1. However, we include the following proper proof adapted from [6] since it provides an *explicit* construction procedure for the tight duals.

Proof of Theorem II.1: Let $\Phi = U \Sigma_\Phi V^*$ be a full SVD of Φ, and let Ψ be an arbitrary dual frame. Following Section I-A, we factor the dual frame as $\Psi = U M_\Psi V^*$, where M_Ψ is given as in (5) with $s_{i,k} \in \mathbb{K}$ for $i = 1, \ldots, n$ and

$k = 1, \dots, r = m - n$. For Ψ to be tight, we need to choose $s_{i,k}$ such that the rows of M_Ψ are orthogonal and have equal norm. This follows from the fact that Ψ is row orthogonal if and only if M_Ψ is row orthogonal.

As the diagonal block of M_Ψ is well-understood, the duality and tightness constraints translate to conditions for the inner products of the $s_i = (s_{i,1}, \dots, s_{i,r}) \in \mathbb{K}^r$, $i = 1, \dots, n$. Indeed, Ψ is a tight dual frame with frame bound c if and only if, for all $1 \le i \le n$, one has

$$c = \frac{1}{\sigma_i^2} + \|s_i\|_2^2 , \tag{8}$$

and, for all $i \neq j = 1, \dots, n$, one has $\langle s_i, s_j \rangle = 0$.

Now assume that $\sigma_n = \sigma_{n-1} = \cdots = \sigma_{p+1} < \sigma_p$ for some $p < n$. As $\sigma_{p+1} < \sigma_i$ for all $1 \le i \le p$, (8) implies that all s_i for $i = 1, \dots, p$ must be nonzero vectors even if $s_{p+1}, \dots, s_n$ are all zero. Furthermore, (8) also determines the norms of $s_1, \dots, s_p$ as a function of $\|s_{p+1}\| = \cdots = \|s_n\|$. The second condition implies that if $s_n \neq 0$, the sequence $(s_i)_{i=1}^n$ is orthogonal, else the sequence $(s_i)_{i=1}^p$.

If $r \ge n$, that is, if $m \ge 2n$, then any choice of s_n allows for an orthogonal system with compatible norms, so tight dual frames with any frame bound above $\frac{1}{\sigma_n}$ exist and can be efficiently constructed. If $r < n$, then no n vectors can form an orthogonal system, one needs to have $s_n = 0$ and hence also $s_j = 0$ for all $j > p$. So no frame bound other than $\frac{1}{\sigma_n}$ is possible. The remaining vectors $\{s_j\}_{j=1}^p$ are all non-zero, so they must form an orthogonal system. For $r \ge n-p+1$, this is possible, and again a solution satisfying the norm constraints can be efficiently constructed. For $r \le n - p$, no such system exists, hence there cannot be a tight dual. ■

We will now derive general conditions on which spectral patterns (now possibly consisting of more than one point) can be achieved by a dual frame of a given frame. The reason that, in the general framework, such an analysis is harder than in the context of tight duals is that in the tight case, the frame operator is a multiple of the identity, hence diagonal in any basis. This no longer holds true if we drop the tightness assumption, so when the orthogonality argument of Theorem II.1 fails, one cannot conclude that there is no dual with a given spectral pattern. However, the orthogonality approach allows to choose a subset of the singular values of the dual frame freely. In particular, if the redundancy of the frame Φ is larger than 2, it follows that for all spectral patterns satisfying a set of lower bounds, which we will later show to be necessary (see Theorem II.4), a dual with that spectrum can be found using a constructive procedure analogous to the proof of Theorem II.1.

Theorem II.2 ([6]). *Let $n, m \in \mathbb{N}$, and let Φ be a frame for $\mathbb{K}^n$ with m frame vectors and singular values $(\sigma_i)_{i=1}^n$. Suppose that $r \le m-n$ and that $I \subset \{1, \dots, n\}$ with $|I| = r$. Then, for any sequence $(q_i)_{i\in I}$ satisfying $q_i \ge 1/\sigma_i$ for all $i \in I$, there exists a dual frame Ψ of Φ such that $\{q_i\}_{i\in I}$ is contained in the spectrum of Ψ. Furthermore, it can be found constructively using a sequence of orthogonalization procedures.*

Proof: The proof is just a slight modification of the proof of Theorem II.1. Again, we choose $(s_i)_{i\in I}$ to be orthogonal and the remaining s_i's to be the zero vector. The non-zero s_i vectors are scaled to satisfy

$$q_i^2 = \frac{1}{\sigma_i^2} + \|s_i\|_2^2 ,$$

where $i \in I$. Hence, by this procedure we obtain a dual frame with spectrum $\{q_i\}_{i\in I} \cup \{\sigma_i^{-1}\}_{i \notin I}$. ■

As a corollary we obtain that using the same simple constructive procedure, one can find dual frames with any frame bound that is possible.

Corollary II.3 ([6]). *Let Φ be a redundant frame for $\mathbb{K}^n$ with singular values $(\sigma_i)_{i=1}^n$. Fix an upper frame bound satisfying $B^\Psi \ge \frac{1}{\sigma_n^2}$ and a lower frame bound $\frac{1}{\sigma_{m-n+1}^2} \ge A^\Psi \ge \frac{1}{\sigma_1^2}$, where we use the convention $\frac{1}{\sigma_{m-n+1}} = \infty$ if $m \ge 2n$. Then a dual frame Ψ of Φ with these frame bounds can be found constructively using a sequence of orthogonalization procedures.*

We are now ready to state the complete characterization of the possible spectra of dual frames.

Theorem II.4 ([6]). *Let $n, m \in \mathbb{N}$, and set $r = m - n$. Let Φ be a frame for $\mathbb{K}^n$ with singular values $(\sigma_i)_{i=1}^n$. Suppose Ψ is any dual frame with singular values $(\sigma_i^\Psi)_{i=1}^n$ (also arranged in a non-increasing order). Then the following inequalities hold:*

$$\frac{1}{\sigma_{n-i+1}} \le \sigma_i^\Psi \qquad \text{for } i = 1, \dots, r, \tag{9}$$

$$\frac{1}{\sigma_{n-i+1}} \le \sigma_i^\Psi \le \frac{1}{\sigma_{n-i+r+1}} \qquad \text{for } i = r+1, \dots, n. \tag{10}$$

Furthermore, for every sequence $(\sigma_i^\Psi)_{i=1}^n$ which satisfies (9) and (10), there is a dual Ψ of Φ with singular values $(\sigma_i^\Psi)_{i=1}^n$.

Proof: The necessity of the conditions follows by r applications of [4, Theorem 7.3.9] on the matrix M_Ψ defined in (5) or from the well-known interlacing inequalities for Hermitian matrices by Weyl. For the existence part, we refer to the proof in [6]. ■

The inequalities (9) and (10), written in terms of the singular values $(\sigma_i^{\widetilde{\Psi}})_{i=1}^n$ of the canonical dual frame $\widetilde{\Psi} := S^{-1}\Phi$, have the following simple form:

$$\sigma_i^{\widetilde{\Psi}} \le \sigma_i^\Psi \qquad \text{for } i = 1, \dots, r,$$

$$\sigma_i^{\widetilde{\Psi}} \le \sigma_i^\Psi \le \sigma_{i-r}^{\widetilde{\Psi}} \qquad \text{for } i = r+1, \dots, n.$$

In terms of eigenvalues of frame operators, Theorem II.4 states that the spectra in the set of all duals exhaust the set $\Lambda \subset \mathbb{R}^n$ defined by

$$\Lambda = \left\{ (\lambda_i) \in \mathbb{R}^n : \lambda_i^{\widetilde{\Psi}} \le \lambda_i \le \lambda_{i-r}^{\widetilde{\Psi}} \text{ for all } i = 1, \dots, n \right\},$$

where $\lambda_i^{\widetilde{\Psi}} = 1/\lambda_{n-i+1}^\Phi$ is the ith eigenvalue of the canonical dual frame operator; we again use the convention that $\lambda_i^{\widetilde{\Psi}} = \infty$ for $i \le 0$. By considering the trace of $M_\Psi M_\Psi^*$, we see that the canonical dual frame is the unique dual frame that minimizes the inequalities in Λ. Therefore, the canonical dual is a minimizer among all duals for any matrix norm related to the spectrum of an operator. In general, it is only a unique minimizer if the matrix norm involves all singular values. Moreover, any other spectrum in Λ will not be associated with a unique dual frame, in particular, if $s_i = (s_{i,1}, \dots, s_{i,r}) \neq 0$ in M_Ψ for some $i = 1, \dots, n$, then replacing s_i by zs_i for any $|z| = 1$ will yield a dual frame with unchanged spectrum.

For a better understanding of the more general framework where Theorem II.2 does not yield a complete characterization of the possible spectral patterns, we will continue by a discussion of the example of a frame of three vectors in $\mathbb{R}^2$.

Example 1. Suppose Φ is a frame in $\mathbb{R}^2$ with 3 frame vectors and frame bounds $0 < A^\Phi \leq B^\Phi$, and let $\Phi = U\Sigma_\Phi V^*$ be the SVD of Φ. Then all dual frames are given as $\Psi = UM_\Psi V^*$, where

$$M_\Psi = \begin{bmatrix} 1/\sigma_1 & 0 & s_1 \\ 0 & 1/\sigma_2 & s_2 \end{bmatrix}$$

for $s_1, s_2 \in \mathbb{R}$. Since the frame operator of the dual frame is given by $S_\Psi = \Psi\Psi^* = UM_\Psi M_\Psi^* U^*$, we can find the eigenvalues of S_Ψ by considering eigenvalues of

$$S := M_\Psi M_\Psi^* = \begin{bmatrix} 1/\sigma_1^2 + s_1^2 & s_1 s_2 \\ s_1 s_2 & 1/\sigma_2^2 + s_2^2 \end{bmatrix}.$$

These are given by

$$\lambda_{1,2} = \frac{1}{2}\operatorname{tr} S \pm \frac{1}{2}R, \quad \text{where} \quad R = \sqrt{(\operatorname{tr} S)^2 - 4\det S}.$$

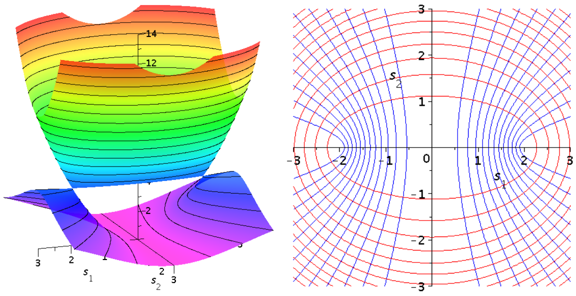
One easily sees that $\operatorname{tr} S$ monotonically grows as a function of $s_1^2 + s_2^2$, whereas for fixed $\operatorname{tr} S$, the term R grows as a function of $s_1^2 - s_2^2$. This exactly yields the two degrees of freedom predicted by the existence part of Theorem II.4. A straightforward calculation shows that $R + (s_1^2 + s_2^2) \geq \frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2} \geq 0$, hence we see that

$$\lambda_1 \geq \tfrac{1}{\sigma_2^2} \quad \text{and} \quad \tfrac{1}{\sigma_2^2} \geq \lambda_2 \geq \tfrac{1}{\sigma_1^2}, \tag{11}$$

which is also the conclusion of the necessity part of Theorem II.4. We remark that the two eigenvalues depend only on quadratic terms of the form s_1^2 and s_2^2. Therefore, if s_1 and s_2 are non-zero, then the choices $(\pm s_1, \pm s_2)$ yield four different dual frames having the same eigenvalues. In this case the level sets of λ_1 as a function of (s_1, s_2) are origin-centered ellipses with major and minor axes in the s_1 and s_2 direction, respectively. Moreover, the semi-major axis is always greater than $s_0 := (\sigma_2^{-2} - \sigma_1^{-2})^{1/2}$. The level sets of λ_2 are origin-centered, East-West opening hyperbolas with semi-major axes greater than s_0. In Figure 1 the possible eigenvalues of the dual frame operator of the frame Φ defined by

$$\Phi = \frac{1}{50}\begin{bmatrix} 90 & -12 & -16 \\ 120 & 9 & 6 \end{bmatrix} \tag{12}$$

are shown as a function of the two parameters s_1 and s_2; Figure 1b shows the level sets and the four intersection points $(\pm s_1, \pm s_2)$ for each allowed spectrum in the interior of Λ. Note that the singular values are $\sigma_1 = 3$ and $\sigma_2 = 1/2$, hence $B^\Phi = 9$ and $A^\Phi = 1/4$.



(a) Graphs of λ_1 and λ_2 (b) Level curves of λ_1 and λ_2

Figure 1: The lower and upper frame bounds of dual frames Ψ (to Φ defined in (12)) as a function of s_1 and s_2. The two graphs in (a) meet at $s_1 = \pm\sqrt{35}/3$ and $s_2 = 0$ which correspond to tight dual frames.

When the difference between the singular values of Ψ goes to zero, the ellipses degenerate to a line segment (or even to a point if $\sigma_1 = \sigma_2$). The limiting case corresponds to tight dual frames so Theorem II.1(ii) applies, and we are forced to set $s_2 = 0$ to achieve row orthogonality of M_Ψ. We then need to pick s_1 such that the two row norms of M_Ψ are equal, thus

$$|s_1| = \sqrt{\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}} = \sqrt{\frac{1}{A^\Phi} - \frac{1}{B^\Phi}} = s_0,$$

which shows that the above lower bound for the semi-major axis is sharp.

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